ZERO INTEGRALS ON CIRCLES AND CHARACTERIZATIONS OF
HARMONIC AND ANALYTIC FUNCTIONS

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ABSTRACT. We determine the kernels of two circular Radon transforms of continuous functions on an annulus and use this to obtain a characterization of harmonic functions in the open unit disc which involves Poisson averages over circles computed at only one point of the disc and to obtain a version of Morera's theorem which involves only the circles which surround the origin.

1. INTRODUCTION

Suppose that \( f \) is a continuous function on the open unit disc \( \Delta \) in the complex plane. For each simple closed curve \( \Gamma \subset \Delta \) bounding a domain \( D \), \( 0 \in D \), let \( F_{\Gamma} \) be the function which is continuous on \( D \cup \Gamma \), harmonic in \( D \) and which coincides with \( f \) on \( \Gamma \). It is known that if \( F_{\Gamma}(0) = f(0) \) for each smooth curve \( \Gamma \) bounding a strictly convex domain then \( f \) is harmonic in \( \Delta \). This is a special case of the main result of [5].

The starting point of the present investigation was the fact that the function \( f \) is not necessarily harmonic if one assumes only that \( F_{\Gamma}(0) = f(0) \) for each circle \( \Gamma \subset \Delta \) surrounding the origin. In fact, for each \( k \in \mathbb{N} \) there is a function \( f \) of class \( \mathcal{C}^k \) on \( \Delta \) such that \( F_{\Gamma}(0) = f(0) \) for each circle \( \Gamma \subset \Delta \) surrounding the origin and which is not harmonic in \( \Delta \) ([5], see also §9).

One of the results of the present paper is that if \( F_{\Gamma}(0) = f(0) \) for each circle \( \Gamma \subset \Delta \) which surrounds the origin then \( f \) is harmonic in \( \Delta \) under the additional assumption that it is infinitely differentiable at the origin, that is, if for each \( n \in \mathbb{N} \) there is a polynomial \( P_n \) of degree \( n \) such that

\[
f(z) = P_n(z, \bar{z}) + Q_n(z) \quad (z \in \Delta)
\]

where \( \lim_{z \to 0} |z|^{-n} Q_n(z) = 0 \). In fact, to get harmonicity it is enough to assume only that \( F_{\Gamma}(0) = f(0) \) for each circle \( \Gamma \) belonging to a family which is only slightly larger than the family of circles in \( \Delta \) centered at the origin. Note that \( f \) is infinitely differentiable at the origin if it is of class \( \mathcal{C}^\infty \) in a neighbourhood of the origin, or more generally, if it belongs to \( \mathcal{C}^\infty(\{0\}) \), that is, if for each \( k \in \mathbb{N} \) there is a neighbourhood of the origin in which \( f \) is of class \( \mathcal{C}^k \).
This characterization of harmonic functions is a consequence of our first main result which describes the Fourier coefficients of the continuous functions on an annulus $\Omega = \{ \zeta \in C: R_1 < |\zeta| < R_2 \}$ which have zero average on each circle $\Gamma \subset \Omega$ that surrounds the origin. The second main result describes the Fourier coefficients of the continuous functions $f$ on $\Omega$ such that $\int_\Gamma f(z) \, dz = 0$ for each circle that surrounds the origin. Its consequence is a version of Morera’s theorem: If $f$ is continuous on $\Delta$ and infinitely differentiable at the origin and if $\int_\Gamma f(z) \, dz = 0$ for each circle $\Gamma \subset \Delta$ which surrounds the origin then $f$ is analytic in $\Delta$. Again, one cannot drop the smoothness requirement at the origin since for each $k \in N$ there is a function $f$ of class $\mathcal{C}^k$ on $\Delta$ such that $\int_\Gamma f(z) \, dz = 0$ for each circle $\Gamma \subset \Delta$ surrounding the origin which is not analytic in $\Delta$ ([3], see also §9).

2. THE MAIN RESULTS

Let $0 \leq R_1 < R_2$ and let $f$ be a continuous function on the annulus $\Omega = \{ \zeta \in C: R_1 < |\zeta| < R_2 \}$. For each $r$, $R_1 < r < R_2$, and for each $n \in Z$, let

$$f_n(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta}) \, d\theta.$$ 

Thus, for each $r$, $R_1 < r < R_2$, $\sum_{n=-\infty}^{\infty} f_n(r) e^{in\theta}$ is the Fourier series of $f(re^{i\theta})$.

Let $\Gamma = \{ \zeta \in C: |\zeta - z| = r \}$ and let $\epsilon > 0$. The $\epsilon$-neighbourhood of the circle $\Gamma$ is the family of all circles of the form $\{ \zeta \in C: |\zeta - w| = \rho \}$ with $|w - z| < \epsilon$ and $|r - \rho| < \epsilon$, $\rho > 0$. Let $\mathcal{F}$ be a family of circles in $\Omega$ which surround the origin such that $\mathcal{F}$ contains a neighbourhood of each circle $\Gamma \subset \Omega$ centered at the origin. Note that $\mathcal{F}$ may be only slightly larger that the family of all circles in $\Omega$ centered at the origin.

If $\Gamma = \{ a + \epsilon e^{i\theta} b: 0 \leq \theta < 2\pi \}$ and if $\int_0^{2\pi} f(a + \epsilon e^{i\theta} b) \, d\theta = 0$ then we will say that $f$ has zero average on $\Gamma$.

Theorem 1. Let $f$ be a continuous function on $\Omega$. The following are equivalent:

(i) $f$ has zero average on each circle $\Gamma \in \mathcal{F}$,

(ii) $f$ has zero average on each circle $\Gamma \subset \Omega$ which surrounds the origin,

(iii) $f_0(r) = 0$ ($R_1 < r < R_2$) and for each $n \in Z$, $n \neq 0$, there are numbers $a_{n0}, a_{n1}, \ldots, a_{n,|n|-1}$ such that

$$f_n(r) = r^{-|n|} (a_{n0} + a_{n1} r + \cdots + a_{n,|n|-1} r^{2(|n|-1)}) \quad (R_1 < r < R_2).$$

Theorem 2. Let $f$ be a continuous function on $\Omega$. The following are equivalent:

(i) $\int_\Gamma f(z) \, dz = 0$ for each circle $\Gamma \in \mathcal{F}$,

(ii) $\int_\Gamma f(z) \, dz = 0$ for each circle $\Gamma \subset \Omega$ which surrounds the origin,

(iii) $f_{-1}(r) = 0$ ($R_1 < r < R_2$) and for each $n \in Z$, $n \neq -1$, there are numbers $b_{n0}, b_{n1}, \ldots, b_{n,|n|+1}$ such that

$$f_n(r) = r^{-|n|} (b_{n0} + b_{n1} r^2 + \cdots + b_{n,|n|+1} r^{2(|n|+1)-1}) \quad (R_1 < r < R_2).$$
Let \( \mathcal{H} \) be a family of circles in \( \Delta \) which surround the origin such that \( \mathcal{H} \) contains a neighbourhood of each circle \( \Gamma \subset \Delta \) centered at the origin.

