TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 318, Number 1, March 1990

MASSEY PRODUCTS IN THE COHOMOLOGY OF GROUPS WITH APPLICATIONS TO LINK THEORY

DAVID STEIN

ABSTRACT. Invariants of links in S^3 are developed using a modification of the Massey product of one-dimensional classes in the cohomology of certain groups. The theory yields two types of invariants, invariants which depend upon a collection of meridians, or basing, of a link, and invariants which do not. The invariants, which are independent of the basing, are compared with John Milnor's $\overline{\mu}$-invariants. For two component links, a collection of ostensibly based invariants is shown to be independent of the basing. If the linking number of the components of such a link is zero, the resulting invariants may be equivalent to the Sato-Levine-Cochran invariants.

INTRODUCTION

Invariants of links in S^3 are developed using a variation of the Massey product of one-dimensional classes in the cohomology of groups. A basing of a link is a set consisting of exactly one meridian of each component of the link. We obtain a collection of invariants, which depend upon a basing of the link, and a refinement of these which do not. The link invariants are compared with Milnor's $\overline{\mu}$-invariants [8, 9]. Porter [10] and Turaev [16] were the first to prove Stallings' conjecture [13] linking the $\overline{\mu}$-invariants and Massey products. For two component links, an infinite family of the based invariants is independent of the basing. These invariants, for two component links such that the linking number of the components is zero, may be equivalent to the Sato-Levine-Cochran invariants [1, 2, 12].

The present point of view has other implications. It has led to an algebraic means of calculating the invariants and a computer program which does so [15]. Invariants, with no indeterminacy, can be defined for disk links, and the indeterminacy of the invariants for links can be studied as one closes disk links and forms links. An "infinite" invariant may exist when certain of the finite invariants are zero. The effect, on the invariants, of certain alterations of the link may be tractable.

The paper contains four sections. In §1, a variant of the Massey product of one-dimensional classes in the cohomology of certain groups is shown to contain exactly one element. In §2, the theory of §1 is applied to link groups.

Received by the editors July 13, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57M25, 55S30.

The resulting invariants of links are compared with Milnor's invariants in §3. In §4, an infinite family of the based invariants of two component links is shown to be independent of the basing.

Most of this work is part of the author's Ph.D. thesis. He thanks his advisor, Professor Jerome Levine, for his guidance.

1. The Massey product invariants

The invariants are defined using a modification of the Massey product.

Definition 1.1 [5, 7]. Let $(\mathscr{A}, \delta)$ be a differential graded algebra, $q > 1$, and $c_1, \dots, c_q \in H^1(\mathscr{A})$. A *defining system*, for the Massey product of $(c_1, \dots, c_q)$, is an equivalence class of upper triangular $(q+1)\times(q+1)$ matrices, $\mathbf{M} = (m_{ij})$, such that

1. $m_{ii} = 1, m_{ij} \in \mathscr{A}^1$, and m_{ii+1} represents c_i;
2. $\delta m_{ij} = \sum_{i<k<j} m_{ik} m_{kj}$ for all $j \neq i+1$, and $(i,j) \neq (1, q+1)$.

Matrices $\mathbf{M}$ and $\mathbf{M}'$, which satisfy conditions 1 and 2, are equivalent if and only if $m_{ij} = m'_{ij}$ for all $(i,j) \neq (1, q+1)$. As δ is a derivation of degree one, $\sum_{i<k<j} m_{ik} m_{kj}$ is a cocycle. Let $[\mathbf{M}]$ be the cohomology class $[\sum_{1<k<q+1} m_{1k} m_{kq+1}]$. The *Massey product* of $(c_1, \dots, c_q)$, denoted by $\langle c_1, \dots, c_q \rangle$, is $\{[\mathbf{M}] | [\mathbf{M}]$ is a defining system for $(c_1, \dots, c_q)\}$. Defining systems for the Massey product of a particular ordered set may not exist, or the Massey product may contain more than one element [5, 7]. In the bar resolution of certain groups, with appropriate coefficients, the defining systems can be restricted so that the Massey products of particular ordered sets contain a unique element. This gives rise to the invariants.

Definition 1.2 [3]. Let q be a positive integer. A *coefficient system* is a collection of ring homomorphisms, $h_{ikj}\colon R_{ik} \otimes R_{kj} \to R_{ij}$ $(1 \le i < k < j \le q+1)$, such that the diagrams below commute.

$$\begin{array}{ccc} R_{ik} \otimes R_{kj} \otimes R_{jl} & \xrightarrow{1 \otimes h_{kjl}} & R_{ik} \otimes R_{kl} \\ {\scriptstyle h_{ikj} \otimes 1} \downarrow & & \downarrow {\scriptstyle h_{ikl}} \\ R_{ij} \otimes R_{jl} & \xrightarrow{h_{ijl}} & R_{il} \end{array}$$

Let $\mathscr{R}$ be a coefficient system. $\mathbf{U}(\mathscr{R}, q+1)$ denotes the group of upper triangular $(q+1)\times(q+1)$ matrices, $\{(a_{ij}) | a_{ij} \in R_{ij}$, and $a_{ii} = 1\}$. The multiplication is matrix multiplication using the h_{ikj}. Let $\overline{\mathbf{U}}(\mathscr{R}, q+1)$ denote the group of equivalence classes of elements in $\mathbf{U}$ under the relation $M \equiv M'$ if and only if $m_{ij} = m'_{ij}$ for all $(i,j) \neq (1, q+1)$. Thus, $\overline{\mathbf{U}} = \mathbf{U}/\mathrm{center}(\mathbf{U})$. One may consider an element of $\overline{\mathbf{U}}$ to be a "matrix" with the upper right-hand entry unspecified.

