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WEAK STABILITY IN THE GLOBAL L^1-NORM FOR SYSTEMS OF HYPERBOLIC CONSERVATION LAWS

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ABSTRACT. We prove that solutions for systems of two conservation laws which are generated by Glimm's method are weakly stable in the global L^1-norm. The method relies on a previous decay result of the author, together with a new estimate for the L^1 Lipschitz constant that relates solutions at different times. The estimate shows that this constant can be bounded by the supnorm of the solution, and is proved for any number of equations. The techniques do not rely on the existence of a family of entropies, and moreover the results would generalize immediately to more than two equations if one were to establish the stability of solutions in the supnorm for more than two equations.

1. INTRODUCTION

We consider the initial value problem for systems of hyperbolic conservation laws

$$u_t + f(u)_x = 0, \tag{1}$$

$$u(x,0) = u_0(x). \tag{2}$$

Here $u = (u', \dots, u^n) \equiv u(x,t)$, $f = (f', \dots, f^n) \equiv f(u)$, $-\infty \le x < +\infty$, and $t \ge 0$. We assume that (1) is strictly hyperbolic and genuinely nonlinear in each characteristic field in a neighborhood of a state $\overline{u} \in \mathbb{R}^n$ [9,19]. In this paper we prove that for systems of two equations, solutions generated by the random choice method are weakly stable in the global L^1-norm. Specifically, we show that if $n = 2$ and $u(x,t)$ denotes a solution generated by Glimm's method [3] satisfying $u(x,0) = u_0(x)$ and $u_0(\pm\infty) = \overline{u}$ then

$$\|u(\cdot,t) - \overline{u}\|_{L^1} \le G(t, \|u_0,(\cdot) - \overline{u}\|_{L^1}), \tag{3}$$

where G is an explicitly constructed smooth function satisfying $G(t,\xi) \to 0$ as $\xi \to 0$ for every fixed $t > 0$. Hence we let $\| \ \|_{L^1}$ denote the L^1-norm. Thus (3) verifies that the global L^1-norm of the solution at time $t > 0$ is controlled by the L^1-norm of the initial data at $t = 0$ through the nonlinear function

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G. For $n = 2$, Glimm's original 1965 paper [3] establishes the stability of the solutions $u(x, t)$ in the total variation norm and in the supnorm for sufficiently small initial data. The stability in the total variation norm leads to the compactness of approximate solutions generated by the random choice method, and this produced the first existence theorem for the Cauchy problem for systems of conservation laws. For systems we still do not have a proof of the uniqueness or the continuous dependence of solutions on initial values for solutions generated by this method. The L^1-norm is the most reasonable L^p-norm in which to expect uniqueness and continuous dependence to hold because it is the norm in which the solutions of a scalar conservation law generate a contractive semigroup [22]. The stability result (3) is the first stability result for systems in the global L^1-norm (see [22]), and although (3) is only proved for two equations, the result does not rely on the existence of a family of entropies. Indeed, the arguments which establish the decay result in [22] as well as the stability result (3) would apply essentially unchanged were one to obtain the supnorm estimate

$$\|u(\cdot, t) - \bar{u}\|_{\sup} \leq C\|u_0(\cdot) - \bar{u}\|_{\sup} \tag{4}$$

for $n > 2$. In fact, (4) has only been established for $n = 2$, and this was accomplished in Glimm's fundamental paper. A simplified proof of this was written in [24]. Statement (3) follows from the decay result proved by the author in [22] together with a new estimate for the growth of the L^1-norm in a solution of (1), (2) constructed by Glimm's method. This estimate is proved for any number of equations, and can be stated as

$$\|u(\cdot, t) - \bar{u}\|_{L^1} \leq C\|u_0(\cdot) - \bar{u}\|_{\sup} t, \tag{5}$$

where $\| \ \|_{\sup}$ denotes the supnorm and C denotes a generic constant (see §2). Estimate (5) is proved in §2 for the approximate solutions generated by Glimm's method. The result in [25] indicates that estimate (5) gives the best coefficient of t for estimates of this type. We note that (5) requires that the sample sequence in the method be equidistributed, but the proof given here is elementary in that it does not require the technical theory of wave tracing. We next derive (3) from the decay estimate in [22] together with estimate (4). In §2 we present the proof of estimate (2).

