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HOMOGENEOUS CONTINUA IN EUCLIDEAN $(n+1)$-SPACE WHICH CONTAIN AN n-CUBE ARE n-MANIFOLDS

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ABSTRACT. Let X be a homogeneous continuum and let E^n be Euclidean n-space. We prove that if X is properly contained in a connected $(n+1)$-manifold, then X contains no n-dimensional umbrella (i.e. a set homeomorphic to the set $\{(x_1, \dots, x_{n+1}) \in E^{n+1} : x_1^2 + \cdots + x_{n+1}^2 \le 1$ and $x_{n+1} \le 0$ and either $x_1 = \cdots = x_n = 0$ or $x_{n+1} = 0\})$. Combining this fact with an earlier result of the author we conclude that if X lies in E^{n+1} and topologically contains E^n, then X is an n-manifold.

The main purpose of this paper is to prove the following theorem.

1. Theorem. *Each homogeneous proper subcontinuum of a connected $(n+1)$-manifold contains no n-dimensional umbrella.*

The results of this paper are related to two classical results: the first one of S. Mazurkiewicz [M], and, the second one of R. H. Bing [B]. Namely, with the help of the result of [P], we give a full generalization of the result of [B] to all finite-dimensional cases (Theorem 7 below, and also, the statement formulated in the title). As it was emphasized in [P], the theorem of [B] may be obtained by combining two other theorems: 1° each homogeneous locally connected nondegenerate plane continuum is a simple closed curve (this is the result of [M]), 2° each homogeneous plane continuum that contains an arc is locally connected (this is the step really done in [B]), and thus 3° each homogeneous plane continuum that contains an arc is a simple closed curve. (Bing's proof did not follow this scheme.) One can easily observe that Theorem 1 implies the result of [M] (for $n = 1$). Thus this paper generalizes step 1°. Step 2° has already been extended in [P] to all finite-dimensional cases. Therefore we get Theorem 7 as a generalization of step 3°.

Finally, let us stress the fact that, similarly as in [P], the ε-push property (Theorem 4) plays a crucial role in the argument of the proof of Theorem 1. Probably, this is the real reason that the results of [P] and of this paper have not been earlier found.

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All spaces considered here will be either Euclidean n-spaces E^n equipped with the usual Euclidean metric d, or (not necessarily compact) n-manifolds with a metric also denoted by d. The open ball of the space with center c and radius ε will be denoted by $B(c,\varepsilon)$. For two subsets A and B of the space we put $d(A,B) = \inf\{d(a,b)\colon a \in A \text{ and } b \in B\}$. If M is an n-manifold, the symbol ∂M denotes its combinatorial boundary. An arc with end points a and b will often be denoted by ab. The symbol I means the unit segment $[0,1]$. Let a set A be homeomorphic to the cube I^n and let ab be an arc. If $A \cap ab = \{a\}$ and $a \notin \partial A$, then the union $A \cup ab$ will be called an n-dimensional umbrella. A set X is said to be homogeneous if for given $x, y \in X$ there is a homeomorphism $h\colon X \to X$ with $h(x) = y$. A mapping means a continuous function. A mapping (a homeomorphism) $f\colon X \to Y$ between subsets X and Y of the same space is called an ε-translation (an ε-homeomorphism) provided $d(x, f(x)) < \varepsilon$ for every $x \in X$. A point x is said to be accessible from a set V if there is an arc xy with $xy \setminus \{x\} \subset V$. A set C separates a set V between two points $p, q \in V$, if p and q lie in distinct components of $V \setminus C$.

We start with two lemmas, which we need to prove Theorem 1.

