ALGEBRAICALLY INVARIANT EXTENSIONS OF
σ-FINITE MEASURES ON EUCLIDEAN SPACE

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ABSTRACT. Let $G$ be a group of algebraic transformations of $\mathbb{R}^n$, i.e., the
group of functions generated by bijections of $\mathbb{R}^n$ of the form $(f_1, \ldots, f_n)$ where
each $f_i$ is a rational function with coefficients in $\mathbb{R}$ in $n$-variables. For a
function $\gamma: G \to (0, \infty)$ we say that a measure $\mu$ on $\mathbb{R}^n$ is $\gamma$-invariant when
$\mu(g[A]) = \gamma(g) \cdot \mu(A)$ for every $g \in G$ and every $\mu$-measurable set $A$. We
will examine the question: "Does there exist a proper $\gamma$-invariant extension of
$\mu$?" We prove that if $\mu$ is $\sigma$-finite then such an extension exists whenever
$G$ contains an uncountable subset of rational functions $H \subset (\mathbb{R}(X_1, \ldots, X_n))^n$
such that $\mu(\{x: h_1(x) = h_2(x)\}) = 0$ for all $h_1, h_2 \in H$, $h_1 \neq h_2$. In particular
if $G$ is any uncountable subgroup of affine transformations of $\mathbb{R}^n$, $\gamma(g)$ is the
absolute value of the Jacobian of $g \in G$ and $\mu$ is a $\gamma$-invariant extension of the
$n$-dimensional Lebesgue measure then $\mu$ has a proper $\gamma$-invariant extension.
The conclusion remains true for any $\sigma$-finite measure if $G$ is a transitive group
of isometries of $\mathbb{R}^n$. An easy strengthening of this last corollary gives also an
answer to a problem of Harazisvili.

0. INTRODUCTION: NOTATION AND HISTORY

Our terminology related to algebra, measure theory, set theory and model
theory follows [La, Ru, Je and CK] respectively.

Throughout the paper a measure on a set $X$ will stand for a nontrivial posi-
tive $\sigma$-additive measure, i.e., a function $\mu: \mathcal{M} \to [0, \infty]$ defined on a $\sigma$-algebra
$\mathcal{M}$ of subsets of $X$ containing all singletons such that

(i) $\mu(\bigcup_{i=0}^{\infty} A_i) = \sum_{i=0}^{\infty} \mu(A_i)$ for all pairwise disjoint sets $A_i$ from $\mathcal{M}$,
(ii) $\mu(\{x\}) = 0$ for all $x \in X$,
(iii) $0 < \mu(A) < \infty$ for some $A \in \mathcal{M}$.

If $\mu: \mathcal{M} \to [0, \infty]$ is a measure on $X$ and $A \subseteq X$ then the inner measure
of $A$ is defined in the standard way: $\mu_*(A) = \sup\{\mu(B): B \subseteq A&B \in \mathcal{M}\}$.

A measure on $X$ is said to be $\sigma$-finite if $X$ is a countable union of sets of
finite measure. A measure $\mu$ is complete if all subsets of every set of $\mu$
measure zero are $\mu$-measurable.

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If $G$ is a group of bijections of a set $X$ then a measure $\mu$ on $X$ is said to be $G$-invariant provided $\mu$ is $\gamma$-invariant where $\gamma(g) = 1$ for all $g \in G$.

For example, if $A_n$ is a group of affine transformations of $\mathbb{R}^n$ then every element of $A_n$ is uniquely represented as a superposition $T \circ L$ where $T$ is a translation and $L$ is a linear transformation of $\mathbb{R}^n$. Let $\gamma: A_n \to (0, \infty)$, where $\gamma(T \circ L)$ is defined as the absolute value of the Jacobian of $L$. Then $m$, the $n$-dimensional Lebesgue measure, is $\gamma$-invariant. Moreover, if $G_n$ is a group of isometries of $\mathbb{R}^n$ then $G_n \subset A_n$ and $m$ is $G_n$-invariant.

We say that a measure $\nu: \mathcal{N} \to [0, \infty]$ on a set $X$ is an extension of a measure $\mu: \mathcal{M} \to [0, \infty]$ defined on the same set $X$ if $\mathcal{M} \subset \mathcal{N}$ and $\nu(A) = \mu(A)$ for every $A \in \mathcal{M}$. Moreover, an extension is proper if $\mathcal{M} \neq \mathcal{N}$.

For a group $G$ of bijections of a set $X$ we say that a set $N \subset X$ is $G$-absolutely negligible if for every $G$-invariant $\sigma$-finite measure $\mu$ on $X$ and for every countable set $\{g_r: r = 0, 1, 2, \ldots \} \subset G$ we have $\mu_*(\bigcup_{r=0}^{\infty} g_r[N]) = 0$ (or, equivalently, if for every $G$-invariant $\sigma$-finite measure $\mu$ on $X$ there exists a $G$-invariant extension $\nu$ of $\mu$ such that $\nu(N) = 0$; compare Proposition 1.2(b)).