In [22] the author proved the following decay result for solutions of (1), (2) evolving from initial data that need not be compactly supported (cf. [22, Theorem (1), p. 44]). The result applies to any solution $u(x, t)$ of (1), (2) generated by the random choice method of Glimm [9, 19, 24]. Let $\| \ \|_{L^1}$ denote the L^1-norm and $\| \ \|_{\sup}$ the supnorm so that

$$\|u_0(\cdot) - \bar{u}\|_{L^1} \equiv \int_{-\infty}^{\infty} \|u_0(x) - \bar{u}\| \, dx, \tag{6}$$

and

$$\|u_0(\cdot) - \bar{u}\|_{\sup} \equiv \sup_{-\infty < x < +\infty} \|u_0(\cdot) - \bar{u}\|. \tag{7}$$

Theorem 1. *Assume that* $n = 2$. *Then for every* $V > 0$ *and* $0 < \sigma < 1$ *there exists constants* $\delta = \delta(V) < 1$ *and* $C(\sigma) > 1$ *such that, if* $u_0(\cdot)$ *satisfies*

$$u_0(\pm\infty) = \bar{u} \tag{8}$$

$$TV\{u_0(\cdot)\} < V, \tag{9}$$

and

$$\|u_0(\cdot) - \bar{u}\|_{\sup} < \delta, \tag{10}$$

then

$$\|u(\cdot, t) - \bar{u}\|_{\sup} \leq F\left(\frac{t}{\|u_0(\cdot) - \bar{u}\|_{L^1}}\right), \tag{11}$$

where F *is the positive decreasing function satisfying* $F(\xi) \to 0$ *as* $\xi \to +\infty$ *given by*

$$F(\xi) \equiv C(\sigma)\{\log(\xi)\}^{1/(2+\sigma)}. \tag{12}$$

(We let C denote a generic constant depending only on f, V and appended arguments, while M denotes a generic constant depending only on f). In fact, (11) actually only holds for all $t > \|u_0(\cdot) - \bar{u}\|_{L^1}$ since $F(\xi)$ is only defined and positive for $\xi > 1$. This represents no real restriction because (11) is only interesting for large values of the argument of F. For our purposes we require that F be everywhere positive and bounded. For this reason, define

$$\widetilde{F}(\xi) = \begin{cases} C\,\mathrm{Min}\{\delta, F(\xi)\} & \text{if } \xi > 1, \\ C\delta \quad \text{if } \xi \leq 1, \end{cases} \tag{13}$$

where C, which depends only on f and V, is large enough so that

$$\|u_0(\cdot, t) - \bar{u}\|_{\sup} \leq C\delta. \tag{14}$$

(Statement (14) is just (4), which is statement (14) on p. 47 of [22]. This estimate was first obtained by Glimm in [9] for systems in which there exists a coordinate system of Riemann invariants; see also [24].) In this case it is clear that F can be replaced by $\widetilde{F}$ in (11). So as not to carry the tilde along, we replace F by $\widetilde{F}$ in (11) and write

$$\|u(\cdot, t) - \bar{u}\|_{\sup} \leq F\left(\frac{t}{\|u_0(\cdot) - \bar{u}\|_{L^1}}\right), \tag{11}$$

where we assume that F is given in (13) so that (11) holds for all values of

$$\xi = \frac{t}{\|u_0(\cdot) - \bar{u}\|_{L^1}} \geq 0,$$

and we can assume that F is bounded:

$$|F(\xi)| \leq C\delta. \tag{15}$$

Fixing the initial data and letting $t \to +\infty$ in (11) gives that solutions decay to the constant state $\bar{u}$ in the supnorm at a rate depending only on the L^1-norm of the initial data.