**Corollary 1.** Let \( f \) be a continuous function in \( \Delta \) which is infinitely differentiable at the origin. If \( F_{\Gamma}(0) = f(0) \) for each circle \( \Gamma \in \mathcal{H} \) then \( f \) is harmonic in \( \Delta \). In particular, if \( F_{\Gamma}(0) = f(0) \) for each circle \( \Gamma \subset \Delta \) surrounding the origin then \( f \) is harmonic in \( \Delta \).

**Corollary 2.** Let \( f \) be a continuous function in \( \Delta \) which is infinitely differentiable at the origin. If \( \int_{\Gamma} f(z) \, dz = 0 \) for each circle \( \Gamma \in \mathcal{H} \) then \( f \) is analytic in \( \Delta \). In particular, if \( \int_{\Gamma} f(z) \, dz = 0 \) for each circle \( \Gamma \subset \Delta \) surrounding the origin then \( f \) is analytic in \( \Delta \).

An easy consequence of Theorems 1 and 2 is the following support theorem for the circular Radon transforms \( f \mapsto \int_{\Gamma} f(z) \, ds \) and \( f \mapsto \int_{\Gamma} f(z) \, dz \) which probably has a simple direct proof:

**Corollary 3.** Let \( f \) be a continuous function on \( \Omega \) such that for each \( k \in \mathbb{N} \) the function \( z \mapsto (\mathbb{R}^2 - |z|)^{-k} f(z) \) is bounded as \( |z| \to \mathbb{R}^2 \). If \( f \) has zero average on each circle \( \Gamma \in \mathcal{G} \) then \( f \) vanishes identically on \( \Omega \). If \( \int_{\Gamma} f(z) \, dz = 0 \) for each circle \( \Gamma \in \mathcal{G} \) then \( f \) vanishes identically on \( \Omega \).

The paper is organized as follows. We first prove that Corollaries 1–3 follow from Theorems 1 and 2 (§3). Then we show that \( \int_{\Gamma} f(z) \, ds = 0 \) or \( \int_{\Gamma} f(z) \, dz = 0 \) for each circle \( \Gamma \in \Omega \) surrounding the origin if and only if the same holds for each function \( re^{i\theta} \mapsto f_n(r)e^{in\theta} \) (§4). This happens if and only if in each case \( f_n \) satisfies a Volterra integral equation of the first kind whose kernel has a weak singularity on the diagonal (§5). We look at the properties of these equations and iterate the kernels to get equations for the functions \( r \mapsto f_n(r)(r_0 - r)^{-1/2} \) with analytic kernels having zeros of order \( |n| \) and \( |n + 1| \) on the diagonal (§6). Since only the structure of bounded solutions of such equations has been studied in detail [9, 12] with only a remark being made in [9] about the general case we revisit [9] to show that the approximation procedure used there for bounded analytic solutions can be used also for unbounded smooth solutions (§§7, 8). We then present examples of functions satisfying \( \int_{\Gamma} f(z) \, ds = 0 \), \( \int_{\Gamma} f(z) \, dz = 0 \) and show that using these examples one gets all solutions of the original integral equations and thus complete the proofs of Theorems 1 and 2 (§9).

### 3. Proofs of the corollaries

**Proposition 1.** Let \( D \subset C \) be an open disc, \( 0 \in D \), and let \( \Gamma \) be its boundary. Assume that \( F \) is a continuous function on \( D \cup \Gamma \) which is harmonic in \( D \). Then \( F(0) = 0 \) if and only if the function \( z \mapsto F(z)/|z|^2 \) has zero average on \( \Gamma \).
Proof. Let \( \Gamma = \{ w + Re^{i\theta} : 0 \leq \theta < 2\pi \} \), \( w = re^{i\alpha} \). If \( 0 \leq \rho < 1 \) and \( 0 \leq \varphi < 2\pi \) then the Poisson formula gives
\[
F(w + \rho Re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{F(w + Re^{i\theta})(1 - \rho^2)}{1 - 2\rho \cos(t - \varphi) + \rho^2} d\theta .
\]
Putting \( \rho = r/R \) and \( \varphi = \alpha + \pi \) we get
\[
F(0) = F(re^{i\alpha} + (r/R)Re^{i(\alpha + \pi)}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{F(re^{i\alpha} + Re^{i\theta})(1 - \rho^2)}{1 - 2(r/R) \cos(t - \alpha - \pi) + r^2/R^2} d\theta
\]
\[
= \left( R^2 - r^2 \right) \frac{1}{2\pi} \int_0^{2\pi} \frac{F(re^{i\alpha} + Re^{i\theta})}{|re^{i\alpha} + Re^{i\theta}|^2} d\theta .
\]
This completes the proof.

Proof of Corollary 1, assuming Theorem 1. Suppose that \( f \) is a continuous function on \( \Delta \) which is infinitely differentiable at the origin and which satisfies \( F_\Gamma(0) = f(0) \) for each circle \( \Gamma \in \mathcal{H} \). It is enough to prove that \( f \) is harmonic in \( R\Delta \) for each \( R < 1 \) so we may assume without loss of generality that \( f \) extends continuously to the closure \( \bar{\Delta} \) of the unit disc.

Let \( g \) be the function which is continuous on \( \bar{\Delta} \), harmonic in \( \Delta \) and which coincides with \( f \) on the unit circle \( b\Delta \) and put \( h = f - g \). The function \( h \) is continuous on \( \bar{\Delta} \) and vanishes identically on \( b\Delta \). Since \( g \) is harmonic in \( \Delta \) it follows that \( h \) is infinitely differentiable at the origin and that \( H_\Gamma(0) = 0 \) for each circle \( \Gamma \in \mathcal{H} \). Proposition 1 now implies that the function \( w(z) = h(z)/|z|^2 \) has zero average on each circle \( \Gamma \in \mathcal{H} \). By Theorem 1 it follows that \( w_0(r) = 0 \) \((0 < r < 1)\) and that for each \( n \in \mathbb{Z}, n \neq 0 \), there are numbers \( a_{ni}, 0 \leq i \leq |n| - 1 \), such that
\[
w_n(r) = r^{-|n|}(a_{n0} + a_{n1}r^2 + \cdots + a_{n,|n|-1}r^{2(|n|-1)}) \quad (0 < r < 1) .
\]
Since \( h_n(r) = r^2w_n(r) \) \((0 < r < 1)\) it follows that
\[
r^{-|n|}h_n(r) = r^{-2|n|}(a_{n0}r^2 + a_{n1}r^4 + \cdots + a_{n,|n|-1}r^{2|n|}) \quad (0 < r < 1) .
\]
Fix \( n \in \mathbb{Z}, n \neq 0 \). Since \( h \) is infinitely differentiable at the origin there is a polynomial \( P_{|n|} \) of degree \( |n| \) such that \( h(z) = P_{|n|}(z, z) + Q_{|n|}(z) \) where \( \lim_{z \to 0} |z|^{-|n|}Q_{|n|}(z) = 0 \). This implies that there is a number \( \alpha_n \) such that
\[
h_n(r) = \alpha_n r^{|n|} + \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} Q_{|n|}(re^{i\theta}) d\theta
\]
and
\[
\lim_{r \to 0} r^{-|n|}h_n(r) = \alpha_n .
\]
Thus \( a_{nj} = 0 \) \((0 \leq j \leq |n| - 2)\) and \( h_n(r) = a_{n,|n|-1}r^{|n|} \) \((0 < r < 1)\). Since \( h \) vanishes identically on \( b\Delta \) it follows that \( h_n(1) = 0 \) which implies that
\( h_n(r) = 0 \) \((0 < r \leq 1)\). As this holds for each \( n \in \mathbb{Z} \) it follows that \( h \) vanishes identically on \( \Delta \), that is, \( f \) coincides with \( g \) on \( \Delta \) which implies that \( f \) is harmonic in \( \Delta \). This completes the proof.