We say that a bijection $g$ of $\mathbb{R}^n$ is an algebraic transformation of $\mathbb{R}^n$ if $g$ is generated by bijections of $\mathbb{R}^n$ from the set $(\mathbb{R}(X_1, \ldots, X_n))^n$. For an algebraic transformation $g$ of $\mathbb{R}^n$ we say that $g$ is defined over the field $L \subset \mathbb{R}$ if $g$ is generated by some bijections of $\mathbb{R}^n$ from $(L(X_1, \ldots, X_n))^n$. For example, the functions

$$f(x, y) = (x^3 + 1, (y + 7)^5), \quad g(x, y) = \left( x, y + \frac{1}{x^2 + 1} \right)$$

and

$$(f^{-1} \circ g)(x, y) = \left( (x - 1)^{1/3}, \left( y + \frac{1}{x^2 + 1} \right)^{1/5} - 7 \right)$$

are algebraic transformations of $\mathbb{R}^2$ defined over $\mathbb{Q}$. Notice also that isometries and, more generally, nonsingular affine transformations of $\mathbb{R}^n$ are algebraic transformations of $\mathbb{R}^n$ that belong to the set $(\mathbb{R}(X_1, \ldots, X_n))^n$.

Now let $G$ be the group of all isometries of $\mathbb{R}^n$ and let $\mu$ be a $G$-invariant $\sigma$-finite measure on $\mathbb{R}^n$. Can we find a proper $G$-invariant extension of $\mu$?

This question has been discussed several times in the literature. In 1935 Szpilrajn proved that Lebesgue measure on $\mathbb{R}^n$ has a proper isometrically invariant extension (see [Sz]). In the same paper, he stated Sierpinski’s question: “Does there exist a maximal isometrically invariant extension of Lebesgue measure on $\mathbb{R}^n$?" A negative answer to this question, i.e., the theorem “every isometrically invariant measure that extends Lebesgue measure on $\mathbb{R}^n$ has a proper isometrically invariant extension,” was proved by several mathematicians. The first result of that kind was obtained independently by Pkhakadze (in 1958, see [P]) and Hulanicki (in 1962, see [Hu]) under the additional set-theoretical assumption that there does not exist a real measurable cardinal less
than or equal to continuum $2^\omega$, i.e., that there is no measure on $\mathbb{R}$ defined on all subsets of $\mathbb{R}$. In 1977, Harazisvili got the full result stated above without any set-theoretical assumptions for the one dimensional case, i.e., for $n = 1$ (see [Ha1]). Finally in 1983, Ciesielski and Pelc generalized Harazisvili's result to all $n$-dimensional Euclidean spaces $\mathbb{R}^n$ (see [CP]; for more historical details of this issue see also [Ci]). In the same paper Ciesielski and Pelc stated the problem of characterizing those groups $G$ of isometries of $\mathbb{R}^n$ for which every $\sigma$-finite $G$-invariant measure has a proper $G$-invariant extension (see [CP, p. 6]). A more technical version of the same problem, i.e., the problem of characterizing those groups $G$ of isometries of $\mathbb{R}^n$ for which $\mathbb{R}^n$ is a union of countable many $G$-absolutely negligible sets, was also stated by Harazisvili in [Ha2].

In the present paper we will consider a generalization of this problem to the case of $\gamma$-invariant measure where $\gamma: G \rightarrow (0, \infty)$ and $G$ is a group of algebraic transformations of $\mathbb{R}^n$. In particular our main theorem (see Abstract, or Theorem 3.1) implies that

"if $G$ is a transitive group of isometries of $\mathbb{R}^n$ then $\mathbb{R}^n$ is a countable union of $G$-absolutely negligible sets."

The above fact has been proved earlier by Harazisvili under the assumption of the continuum hypothesis (see [Ha2]). He also asked whether it is possible to remove this assumption from his theorem. Our results give an affirmative answer to this question.

The proof of our main theorem 3.1 uses a generalization of the technique of Ciesielski and Pelc [CP, Theorem 2.1, pp. 4–6]. The author wishes to thank Jan Mycielski for numerous important remarks about former versions of this paper. In particular it was Mycielski's suggestion to replace in the proof of [CP, Theorem 2.1] the linear basis of $\mathbb{R}$ over $\mathbb{Q}$ by a transcendence basis of $\mathbb{R}$ over $\mathbb{Q}$ and to study in this way algebraic transformations of $\mathbb{R}^n$. Compare also the paper of Weglorz [We, Theorem 2.4] which was influenced by Mycielski in a similar way.

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1. Measure theoretic preliminaries

In what follows we will need the following proposition essentially due to Szpilrajn (see [Sz, §2]).

**Proposition 1.1.** Let $\gamma: G \rightarrow (0, \infty)$ where $G$ is a group of bijections of a set $X$ and let $\mu: \mathcal{M} \rightarrow [0, \infty]$ be a $\gamma$-invariant measure on $X$. If a family $\mathcal{A}$ of subsets of $X$ is such that

1. $\mathcal{A}$ is closed under countable union,
2. if $A \in \mathcal{A}$ and $g \in G$ then $g[A] \in \mathcal{A}$,
3. every $A \in \mathcal{A}$ has $\mu$ inner measure zero,
then $\mu$ has a $\gamma$-invariant extension $\nu: \mathcal{N} \rightarrow [0, \infty]$ such that $\mathcal{A} \subset \mathcal{N}$ and $\nu(A) = 0$ for every $A \in \mathcal{A}$.

The construction of such an extension is very simple. If $\mathcal{F}$ is an ideal of subsets of $X$ generated by the family $\mathcal{A}$, and $\mathcal{N}$ stands for a $\sigma$-algebra generated by $\mathcal{M} \cup \mathcal{F}$ then all elements of $\mathcal{N}$ are of the form $(M \cup I_1)I_2$ where $M \in \mathcal{M}$ and $I_1, I_2 \in \mathcal{F}$. It is easy to see that $\nu: \mathcal{N} \rightarrow [0, \infty]$ such that $\nu((M \cup I_1)I_2) = \mu(M)$ is a well-defined $\gamma$-invariant measure on $X$ extending $\mu$.