The purpose of the present paper is to use (11) to prove the following theorem which is a restatement of (3) and states that the constant state is weakly stable in the global L^1-norm:

Theorem 2. *Assume the hypotheses of Theorem* 1, *and assume that*

$$\|u_0(\cdot) - \overline{u}\|_{L^1} < \infty. \tag{16}$$

Then

$$\|u(\cdot, t) - \bar{u}\|_{L^1} \leq G(t, \|u_0(\cdot) - \bar{u}\|_{L^1}), \tag{17}$$

where $G(t, \xi) \to 0$ *as* $\xi \to 0$ *for fixed* $t > 0$. *Specifically,* G *is given by*

$$G(t, \xi) = \xi + C\int_0^t F\left(\frac{\tau}{\xi}\right)\, d\tau, \tag{18}$$

where C *is a constant depending only on* f *and* V.

Since F is bounded and tends to zero as its argument tends to infinity, $G(t, \xi) \to 0$ as $\xi \to 0$, so that (17) gives a rate at which the global L^1-norm of the solution at time $t > 0$ tends to zero with the L^1-norm of the initial data. Thus (17) verifies the weak stability of the constant state in the global L^1-norm for systems of two equations.

We note that, as in [22], the only reason our analysis fails to obtain Theorems 1 and 2 for $n > 2$ is that the supnorm estimate (14) is required, and has only been obtained in the presence of a coordinate system of Riemann invariants (see [24]). Note that (ii) gives directly the weak stability of the constant state in L^1_{loc} (see [22]).

We now deduce Theorem 2 from Theorem 1 together with the following lemma whose proof is the subject of §2. The author believes that this lemma, which gives an improved estimate for the growth of the L^1-norm in the solution of an arbitrarily large system of conservation laws, is interesting in its own right. For convenience, let

$$I(t) \equiv \|u(\cdot, t) - \overline{u}\|_{L^1} = \int_{-\infty}^{\infty} \|u(x, t) - \overline{u}\|\, dx \tag{19}$$

where $\|u - v\|$ denotes the sum of the differences in the Riemann invariants between u and v (see §2, (2.4)). Let $u(x, t)$ denote a weak solution of (1.1), (1.2) generated by the random choice method from initial data $u_0(x)$. Assume $I(0) < \infty$ and that the sample sequence in the random choice method is equidistributed.

Lemma 1. *Assume that* $n = 2$ *and that* u *satisfies the hypotheses of Theorem* 1. *Then there exists a constant* $M > 0$ *depending only on* f *such that*

$$I(t) \leq I(s) + MV\delta|t - s|, \tag{20}$$

for all $0 \leq s < t < \infty$.

In the case $n > 2$ we obtain (20) under the corresponding condition on $u_0(\cdot)$ required in the proof of convergence of the random choice method (see [9, 19]).

Lemma 2. *Assume $n > 2$, and that $u(x,t)$ is a weak solution of the random choice method satisfying*

$$TV\{u(\cdot,t)\} < V, \quad \|u(\cdot,t) - \overline{u}\|_{\sup} < \delta, \tag{21}$$

for all $t \geq 0$. Then there exists a constant $M > 0$ depending only on f such that

$$I(t) \leq I(s) + MV\delta|t-s|, \tag{22}$$

for all $0 \leq s < t < +\infty$.

We note that in the above settings the estimate

$$\|u(\cdot,t) - u(\cdot,s)\|_{L^1} \leq MV|t-s|, \tag{23}$$

which gives the Lipschitz continuity of the solution u in L^1, was proved in [9] and was required for the compactness argument in Glimm's convergence proof. In contrast to (21), one can show by simple examples that $\|u(\cdot,t) - u(\cdot,s)\|_{L^1}$ cannot in general be bounded by a constant times the supnorm times t, but only by the total variation times t as in (23). Moreover, the negative result in [25] indicates that estimate (22) is best possible.