**Proof of Corollary 2, assuming Theorem 2.** Suppose that \( f \) is a continuous function in \( \Delta \) which is infinitely differentiable at the origin and which satisfies \( \int_{\Gamma} f(z) \, dz = 0 \) for each circle \( \Gamma \in \mathcal{H} \). By Theorem 2, \( f_n(r) = 0 \) \((0 < r < 1)\) and for each \( n \in \mathbb{Z}, n \neq -1 \), there are numbers \( b_n, 0 \leq i \leq |n+1|-1 \), such that

\[
 f_n(r) = r^{-|n|}(b_{n0} + b_{n1}r^2 + \cdots + b_{n,|n+1|-1}r^{2(|n+1|-1)}) \quad (0 < r < 1).
\]

As in the proof of Corollary 1, since \( f \) is infinitely differentiable at the origin it follows that for each \( n \in \mathbb{Z}, n \neq 0 \), \( \lim_{r \to 0} r^{-|n|} f_n(r) \), exists, that is,

\[
 \lim_{r \to 0}(b_{n0}r^{-2|n|} + b_{n1}r^{-2|n|+2} + \cdots + b_{n,|n+1|-1}r^{-2|n|+2|n+1|-1})
\]

exists. It follows that if \( n < -1 \) then \( f_n(r) = 0 \) \((0 < r < 1)\) and if \( n \geq 0 \) then \( f_n(r) = b_{n,|n+1|-1}r^n \) \((0 < r < 1)\). Now, for each \( r, 0 < r < 1 \), \( \sum_{k=\infty} f_n(r)e^{i\theta k} \) is the Fourier series of \( f(re^{i\theta}) \) so on \([0, 2\pi]\) the function \( f(re^{i\theta}) \) is the uniform limit of its Cesàro means

\[
 \sigma_m(re^{i\theta}) = m^{-1} \left( f_0(r) + \sum_{k=-1}^{m} f_k(r)e^{i\theta k} + \cdots + \sum_{k=-(m-1)}^{m-1} f_k(r)e^{i\theta k} \right)
\]

\[
 = m^{-1} \left( b_{00} + \sum_{k=0}^{m-1} b_{k,k}(re^{i\theta})^k + \cdots + \sum_{k=0}^{m-1} b_{k,k}(re^{i\theta})^k \right).
\]

The usual proof of the Fejér theorem [8] shows that the convergence is also uniform in \( r, \rho_1 \leq r \leq \rho_2 \), for each \( \rho_1, \rho_2, 0 < \rho_1 < \rho_2 < 1 \), since \( f \) is uniformly continuous in \( \{\zeta: \rho_1 \leq |\zeta| \leq \rho_2\} \) [3]. Since each \( \sigma_m \) is analytic in \( \Delta \) it follows that \( f \) is analytic in \( \Delta \setminus \{0\} \) and being continuous at 0, \( f \) is analytic in \( \Delta \). This completes the proof.

**Remark.** The referee has kindly pointed out that if \( f \in \mathcal{C}^{\infty}({\{0}\}) \) then the second part of Corollary 2 can be easily derived from the following consequence of an old result of A. M. Cormack: If \( g \in \mathcal{C}^{\infty}(\Delta) \) has zero average on each circle \( \Gamma \subset \Delta \) which passes through the origin then \( g \) vanishes identically (for the proof see [1]). The reason is that once \( f \in \mathcal{C}^{\infty}({\{0}\}) \) then it can be uniformly approximated by \( \mathcal{C}^{\infty} \) functions with the same vanishing properties as \( f \). Let \( f \in \mathcal{C}^{\infty}(\Delta) \) and let \( \int_{\Gamma} f(z) \, dz = 0 \) for each circle \( \Gamma \) surrounding the origin. By continuity we have that \( \int_{bD} f(z) \, dz = 0 \) for every disc \( D \subset \Delta, 0 \in bD \). By Green’s formula, \( \int_D \partial f / \partial z \, dz \wedge dz = 0 \). It is easy to conclude that \( \int_{bD} \partial f / \partial z \, ds = 0 \) for all such discs \( D \). It follows that \( \partial f / \partial z \) vanishes identically so \( f \) is analytic on \( \Delta \).

**Proof of Corollary 3, assuming Theorems 1 and 2.** The assumption implies that for each \( n \in \mathbb{Z}, k \in \mathbb{N} \), the function \( r \mapsto (R_2 - r)^{-k} f_n(r) \) is bounded as
$r \to R_2$. If $f$ has zero average on each circle $\Gamma \in \mathcal{C}$ then by Theorem 1 each $f_n$ extends to an analytic function in a neighbourhood of $(0, R_2)$ which is possible only if $f_n$ vanishes identically so $f$ vanishes identically in $\Omega$. In the same way, using Theorem 2, we see that if $\int_{\Gamma} f(z) \, dz = 0$ for each $\Gamma \in \mathcal{C}$ then $f$ vanishes identically in $\Omega$. This completes the proof.

4. Zero integrals and Fourier coefficients

Let $\Gamma \subset C$ be a circle which surrounds the origin and whose center is not the origin. Let $A$ be the closed annulus obtained by rotating $\Gamma$ around the origin, that is,

$$A = \bigcup_{0 \leq \alpha < 2\pi} e^{i\alpha} \Gamma.$$

Let $R_1$, $R_2$ be such that $A = \{\zeta \in C : R_1 \leq |\zeta| \leq R_2\}$. Let $f$ be a continuous function on $A$. For each $n \in \mathbb{Z}$, let

$$\Phi_n(z) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-in\theta} f(e^{i\theta}z) \, d\theta \quad (R_1 \leq |z| \leq R_2).$$

Note that if $z = re^{i\alpha}$ then $\Phi_n(z) = f_n(r)e^{in\alpha}$.

**Lemma 1.** The following are equivalent:

1. $f$ has zero average on each circle $e^{i\alpha} \Gamma$, $0 \leq \alpha < 2\pi$,
2. for each $n \in \mathbb{Z}$ the function $\Phi_n$ has zero average on $\Gamma$.

**Proof.** Suppose that (i) holds. Let $\Gamma = \{a + e^{i\theta}b : 0 \leq \theta < 2\pi\}$. Let $n \in \mathbb{Z}$. By the assumption, $\int_{0}^{2\pi} f(e^{i\alpha}(a + e^{i\theta}b)) \, d\theta = 0 \quad (0 \leq \alpha < 2\pi)$ so by the Fubini theorem

$$\int_{0}^{2\pi} \left[ \frac{1}{2\pi} \int_{0}^{2\pi} e^{-in\alpha} f(e^{i\alpha}(a + e^{i\theta}b)) \, d\alpha \right] \, d\theta = 0$$

which proves (ii). Conversely, assume that (ii) holds. Note first that (ii) implies that for each $n \in \mathbb{Z}$ the function $\Phi_n$ has zero average on each circle $e^{i\alpha} \Gamma$, $0 \leq \alpha < 2\pi$. For each $r$, $R_1 \leq r \leq R_2$, $\sum_{-\infty}^{\infty} f_n(r)e^{in\theta} = \sum_{-\infty}^{\infty} \Phi_n(re^{i\theta})$ is the Fourier series of $f(re^{i\theta})$. As in the proof of Corollary 2 we see that its Cesàro means

$$\sigma_m(re^{i\theta}) = m^{-1} \left( \Phi_0(re^{i\theta}) + \sum_{k=-1}^{1} \Phi_k(re^{i\theta}) + \cdots + \sum_{k=-(m-1)}^{m-1} \Phi_k(re^{i\theta}) \right)$$

converge to $f(re^{i\theta})$ uniformly on $A$ which implies (i). This completes the proof.