Let G be a group, $q > 1$, and for $1 \le i \le q$, let $u_i \in H^1(G, \mathbf{Z})$. Dwyer [3] shows that defining systems for the Massey product of $(u_1, \dots, u_q)$,

with coefficients in $\mathbf{Z}$, are in one-to-one correspondence with homomorphisms, $\overline{D}\colon G \to \overline{\mathbf{U}}(\mathbf{Z}, q+1)$, such that for each $1 \le i \le q$, the homomorphism $\overline{D}_{ii+1}$ is a representative of $-u_i$. Let $1 \to N \to F \to G \to 1$ be a presentation of G, and let $h\colon H_2(G, \mathbf{Z}) \to (N \cap F \sharp F)/(F \sharp N)$ be the Hopf isomorphism. Given $\overline{D}$, there is a homomorphism D such that the diagram below commutes.

$$\begin{tikzcd} & \mathbf{U}(\mathbf{Z}, q+1) \arrow[d, "\text{proj}"] \\ F \arrow[r] \arrow[ur, "D"] & G \arrow[r, "\overline{D}"] & \overline{\mathbf{U}}(\mathbf{Z}, q+1) \end{tikzcd}$$

Let $[\overline{D}]$ be the element of $\langle u_1, \dots, u_q \rangle$ obtained from the defining system corresponding to $\overline{D}$, and let $w \in H_2(G, \mathbf{Z})$. Dwyer shows that $\langle [\overline{D}], w \rangle = D_{1q+1}(h(w))$. There is the following naturality result.

Proposition 1.3. *Let $f\colon G \to H$ be a group homomorphism, $c_1, \dots, c_q \in H^1(H, \mathbf{Z})$, and $N\colon H \to \overline{\mathbf{U}}(\mathbf{Z}, q+1)$ a homomorphism such that N_{ii+1} represents $-c_i$. If $M\colon G \to \overline{\mathbf{U}}(\mathbf{Z}, q+1)$ is a homomorphism such that $M = N \circ f$, then $[M] = f^*[N]$. Note that M_{ii+1} represents $-f^*(c_i)$.*

We establish the formula for a homomorphism from a free group to $\overline{\mathbf{U}}(\mathbf{Z}, q)$, given the values of the homomorphism on the generators of the group. The formula utilizes the following symbols. $F = \langle x_1, \dots, x_n \rangle$ is the free group on $\{x_1, \dots, x_n\}$. $\mathbf{Z}[[t_1, \dots, t_n]]$ is the power series ring in n noncommuting variable with integer coefficients. $\mathscr{M}\colon F \to \mathbf{Z}[[t_1, \dots, t_n]]$ is the Magnus homomorphism defined by $x_i \to 1 + t_i$.

$$I(n) = \{(\tau_1, \dots, \tau_m) | \forall\ 1 \le l \le m, r_l \in \mathbf{Z} \text{ and } 1 \le r_l \le n\},$$

and $I(n)_s = \{(r_1, \dots, r_s) | (r_1, \dots, r_s) \in I(n)\}$. For $I = (r_1, \dots, r_m) \in I(n)$, let $t_I = t_{r_1} t_{r_2} \cdots t_{r_m} \in \mathbf{Z}[[t_1, \dots, t_n]]$. If $I \ne \varnothing$, and if $\omega \in F$, $m(I : \omega)$ is the coefficient of t_I in $\mathscr{M}(\omega)$, and if $I = \varnothing$, $m(I : \omega) = 1$. $S_{ij} = \{(c_1, \dots, c_s) | \forall l, c_l$ is a positive integer, and $\sum_1^s c_l = j - i\}$. For $i \le p \le j$, $S_{ij}^p = S_{ip} \times S_{pj}$. If, for each $1 \le i \le s$, $f_i\colon \{x_1, \dots, x_n\} \to \mathbf{Z}$ is a function, and if $I = (r_1, \dots, r_s) \in I(n)_s$,

$$(f_1, \dots, f_s)(I) = f_1(x_{r_1}) f_2(x_{r_2}) \cdots f_s(x_{r_s}).$$

Lemma 1.4 [10]. $m(I : \nu\omega) = \sum_{(I_1, I_2) = I} m(I_1 : \nu) m(I_2 : \omega)$.