Theorem 2 now follows from Theorem 1 together with Lemma 1 as follows. Assume that u satisfies the conditions of Theorem 2, and fix $T > 0$. Define for each $N > 0$ the partition of $[0,T]$ given by

$$t_k = k\Delta t, \quad \Delta t = T/N. \tag{24}$$

By (11) we know that

$$\|u(\cdot,t) - \overline{u}\|_{\sup} \leq F(t/I(0)). \tag{25}$$

Since F is nonincreasing we know that for $t \in [t_k, t_{k+1}]$,

$$\|u(\cdot,t) - \overline{u}\|_{\sup} \leq F\left(t_k/I(0)\right), \tag{26}$$

so by (20) of Lemma 1,

$$|I(t_{k+1}) - I(t_k)| \leq MF\left(t_k/I(0)\right)\Delta t. \tag{27}$$

Here M denotes a generic constant depending only on f and V. Thus we can write

$$|I(T) - I(0)| \leq \sum_{i=0}^{N-1} |I(t_{k+1}) - I(t_k)| \leq \sum_{i=0}^{N-1} MF\left(\frac{t_k}{I(0)}\right)\Delta t. \tag{28}$$

Taking the limit $N \to +\infty$ in (28) gives

$$|I(T) - I(0)| \leq M\int_0^T F\left(\frac{\tau}{I(0)}\right) d\tau. \tag{29}$$

Therefore we conclude that

$$\begin{aligned} I(T) &\leq I(0) + |I(T) - I(0)| \\ &\leq I(0) + M\int_0^T F\left(\frac{\tau}{I(0)}\right) d\tau = G(T, I(0)), \end{aligned} \tag{30}$$

which verifies the conclusion (17) of Theorem 2. The next section is devoted to the proof of Lemmas 1 and 2.

2. The L^1-derivative estimate

Let $u(x,t)$ denote a weak solution of (1), (2) generated by Glimm's method. Assume that $u_0(\pm\infty) = \overline{u}$, and without loss of generality assume $\overline{u} = 0$. Here we present a proof of Lemma 1 which is the estimate

$$\|u(\cdot,t)\|_{L^1} \leq MV\delta t \tag{2.1}$$

in the case $n = 2$. The case $n > 2$, stated in Lemma 2, can be proved in the same manner. The reason the stronger assumptions (21) must replace assumptions (8)–(10) when $n > 2$ is due only to the stronger assumptions required for Glimm's existence proof when $n > 2$.

Let λ_i denote the eigenvalues and R_i the corresponding eigenvectors of the $n \times n$ matrix df. Assume without loss of generality that δ is so small that $\lambda_i(u) < \lambda_j(v)$ for all $i < j$, $u, v \in U$, where U is a neighborhood of $\overline{u} = 0$ in which the solutions of Lemmas 1 or 2 take their values. Let

$$z \equiv (z^1, \ldots, z^n) \equiv z(u) \tag{2.2}$$

denote an approximate coordinate system of Riemann invariants which we assume, without loss of generality, is defined and regular in U. By an approximate coordinate system of Riemann invariants we mean that

$$z(0) = 0, \quad \left.\frac{\partial}{\partial z^i}\right|_{z=0} = R_i(0). \tag{2.3}$$

Let

$$|z| = |z^1| + |z^2|, \tag{2.4A}$$

and also let

$$\|u\| = |z(u)| = |z^1(u)| + |z^2(u)|. \tag{2.4B}$$

Define the L^1-norm of a function $u: \mathbb{R} \to U$ by

$$\|u(\cdot)\|_{L^1} \equiv \int_{-\infty}^{\infty} \|u(x)\|\, dx. \tag{2.5}$$

We now restrict to the case $n = 2$ so that our notation requires keeping track of only two families of waves. Our reason for using an approximate coordinate system of Riemann invariants instead of a full coordinate system of Riemann invariants which exists for $n = 2$ is only to see that our argument generalizes to the case $n > 2$. We first give a precise statement of the existence theorem of Glimm for $n = 2$ as modified in [24]:

Theorem (Glimm). *For every $V > 0$ there exists a constant $\delta << 1$ depending only on f and V such that if*

$$\|u_0(\cdot)\|_{\sup} < \delta, \tag{2.6}$$

and

$$TV\{u_0(\cdot)\} < V, \tag{2.7}$$

then there exists a solution generated by the random choice method satisfying

$$\|u(\cdot, t)\|_{\sup} < C\delta, \tag{2.8}$$

$$TV\{u(\cdot)\} < MV, \tag{2.9}$$

$$\|u(\cdot, t) - u(\cdot, s)\|_{L^1} < C|t-s|. \tag{2.10}$$

Here, as before, C denotes a generic constant depending only on f and V, while M denotes a generic constant depending only on f. We now give a proof of Lemma 1 which we restate more precisely as follows:

Lemma 1. *For every* $V > 0$ *there exists a* $\delta << 1$ *such that if* $u(x,t)$ *is any weak solution of* (1.1), (1.2) *constructed by Theorem* (*Glimm*) *satisfying* $u_0(\pm\infty) = 0$, *then*

$$I(t) \le I(0) + MV\delta t \tag{2.11}$$

for all $t \ge 0$. *Here*

$$I(t) \equiv \int_{-\infty}^{\infty} \|u(x,t)\|\, dx. \tag{2.12}$$

Proof. First, for simplicity assume that $\lambda_1(u) < 0 < \lambda_2(v)$ for all $u, v \in U$. We prove Lemma 1 by obtaining (2.11) for an approximate solution generated by the random choice method. We first establish notation. Let Δx be a mesh length in x and Δt a mesh length in t. Define the grid of mesh points $x_i = i\Delta x$, $t_j = j\Delta t$, and let $\mathbf{a} \equiv \{a_j\}_{j=0}^{\infty}$ be any fixed equidistributed sequence of numbers, $0 < a_j < 1$. Let $u_{\Delta x}(x,t)$ be the approximate solution of the random choice method defined as follows:

Let

$$u_{\Delta x}(x, 0) = u_0(x_i + a_0\Delta x) \tag{2.13}$$

define the solution at $t = 0$. Then assuming that the solution is defined and is constant on intervals $x_i < x < x_{i+1}$, $t = t_j$, define $u_{\Delta x}(x,t)$ for $t_j \le t < t_{j+1}$ to be the solution of (1) obtained by solving for time Δt the Riemann problems posed at $t = t_j$. Assume that

$$\frac{\Delta x}{\Delta t} \equiv \Lambda \ge 2 \sup_{\substack{u \in U \\ i=1,2}} |\lambda_i(u)|,$$

so that waves do not interact in $[t_j, t_{j+1})$ when the values of $u_{\Delta x}$ lie in U [9]. Finally, complete the definition of $u_{\Delta x}$ by defining

$$u_{\Delta x}(x, t_{j+1}+) = u_{\Delta x}(x_i + a_{j+1}\Delta x, t_{j+1}-) \tag{2.14}$$

for $x_i \le x < x_{i+1}$. Theorem (Glimm) is obtained by proving that if u_0 satisfies (2.6) and (2.7), then (see [24])

$$(2.8)_{\Delta x} \qquad \|u_{\Delta x}(\cdot\,,t)\|_{\sup} < C\delta\,,$$

$$(2.9)_{\Delta x} \qquad TV\{u_{\Delta x}(\cdot\,,t)\} < M\delta\,,$$

$$(2.10)_{\Delta x} \qquad \|u_{\Delta x}(\cdot\,,t) - u_{\Delta x}(\cdot\,,s)\|_{L^1} \le C\{|t-s| + \Delta x\}\,,$$