In almost the same way we prove the following lemma.

**Lemma 1'**. The following are equivalent:

1. $\int_{\Gamma} f(z) \, dz = 0$ for each $\alpha$, $0 \leq \alpha < 2\pi$,
2. $\int_{\Gamma} \Phi_n(z) \, dz = 0$ for each $n \in \mathbb{Z}$.
5. The integral equations for the Fourier coefficients

For each \( \rho, R, 0 < \rho < R \), let \( \Gamma_{\rho R} \) be the circle whose diameter is the segment \([-R, \rho]\) on the real axis. Further, for each \( n \in \mathbb{N} \cup \{0\} \) let \( T_n \) be the Čebyšev polynomial of the first kind of degree \( n \) [10]. We have \( T_n(\cos \alpha) = \cos n\alpha \) for each \( \alpha \).

Lemma 2. Let \( n \in \mathbb{Z} \) and let \( f_n \) be a continuous function on \([\rho, R]\). Let \( \Phi_n(re^{i\theta}) = f_n(r)e^{in\theta} \) \( (\rho \leq r \leq R, \ 0 \leq \theta < 2\pi) \) and let \( s = \rho/R \). Write

\[
P_n(s, t) = T_{|n|}\left(\frac{s-t^2}{t(1-s)}\right)(t^2-s^2)^{-1/2}(1-t^2)^{-1/2}t \quad (s < t < 1),
\]

\[
Q_n(s, t) = \left[2tT_{|n+1|}\left(\frac{s-t^2}{t(1-s)}\right) + (1-s)T_{|n|}\left(\frac{s-t^2}{t(1-s)}\right)\right]
\times (t^2-s^2)^{-1/2}(1-t^2)^{-1/2}t \quad (s < t < 1).
\]

Then

(i) \( \Phi_n \) has zero average on \( \Gamma_{\rho R} \) if and only if

\[
\int_0^1 f_n(Rt)P_n(s, t) \, dt = 0,
\]

(ii) \( \int_{\Gamma_{\rho R}} \Phi_n(z) \, dz = 0 \) if and only if

\[
\int_0^1 f_n(Rt)Q_n(s, t) \, dt = 0.
\]

Proof. In polar coordinates \( r \), the circle \( \Gamma_{\rho R} \) is given by the equation

\[
\cos \phi = \frac{R\rho - r^2}{r(R-\rho)}.
\]

Differentiating this we get

\[
\frac{d\phi}{dr} \sin \phi = \frac{R\rho + r^2}{(R-\rho)r^2} \quad (\rho < r < R).
\]

If \( 0 < \phi < \pi \) then (5.1) gives

\[
\sin \phi = (1 - \cos^2 \phi)^{1/2} = (r^2 - \rho^2)^{1/2}(R^2 - r^2)^{1/2}r^{-1}(R-\rho)^{-1}
\]

so

\[
\frac{d\phi}{dr} = \frac{R\rho + r^2}{r(r^2 - \rho^2)^{1/2}(R^2 - r^2)^{1/2}} \quad (\rho < r < R).
\]

Similarly, if \( -\pi < \phi < 0 \),

\[
\frac{d\phi}{dr} = -\frac{R\rho + r^2}{r(r^2 - \rho^2)^{1/2}(R^2 - r^2)^{1/2}} \quad (\rho < r < R).
\]
In both cases we have
\[ \left(1 + \left(r \frac{d \varphi}{dr}\right)^2\right)^{1/2} = r(R + \rho)(r^2 - \rho^2)^{-1/2}(R^2 - r^2)^{-1/2} \quad (\rho < r < R). \]

Now, writing
\[ \varphi(r) = \arccos \frac{R\rho - r^2}{r(R - \rho)} \quad (\rho < r < R) \]
we have \( \Gamma_{\rho R} = \{re^{i\varphi(r)} : \rho \leq r \leq R\} \cup \{re^{-i\varphi(r)} : \rho \leq r \leq R\} \) so if \( \Psi \) is a continuous function on \( \Gamma_{\rho R} \) its average on \( \Gamma_{\rho R} \) is
\[ \pi^{-1} \int_{\rho}^{R} [\Psi(re^{i\varphi(r)}) + \Psi(re^{-i\varphi(r)})](r^2 - \rho^2)^{-1/2}(R^2 - r^2)^{-1/2} r \, dr. \]

To prove (i), let \( \Psi = \Phi_n \) and put \( r = tR \). To prove (ii), observe that \( \int_{\Gamma_{\rho R}} \Phi_n(z) \, dz = 0 \) if and only if the average of \( \Psi(z) = \Phi_n(z)(z - (\rho - R)/2) \) on \( \Gamma_{\rho R} \) is zero and put \( r = tR \). This completes the proof.

**Lemma 3.** Let \( n \in \mathbb{Z}, \) let \( 0 < R_1 < R_2 \) and assume that \( f_n \) is a continuous function on \([R_1, R_2]\). Let \( \Psi_n(re^{i\vartheta}) = f_n(r)e^{in\vartheta} \) \( (R_1 \leq r \leq R_2, \ 0 \leq \vartheta < 2\pi) \). Let \( R_1/R_2 < q < 1 \).

(i) If for each \( \rho, R, R_1 \leq \rho < R \leq R_2 \), the function \( \Psi_n \) has zero average on \( \Gamma_{\rho R} \) then on \([q, 1]\) the function \( t \mapsto f_n(tR_2) \) can be uniformly approximated by functions \( \Phi \) of class \( \mathcal{C}^\infty \) which satisfy

\[ \int_{s}^{1} \Phi(t)P_n(s, t) \, dt = 0 \quad (q \leq s < 1). \]

(ii) If for each \( \rho, R, R_1 \leq \rho < R \leq R_2 \), \( \int_{\Gamma_{\rho R}} \Psi_n(z) \, dz = 0 \), then on \([q, 1]\) the function \( t \mapsto f_n(tR_2) \) can be uniformly approximated by functions \( \Phi \) of class \( \mathcal{C}^\infty \) which satisfy

\[ \int_{s}^{1} \Phi(t)Q_n(s, t) \, dt = 0 \quad (q \leq s < 1). \]

**Proof.** Suppose that for each \( \rho, R, R_1 \leq \rho < R \leq R_2 \), \( \Psi_n \) has zero average on \( \Gamma_{\rho R} \). By Lemma 2
\[ \int_{s}^{1} f_n(tR)P_n(s, t) \, dt = 0 \quad (R_1/R \leq s < 1) \]
holds for each \( R, R_1 < R \leq R_2 \). In particular, if \( R_1 < R_0 < R_2 \) then for each \( R, R_0 < R < R_2 \), the function \( t \mapsto \Phi(t) = f_n(tR) \) satisfies

\[ \int_{s}^{1} \Phi(t)P_n(s, t) \, dt = 0 \quad (R_1/R_0 < s < 1). \]