Lemma 1.5. *For each $1 \le i < j \le q+1$, let $a_{ij}\colon \{x_1, \dots, x_n\} \to \mathbf{Z}$ be a function. For each $\omega \in F$, define*

$$(1) \qquad a_{ij}(\omega) = \sum_{s=1}^{j-i} \sum_{c_1, \dots, c_s \in S_{ij}} \sum_{I \in I(n)_s} (a_{ii+c_1} a_{i+c_1 i+c_1+c_2} \cdots a_{j-c_s j})(I) m(I : \omega).$$

Let $A = (a_{ij})$; of course, $a_{ii}(\omega) = 1$. Then, $A\colon F \to \mathbf{U}(\mathbf{Z}, q+1)$ is a homomorphism. Consequently, if $A\colon F \to \mathbf{U}(\mathbf{Z}, q+1)$ is a homomorphism, with coordinate functions a_{ij}, then $A(\omega)$ is given by the above formula.

Proof. It suffices to show that $a_{ij}(\nu\omega) = \sum_{p=i}^{j} a_{ip}(\nu)a_{pj}(\omega)$.

$$a_{ij}(\nu\omega) = \sum_{s=1}^{j-i} \sum_{c_1,\dots,c_s \in S_{ij}} \sum_{I \in I(n)_s} \left((a_{i\,i+c_1} \cdots a_{j-c_s\,j})(I) \times \left(\sum_{(I_1,I_2)=I} m(I_1:\nu)m(I_2:\omega) \right) \right),$$

$$\begin{aligned}
&\sum_{p=i}^{j} a_{ip}(\nu)a_{pj}(\omega) \\
&\quad = \sum_{p=i}^{j} \left[\left(\sum_{s_1=1}^{p-i} \sum_{c_1\cdots c_{s_1} \in S_{ip}} \sum_{I \in I(n)_{s_1}} (a_{i\,i+c_1} \cdots a_{p-c_{s_1}\,p})(I) m(I:\nu) \right) \right. \\
&\qquad \left. \times \left(\sum_{s_2=1}^{j-p} \sum_{d_1,\dots,d_{s_2} \in S_{pj}} \sum_{J \in I(n)_{s_2}} (a_{p\,p+d_1} \cdots a_{j-d_{s_2}\,j})(J) m(J:\omega) \right) \right] \\
&\quad = \sum_{p=i}^{j} \sum_{s=1}^{j-i} \sum_{c_1,\dots,c_s \in S_{ij}^p} \sum_{I \in I(n)_s} ((a_{i\,i+c_1} \cdots a_{j-c_s\,j})(I) \\
&\qquad \times m(r_1,\dots,r_l:\nu)m(r_{l+1},\dots,r_s:\omega)).
\end{aligned}$$

Here $I = (r_1,\dots,r_s)$, and $p = c_1 + \cdots + c_l$. The sequence $(c_1,\dots,c_s) \in S_{ij}$ is in S_{ij}^p for $p = i + c_1, i + c_1 + c_2, \dots, i + c_1 + \cdots + c_s$. Thus

$$\begin{aligned}
\sum_{p=i}^{j} a_{ip}(\nu)a_{pj}(\omega) &= \sum_{s=1}^{j-i} \sum_{c_1,\dots,c_s \in S_{ij}} \sum_{I \in I(n)_s} \left[(a_{i\,i+c_1} \cdots a_{j-c_s\,j})(I) \right. \\
&\qquad \left. \times \left(\sum_{(I_1,I_2)=I} m(I_1:\nu)m(I_2:\omega) \right) \right] \\
&= a_{ij}(\nu\omega).
\end{aligned}$$

The Massey product invariants of a group are defined relative to a subset.

Definition 1.6. Let G be a group, $\overline{D}\colon G \to \overline{\mathbf{U}}(\mathbf{Z}, q+1)$ a homomorphism, and $V \subset G$. $\overline{D}$ is *zero* on V if for each $v \in V$ and each (i,j) such that $j \neq i+1, \overline{D}_{ij}(v) = 0$.

Let d be an integer, $C = (c_1,\dots,c_q) \in H^1(G,\mathbf{Z}/d) \times \cdots \times H^1(G,\mathbf{Z}/d)$, and $V \subset G$. An integer $L_V(C)$, a coefficient system $R_V(C)$, and a subset $T_V(C)$ of $H^2(G,\mathbf{Z}/d)$ are defined recursively. For $1 \le a \le b \le q$, let $C_a^b = (c_a,\dots,c_b)$. If $C = (c_1,c_2), L_V(C) = d, R_{12} = R_{23} = R_{13} = \mathbf{Z}/d$, and $T_V(C) = \{c_1 \cdot c_2\}$. For $q > 2$ define $L_V(C)$ to be the

$$\gcd\{\langle k,w\rangle, L_V(C_1^{q-1}), L_V(C_2^q) | k \in T_V(C_1^{q-1}) \cup T_V(C_2^q) \text{ and } w \in H_2(G,\mathbf{Z}/d)\}.$$

$R_V(C)_{ij} = \mathbf{Z}/a_{ij}$ where

1. $a_{1q+1} = L_V(C)$;
2. $a_{ii+1} = d$;
3. $a_{ij} = \gcd\{L_V(C_i^{j-1}), \langle k, w\rangle | k \in T_V(C_i^{j-1})$, and $w \in H^2(G, \mathbf{Z}/d)\}$ for other (i, j).