In the proof of the next proposition, we use a method which goes back to Harazisvili’s paper [Ha1] (see also [CP, Proposition 1.9, p. 4]).

**Proposition 1.2.** Let $G$ be a group of bijections of $X$, $\gamma: G \rightarrow (0, \infty)$ and let $\mu$ be a $\gamma$-invariant $\sigma$-finite measure on $X$.

(a) If $N \subset X$ is such that there is an uncountable set $H \subset G$ such that $\mu_\gamma(h_1[N] \cap h_2[N]) = 0$, for distinct $h_1, h_2 \in H$, then $\mu_\gamma(N) = 0$.

(b) If $N \subset X$ is such that for every countable set $\{g_r: r = 0, 1, 2, \ldots\} \subset G$ we have $\mu_\gamma(\bigcup_{r=0}^{\infty} g_r[N]) = 0$ then there exists a $\gamma$-invariant extension $\nu$ of $\mu$ such that $\nu(N) = 0$.

(c) Moreover if $X = \bigcup_{k=0}^{\infty} N_k$ where each $N_k$ satisfies the assumption of (b) then $\mu$ has a proper $\gamma$-invariant extension.

**Proof.** (a) If $M \in \mathcal{M}$ is a subset of $N$ then $\mu(h_1[M] \cap h_2[M]) = 0$ for every distinct $h_1, h_2$ from $H$. But $\mu(h[M]) = \gamma(h) \cdot \mu(M)$ and $\gamma(h) \neq 0$ for every $h$ from $H$. Hence, $\sigma$-finiteness of $\mu$ implies that $\mu(M) = 0$ and so $\mu_\gamma(N) = 0$.

(b) By Proposition 1.1 it is enough to notice that every element of the family $\mathcal{A} = \{\bigcup_{r=0}^{\infty} g_r[N]: g_r \in G \text{ for } r = 0, 1, 2, \ldots\}$ has $\mu$ inner measure 0.

(c) By part (b), for each $k = 0, 1, 2, \ldots$ there is a $\gamma$-invariant extension $\nu_k$ of $\mu$ such that $\nu_k(N_k) = 0$. But all $N_k$’s cannot have $\mu$ measure zero. So some $\nu_k$ must be a proper extension of $\mu$.

In what follows, we will also use the following well-known fact. For the complex case the proof (using the Jensen’s Inequality) can be found in [GR, p. 9]. The direct proof follows also from Fubini’s theorem.

**Proposition 1.3.** If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a nonzero real analytic function then the set $Z = \{a \in \mathbb{R}^n: f(a) = 0\}$ has Lebesgue measure zero. In particular, if $h, g \in (\mathbb{R}(X_1, \ldots, X_n))^n$ are different algebraic transformations of $\mathbb{R}^n$ then the set $\{a \in \mathbb{R}^n: h(a) = g(a)\}$ has Lebesgue measure zero.

2. Algebraic preliminaries

A field $L \subset \mathbb{R}$ is said to be algebraically closed in $\mathbb{R}$ if $L = M \cap \mathbb{R}$ where $M \subset \mathbb{C}$ is an algebraic closure of $L$. Notice, that an algebraically closed field in $\mathbb{R}$ is real closed (i.e. satisfies the theory of real closed fields) in the sense defined in [CK or Ro]. The smallest field algebraically closed in $\mathbb{R}$ containing $L \subset \mathbb{R}$ is called a real closure of $L$ and it will be denoted by $\text{cl}_\mathbb{R}(L)$. The algebraic closure of a field $K$ will be denoted by $\text{cl}(K)$. 

The next proposition will be used only in the case of algebraic transformation \( g \) such that \( g^{-1} \in (\mathbb{R}(X_1, \ldots, X_n))^n \). In this case this is a well-known fact and can be proved using standard algebraic technic. However we like to prove it in more general form (that possibly can be used to answer Problem 3 stated in the end of the paper). For this we will need the following model-theoretic definition (compare e.g. [CK]).

A model \( L \) is said to be an elementary submodel of a model \( R \) if \( L \subseteq R \) and for every first order formula \( \varphi(x_1, \ldots, x_m) \) and any parameters \( a_1, \ldots, a_m \) from \( L \) the model \( L \) satisfies \( \varphi(a_1, \ldots, a_m) \) if and only if \( R \) satisfies \( \varphi(a_1, \ldots, a_m) \).

A theory \( T \) is said to be model complete if and only if for all models \( L \) and \( R \) of \( T \), if \( L \subseteq R \) then \( L \) is an elementary submodel of \( R \).

We need the following important theorem of A. Robinson (see [CK, p. 110] or [Ro, §3.3]).

**Theorem 2.1.** The theory \( T \) of real closed fields is model complete. In particular if \( L \subseteq R \) is a real closed field then \( L \) is an elementary submodel of \( R \).

As a corollary of this fact we easily obtain

**Proposition 2.1.** If \( g \) is an algebraic transformation of \( \mathbb{R}^n \) defined over a real closed field \( L \subseteq \mathbb{R} \) then

\[
(2.1) \quad g[L^n] = L^n.
\]

**Proof.** A first order formula \( \varphi(x_1, \ldots, x_n, y_1, \ldots, y_n) \) defined by \( g(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \) has as its parameters only elements from \( L \). If \( a = (a_1, \ldots, a_n) \in L^n \) then \( \mathbb{R} \) satisfies \( \exists y_1 \cdots \exists y_n \varphi(a_1, \ldots, a_n, y_1, \ldots, y_n) \) and so does \( L \) (by Theorem 2.1), i.e. \( g(a_1, \ldots, a_n) \in L^n \). This proves \( g[L^n] \subseteq L^n \). To show the converse inclusion it is enough to consider the formula \( \exists x_1 \cdots \exists x_n \varphi(x_1, \ldots, x_n, a_1, \ldots, a_n) \).