Choose \( R_0 < R_2 \) so close to \( R_2 \) that \( R_1/R_0 < q \) and choose \( \delta > 0 \) so small that \((1 - \delta)R_2 < R_0 \).
Let $\chi$ be a nonnegative $C^\infty$ function on $R$ with support in $(1 - \delta, 1)$ and such that $\int \chi(\omega) \, d\omega = 1$. Since each function $t \mapsto f_n(\omega t R_n)$, $1 - \delta < \omega < 1$, satisfies (5.4) it follows by the Fubini theorem that also the function $t \mapsto f(\omega t R) \, d\omega$ satisfies (5.4). The function $\Phi$ is of class $C^\infty$ on $R \setminus \{0\}$ and if $\delta$ is chosen small enough then $|\Phi(t) - f_n(t R)|$ will be uniformly small on $[R_1/R_0, 1]$. This completes the proof of (i). We prove (ii) in the same way with $P_n$ replaced by $Q_n$.

6. Properties of the integral equations

To prove Theorems 1 and 2 we will need all smooth solutions of (5.2) and (5.3). We first mention two trivial special cases:

**Proposition 2.** Let $0 < R_1 < R_2$ and assume that $\Phi$ is a continuous function on $(R_1, R_2)$. If $\Psi(r e^{i\theta}) = \Phi(r)$ $(R_1 < r < R_2$, $0 \leq \theta < 2\pi)$ and if $\Psi$ has zero average on each circle $|z| = r$, $R_1 < r < R_2$, then $\Phi(r) = 0$ $(R_1 < r < R_2)$. If $\Psi(r e^{i\theta}) = \Phi(r) e^{-i\theta}$ $(R_1 < r < R_2$, $0 \leq \theta < 2\pi)$ and if $\int_{|z|=r} \Psi(z) \, dz = 0$ for each $r$, $R_1 < r < R_2$, then $\Phi(r) = 0$ $(R_1 < r < R_2)$.

This shows that we will only have to consider (5.2) if $n \neq 0$ and (5.3) if $n \neq -1$.

Let $P_n$ and $Q_n$ be as in Lemma 3.

**Lemma 4.** Let $0 < \tau < 1$ and let $\Phi$ be a continuous function on $[1 - \tau, 1]$.

(i) If $n \in \mathbb{Z}$, $n \neq 0$, and if $\Phi$ satisfies

\[(6.1) \quad \int_{s}^{1} \Phi(t) P_n(s, t) \, dt = 0 \quad (1 - \tau < s < 1),\]

then

\[\int_{0}^{p} K_n(p, t) g(t) \, dt = 0 \quad (0 < p < \tau)\]

where $g(t) = h(1 - t)$, $h(t) = t^{-|n|+1}(1 - t^2)^{-1/2} \Phi(t)$ and where $K_n$ is analytic in a neighbourhood of zero and is of the form

\[K_n(p, t) = \sum_{i=0}^{|n|} b_i t^{|n| - i} p^i + \text{higher order terms in } t, p\]

where $\sum_{i=0}^{|n|} b_i \neq 0$.

(ii) If $n \in \mathbb{Z}$, $n \neq -1$, and if $\Phi$ satisfies

\[(6.2) \quad \int_{s}^{1} \Phi(t) Q_n(s, t) \, dt = 0 \quad (1 - \tau < s < 1),\]

then

\[\int_{0}^{p} L_n(p, t) g(t) \, dt = 0 \quad (0 < p < \tau)\]
where $g(t) = h(1 - t)$, $h(t) = t^{-(n+1)+1}(1-t^2)^{-1/2}\Phi(t)$ and where $L_n$ is analytic in a neighbourhood of zero and is of the form

$$L_n(p, t) = \sum_{i=0}^{[n+1]} c_i t^{[n+1]-i} p^i + \text{higher order terms in } t, p$$

where $\sum_{i=0}^{[n+1]} c_i \neq 0$.

**Proof.** Let $n \in \mathbb{Z}, n \neq 0$, and let $\Phi$ satisfy (6.1). Let $g$ be as in (i). Multiplying (6.1) by $(1-s)^n$ we get

$$(6.3) \quad \int_0^1 L(s, t)h(t)(t-s)^{-1/2} dt = 0 \quad (1 - \tau < s < 1)$$

where

$$L(s, t) = (t+s)^{-1/2}\sum_{i=0}^{[n]} p_i(s - t^2)^{[n]-i} (t(1-s))^i$$

and where $T_{[n]}(x) = \sum_{i=0}^{[n]} p_i x^{[n]-i}$. Put $Q(s, t) = L(1-s, 1-t)$ and replace $t$ by $1-t$ and $s$ by $1-s$ in (6.3) to get

$$(6.4) \quad \int_0^1 Q(s, t)g(t)(t-s)^{-1/2} dt = 0 \quad (0 < s < \tau)$$

where

$$Q(s, t) = 2^{-1/2}\sum_{i=0}^{[n]} p_i (2t-s)^{[n]-i} s^i + \text{higher order terms in } s, t.$$ 

Since $T_{[n]}(\cos x) = \cos nx$ we have $\sum_{i=0}^{[n]} p_i = 1$ so $Q(s, s) = 2^{-1/2} s^{[n]} + \text{higher order terms in } s$. Thus we have shown that

$$Q(s, t) = \sum_{i=0}^{[n]} a_i t^{[n]-i} s^i + \text{higher order terms in } s, t,$$

where $\sum_{i=0}^{[n]} a_i \neq 0$. One completes the proof of (i) by iterating the kernel as in [9, p. 155].

To prove (ii) let first $n \geq 0$. Let

$$T_n(x) = \sum_{i=0}^{n} p_i x^{n-i}, \quad T_{n+1}(x) = \sum_{i=0}^{n+1} q_i x^{n+1-i}.$$

Multiplying (6.2) by $(1-s)^{n+1}$ we get (6.3) where

$$L(s, t) = (t+s)^{-1/2}\left[2t\sum_{i=0}^{n+1} q_i(s - t^2)^i (t(1-s))^{n+1-i} \right. \left. + t(1-s)^2 \sum_{i=0}^{n} p_i(s - t^2)^i (t(1-s))^{n-i}\right].$$
As before, this implies (6.4) where

\[ Q(s, t) = 2^{-1/2} \left[ 2 \sum_{i=0}^{n+1} q_i (2t - s)^i s^{n+1-i} + \text{higher order terms in } s, t \right] \]
\[ + 2^{-1/2} \left[ \sum_{i=0}^{n} p_i (2t - s)^i s^{n-i} + \text{higher order terms in } s, t \right] \]
\[ = 2^{1/2} \sum_{i=0}^{n+1} q_i (2t - s)^i s^{n+1-i} + \text{higher order terms in } s, t . \]