Note that if $s \le i < j \le t$, then $a_{st}|a_{ij}$. Let $h_{ikj}: R_{ik} \otimes R_{kj} \to R_{ij}$ be the tensor product of the projection maps. Thus, $R_V(C)$ is a coefficient system. Define $T_V(C) = \{[\mathbf{M}] \in H^2(G, \mathbf{Z}/L_V(C)) | \mathbf{M}$ is a defining system for the massey product of $(c_1, \dots, c_q)$, with coefficients in $R_V(C)$, which is zero on $V\}$.

The subsets $T_V(C)$ contain exactly one element provided V satisfies a certain condition. The lower central series $G_1 \supset G_2 \supset \cdots \supset G_k \supset \cdots$ of G is defined by $G = G_1$, and $G_k = [G, G_{k-1}]$. If H is a subgroup of G, and d is an integer, $G\sharp_d H$ is the normal subgroup of G generated by $\{[g_1, h]g_2^k | g_1, g_2 \in G, h \in H$, and $d|k\}$ [13].

Definition 1.7. Let $A = \langle x_1, \dots, x_n\rangle$ be the free group on n generators. A subset $V = \{v_1, \dots, v_n\}$ of G has property $M(d)$ provided there is a presentation $1 \to N \to F \to G \to 1$ of G such that, $\forall q > 1$, there is a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 & \to & N & \to & F & \to & G & \to & 1 \\ & & & & {\scriptstyle n_q}\downarrow & & \downarrow{\scriptstyle a_q = \mathrm{proj}} & & \\ 1 & \to & N(q) & \to & A & \xrightarrow{p_q} & G/G_q & \to & 1 \end{array} \tag{2}$$

and the following propositions are true.

1. $\forall q > 1$, $p_q(x_i) = \mathrm{proj}(v_i)$.
2. $\exists\{r_{1q}, \dots, r_{mq}\} \subset A\sharp_d A$ such that $N(q)$ is the normal closure of $\{r_{1q}, \dots, r_{mq}\} \cup A_q$.
3. $\forall i$ and $q, \exists w_{iq} \in N \cap F\sharp_d F$ such that $n_q(w_{iq}) \equiv r_{iq} \pmod{A_q}$.

Lemma 1.8. *Suppose $V = \{v_1, \dots, v_n\} \subset G$ satisfies $M(d)$; then*

1. $H_1(G, \mathbf{Z}/d) \cong (\mathbf{Z}/d)^n$.
2. *If $D|d$, then $H^2(G, \mathbf{Z}/D) \cong \hom(H_2(G, \mathbf{Z}/d), \mathbf{Z}/D)$.*
3. *$H_2(G/G_q, \mathbf{Z}/d) \cong (N(q) \cap A\sharp_d A)/A\sharp_d N(q)$, and this quotient of subgroups of A is generated by $\{r_{iq} | 1 \le i \le m\} \cup A_q$.*

Proof. Statement 1 follows from condition 2 of the definition, with $q = 2$. Statement 2 follows from 1 and the universal coefficient theorem. The isomorphism stated in 3 is the $\bmod d$ Hopf isomorphism. Condition 2 of the definition implies that $N(q) = N(q) \cap A\sharp_d A$, and the other half of statement 3 follows.

Theorem 1.9. *If $V = \{v_1, \dots, v_n\}$ satisfies $M(d)$, then for all $C = (c_1, \dots, c_q) \in H^1(G, \mathbf{Z}/d) \times \cdots \times H^1(G, \mathbf{Z}/d)$:*

1. *$T_V(C)$ contains exactly one element;*

2. $\forall w \in H_2(G, \mathbf{Z}/\mathbf{d})$, *and* $r > 1$:

$$\langle T_V(C), w\rangle = (-1)^q \sum_{I \in I(n)_q} (a_{q+r}^{*-1}(c_1) \cdots a_{q+r}^{*-1}(c_q))(I) m(I : h(a_{q+r_*}(w)));$$

3.

$$L_V(C) = \gcd\left\{ d, \sum_{I \in I(n)_{b-a+1}} (a_{q+r}^{*-1}(c_a) \cdots a_{q+r}^{*-1}(c_b))(I) m(I : r_{jq+r}) | \right.$$
$$\left. 1 \le j \le m, 1 \le a < b \le q+1, \text{ and } (a,b) \ne (1, q+1) \right\}.$$

Proof. For $r \ge 0$ and any coefficient system $\mathscr{R}$ the lower central subgroups, $\overline{\mathbf{U}}(\mathscr{R}, q)_{q+r-1}$, and $\mathbf{U}(\mathscr{R}, q)_{q+r}$ are 1. Thus a homomorphism $M: G \to \overline{\mathbf{U}}(\mathscr{R}, q)$ induces a homomorphism $M': G/G_{q+r} \to \overline{\mathbf{U}}(\mathscr{R}, q)$ and by Proposition 1.3, $a_{q+r}^*([M']) = [M]$. Furthermore, $[M']$ vanishes on A_{q+r}, and thus, Lemma 1.8 implies that

$$\begin{aligned}\langle [M], H_2(G, \mathbf{Z}/d)\rangle &= \langle [M'], a_{q+r_*}(H_2(G, \mathbf{Z}/d))\rangle \\ &= \langle [M'], \{r_{1q+r}, \dots, r_{mq+r}\}\rangle.\end{aligned}$$

Lemma 1.8 also implies that

$$a_{q+r}^*: H^1(G/G_{q+r}, \mathbf{Z}/d) \to H^1(G, \mathbf{Z}/d)$$

is an isomorphism.