2. Lemma. *If a point $c \in C \subset E^{n+1}$ has a neighborhood (in a set C) homeomorphic to E^n, then the number of components of $E^{n+1} \setminus C$ containing c in their closures is either one or two. Moreover, c is accessible from each of these components.*

Proof. Let a neighborhood A of c in C be homeomorphic to I^n, and let a ball $B(c,\xi) \subset E^{n+1}$ be such that $C \cap B(c,\xi) = A \cap B(c,\xi)$ and $\partial A \cap B(c,\xi) = \varnothing$. Further, let $\{A_0, A_1, \ldots\}$ be the family (finite or infinite) of all components of $A \cap B(c,\xi)$ with $c \in A_0$. By Proposition 3 of [P] the set A_0 separates $B(c,\xi)$ into exactly two components U_1^0 and U_2^0. By the local connectedness of A there is a ball $B(c,\tau)$ with $A \cap B(c,\tau) \subset A_0$. Since no pair of points of $U_i^0 \cap B(c,\tau)$ is separated by any A_n in $B(c,\xi)$ for $i \in \{1,2\}$ and $B(c,\xi)$ is homeomorphic to E^{n+1}, we see by Proposition 4 of [P] that A also does not separate $B(c,\xi)$ between such points. This implies that $c \in \operatorname{cl} U_i$ for the component U_i of $B(c,\xi) \setminus A$ containing $U_i^0 \cap B(c,\tau)$, for $i \in \{1,2\}$, and c does not lie in the closure of any other component of $B(c,\xi) \setminus A$. Thus the number of components of $E^{n+1} \setminus C$ containing c in their closures is at most two, and, in fact, not less than one. Moreover, since A_0 is an ANR-set, the point c is accessible from both U_1^0 and U_2^0 (see [Bo, p. 217]). Finally, the desired accessibility of c follows by the previous argument.

3. Lemma. *Let a set $A \subset E^{n+1}$ be homeomorphic to I^n. Given a point $c \in E^{n+1}$ and a number $\varepsilon > 0$ such that $d(c, A) < \varepsilon < d(c, \partial A)$, let A_1 denote a component of $A \cap B(c,\varepsilon)$. For two given points p and q of distinct components of $B(c,\varepsilon) \setminus A_1$ let pa and aq be arcs in $B(c,\varepsilon)$ such that $(pa \cup aq) \cap A_1 = \{a\}$*

(compare Proposition 3 *of* [P] *and Lemma* 2*). Then for every* $\delta > 0$ *such that*

$$\delta < \tfrac{1}{2}\delta_0 = \tfrac{1}{2}\min\{d(pa \cup aq, E^{n+1}\backslash B(c,\varepsilon)), d(\{p,q\}, A_1)\}$$

and for every δ*-homeomorphism* $h\colon A_1 \to h(A_1) \subset E^{n+1}$ *the component* A_2 *of* $h(A_1) \cap B(c, \varepsilon-\delta)$ *containing the point* $a' = h(a)$ *separates* $B(c, \varepsilon-\delta)$ *between* p *and* q.

Moreover, if $\delta < \delta_0/4$, *there are arcs* pa' *and* $a'q$ *in* $B(c, \varepsilon-\delta-\delta_0/4)$ *such that* $(pa' \cup a'q) \cap A_2 = \{a'\}$.

Proof. By Proposition 5 of [P] the set $h(A_1)$ separates the ball $B(c, \varepsilon-\delta)$ between p and q. Therefore a component A_2 of $h(A_1) \cap B(c, \varepsilon-\delta)$ so does (see Proposition 4 of [P]). Again by Proposition 5 of [P], noting that A_2 is closed in $B(c, \varepsilon-\delta)$, we see that the set $h^{-1}(A_2) \subset A_1$ separates $B(c, \varepsilon-2\delta)$ between p and q, and thus it intersects the set $pa \cup aq$. By the assumption the only point of this intersection is a, thus $a' = h(a) \in A_2$.