so that in particular $u_{\Delta x} \in U$ for all x and $t \ge 0$, and thus can be defined consistently by the above procedure. The weak solution to (1.1), (1.2) constructed by the method of Glimm is given by

$$(2.15) \qquad u = \lim_{\Delta x \to 0} u_{\Delta x}$$

for some sequence of mesh lengths $\Delta x \to 0$, and the convergence is in L^1_{loc} at each time, uniform on compact time intervals. Without loss of generality, assume that Δx is in this sequence. For fixed Δx, let γ^p_{ij} denote the p-wave in the solution of the Riemann problem posed at x_i, t_j in the approximate solution $u_{\Delta x}$, $p = 1,2$ [22]. If the wave γ^p_{ij} has left state u_L and right state u_R, then let γ^p_{ij} also refer to the vector

$$(2.16) \qquad \gamma^p_{ij} \equiv z(u_R) - z(u_L)\,,$$

and let the strength of the wave also be the strength as measured in the approximate coordinate system of Riemann invariants, so that

$$(2.17) \qquad I_{\Delta x}(t_j+) = \sum_{i=-\infty}^{\infty} |z_{ij}|\Delta x \equiv I_j\,,$$

where

$$(2.18) \qquad z_{ij} \equiv z(u_{\Delta x}(x_i + a_j\Delta x\,, t_j-))$$

are the z-coordinates of the constant states appearing in the function $u_{\Delta x}$ at time t_j+. Now because $u_{\Delta x} \to u$ in L^1_{loc} at each time, Lemma 1 is a direct consequence of the following lemma:

Lemma 2.1. *Let $t > 0$ and* **a** *be fixed. Let $J \equiv J(\Delta x)$ denote the integer such that $t_{J-1} \le t < t_J$. Then for Δx sufficiently small,*

$$(2.19) \qquad |I_J - I_0| \le MV\delta t_J.$$

Note that $|I_{\Delta x}(t) - I_J| = o(\Delta x)$ by $(2.1)_{\Delta x}$, so that taking the limit $\Delta x \to 0$ in (2.19) clearly yields (2.11) of Lemma 1. Thus it remains only to verify (2.19). We obtain (2.19) by estimating the increase in the L^1-norm over each time step of $u_{\Delta x}$. We derive two estimates at each t_j depending on where the sample points $x_i + a_j\Delta x$ fall relative to the waves in the solutions of the Riemann problems posed at time t_{j-1}. The idea is that if sample points $x_i + a_j\Delta x$ fall on the same side of the waves in the same family in each rectangle at $t = t_j$, then the estimate

$$|I_j - I_{j-1}| \le MV\delta|t_j - t_{j-1}| \tag{2.20}$$

is obtained. But when the sample points fall on various sides or within the waves of the same family at $t = t_j$, then estimate (2.20) may fail. The second estimate states that this can only happen $M\delta J$ times between $t = 0$ and $t = t_j$ because the sequence $\mathbf{a}$ is equidistributed. These two estimates are enough to obtain (2.19).

To be precise, let

$$\lambda_p \equiv \lambda_p(0), \qquad p = 1, 2. \tag{2.21}$$

Since the eigenvalues λ_p depend smoothly on U, it is clear that for $u = u_{\Delta x}(x, t)$,

$$|\lambda_p(u) - \lambda_p| \le M\delta, \qquad p = 1, 2, \tag{2.22}$$

for some M depending only on f. Now consider the 2-wave emanating from the mesh point (x_i, t_{j-1}) in the approximate solution $u_{\Delta x}$. By (2.22), the states in this wave will intersect the times $t = t_j$ within an x-interval estimated by $(x_i + (\lambda_2 - M\delta)\Delta t, x_i + (\lambda_2 + M\delta)\Delta t)$. Thus at $t = t_j$, a sample point $x_i + a_j\Delta x$ can fall within the 2-wave γ_{ij}^2 only if

$$a_j \in \left(\frac{\lambda_2 - M\delta}{\Lambda}, \frac{\lambda_2 + M\delta}{\Lambda}\right) \equiv \mathcal{T}_2 \subseteq [0, 1]. \tag{2.23}$$