The proof is by induction on q. Let $q = 2$, and let M be a defining system for $\langle c_1, c_2\rangle$.

$$\begin{aligned}\langle T_V(C), w\rangle &= \langle [M], w\rangle = \langle [M'], a_{2+r_*}(w)\rangle \\ &= \sum_{I \in I(n)_2} (a_{2+r}^{*-1}(c_1) a_{2+r}^{*-1}(c_2))(I) m(I : h(a_{2+r_*}(w))).\end{aligned}$$

The last equality follows from Lemma 1.5.

Let $C = (c_1, \dots, c_q)$. $L_V(C) = \gcd\{\langle k, w\rangle | k \in T_V(C_a^b), 1 \le a < b \le q, (a,b) \ne (1,q)$, and $w \in H_2(G, \mathbf{Z}/d)\}$. By induction and the relationships mentioned at the beginning of the proof, statement 3 of the theorem holds.

We show that $T_V(C)$ contains at least one element. Let $\mathscr{R}$ denote $R_V(C)$. Define $\widetilde{M}: A \to \overline{\mathbf{U}}(\mathscr{R}, q+1)$ by

$$\widetilde{M}_{ij}(x_s) = \begin{cases} -a_{q+r}^{*-1}(c_i)(p_{q+r}(x_s)), & \text{if } j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\widetilde{M}_{ij}(A_{q+r}) = 0$, and $\widetilde{M}_{ij}(r_{lq+r}) \equiv 0 \pmod{a_{ij}}$. Thus $\widetilde{M} = M' \circ p_{q+r}$. Let $M = M' \circ a_{q+r}$. M is zero on V, and thus $[M] \in T_V(C)$. $\langle [M], w\rangle$ is evidently given by statement 2 of the theorem.

We show that $T_V(C)$ contains at most one element. Let N be another defining system for $\langle c_1, \dots, c_q\rangle$ which is zero on V. Then N' is zero on $\{p_{q+r}(x_1), \dots, p_{q+r}(x_n)\}$. Lemma 1.5 implies that $N' = M'$, and thus, $M = N$.

For the present work, a link is a collection of smoothly embedded, ordered, oriented circles in S^3.

Proposition 1.10. *Let $L \subset S^3$ be a link with components $K_1, \dots, K_n$. For $1 \le i \le n$, let v_i be a meridian of K_i, and let $V = \{v_1, \dots, v_n\} \subset \pi_1(S^3 - L, *)$.*

1. *V satisfies property $M(0)$.*
2. *Let N be the 3-manifold obtained by framed surgery on L, and let $d = \gcd\{lk(K_i, K_j) | 1 \le i \le j \le n\}$. $lk(K_i, K_j)$ is the linking number of K_i and K_j if $i \ne j$, while $lk(K_i, K_i)$ is the framing number of K_i. Then $V \subset \pi_1(N, *)$ satisfies $M(d)$.*

See [9] or §2 for a proof of 1, and see §4 for a proof of 2.

$T_V(C)$ is a based invariant of a link, as it generally depends upon the choice of meridians. One eliminates this dependence by increasing the indeterminacy. Suppose $V = \{v_1, \dots, v_n\} \subset G$ satisfies $M(d)$, then $\{v_1, \dots, v_n\}$ and $\{v_1^*, \dots, v_n^*\}$ freely generate $H_1(G, \mathbf{Z}/d)$ and $H^1(G, \mathbf{Z}/d)$, respectively, over $\mathbf{Z}/d$. Let $I = (l_1, \dots, l_q)$ denote $(v_{l_1}^*, \dots, v_{l_q}^*)$.

Definition 1.11. 1. $\Lambda_V(I) = \gcd\{\langle T_V(J), w\rangle | J$ is a proper subsequence of I, and $w \in H_2(G, \mathbf{Z}/d)\}$.

2. $\sigma_V(I)$ is the image of $T_V(I)$ in $H^2(G, \mathbf{Z}/\Lambda_V(I))$.

Definition 1.12. Subsets $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$ of G are conjugate if, $\forall u_i \in U, \exists g_i \in G$ such that $v_i = g_i u_i g_i^{-1}$.