Let $\delta < \delta_0/4$. Put $\delta_1 = \varepsilon-\delta-\delta_0/4$. Suppose there is no arc pa' in $B(c, \delta_1)$ with $pa' \cap A_2 = \{a'\}$. Let A_3 be the component of $B(c, \delta_1) \cap A_2$ containing a'. Thus there is an arc $Z \subset B(c, \delta_1)$ with end points p and a' such that $Z \cap A_3 = \{a'\}$ (see Proposition 3 of [P] and Lemma 2). Let a point $z \in Z\backslash\{a'\}$ be such that the arc $za' \subset Z$ intersects A_2 in the single point a'. Thus, by the above assumption, A_2 separates the ball $B(c, \delta_1)$ between p and z. Therefore some component A_4 of $B(c, \delta_1) \cap A_2$ separates $B(c, \delta_1)$ between p and z (see Proposition 4 of [P]) and we have $A_4 \neq A_3$. Hence A_4 separates $B(c, \delta_1)$ between p and a'. By the proved part of the conclusion of this lemma the set A_3 separates the ball $B(c, \delta_1)$ between p and q (for h is also a $(\delta + \delta_0/4)$-homeomorphism). Therefore the set A_4 separates $B(c, \delta_1)$ between p and q. Thus, by Proposition 5 of [P], the set $h^{-1}(A_4)$ separates $B(c, \delta_1-\delta)$ between p and q. By the assumption on δ we have $pa \cup aq \subset B(c, \varepsilon-\delta_0) \subset B(c, \delta_1-\delta)$, therefore the set $h^{-1}(A_4) \subset A_1$ intersects the arc $pa \cup aq$ in a point distinct from a (for $a' = h(a) \notin A_4$), a contradiction. The argument for the existence of an arc $a'q$ runs similarly.

Now recall the theorem (the so-called ε-push property) which is a corollary to the well-known Effros theorem.

4. Theorem (Lemma 4 of [H, p. 37]). *Let* X *be a homogeneous metric continuum. Then for every* $\varepsilon > 0$ *there is* $\delta > 0$ *(the so-called Effros number for the number* ε*) such that for two given points* $x, y \in X$ *with* $d(x,y) < \delta$ *there is an* ε*-homeomorphism* $h\colon X \to X$ *sending* x *to* y.

Proof of Theorem 1. Let P be a homogeneous proper subcontinuum of a connected $(n+1)$-manifold M. Suppose, on the contrary, P contains an n-dimensional umbrella. By the intrinsic invariance of open sets in the Euclidean spaces, since ∂M is either an n-manifold or the empty set, P cannot be contained in ∂M. Because P is a boundary set in M, there is a point $a \in P\backslash\partial M$

accessible from $M\backslash P$. Let d be a metric on M such that the ball $B(a,1)\subset M$ is isometric to the appropriate ball of Euclidean $(n+1)$-space. By the homogeneity of P there is an n-dimensional umbrella $T\subset B(a,1)\cap P$ such that $T = A\cup ab$, where the set A is homeomorphic to I^n, ab is an arc with ends a and b, and $A\cap ab=\{a\}\subset A\backslash\partial A$. Let a number $\varepsilon>0$ be such that $\partial A\cap B(a,\varepsilon)=\varnothing$. Without loss of generality we may assume that $ab\subset B(a,\varepsilon)$. Let A_1 be the component of $A\cap B(a,\varepsilon)$ containing a. Then A_1 separates the ball $B(a,\varepsilon)$ into exactly two components (see Proposition 3 of [P]): V_b with $ab\backslash\{a\}\subset V_b$, and V_c with some point $c\in V_c$. By the accessibility of a there is an arc $ap\subset B(a,\varepsilon)$ with $ap\cap P=\{a\}$. By Lemma 2 there is an arc $ac\subset B(a,\varepsilon)$ with $ac\backslash\{a\}\subset V_c$.

(1) For any arc ax with $ax\backslash\{a\}\subset V_c$ we have $P\cap(ax\backslash\{a\})\neq\varnothing$.

In fact, let $cx\subset V_c$ be an arc (may be degenerate if $c=x$), and $\{z_m\}$ be a sequence of points of $ab\backslash\{a\}$ converging to a. By the ε-push property (Theorem 4) there are ξ_m-homeomorphisms $g_m\colon P\to P$ with $g_m(z_m)=a$ and $\lim\xi_m=0$. By Proposition 5 of [P] the sets $g_m(A_1)$ separate the balls $B(a,\varepsilon-\xi_m)$ between c and b, and, do not intersect cx and some fixed ball $B(b,\xi)\subset V_b$ for sufficiently great m. Since $g_m(b)\in B(b,\xi)$, the sets $B(b,\xi)\cup g_m(bz_m)\cup ax\cup cx$ are connected for almost all m. Noting $g_m(bz_m)\cap g_m(A_1)=\varnothing$, where $bz_m\subset ab$, we see that the set $g_m(A_1)\subset P$ intersects $ax\backslash\{a\}$ for large m.