Similarly, the sample points at $t = t_j$ can fall within the 1-waves emanating from the mesh point (x_{i+1}, t_{j-1}) only if

$$a_j \in \left(1 - \frac{\lambda_1 + M\delta}{\Lambda}, 1 - \frac{\lambda_1 - M\delta}{\Lambda}\right) \equiv \mathcal{T}_1 \subseteq [0, 1]. \tag{2.24}$$

In particular, if $a_j \in [0, 1] \setminus \mathcal{T} \equiv \mathcal{J}$, where $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, then the sample points $x_i + a_j\Delta x$ must fall to the same side of the waves $\gamma_{i,j-1}^2$ and $\gamma_{i+1,j-1}^1$ for all values of i. We assume that δ is sufficiently small so that $\mathcal{J}$ consists of the three disjoint sets

$$\mathcal{J}_1 = \left[0, \frac{\lambda_2 - M\delta}{\Lambda}\right),$$
$$\mathcal{J}_2 = \left[\frac{\lambda_2 + M\delta}{\Lambda}, 1 - \frac{\lambda_1 + M\delta}{\Lambda}\right],$$
$$\mathcal{J}_3 = \left(1 - \frac{\lambda_1 - M\delta}{\Lambda}, 1\right].$$

The following lemma is a restatement of (2.20):

Lemma 2.2. *If* $a_j \in \mathscr{J}$ *, then we have* (2.20):

$$|I_j - I_{j-1}| \le MV\delta\Delta t. \tag{2.20}$$

Proof. We do the case $a_j \in \mathscr{J}_3$. The case $a_j \in \mathscr{J}_2$ is trivial and $a_j \in \mathscr{J}_1$ is similar. Since j is fixed, let $\alpha_i = \gamma^1_{i,j-1}$, $\beta_i = \gamma^2_{i,j-1}$, and $z_i = z_{i,j-1}$. Then we have

$$I_{j-1} = \sum_{i=-\infty}^{\infty} |z_i|\Delta x, \tag{2.25}$$

and

$$I_j = \sum_{i=-\infty}^{\infty} |z_i + \alpha_{i+1}|\Delta x, \tag{2.26}$$

where we have used the fact that

$$z_{ij} = z_i + \alpha_{i+1} \tag{2.27}$$

because $j \in \mathscr{J}_1$. Thus

$$I_j - I_{j-1} = \sum_{i=-\infty}^{\infty} \{|z_i + \alpha_{i+1}| - |z_i|\}\Delta x. \tag{2.28}$$

Since $\big||z_i + \alpha_{i+1}| - |z_i|\big| \le M|\alpha_{i+1}|$, it is easy to see that

$$|I_j - I_{j-1}| \le MV\Delta t. \tag{2.29}$$

But by our definition of α_i and β_i as vectors in z-space, we also have $z_i = z_{i-1} + \alpha_1 + \beta_i$. Now since p-waves lie on integral curves of R_p to within errors which are quadratic in the wave strength, (2.3) implies that

$$|\alpha_i^2| \le M\delta|\alpha_i|, \tag{2.30}$$

$$|\beta_i^1| \le M\delta|\beta_i|, \tag{2.31}$$

where α_i^p, β_i^p denote the p-components of the vectors α_i, β_i, respectively, $p = 1, 2$. These estimates imply that

$$|z_i + \alpha_{i+1}| - |z_i| = |z_i^1 + \alpha_{i+1}^1| - |z_i^1| \pm M\delta|\alpha_{i+1}|, \tag{2.32}$$