Theorem 1.13. *If U and V are conjugate, and U and V satisfy $M(d)$, then for all $I \in I(n), \Lambda_V(I) = \Lambda_U(I)$, and $\sigma_V(I) = \sigma_U(I)$.*

This theorem follows from two lemmas. For $1 \le i < j \le q+1, (i,j) \ne (1, q+1)$, define $b_{ij} = \gcd\{\langle T_V(J), w\rangle | J$ is a subsequence of $(l_i, \dots, l_{j-1})$, and $w \in H^2(G, \mathbf{Z}/d)\}$, and define $b_{1q+1} = \gcd\{\langle T_V(J), w\rangle | J$ is a proper subsequence of $(l_1, \dots, l_q)$, and $w \in H^2(G, \mathbf{Z}/d)\}$. Note that if $i \le s < t \le j$, then $b_{ij} | b_{st}$. Define $R_{ij} = \mathbf{Z}/b_{ij}$, and let $h_{ikj}\colon R_{ik} \otimes R_{kj} \to R_{ij}$ be the tensor product of the projection maps. Let $\mathscr{R} = (R_{ij})$. Then $\mathscr{R}$ is a coefficient system.

Definition 1.14. Given I as above, a homomorphism $M = (m_{ij})\colon G \to \overline{\mathbf{U}}(\mathscr{R}, q+1)$ is semizero on V for I if

1. m_{ii+1} represents $-v_{l_i}^*, 1 \le i \le q$, and
2. if $p \notin \{l_i, l_{i+1}, \dots, l_{j-1}\}$, then $\forall i \le s < t \le j, m_{st}(v_p) = 0$.

Lemma 1.15. *If M is semizero on V for I, then $[M] = \sigma_V(I)$.*

Proof. Suppose M is semizero on V, and $M' \colon G/G_{q+r} \to \overline{\mathbf{U}}(\mathscr{R}, q+r)$ is the induced homomorphism. Then

$$\begin{aligned}\langle [M], w\rangle &= \langle [M'], a_{q+r_*}(w)\rangle \\ &= \sum_{s=1}^{q} \sum_{c_1,\dots,c_s \in S_{1q+1}} \sum_{J \in I(n)_s} (m'_{11+c_1} \cdots m'_{q+1-c_s\, q+1})(J) m(J \colon a_{q+r_*}(w)).\end{aligned}$$

$m'_{1k_1} \cdots m'_{k_r k_{q+1}}(j_1, \dots, j_{r+1}) = 0$, unless

$$\begin{aligned} j_1 &\in \{l_1, \dots, l_{k-1}\}, \\ j_2 &\in \{l_k, \dots, l_{k_2-1}\}, \\ &\vdots \\ j_{r+1} &\in \{l_k, \dots, l_q\}. \end{aligned}$$

If this is the case, and $J \neq I$, $m(J : a_{q+r_*}(w)) \equiv 0 \pmod{b_{1\,q+1}}$. Thus

$$\langle [M], w\rangle \equiv m(I : a_{q+r_*}(w)) = \langle \sigma_V(I), w\rangle.$$

Lemma 1.16. *If $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$ are conjugate subsets of G, and M is semizero on U for I, then M is semizero on V for I.*

Proof. Let M be semizero on V for I. Since, for each $1 \le i \le n$, v_i is a conjugate of u_i, $v_i^* = u_i^*$. Thus $[m_{ii+1}] = -u_{li}^*$.

For each $1 \le i < j \le q+1$, $(i, j) \neq (1, q+1)$, deleting rows 1 through $i-1$ and rows $j+1$ through $q+1$, and deleting columns 1 through $i-1$ and columns $j+1$ through $q+1$ defines a homomorphism, $\Phi_{ij} \colon \overline{\mathbf{U}}(\mathbf{Z}, q+1) \to \mathbf{U}(\mathbf{Z}, j-i+1)$. Precisely, $(\Phi_{ij}(B))_{st} = b_{s+i-1, i+t-1}$. Φ_{ij} is defined similarly for the matrix groups with coefficients in a coefficient system.

Suppose $p \notin \{l_i, l_{i+1}, \dots, l_{j-1}\}$. Then $(\Phi_{ij} \circ M)(v_p) = 1$. Thus

$$\begin{aligned}(\Phi_{ij} \circ M)(u_p) &= (\Phi_{ij} \circ M)(g_p v_p g_p^{-1}) \\ &= (\Phi_{ij} \circ M)(g_p)(\Phi_{ij} \circ M)(v_p)(\Phi_{ij} \circ M)(g_p^{-1}) \\ &= 1.\end{aligned}$$

Therefore, M is semizero on U for I. Theorem 1.13 follows from these two lemmas.

2. The invariants for link groups

In this section Milnor's construction of the $\overline{\mu}$-invariants is outlined, and one sees that any collection of meridians of a link group satisfies $M(0)$. As any two collections of meridians are conjugate, the σ-invariants, defined above, are independent of the basing. The based invariants are shown to be invariants of based P.L. I-equivalence, while the link invariants are invariants of smooth cobordism and P.L. isotopy.