By (1) we see that there is a sequence $\{a_m\}$ converging to a with $a_m\in P\cap ac\backslash\{a\}$, and also

$$ap\backslash\{a\}\subset V_b. \tag{2}$$

Let $pb\subset V_b$ be an arc, and let τ be a number such that

$$0<\tau<\tfrac{1}{2}\min\{d(ap\cup pb\cup ab\cup ac, M\backslash B(a,\varepsilon)), d(pb\cup\{c\},A_1)\}.$$

Further, let $\psi>0$ be an Effros number for the number $\tau/4$ (see Theorem 4), and let $\varphi>0$ be an Effros number for the number ψ. Now, find a_k with $d(a,a_k)<\varphi$, and let $f\colon P\to P$ be a ψ-homeomorphism such that $f(a)=a_k$. Let A_2 denote the component of $B(a,\varepsilon-\tau/4)\cap f(A_1)$ containing $f(a)$. By Lemma 3 this component separates the ball $B(a,\varepsilon-\tau/4)$ between c and b, and also between c and p (for obviously we have $\psi\le\tau/4$). Moreover, the same lemma guarantees the existence of an arc $ca_k\subset B(a,\varepsilon-\tau/4-\tau/2)=B(a,\varepsilon-3\tau/4)$ with $ca_k\cap A_2=\{a_k\}$. Let r be the first point of the arc $f(ab)$ (in the ordering from $f(a)$ to $f(b)$) intersecting A_1. Since $d(a,a_k)<\psi$, we may find a point $q\in a_kr\subset f(ab)$ with $a_k\neq q\neq r$ and $d(a,q)<\psi$.

Now consider the component U of $B(a,\varepsilon-\tau/4)\backslash(A_1\cup A_2)$ containing q. Observe that this component contains no point of the set $pb\cup\{c\}$. The point a_k lies in the closures of U and of the component U_c of $B(a,\varepsilon-\tau/4)\backslash(A_2\cup\mathrm{Bd}\,U)$ containing c, for there exist the arcs ca_k and $a_kq\subset f(ab)$. Since a_k has a neighborhood in $A_2\cup\mathrm{Bd}\,U$ homeomorphic to E^n, there is, by Lemma 2, no other component of $B(a, \varepsilon - \tau/4) \backslash (A_2 \cup \operatorname{Bd} U)$ containing a_k in its closure, in particular, a_k does not lie in the closure of the component U_b containing b. Let B be a connected open neighborhood of a_k in A_2 with a positive distance from $A_1 \cup \operatorname{Bd} U_b$. Then

(3) the set $(A_2 \cup \operatorname{Bd} U) \backslash B$ separates the ball $B(a, \varepsilon - \tau/4)$ between c and b.

For every $x \in qb' \cap \operatorname{Bd} U \subset A_1 \backslash A_2$, where $b' = f(b)$ and $qb' \subset f(ab)$, find an open connected neighborhood B_x of x in $A_1 \backslash A_2$ such that

$$\operatorname{cl}\left(\bigcup\{B_x : x \in qb' \cap \operatorname{Bd} U\}\right) \cap \operatorname{cl} A_2 = \varnothing.$$

Put $B_1 = \bigcup\{B_x : x \in qb' \cap \operatorname{Bd} U\}$. Since $A_2 \subset (A_2 \cup \operatorname{Bd} U) \backslash B_1$, we get

(4) the set $(A_2 \cup \operatorname{Bd} U) \backslash B_1$ separates the ball $B(a, \varepsilon - \tau/4)$ between c and b.

We also have

(5) the set $(A_2 \cup \operatorname{Bd} U) \backslash (B \cup B_1)$ does not separate the ball $B(a, \varepsilon - \tau/4)$ between c and b.

For, the segment between b and $f(b)$ does not intersect $A_1 \cup A_2$, and, the arc $ca_k \cup f(ab)$ does not intersect $(A_2 \cup \operatorname{Bd} U) \backslash (B \cup B_1)$.

Now, find a $(\tau/4)$-homeomorphism $h: P \to P$ such that $h(q) = a$. Then we obtain

(6) the set $h(A_2 \cup \operatorname{Bd} U) \backslash h(B \cup B_1)$ does not separate the ball $B(a, \varepsilon - 2\tau/4)$ between c and b.