Again, \[ Q(s, s) = 2^{1/2} s^{n+1} + \text{higher order terms in } s , \] so

\[ Q(s, t) = \sum_{i=0}^{n+1} a_i t^{n+1-i} s^i + \text{higher order terms in } s, t \]

where \( \sum_{i=0}^{n+1} a_i \neq 0 \). Together with [9, p. 155] this completes the proof of (ii) if \( n \geq 0 \). Now, let \( n \leq -2 \). Write \( n = -m \). Since \( T_m(x) = 2x T_{m-1}(x) - T_{m-2}(x) \) [10] we have

\[
2 t T_{m-1} \left( \frac{s - t^2}{t(1-s)} \right) + (1-s) T_m \left( \frac{s - t^2}{t(1-s)} \right) 
\]
\[
= 2 t \sum_{i=0}^{m-1} p_i (s-t^2)^{m-1-i} (t(1-s))^i 
- (1-s)^2 t \sum_{i=0}^{m-2} q_i (s-t^2)^{m-2-i} (t(1-s))^i 
\]

so multiplying (6.2) by \( (1-s)^{m-1} \) we get (6.3) where

\[ L(s, t) = (t + s)^{-1/2} \left[ 2^{s-1} \sum_{i=0}^{m-1} p_i (s-t^2)^{m-1-i} (t(1-s))^i \right] 
- (1-s)^2 t \sum_{i=0}^{m-2} q_i (s-t^2)^{m-2-i} (t(1-s))^i \]

and where \( T_{m-1}(x) = \sum_{i=0}^{m-1} p_i x^{m-1-i} \) and \( T_{m-2}(x) = \sum_{i=0}^{m-2} q_i x^{m-2-i} \). As before, (6.1) follows where

\[ Q(s, t) = 2^{1/2} \sum_{i=0}^{m-1} p_i (2t - s)^{m-1-i} s^i + \text{higher order terms in } s, t . \]

Again, \[ Q(s, s) = 2^{1/2} s^{m-1} + \text{higher order terms in } s , \] so

\[ Q(s, t) = \sum_{i=0}^{m-1} a_i t^{m-1-i} s^i + \text{higher order terms in } s, t \]

where \( \sum_{i=0}^{m-1} a_i \neq 0 \). One completes the proof of (ii) as in [9, p. 155].
7. The integro-differential equation

Fix $n \in \mathbb{N}$ and let $K$ be analytic in a neighbourhood of zero and of the form

$$K(x,s) = \sum_{i=0}^{n} \beta_i s^{n-i} x^i + \text{higher order terms in } s, x$$

where $\sum_{i=0}^{n} \beta_i \neq 0$. For the proofs of Theorems 1 and 2 we have to prove that if $\tau > 0$ is small then the dimension of the space of all smooth solutions $g$ of the equation

$$\int_{0}^{x} K(x,s) g(s) \, ds = 0 \quad (0 < x < \tau) \tag{7.1}$$

such that $s^{1/2} g(s)$ is bounded on $(0, \tau)$, does not exceed $n$. For continuous functions $g$ on $[0, \tau)$ this follows from [12]. For bounded analytic functions $g$ this also follows from [9] where also a remark about the unbounded case was made. We follow [9] to show that the approximation procedure used there works also in our case.

Write $K(s,x) = \sum_{i=0}^{\infty} a_i(s)(x-s)^i/i!$ where $a_i$ are analytic in a neighbourhood of zero and where the series converges in a neighbourhood of zero. By the properties of $K$,

$$(a_0 \text{ has zero of order } n \text{ at the origin and each } a_i, 1 \leq i \leq n-1, \begin{cases} 
\text{has zero of order at least } n-i \text{ at the origin.} 
\end{cases}) \tag{7.2}$$

Suppose that $g$ is a smooth solution of (7.1) such that

$$|g(s)| \leq C s^{-1/2} \quad (0 < s < \tau) \tag{7.3}$$

for some constant $C$. Since $g$ is smooth on $(0, \tau)$ and satisfies (7.1) and (7.3) one can differentiate (7.1) $n+1$ times to see that $g$ satisfies

$$(Dg)(x) = \int_{0}^{x} K_n(x,s) g(s) \, ds \quad (0 < x < \tau) \tag{7.4}$$

where $K_n$ is analytic in a neighbourhood of zero and where

$$Dg = (a_0 g)^{(n)} + (a_1 g)^{(n-1)} + \cdots + (a_n g).$$

By (7.2) zero is a singular point of $D$ which is of Fuchsian type, that is, it is a regular singular point of $D$.

**Lemma 5.** There are $k$, $0 \leq k \leq n$, complex numbers $r_i$, $1 \leq i \leq n$, $\Re r_i > 0$ $(1 \leq i \leq k)$, $\Re r_i \leq 0$ $(k + 1 \leq i \leq n)$, and functions $\Omega_i$, $H_i$, of the form

$$\Omega_i(x) = \sum_{j=0}^{k_i} \Omega_{ij}(x)(\log x)^j, \quad H_i(x) = \sum_{j=0}^{k_i} H_{ij}(x)(\log x)^j \tag{7.5}$$

where each $\Omega_{ij}$ and each $H_{ij}$ is analytic in a neighbourhood of zero such that for small $\tau > 0$ the following holds:
If \( \Psi \) is a continuous function on \([0, \tau]\) such that \( |\Psi(x)| \leq Cx^n \) \((0 < x < \tau)\) then

\[
y(x) = \sum_{i=1}^{k} \Omega_i(x)x^{r_i} \int_{\tau/2}^{x} t^{-r_i-1}H_i(t)\Psi(t) \, dt
\]

\[
+ \sum_{i=k+1}^{n} \Omega_i(x)x^{r_i} \int_{0}^{x} t^{-r_i-1}H_i(t)\Psi(t) \, dt
\]

satisfies \( (Dy)(x) = \Psi(x) \) \((0 < x < \tau)\).

**Proof.** Each nonzero solution of \( Dy = 0 \) has the form

\[
y(x) = x' \sum_{j=0}^{m} \Phi_j(x)(\log x)^j
\]

where each \( \Phi_j \) is analytic in a neighborhood of zero and at least one of the numbers \( \Phi_j(0), \ 0 \leq j \leq m \), is different from zero [2]. Choose a fundamental system \( y_i(x) = x'^{r_i} \Omega_i(x), \ 1 \leq i \leq n \), where \( \Omega_i \) are as in (7.5). The Wronskian has the form \( W(x) = x'^{r_i+\cdots+r_n-n(n-1)/2}R(x) \) where \( R \) is analytic in a neighborhood of zero and satisfies \( R(0) \neq 0 \) [2, p. 77]. We complete the proof by using the variation of constants.

**8. Successive approximations and the dimension of the space of solutions**

We keep the notation from §7. Define the operator

\[
(L_n \phi)(x) = \sum_{i=1}^{k} \Omega_i(x)x^{r_i} \int_{\tau/2}^{x} t^{-r_i-1}H_i(t) \left[ \int_{0}^{t} K_n(t,s)\phi(s) \, ds \right] \, dt
\]

\[
+ \sum_{i=k+1}^{n} \Omega_i(x)x^{r_i} \int_{0}^{x} t^{-r_i-1}H_i(t) \left[ \int_{0}^{t} K_n(t,s)\phi(s) \, ds \right] \, dt.
\]

If \( \tau > 0 \) is small and if \( \phi \) is smooth on \((0, \tau)\) and such that \( \phi(x)x^{1/2} \) is bounded on \((0, \tau)\) then \( L_n \phi \) is well defined and smooth on \((0, \tau)\). Further, if \( \tau \) is small then all the functions involved in the definition of \( L_n \) are analytic in \( \Sigma_{\tau} = \{ re^{i\alpha} : 0 < r < \tau, |\alpha| < \pi/4 \} \) so if \( \phi \) is analytic in \( \Sigma_{\tau} \) and such that \( \phi(x)x^{1/2} \) is bounded on \( \Sigma_{\tau} \) then \( L_n \phi \) is well defined and analytic in \( \Sigma_{\tau} \).