J. Milnor defines the $\overline{\mu}$-invariants [8, 9]. Let L be an n-component link, and for each $1 \leq i \leq n$, let r_i be the number of crossings over the ith component in some projection of L. Then G, the fundamental group of the complement of L, has a presentation of the form $\langle x_{ij} | w_{ij} ; 1 \leq i \leq n, 1 \leq j \leq r_i \rangle$, where

$$\begin{aligned} w_{ij} &= x_{ij+1}^{-1} v_{ij}^{-1} x_{i1} v_{ij} \qquad (j \neq r_i), \\ w_{ir_i} &= x_{i1}^{-1} v_{ir_i}^{-1} x_{i1} v_{ir_i}. \end{aligned} \tag{3}$$

(x_{i1}, v_{ir_i}) represents a meridian-longitude pair of the ith component [11]. Let F be the free group $\langle x_{ij} \rangle$, and let A be the free group $\langle x_1, \dots, x_n \rangle$. For $q \geq 2$, Milnor defines homomorphisms $n_q \colon F \to A$, so that

1. $n_q(x_{i1}) = x_i$;
2. for each q, the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "p"] \arrow[d, "n_q"'] & G \arrow[r] \arrow[d, "\mathrm{proj}"] & 1 \\ A \arrow[r, "p_q"] & G/G_q \arrow[r] & 1 \end{tikzcd} \tag{4}$$

He shows that, for $j \neq r_i$, $n_q(v_{ij}) \equiv 0 \pmod{A_q}$, and letting $v_{iq} = n_q(v_{ir_i})$, that $\ker(p_q)$ is the normal subgroup generated by $\{[x_i, v_{iq}], A_q \mid 1 \leq i \leq n\}$. Thus $\{p(x_{11}), \dots, p(x_{n1})\}$, and hence, any collection of meridians satisfies $M(0)$. He also shows that, $\forall r \geq 1, v_{iq} \equiv v_{iq+r} \pmod{A_q}$. Thus, if $s < q$, and $I \in I(n)_s$, $m(I : v_{iq}) = m(I : v_{iq+r})$. He then defines the $\overline{\mu}$-invariants of a link. Let $r \geq 2$, and $I = (i_1, i_2) \in I(n)$. $\overline{\mu}(I) = m(i_1 : v_{i_2 r})$. Let $q > 2$, $r \geq 0$, and let $I = (i_1, \dots, i_q)$.

$$\begin{aligned} \Delta(I) = \gcd\{ & m(s_a, \dots, s_b : v_{s_{b+1}q+r}) \mid (s_a, \dots, s_{b+1}) \text{ is a cyclic permutation} \\ & \text{of a proper subsequence of } (i_1, \dots, i_q)\}, \\ \overline{\mu}(I) = \text{the residue of } & m(i_1, \dots, i_{q-1} : v_{i_q q+r}) \pmod{\Delta(I)}. \end{aligned}$$

If V is a collection of meridians of a link group of an n-component link, then V satisfies $M(0)$. Thus, for $q > 1$ and C, a q-tuple of cohomology classes, or I, an element of $I(n)_q$, the invariants $T_V(C)$ and $\sigma_V(I)$ are defined. As any two collections of meridians are conjugate, one defines $\sigma(I) = \sigma_V(I)$.

We show that the invariants are invariants of the equivalence classes of links stated above. Let $N(L)$ be a tubular neighborhood of L, $X = \overline{S^3 - N(L)}$, and $G = \pi_1(X, x)$. The boundary of X is the disjoint union of n torii, $T_1, \dots, T_n$, which are ordered and oriented compatibly with the components of L. Let $\tau_j \in H_2(X)$ be the class represented by T_j. Let $\hbar \colon H_2(X) \to H_2(G)$ be the Hopf map, and let $\omega_j = \hbar(\tau_j)$.

Let L^0 and L^1 be n-component links, and let $V^0 = \{v_1^0, \dots, v_n^0\}$ and $V^1 = \{v_1^1, \dots, v_n^1\}$ be collections of meridians of L^0 and L^1 respectively.

Definition 2.1. (L^0, V^0) and (L^1, V^1) are based P.L. I-equivalent if

1. there is a proper P.L. embedding $\Phi\colon \coprod_{j=1}^n S^1 \times I \to S^3 \times I$ such that, for $i = 0, 1, \Phi_i$, which is $\Phi|\coprod S^1 \times \{i\}$, is L^i, and

2. for all $1 \le j \le n$ and $q > 1$, $\iota_\#^0(v_j^0) \pmod{G_q}$. Here X is the complement of a regular neighborhood of the image of Φ, $X^i = X \cap (S^3 \times \{i\})$, $\iota^i\colon X^i \to X$ is inclusion, $G = \pi_1(X, x)$, and $G^i = \pi_1(X^i, x^i)$.