### 3. The main theorem

From now on let \( \mathcal{B} \) denote a transcendence base of \( \mathbb{R} \) over \( \mathbb{Q} \).

Now we are ready to prove our main lemma.

**Lemma 3.1.** Let \( H \subseteq (\mathbb{R}(X_1, \ldots, X_n))^n \) be an uncountable set of algebraic transformations of \( \mathbb{R}^n \). Then there exists an uncountable set \( H' \subseteq H \), a finite set \( A \subseteq \mathcal{B} \) and, for every \( h \in H' \), a finite set \( A_h \subseteq \mathcal{B}\setminus A \) with the following properties:

1. each \( h \in H' \) (and so \( h^{-1} \)) is defined over the field \( \text{cl}_R(\mathbb{Q}(A \cup A_h)) \);
2. \( A_{h_1} \cap A_{h_2} = \emptyset \) for distinct \( h_1, h_2 \in H' \);
3. for every \( h_1, h_2 \in H' \) if \( L = \text{cl}_R(\mathbb{Q}(\mathcal{B}\setminus (A_{h_1} \cup A_{h_2}))) \) then \( a \in h_1^{-1}[L^n] \cap h_2^{-1}[L^n] \) implies \( h_1(a) = h_2(a) \), i.e., \( h_1^{-1}[L^n] \cap h_2^{-1}[L^n] \subseteq \{a : h_1(a) = h_2(a)\} \).
Proof. In the definition of each \( h \in H \) we use only finitely many parameters (i.e. coefficients) so for every \( h \in H \) there exists a finite set \( B_h \subset \mathcal{B} \) such that

\[
h = (h_1, \ldots, h_n) \in [\text{cl}\mathcal{R}(\mathbb{Q}(B_h)(X_1, \ldots, X_n))]^n.
\]

Using for the family \( \{B_h; h \in H\} \) the \( \Delta \)-system argument (see e.g. [Je, Lemma 22.6, p. 226]) we can find an uncountable set \( H_0 \subset H \), a finite set \( A \subset \mathcal{B} \), a natural number \( m \) and, for every \( h \in H_0 \), a set \( A_h \) such that

(i) \( B_h = A \cup A_h \), and \( A \cap A_h = \emptyset \),
(ii) \( A_h \cap A_h' = \emptyset \) for distinct \( h_1, h_2 \in H_0 \),
(iii) \( A_h \) has exactly \( m \) elements.

Thus for the family \( H_0 \), the sets \( A, A_h \ (h \in H_0) \) already satisfy (1) and (2). Therefore it is enough to find an uncountable \( H' \subset H_0 \) which satisfies (3). We will do this in such a way that all elements of \( H' \) will have the same definitions with parameters from \( \mathcal{B} \).

Let \( Z = \{Z_1, \ldots, Z_m\} \) be a set of variables and, for \( h \in H_0 \), let \( \sigma'_h : A_h \rightarrow Z \) be a bijection. Then we can extend \( \sigma'_h \) to a field isomorphism \( \sigma''_h \) from \( \text{cl}(\mathcal{Q}(\mathcal{B})) = \mathcal{C} \) to \( \text{cl}(\mathcal{Q}(\mathcal{B} \setminus A_h)(Z)) \) in such a way that \( \sigma''_h(a) = a \) for every \( a \in \text{cl}(\mathcal{Q}(\mathcal{B} \setminus A_h)) \). Let us extend \( \sigma''_h \) to \( \sigma'_h : [\text{cl}(\mathcal{Q}(\mathcal{B}))(X_1, \ldots, X_n)]^n \rightarrow [\text{cl}(\mathcal{Q}(\mathcal{B} \setminus A_h)(Z))(X_1, \ldots, X_n)]^n \). But \( \sigma_h(h) \in [\text{cl}(\mathcal{Q}(A \cup Z))(X_1, \ldots, X_n)]^n \) and the field \( \text{cl}(\mathcal{Q}(A \cup Z)) \) is countable.

Define \( H' \subset H_0 \) as an uncountable set with the property

\[
\sigma_{h_1}(h_1) = \sigma_{h_2}(h_2) \quad \text{for every } h_1, h_2 \in H'.
\]

We prove that \( H' \) satisfies (3).

Let \( a \in h_1^{-1}[L^n] \cap h_2^{-1}[L^n] \), where \( L = \text{cl} \mathcal{R}(\mathcal{Q}(\mathcal{B} \setminus A_{h_1} \cup A_{h_2})) \) and \( h_1, h_2 \in H' \). Notice that \( a \in L^n \) as, by Proposition 2.1, (1) and (2),

\[
a \in h_1^{-1}[L^n] \cap h_2^{-1}[L^n] \subset h_1^{-1}[\text{cl} \mathcal{R}(\mathcal{Q}(\mathcal{B} \setminus A_{h_2}))]^n \cap h_2^{-1}[\text{cl} \mathcal{R}(\mathcal{Q}(\mathcal{B} \setminus A_{h_1}))]^n = \text{cl} \mathcal{R}(\mathcal{Q}(\mathcal{B} \setminus A_{h_1}))^n \cap \text{cl} \mathcal{R}(\mathcal{Q}(\mathcal{B} \setminus A_{h_2}))^n = L^n.
\]

Put \( h_1(a) = b_1 \) and \( h_2(a) = b_2 \). Thus \( b_1, b_2 \in L^n \). We have to prove that \( b_1 = b_2 \). But, by (*) and the fact that \( \sigma_{h_1}(c) = c = \sigma_{h_2}(c) \) for every \( c \in L^n \),

\[
b_1 = \sigma_{h_1}(b_1) = \sigma_{h_1}(h_1(a)) = \sigma_{h_1}(h_1)(\sigma_{h_1}(a)) = \sigma_{h_1}(h_1)(a)
= \sigma_{h_2}(h_2)(a) = \sigma_{h_2}(h_2)(\sigma_{h_2}(a)) = \sigma_{h_2}(h_2(a)) = \sigma_{h_2}(b_2) = b_2.
\]

This finishes the proof of Lemma 3.1.