and

$$|z_i^1| = |z_{i-1}^1 + \alpha_i^1 + \beta_i^1| = |z_{i-1}^1 + \alpha_i^1| \pm M\delta|\beta_i|, \tag{2.33}$$

where for clarity we let $\pm M$ denote big $0(1)$. Putting (2.32) and (2.33) together gives

$$|z_i + \alpha_{i+1}| - |z_i| = |z_i^1 + \alpha_{i+1}^1| - |z_{i-1}^1 + \alpha_i^1| \pm M\delta(|\alpha_{i+1}| + |\beta_i|). \tag{2.34}$$

Therefore,

$$\begin{aligned}\left|\sum_{i=-\infty}^{\infty}\{|z_i+\alpha_{i+1}|-|z_i|\}\right| &\\ \le \left|\sum_{i=-\infty}^{\infty}\{|z_i^1+\alpha_{i+1}^1|-|z_{i-1}^1+\alpha_i^1|\}\right| &\\ + M\delta\sum_{i=-\infty}^{\infty}\{|\alpha_{i+1}|+|\beta_i|\} &\\ \le MV\delta, &\end{aligned} \tag{2.35}$$

because the first sum on the R.H.S. of (2.35) is a collapsing sum. Thus we use (2.35), (2.28) to conclude that

$$|I_j - I_{j-1}| \le MV\delta\Delta x,$$

which is (2.20). This completes the proof of Lemma 2.2. We now complete the proof of Lemma 1 by giving the

Proof of Lemma 2.2. For a given integer $N > 0$, let

$$\mathscr{N} = \{j \in [0, N]: a_j \in \mathscr{T}\}, \tag{2.36}$$

$$m = \text{ Card } \mathscr{N}. \tag{2.37}$$

Since $\mathbf{a}$ is equidistributed, we can choose N so large that

$$m/N \le |\mathscr{T}| + M\delta \equiv M\delta, \tag{2.38}$$

where we have used (2.23) and (2.24) to obtain $|\mathscr{T}| \le M\delta$, $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$. Here $|\mathscr{T}| = |\mathscr{T}_1| + |\mathscr{T}_2|$ denotes the sum of the length of the interval $\mathscr{T}_1$ and $\mathscr{T}_2$, and M is again a generic constant depending only on f.

Now we write

$$\begin{aligned}|I(t_N) - I(0)| &= \left|\sum_{j=1}^{N}(I_j - I_{j-1})\right| \le \sum_{j=1}^{N}|I_j - I_{j-1}| \\ &= \sum_{j\in\mathscr{N}}|I_j - I_{j-1}| + \sum_{j\in[1,N]/\mathscr{N}}|I_j - I_{j-1}|.\end{aligned} \tag{2.39}$$

By (2.29) and (2.38), if N is sufficiently large, then

$$\sum_{j\in\mathscr{N}}|I_j - I_{j-1}| \le mMV\Delta t \le MV\delta N\Delta t = MV\delta t_N. \tag{2.40}$$

But for $j \in [1, N]\setminus\mathscr{N}$ Lemma (2.2) applies and we can estimate

$$\sum_{j\in[0,N]\setminus\mathscr{N}}|I_j - I_{j-1}| \le NMV\delta\Delta t \le MV\delta t_N. \tag{2.41}$$

Putting (2.40) and (2.41) into (2.39) gives the result that for sufficient large N,

$$|I(t_N) - I(0)| \le MV\delta t_N. \tag{2.42}$$

Now for fixed t, we can choose Δx sufficiently small so that $J > N$, where N is sufficient for (2.42), and we can conclude that for Δx sufficiently small,

$$|I_J - I_0| \leq MV\delta t_J,$$

which is (2.20) of Lemma 2.2. This completes the proof of Lemma (2.2), and the proofs of Lemma 2.1 and Theorem 2 are complete.

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