In fact, if not, then the set

$$h^{-1}(h(A_2 \cup \operatorname{Bd} U) \backslash h(B \cup B_1)) = (A_2 \cup \operatorname{Bd} U) \backslash (B \cup B_1)$$

would separate $B(a, \varepsilon - 3\tau/4)$ between c and b (see Proposition 5 of [P]), an impossibility, for the segment between b and $f(b)$, as well as the arc $ca_k \cup f(ab)$, lie in $B(a, \varepsilon - 3\tau/4) \backslash ((A_2 \cup \operatorname{Bd} U) \backslash (B \cup B_1))$.

By Proposition 5 of [P] and by (3) and (4) we have

(7) each of sets $h(A_2 \cup \operatorname{Bd} U) \backslash h(B)$ and $h(A_2 \cup \operatorname{Bd} U) \backslash h(B_1)$ separates the ball $B(a, \varepsilon - 2\tau/4)$ between c and b.

The following statement contradicts the previous one, so it completes the proof of Theorem 1.

(8) One of the sets $h(A_2 \cup \operatorname{Bd} U) \backslash h(B)$ and $h(A_2 \cup \operatorname{Bd} U) \backslash h(B_1)$ fails to separate the ball $B(a, \varepsilon - 2\tau/4)$ between c and b.

Indeed, by (6) there is an arc cb in $B(a, \varepsilon - 2\tau/4) \backslash (h(A_2 \cup \operatorname{Bd} U) \backslash h(B \cup B_1))$. By (7) this arc intersects the set $h(B \cup B_1)$. Since $d(B, B_1) > 0$, we have

$d(h(B), h(B_1)) > 0$. This implies that going from c to b along the arc cb, we may find a point $y \in cb$ such that either $y \in h(B)$ and $cy \cap h(B_1) = \emptyset$ or $y \in h(B_1)$ and $cy \cap h(B) = \emptyset$, where $cy \subset cb$. If $y \in h(B)$, then let $ya' \subset h(B)$ be an arc (where $a' = h(a_k) = hf(a)$), and put $J(y, a) = ya' \cup h(a_k q)$, where $a_k q \subset f(ab)$. If $y \in h(B_1)$, then $y \in h(B_x)$ for some $x \in qb' \cap \operatorname{Bd} U$, where $b' = f(b)$ and $qb' \subset f(ab)$. Let $yx' \subset h(B_x)$ be an arc, where $x' = h(x)$, and put $x'a = h(xq)$, where $xq \subset f(ab)$. Then put $J(y, a) = yx' \cup x'a$. Thus in the former case we get $J(y, a) \cap (h(A_2 \cup \operatorname{Bd} U) \backslash h(B)) = \emptyset$, and, in the latter case we have $J(y, a) \cap (h(A_2 \cup \operatorname{Bd} U) \backslash h(B_1)) = \emptyset$. But since both considered homeomorphisms are $(\tau/4)$-homeomorphisms, the set $h(A_2 \cup \operatorname{Bd} U)$ does not intersect the arc pb. By the assumption on pa the set $h(A_2 \cup \operatorname{Bd} U) \subset P$ does not intersect the arc pa. Therefore the connected set $cy \cup J(y, a) \cup pa \cup pb \subset B(a, \varepsilon - 2\tau/4)$ does not intersect either $h(A_2 \cup \operatorname{Bd} U) \backslash h(B)$ or $h(A_2 \cup \operatorname{Bd} U) \backslash h(B_1)$. Thus we have (8).

The proof of Theorem 1 is complete.

A simple proof of the next fact is left to the reader.

5. Fact. *A homogeneous locally connected continuum that topologically contains the cube I^n and contains no n-dimensional umbrella, is an n-manifold.*

Further, we get the following immediate consequence of Theorem 1.

6. Corollary. *A proper homogeneous locally connected subcontinuum of a connected $(n+1)$-manifold, that topologically contains the cube I^n, is an n-manifold.*

It was proved in [P] that each homogeneous subcontinuum of E^{n+1}, which topologically contains I^n is locally connected. Thus we have the conclusion that forms the title of the paper.

7. Theorem. *Each homogeneous continuum that lies in the Euclidean space E^{n+1} and topologically contains the cube I^n is an n-manifold.*

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