As a next step we will prove an essential part of the assumptions of Proposition 1.2.

Lemma 3.2. If \( G \) is a group of algebraic transformations of \( \mathbb{R}^n \) and \( H \subset (\mathbb{R}(X_1, \ldots, X_n))^n \) is an uncountable subset of \( G \) then there exists a countable
family of sets \{N_k: k = 0, 1, 2, \ldots\} such that \( R^n = \bigcup_{k=0}^{\infty} N_k \) and that each \( N_k \) satisfies the condition:

for every countable set \( \{g_r: r = 0, 1, 2, \ldots\} \subset G \) there is an uncountable set \( H_0 \subset H \) such that for every distinct \( h_1, h_2 \in H_0 \)

\[
(3.1) \quad h_1^{-1} \left[ \bigcup_{r=0}^{\infty} g_r[N_k] \right] \cap h_2^{-1} \left[ \bigcup_{r=0}^{\infty} g_r[N_k] \right] \subset \{a \in R^n: h_1(a) = h_2(a)\}.
\]

Proof. Let \( \mathcal{B} \) be a transcendence base of \( R \) over \( Q \) and let \( H' \subset H \), \( A \) and \( A_h \) be as in Lemma 3.1. We choose an increasing sequence \( \mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \) of subsets of \( \mathcal{B} \) in such a way that \( \mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}_k \) and for every \( k \) the set

\[
(*) \quad H^k = \{h \in H': A_h \subset \mathcal{B}_{k+1} \setminus \mathcal{B}_k\}
\]

is uncountable.

Define \( N_k = [\text{cl}_R(Q(\mathcal{B}_k))]^n \). Then \( \bigcup_{k=0}^{\infty} N_k = R^n \).

Let us fix \( \{g_r: r = 0, 1, 2, \ldots\} \subset G \) and a natural number \( k \). Choose also a countable set \( \mathcal{A} \subset \mathcal{B} \) such that \( A \subset \mathcal{A} \) and every \( g_r \) is defined over \( \text{cl}_R(Q(\mathcal{A})) \). Let \( H_0 = \{h \in H^{k+1}: A_h \cap \mathcal{A} = \emptyset\} \).

By (*) the set \( H_0 \) is uncountable.

Let us fix arbitrary distinct \( h_1, h_2 \in H_0 \) and let \( L = \text{cl}_R(Q(\mathcal{B}(A_h \cup A_{h_2})) \).

Then, by (*) and definitions of \( H_0 \) and \( N_k \), we can conclude that \( N_k \subset L^n \) and the \( g_r \)'s are defined over \( L \). Hence, by Proposition 2.1,

\[
\begin{align*}
&h_1^{-1} \left[ \bigcup_{r=0}^{\infty} g_r[N_k] \right] \cap h_2^{-1} \left[ \bigcup_{r=0}^{\infty} g_r[N_k] \right] \subset h_1^{-1} \left[ \bigcup_{r=0}^{\infty} g_r[L^n] \right] \\
&\cap h_2^{-1} \left[ \bigcup_{r=0}^{\infty} g_r[L^n] \right] = h_1^{-1}[L^n] \cap h_2^{-1}[L^n]
\end{align*}
\]

and, by (3) of Lemma 3.1, \( h_1^{-1}[L^n] \cap h_2^{-1}[L^n] \subset \{a: h_1(a) = h_2(a)\} \).

Therefore

\[
\left[ \bigcup_{r=0}^{\infty} g_r[N_k] \right] \cap h_2^{-1} \left[ \bigcup_{r=0}^{\infty} g_r[N_k] \right] \subset h_1^{-1}[L^n] \cap h_2^{-1}[L^n] \subset \{a: h_1(a) = h_2(a)\}.
\]

This finishes the proof of Lemma 3.2.

Theorem 3.1. Let \( G \) be a group of algebraic transformations of \( R^n \), \( \gamma: G \to \{(0, \infty) \) and let \( \mu \) be a \( \gamma \)-invariant \( \sigma \)-finite measure on \( R^n \). If \( G \) has an uncountable subset \( H \subset (R(X_1, \ldots, X_n))^n \) with the property

\[
(3.2) \quad \mu_\ast(\{a: h_1(a) = h_2(a)\}) = 0 \quad \text{for every} \quad h_1, h_2 \in H, h_1 \neq h_2
\]

then \( \mu \) has a proper \( \gamma \)-invariant extension.

Proof. By (3.2) and Lemma 3.2 we have \( R^n = \bigcup_{k=0}^{\infty} N_k \) where, by Proposition 1.2(a), \( \mu_\ast(\bigcup_{r=0}^{\infty} g_r[N_k]) = 0 \) for every countable set \( \{g_r: r = 0, 1, 2, \ldots\} \subset G \)
and every \( k = 0, 1, 2, \ldots \). Hence, by Proposition 1.2(c), \( \mu \) has a proper \( \gamma \)-invariant extension.