Using elementary estimates of the integrals we get

**Lemma 6.** There is a \( \tau_0 > 0 \) such that for each \( \tau, \ 0 < \tau < \tau_0 \), and for each positive constant \( C \) the following hold:

(i) if \( \phi \) is smooth on \((0, \tau)\) and if \( |\phi(x)| \leq Cx^{-1/2} \) \((0 < x < \tau)\) then \( L_n \phi \) is bounded on \((0, \tau)\)

(ii) if \( \phi \) is smooth on \((0, \tau)\) and if \( |\phi(x)| \leq C \) \((0 < x < \tau)\) then \(|(L_n \phi)(x)|\) \leq C/2 \((0 < x < \tau)\)
(iii) if \( \varphi \) is analytic in \( \Sigma_r \) and if \( |\varphi(x)| \leq C|x|^{-1/2} \) \((x \in \Sigma_r)\) then \( L_n \varphi \) is bounded in \( \Sigma_r \).

(iv) if \( \varphi \) is analytic in \( \Sigma_r \) and if \( |\varphi(x)| \leq C \) \((x \in \Sigma_r)\) then \( |(L_n \varphi)(x)| \leq C/2 \) \((x \in \Sigma_r)\).

Now we use successive approximations:

**Lemma 7.** If \( \tau > 0 \) is small enough then the following are equivalent (8.1)

(i) \( g \) is a smooth solution of (7.4) such that \( g(s)s^{1/2} \) is bounded on \((0, \tau)\),

(ii) \( g = g_0 + L_n g_0 + L_n^2 g_0 + \cdots \), where \( g_0 \) satisfies \( Dg_0 = 0 \) on \((0, \tau)\) and is such that \( g_0(x)x^{1/2} \) is bounded on \( \Sigma_r \) and where the series converges uniformly on \( \Sigma_r \).

**Proof.** Let \( \tau > 0 \) be so small that Lemma 6 holds and that in \( 2\tau \Delta \) every solution of \( Dv = 0 \) has the form \( x^r \sum_{j=0}^m P_j(x)(\log x)^j \) where \( P_j \), \( 0 \leq j \leq m \), are analytic in \( 2\tau \Delta \). Suppose that (i) holds. By Lemma 6, \( g_0 = g - L_n g \) is smooth on \((0, \tau)\) and such that \( g_0(s)s^{1/2} \) is bounded on \((0, \tau)\). By the definition of \( L_n \) we have

\[
(Dg_0)(x) = (Dg)(x) - (D(L_n g))(x)
\]

\[
= (Dg)(x) - \int_0^x K_n(x, s) g(s) \, ds = 0 \quad (0 < x < \tau)
\]

since \( g \) satisfies (7.4). Since \( g_0(x)x^{1/2} \) is bounded on \((0, \tau)\) it follows that it is bounded in \( \Sigma_r \). Since \( g_0 \) is analytic there Lemma 6 implies that the series (8.1) converges uniformly in \( \Sigma_r \) and since \( g_0 = g - L_n g \) it follows by Lemma 6(ii) that the sum of the series (8.1) is \( g \).

Conversely, suppose that (ii) holds. Write \( L_n g_0 = g_i \) so that \( g = \sum_{i=0}^\infty g_i \). By the uniform convergence of the series in \( \Sigma_r \) we have \( Dg = \sum_{i=0}^\infty Dg_i \) on \((0, \tau)\) and

\[
\int_0^x K_n(x, s) g(s) \, ds = \sum_{i=0}^\infty \int_0^x K_n(x, s) g_i(s) \, ds \quad (0 < x < \tau).
\]

Since for each \( i \),

\[
(Dg_{i+1})(x) = \int_0^x K_n(x, s) g_i(s) \, ds \quad (0 < x < \tau)
\]

it follows that \( g \) is a solution of (7.4). That \( g(s)s^{1/2} \) is bounded on \((0, \tau)\) follows from Lemma 6. This completes the proof.

**Lemma 8.** If \( \tau \) is small enough then the dimension of the space of all smooth solutions \( g \) of (7.4) such that \( g(s)s^{1/2} \) is bounded on \((0, \tau)\) does not exceed \( n \).

**Proof.** Let \( \tau > 0 \) be so small that Lemmas 6 and 7 hold. If \( g_0 \) is a solution of \( Dg_0 = 0 \) such that \( g_0(x)x^{1/2} \) is bounded in \( \Sigma_r \) then by Lemma 6 the series (8.1) converges uniformly in \( \Sigma_r \). Let \( y_1, y_2, \ldots, y_m \) be the basis of the space.
of all solutions of $Dy = 0$ for which $y(x)x^{1/2}$ is bounded on $\Sigma_x$. For each $i$, $1 \leq i \leq m$, let $\Phi_i = y_i + L_n y_i + \ldots$. By Lemma 7 a smooth function $g$ on $(0, \tau)$ such that $g(s)s^{1/2}$ is bounded on $(0, \tau)$ satisfies (7.4) if and only if it is a linear combination of the functions $\Phi_i$, $1 \leq i \leq m$. This completes the proof.

Now, using Lemma 4 and §7 we get the following consequence:

**Lemma 9.** Let $n \in N$, $n \neq 0$. There is some $q_0 < 1$ such that for each $q$, $q_0 < q < 1$, the dimension of the space of smooth functions $\Phi$ on $[q, 1]$ which satisfy (7.2) does not exceed $|n|$.

Let $n \in N$, $n \neq -1$. There is some $q_0 < 1$ such that for each $q$, $q_0 < q < 1$, the dimension of the space of smooth functions $\Phi$ on $[q, 1]$ which satisfy (5.3) does not exceed $|n + 1|$.

### 9. Examples and the Proofs of Theorems 1 and 2

**Proposition 3.** Let $n \in N$. Each of the functions $z \mapsto z^{n-k}z^{-k}$, $1 \leq k \leq n$, has zero average on each circle that surrounds the origin.

**Proof [5].** Fix $k$, $1 \leq k \leq n$, and let $\varphi(z) = z^{n-k}z^{-k}$. Let $0 \leq |a| < |b|$. Then

$$\int_0^{2\pi} \varphi(a + e^{i\theta}b) d\theta = \int_0^{2\pi} (a + e^{i\theta}b)^{n-k}(a + e^{-i\theta}b)^{-k} d\theta$$

$$= -i \int_0^{2\pi} (ae^{i\theta} + b)^{-k} (e^{i\theta})^{k-1} (a + e^{i\theta}b)^{n-k} ie^{i\theta} d\theta$$

$$= -i \int_{b\Delta} (aw + b)^{-k} w^{k-1} (a + bw)^{n-k} dw.$$  

Since $|b| > |a|$, since $k \geq 1$ and since $n - k \geq 0$ the integrand in the last integral is analytic in a neighbourhood of $\Delta$ so the last integral is zero. This completes the proof.