**Corollary 3.1.** Let \( G \) be a group of algebraic transformations of \( \mathbb{R}^n \), \( \gamma : G \to (0, \infty) \) and let \( \mu \) be a \( \gamma \)-invariant \( \sigma \)-finite measure on \( \mathbb{R}^n \). If at least one of the following conditions holds

(C1) \( G \) contains uncountably many translations;
(C2) \( \mu \) extends the \( n \)-dimensional Lebesgue measure and the set \( G \cap (\mathbb{R}(X_1, \ldots, X_n))^n \) is uncountable;

then \( \mu \) has a proper \( \gamma \)-invariant extension.

**Proof.** It is enough to show that both (C1) and (C2) imply (3.2).

If (C1) holds and \( H \) is an uncountable set of translations then for every \( h_1, h_2 \in H \), \( h_1 \neq h_2 \) the set \( \{ a : h_1(a) = h_2(a) \} \) is empty, so (3.2) is satisfied.

If (C2) holds then (3.2) is implied by Proposition 1.3.

To solve Harazisvili’s problem we will need the following lemma due to Harazisvili (see [Ha2, Remark 2, p. 507]).

**Lemma 3.3.** Let \( G \) be a transitive group of isometries of \( \mathbb{R}^n \), i.e., such that for every \( a, b \in \mathbb{R}^n \) there exists \( g \in G \) with the property \( g(a) = b \). If \( A \subset \mathbb{R}^n \) is a countable union of proper affine hyperplanes of \( \mathbb{R}^n \) then \( A \) is \( G \)-absolutely negligible.

**Proof.** For \( k \leq n \) let \( \mathcal{A}_k \) denote the family of countable unions of affine hyperplanes of \( \mathbb{R}^n \) of dimension less than \( k \). We prove by induction on \( k \leq n \) that elements of \( \mathcal{A}_k \) are \( G \)-absolutely negligible.

So let \( k < n \) be such that the elements of \( \mathcal{A}_k \) are \( G \)-absolutely negligible.

Let us fix an arbitrary \( A \in \mathcal{A}_{k+1} \), a \( G \)-invariant \( \sigma \)-finite measure \( \mu \) on \( \mathbb{R}^n \) and a countable set \( \{ g_r : r = 0, 1, 2, \ldots \} \subset G \). By Proposition 1.2(a) it is enough to find a sequence \( \{ h_r : \zeta < \omega_i \} \subset G \) such that for every \( \zeta < \eta < \omega_i \)

\[
\mu_* \left( h_\zeta \left( \bigcup_{r=0}^{\infty} g_r[A] \right) \cap h_\eta \left[ \bigcup_{r=0}^{\infty} g_r[A] \right] \right) = 0.
\]

We will construct it by transfinite induction.

So let us assume that for some \( \xi < \omega_i \) we have already constructed \( \{ h_r : \zeta < \xi \} \subset G \) such that the condition (a) is satisfied for every \( \zeta < \eta < \xi \). Let \( A_i \) and \( H_j \) \((i, j = 0, 1, 2, \ldots)\) be affine hyperplanes of \( \mathbb{R}^n \) of dimensions less than or equal to \( k \) and such that

\[
\bigcup_{r=0}^{\infty} g_r[A] = \bigcup_{i=0}^{\infty} A_i \quad \text{and} \quad \bigcup_{\zeta < \xi} h_\zeta \left[ \bigcup_{r=0}^{\infty} g_r[A] \right] = \bigcup_{j=0}^{\infty} H_j.
\]

We have to find \( h_\zeta \) such that

\[
\mu_* \left( h_\zeta \left[ \bigcup_{i=0}^{\infty} A_i \right] \cap \bigcup_{j=0}^{\infty} H_j \right) = 0.
\]
But if $h_\xi[A_i] \neq H_j$ then $h_\xi[A_i] \cap H_j \in \mathcal{A}_k$, i.e., by inductive hypothesis, it is enough to construct $h_\xi \in G$ such that

(b) \quad h_\xi[A_i] \neq H_j \quad \text{for every } i, j = 0, 1, 2, \ldots.

Let $w \in \mathbb{R}^n$ represents a vector in $\mathbb{R}^n$ such that $w$ is not parallel to any $H_j$ $(j = 0, 1, 2, \ldots)$. Then for different reals $a, b$ the distances

$$\text{dist}(0, a \cdot w + H_j) \neq \text{dist}(0, b \cdot w + H_j) \quad \text{for every } j = 0, 1, 2, \ldots.$$ 

So we can choose $b \in \mathbb{R}$ such that

(c) \quad \text{dist}(0, -b \cdot w + H_j) \neq \text{dist}(0, A_i) \quad \text{for every } i, j = 0, 1, 2, \ldots.

Now let $h_\xi \in G$ be such that $h_\xi(0) = b \cdot w$. We prove that such $h_\xi$ satisfies (b).

By way of contradiction let us assume that for some $i$ and $j$

(d) \quad h_\xi[A_i] = H_j.

But $h_\xi = T \circ L$, where $L$ is an isometry of $\mathbb{R}^n$ preserving origin and $T$ is a translation such that $T(x) = x + b \cdot w$ for every $x \in \mathbb{R}^n$. Hence, by (d), $L[A_i] = T^{-1}[H_j] = -b \cdot w + H_j$ and so

$$\text{dist}(0, -b \cdot w + H_j) = \text{dist}(0, L[A_i]) = \text{dist}(0, A_i)$$ 

contradicting (c).

Thus we proved that $h_\xi$ satisfies (b). This finishes the proof of the lemma.

**Theorem 3.2.** If $G$ is a transitive group of isometries of $\mathbb{R}^n$ then $\mathbb{R}^n$ is a countable union of $G$-absolutely negligible sets. In particular every $\sigma$-finite $G$-invariant measure on $\mathbb{R}^n$ has a proper $G$-invariant extension.

**Proof.** Let $\{N_k: k = 0, 1, 2, \ldots\}$ be the family given in Lemma 3.2 where $H = G$. Then by Lemma 3.3 and Proposition 1.2(a) we have $\mu_\ast(\bigcup_{r=0}^{\infty} g_r[N]) = 0$ for every countable set $\{g_r: r = 0, 1, 2, \ldots\} \subset G$ and every $k = 0, 1, 2, \ldots$. Hence each $N_k$ is $G$-absolutely negligible.

**Generalizations, examples and problems**

1. Let us remark first that although we have stated Theorem 3.1 only for measures on $\mathbb{R}^n$ the theorem can be generalized for measures on $K^n$ where $K$ is either a real closed or algebraically closed field, since the theory of algebraic closed fields is also model complete (see [CK, p. 110]). Moreover, in the case of algebraically closed fields, the assumptions that $H \subset (K(X_1, \ldots, X_n))^n$ may be dropped.

2. If $X \subset K^n$ where $K$ is as above and we define algebraic transformations on $X$ in natural way, i.e., by functions generated by bijections of $X$ from $(K(X_1, \ldots, X_n))^n$, then we can prove Theorem 3.1 for measures on $X$. In particular we can conclude that it does not exist a maximal isometrically invariant extension of Lebesgue measure on $n$-dimensional sphere $S^n$.
3. Theorem 3.1 and its generalizations as in 1 and 2 can be also proved for complex measures (see [Ru, Chapter 6]).

4. For the cardinal number \( \kappa \) we say that a measure \( \mu \) on a set \( X \) is \( \kappa \)-finite if \( X \) is a union of \( \kappa \) many sets of finite measure. Theorem 3.1 can be also generalized in the following way:

"Let \( \kappa \) be a cardinal number, \( G \) be a group of algebraic transformations of \( \mathbb{R}^n \), \( \gamma: G \rightarrow (0, \infty) \) and let \( \mu \) be a \( \gamma \)-invariant \( \kappa \)-finite measure on \( \mathbb{R}^n \). If \( G \) has a subset \( H \subset (\mathbb{R}(X_1, \ldots, X_n))^n \) of power greater than \( \kappa \) with the property

\[
\{a: h_1(a) = h_2(a)\} = \emptyset \quad \text{for every } h_1, h_2 \in H, \ h_1 \neq h_2,
\]

then \( \mu \) has a proper \( \gamma \)-invariant extension."

5. In 4 condition (*) can be replaced by the original condition (3.2) if we assume in addition that the measure \( \mu \) is \( \kappa^+ \)-additive.

6. We can also generalize the results from 4 and 5 in the way described in 1 and 2.

7. By 4, if in particular \( \kappa \) is less than continuum \( 2^{\omega} \), \( G \) is a group of all isometries of \( \mathbb{R}^n \) and \( \mu \) is a \( \kappa \)-finite \( G \)-invariant measure then there exists a proper \( G \)-invariant extension of \( \mu \). However for \( \kappa \) equal to continuum \( 2^{\omega} \) this cannot be proved as it was shown in [CP, Theorem 3.1].

8. An interesting example, suggested to the author by Jan Mycielski, can be obtained by considering a hyperbolic space \( H^n \) for \( n \approx 2 \). If we identify \( H^n \) with the model \( \{(a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}: a_{n+1} > 0\} \) then the group \( G \) of all isometries of \( H^n \) is a group of algebraic transformations of \( \mathbb{R}^n \) and contains uncountably many translations. Moreover \( G \) is not a subgroup of a group of affine transformations of \( \mathbb{R}^n \) (see [MW or Be]). Let \( \nu \) be the hyperbolic invariant measure on \( H^n \) induced by the Haar measure on \( G \). So \( \nu \) is a \( G \)-invariant \( \sigma \)-finite measure on \( H^n \). Using the previous remarks and Corollary 3.1 we may conclude that the measure \( \nu \) does not have a maximal \( G \)-invariant extension.

9. Now we discuss the assumptions of Theorem 3.1, in particular condition (3.2).

First we prove that uncountability of \( H \subset G \) is essential (compare [Pe, Proposition 2.3, p. 14]).

Let \( G_0 \) be a group of all translations of \( \mathbb{R}^1 \) by rational numbers and let \( V \) be a Vitali set, i.e., \( V \cap H \) is a one element set for each orbit \( H \) of \( G_0 \). If we assume that there is a real measurable cardinal less than or equal to continuum (see [Jel]) then there is a measure \( \nu_0: \mathcal{P}(V) \rightarrow [0, 1] \), where \( \mathcal{P}(V) \) is a family of all subsets of the set \( V \). Define a measure \( \mu: \mathcal{P}(\mathbb{R}^1) \rightarrow [0, \infty] \) by

\[
\mu(A) = \sum_{g \in G_0} \nu_0(g^{-1}[g[V] \cap A]).
\]

It is easy to see that \( \mu \) is \( G_0 \)-invariant and \( \sigma \)-finite. But \( \mu \) is defined on all subsets of \( \mathbb{R}^1 \) so it cannot have any proper extension.
10. It can be also proved that if there is a real measurable cardinal less than or equal to the continuum then for every countable group $G$ of bijections of $\mathbb{R}^1$ there exists a $G$-invariant measure defined on $\mathcal{P} (\mathbb{R}^1)$, however this needs a little more careful definition.