**Example [5].** If $n \in N$ then the function

$$\varphi(z) = \begin{cases} z^{-1}z^{n+2} & (z \neq 0), \\ 0 & (z = 0) \end{cases}$$

is of class $C^\infty$ on $C$. By Propositions 1 and 3 we have $\Phi_\Gamma(0) = \varphi(0)$ for every circle $\Gamma$ that surrounds the origin, yet $\varphi$ is not harmonic in $\Delta$.

**Proposition 4.** Let $0 < a < b \leq 1$ and let $n \in N$. The uniform limit on $(a, b)$ of a sequence of polynomials of the form $a_0 + a_1x^2 + \ldots + a_nx^{2n}$ is a polynomial of the same form.

The proof is easy and we omit it.

**Proof of Theorem 1.** It is enough to prove the equivalence of (ii) and (iii). Let $f$ be a continuous function on $\Omega$. Assume that $f$ satisfies (iii). By Proposition
3 for each \( n \in \mathbb{Z} \) the function \( re^{i\theta} \mapsto f_n(r)e^{in\theta} \) has zero average on each circle \( \Gamma \subset \Omega \) surrounding the origin so by Lemma 1 \( f \) satisfies (ii).

Suppose that \( f \) satisfies (ii). By Proposition 2, \( f_0(r) = 0 \) \( (R_1 < r < R_2) \). Let \( n \in \mathbb{Z}, n \neq 0 \). By Proposition 3 and Lemma 2 each of the functions \( \Phi_k(t) = t^{|n|-2k}, 1 \leq k \leq n \), satisfies (5.2). By Lemma 9 it follows that if \( q_0 < q < 1 \) then each smooth solution of (5.2) is a linear combination of the functions \( \Phi_k, 1 \leq k \leq n \). By Proposition 4 the uniform limit on \([q, 1]\) of a sequence of linear combinations of \( \Phi_k, 1 \leq k \leq n \), is again a linear combination of \( \Phi_k, 1 \leq k \leq n \). If \( R_1 < R < R_2 \) and if \( q > R_1/R \) then it follows by Lemma 3 that on \([q, 1]\) the function \( t \mapsto f_n(tR) \) is a linear combination of \( \Phi_k, 1 \leq k \leq n \). As this holds for every \( R, R_1 < R < R_2 \), the proof is complete.

**Proposition 5.** If \( n \in \mathbb{Z}, n \geq 0 \), then each of the functions \( F_k(z) = z^{n-k} \bar{z}^{-k} \), \( 0 \leq k \leq n \), satisfies \( \int_{\Gamma} F_k(z) \, dz = 0 \) for each circle \( \Gamma \) surrounding the origin. If \( n \in \mathbb{Z}, n \leq -2 \) then each of the functions \( G_k(z) = z^k z^{n+k} \), \( 0 \leq k \leq -n - 2 \), satisfies \( \int_{\Gamma} G_k(z) \, dz = 0 \) for each circle \( \Gamma \) surrounding the origin.

**Proof.** Let \( n \geq 0 \) and let \( 0 \leq k \leq n \). Let \( 0 < |a| < |b| \) and let \( \Gamma = \{a + e^{i\theta} b: 0 \leq \theta < 2\pi\} \). If \( F_k(z) = z^{n-k} \bar{z}^{-k} \) then

\[
\int_{\Gamma} F_k(z) \, dz = \int_0^{2\pi} (a + e^{i\theta} b)^{n-k} (\bar{a} + e^{-i\theta} \bar{b})^{-k} ib e^{i\theta} \, d\theta
\]

\[
= b \int_0^{2\pi} e^{ik\theta} (a + e^{i\theta} b)^{n-k} (\bar{a}e^{i\theta} + \bar{b})^{-k} ie^{i\theta} \, d\theta
\]

\[
= b \int_{b\Delta} w^k (a + wb)^{n-k} (\bar{a}w + \bar{b})^{-k} \, dw = 0
\]

since the integrand in the last integral is analytic in a neighbourhood of \( \overline{\Delta} \). Let \( n \leq -2 \). Write \( n = -m \) and let \( G_k(z) = z^k z^{-m+k} \), \( 0 \leq k \leq m - 2 \). We have

\[
\int_{\Gamma} G_k(z) \, dz = \int_0^{2\pi} (a + e^{-i\theta} \bar{b})^k (a + e^{i\theta} b)^{-m+k} ib e^{i\theta} \, d\theta
\]

\[
= - \int_0^{2\pi} (a + e^{i\theta} b)^k (a + e^{-i\theta} \bar{b})^{-m+k} ib e^{-i\theta} \, d\theta
\]

\[
= - \int_0^{2\pi} (a + e^{i\theta} b)^k e^{i(m-k)\theta} (\bar{a}e^{i\theta} + \bar{b})^{-m+k} ib e^{-i\theta} \, d\theta
\]

\[
= - b \int_{b\Delta} (a + wb)^k w^{m-k-2} (\bar{a}w + \bar{b})^{-m+k} \, dw = 0
\]

since the integrand in the last integral is analytic in a neighbourhood of \( \overline{\Delta} \).

**Example [3].** If \( n \in \mathbb{N} \) then the function

\[
\varphi(z) = \begin{cases} 
  z^{-1} z^{n+2} & (z \neq 0), \\
  0 & (z = 0)
\end{cases}
\]
is of class $\mathcal{C}^n$ on $C$. By Proposition 5 we have $\int_C \varphi(z) \, dz = 0$ for every circle $\Gamma$ that surrounds the origin, yet $\varphi$ is not analytic in $\Delta$.

**Proof of Theorem 2.** It is enough to prove the equivalence of (ii) and (iii). Let $f$ be a continuous function on $\Omega$. If $f$ satisfies (iii) then by Proposition 5 for each $n \in \mathbb{Z}$ the function $re^{i\theta} \mapsto \Psi_n(re^{i\theta}) = f_n(r)e^{in\theta}$ satisfies $\int_\Gamma \Psi_n(z) \, dz = 0$ for each circle $\Gamma \subset \Omega$ surrounding the origin so by Lemma 1 $f$ satisfies (ii).

Suppose that $f$ satisfies (ii). By Proposition 2, $f_{-1}(r) = 0 \ (R_1 < r < R_2)$. Let $n \in \mathbb{Z}$, $n \neq -1$. If $n \geq 0$ then by Proposition 5 and Lemma 2 each of the functions $\Phi_k(t) = t^{-|n|+2k}$, $0 \leq k \leq n$, satisfies (5.3). As in the proof of Theorem 1 it follows that if $R_1 < R < R_2$ and if $q > R_1/R$ then on $[q, 1]$ the function $t \mapsto f_n(tR)$ is a linear combination of $\Phi_k$, $0 \leq k \leq n$. As this holds for every $R$, $R_1 < R < R_2$, (iii) follows for $n \geq 0$. If $n \leq -2$ then by Proposition 5 and Lemma 2 each of the functions $\Phi_k(t) = t^{-|n|+2k}$, $0 \leq k \leq |n| - 2$, satisfies (5.3) and again (iii) follows for $n \leq -2$. This completes the proof.

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**References**


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