11. The group $G_0$ defined in 9 is related to an interesting open problem of Andrzej Pelc (see [Pe, p. 27]).

**Problem 1.** Let $\mu$ be a $G_0$-invariant extension of Lebesgue measure on $\mathbb{R}^1$. Does there exist a proper $G_0$-invariant extension of $\mu$?

12. The next example shows that we have to assume about $G$ something more than only uncountability.

**Example.** Let $G'$ be the group of all rotations of $\mathbb{R}^2$ about the origin and let $\nu : \mathcal{P}(\mathbb{R}^2) \to [0, \infty]$ be such that $\nu(A) = 1$ when $(0, 0) \in A$ and $\nu(A) = 0$ otherwise. $\nu$ does not vanish at points, but still it is a $G'$-invariant measure. To correct this let $\mu$ and $G_0$ be as in Example 2 and let $\mu_1: \mathcal{P}(\mathbb{R}^3) \to [0, \infty]$ be a product measure of $\nu$ and $\mu$, i.e., $\mu_1(A) = \mu(\{x : (0, 0, x) \in A\})$. Then $\mu_1$ is $\sigma$-finite and $G_1$-invariant, where the group $G_1 = \{(g', g'') : g' \in G'$ and $g'' \in G_0\}$ is uncountable. It is also obvious that $\mu_1$ does not have any proper extension.

13. The reason that this example works is that $\mu_1$ is concentrated on a set $S = \{0\} \times \{0\} \times \mathbb{R}$ while $g[S] = S$ for every $g \in G_1$ and the group $\{g|_S : g \in G_1\}$ is countable. This suggests the following

**Definition.** Let $G$ be a group of bijections of a set $X$ and $\mu$ be a $G$-invariant measure on $X$. We say that $G$ is $\mu$-essentially countable if there is a set $S \subset X$ such that $\mu(X \setminus S) = 0$, $g[S] = S$ for all $g \in G$ and the group $\{g|_S : g \in G\}$ is countable.

**Problem 2.** Let $G$ be a group of algebraic transformations of $\mathbb{R}^n$ and $\mu$ be a $G$-invariant $\sigma$-finite measure of $\mathbb{R}^n$ such that $G$ is not $\mu$-essentially countable. Does $\mu$ have a proper $G$-invariant extension?

Recently the author has been informed that Piotr Zakrzewski proved the following result connected with the Problem 2: "If $G$ is a group of isometries of $\mathbb{R}^n$ and $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ is $G$-invariant then the group $G$ is $\mu$-essentially countable."

14. In the next example we will construct a $\gamma$-invariant measure $\mu$ on $\mathbb{R}^1$ where $\gamma$ will not be given in a classical way by Jacobian.

**Example.** Let $G_0 = \{x^{3^n} : n \in \mathbb{Z}\}$ be a group of transformations of $\mathbb{R}^1$ and let $V \subset \mathbb{R}^1 \setminus \{0\}$ be such that $(V \cup \{0\}) \cap H$ contains exactly one element for every orbit $H$ of $G$. Let $\mu_0 : \mathcal{P}(V) \to [0, 1]$ be a measure. For $n \in \mathbb{Z}$ let $g_n(x) = x^{3^n}$ and let $\mu_n : \mathcal{P}(g_n[V]) \to [0, 2^n]$ be defined by $\mu_n(g_n(A)) = 2^n \cdot \mu_0(A)$. Define
\[ \mu : \mathcal{P}(\mathbb{R}^1) \to [0, \infty) \text{ by} \]

\[ \mu(A) = \sum_{n \in \mathbb{Z}} \mu_n(g_n[A_n]) = \sum_{n \in \mathbb{Z}} 2^n \cdot \mu_0(A_n) \]

where \( A_n \subset V \) are such that \( A \setminus \{0\} = \bigcup_{n \in \mathbb{Z}} g_n[A_n] \).

It is easy to see that \( \mu \) is a \( \sigma \)-finite measure. Moreover,

\[ \mu(g_m[A]) = \mu \left( \bigcup_{n \in \mathbb{Z}} (g_m \circ g_n)[A_n] \right) = \sum_{n \in \mathbb{Z}} 2^{m+n} \cdot \mu_0(A_n) = 2^m \cdot \mu(A), \]

i.e., \( \mu \) is \( \gamma_0 \)-invariant where \( \gamma_0 : G_0 \to (0, \infty) \) is defined by \( \gamma_0(g_n) = 2^n \). It is easy to see that \( \gamma_0 \) has little to do with a classical Jacobian.

Our group \( G_0 \) is countable. But if we consider a measure \( \nu \) being a product measure of \( \mu \) and a one-dimensional Lebesgue measure \( m \) then \( \nu \) is a \( \sigma \)-finite \( \gamma \)-invariant where \( \gamma : G \to (0, \infty), \ G = \{(g_n, i) : g_n \in G_0 \text{ and } i \text{ is an isometry of } \mathbb{R}^1 \}, \text{ and } \gamma(g_n, i) = 2^n \). It is also obvious that \( G \) is uncountable. Moreover about \( \nu \) we can prove that if \( f \) is a homeomorphism of \( \mathbb{R}^2 \) and the system \( \langle \mathbb{R}^2, \mu_f, G_f, \gamma_f \rangle \) is induced by \( f \) from the system \( \langle \mathbb{R}^2, \mu, G, \gamma \rangle \) then \( G \) is not a subgroup of affine transformations of \( \mathbb{R}^2 \).

15. **Problem 3.** Is the assumption \( H \subset (\mathbb{R}(X_1, \ldots, X_n))^n \) essential in Theorem 3.1?

**References**


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