Theorem 2.2. 1. *If (L^0, V^0) and (L^1, V^1) are based P.L. I-equivalent, then for all q and all $C = (c_1, \dots, c_q) \in H^2(X) \times \cdots \times H^2(X), R_{V^0}(\iota^{0^*}(C)) = R_{V^1}(\iota^{1^*}(C))$, and for all $1 \le j \le n$,*

$$\langle T_{V^0}(\iota^{0^*}(C)), \omega_j^0\rangle = \langle T_{V^1}(\iota^{1^*}(C)), \omega_j^1\rangle.$$

2. *If L^0 and L^1 are P.L. isotopic or cobordant, then for all indexing sequences I of n, $\Lambda^0(I) = \Lambda^1(I)$, and $\langle \sigma^0(I), \omega_j^0\rangle = \langle \sigma^1(I), \omega_j^1\rangle$.*

Proof of 1. Note that, for $1 \le j \le n$, $\iota_*^0(\nu_j^0) = \iota_*^1(\nu_j^1)$, and $\iota_*^0(\tau_j^0) = \iota_*^1(\tau_j^1)$. One inducts on q. For $q = 2$, the result follows by the naturality of the cup product. By induction, $R_{V^0}(\iota^{0^*}(C)) = R_{V^1}(\iota^{1^*}(C))$. Denote this coefficient system by $\mathscr{R}$. Stallings' theorem [13] implies that

$$\overline{\iota_\#^i}\colon G^i/G_q^i \to G/G_q$$

is an isomorphism for all $q > 1$. Thus a defining system $\Upsilon^0\colon G^0 \to \overline{\mathbf{U}}(\mathscr{R}, q+1)$ for C^0 induces the commutative diagram:

$$\begin{tikzcd}
G^0 \arrow[d, "\iota^0"'] \arrow[dr, "\Upsilon^0"] & \\
G \arrow[r, "\Upsilon"] & \overline{\mathbf{U}}(\mathscr{R}, q+1). \\
G^1 \arrow[u, "\iota^1"] \arrow[ur, "\Upsilon^1"'] &
\end{tikzcd}$$

Υ and Υ^1 are defining systems for C and C^1 respectively. If L^0 and L^1 are based P.L. I-equivalent, and Υ^0 is zero on V^0, then Υ is zero on $\iota_\#^0(V^0)$, and Υ^1 is zero on V^1. Thus,

$$\begin{aligned}
\langle T_{V^0}(C^0), \omega_j^0\rangle &= \langle [\Upsilon^0], \omega_j^0\rangle \\
&= \langle [\Upsilon], \iota_*^0(\omega_j^0)\rangle \quad \text{(by Proposition 1.3)} \\
&= \langle [\Upsilon], \iota_*^1(\omega_j^1)\rangle = \langle [\Upsilon^1], \omega_j^1\rangle = \langle T_{V^1}(C^1), \omega_j^1\rangle,
\end{aligned}$$

which proves statement 1.

Proof of 2. Suppose that L^0 and L^1 are P.L. isotopic or cobordant and C is a collection of the canonical generators of $H^1(X)$, then C^0 and C^1 are represented by the same sequence $I \in I(n)_q$. Furthermore, for each $1 \le j \le n$,

there is an element $g_j \in G$ such that $\iota_\sharp^0(v_j^0) = g_j \cdot \iota_\sharp^1(v_j^1) \cdot g_j^{-1}$. Thus, if Υ^0 is zero on V^0, then Υ^1 is semizero on V^1. Statement 2 follows using the argument above and Lemma 1.16.

3. Comparison of the $\overline{\mu}$ and σ invariants

Let L be an n component link. In this section the $\overline{\mu}$ and σ invariants of L are compared. We prove

Theorem 3.1. *If $I = (i_1, \dots, i_q) \in I(n)$, and $i_1 \neq i_q$, then*

1. $\Delta(I) | \Lambda(I)$,
2. *if $i_\alpha \neq i_\beta$ for all $1 \leq \alpha < \beta \leq q$, then $\Delta(I) = \Lambda(I)$,*
3. $\langle \sigma(I), \omega_{i_1} \rangle \equiv -\langle \sigma(I), \omega_{i_q} \rangle \pmod{\Lambda(I)}$, *and* $(-1)^q \langle \sigma(I), \omega_{i_q} \rangle \equiv \overline{\mu}(I) \pmod{\Delta(I)}$.

Each of the $\overline{\mu}(I)$ compares with a $\sigma(I')$, where I' is a cyclic permutation of I, since the $\overline{\mu}$-invariants are invariant under cyclic permutation of the indices, and $\overline{\mu}(i, i, \dots, i) = 0$ for all $1 \leq i \leq n$ [9]. The σ-invariants are not, in general, invariant under cyclic permutation of the indices, and thus, there are links and sequences for which Δ is a proper divisor of Λ. We present two links which are distinguished by $\sigma(1,2,3,1,4)$, although they have identical $\overline{\mu}$-invariants for sequences of lenght less than or equal to 5. Stallings [13] conjectured that there is a relationship between the $\overline{\mu}$-invariants and Massey products. Porter [10] and Turaev [16] were the first to prove this. The proof of Theorem 3.1 utilizes a refined expression of Δ and several lemmas. As above, let $G = \pi_1(S^3 - L)$. We refer to the presentation $\langle x_1, \dots, x_n | [x_1, v_{1q}], \dots, [x_n, v_{nq}] \rangle$ of G/G_q described in §2. Let $I = (i_1, \dots, i_q)$, and $r > 0$.

Definition 3.2. For $I = (i_1, i_2)$, define $\Delta'(I) = 0$. For $q > 2$, define $\Delta'(I) = \gcd\{m(t_1, \dots, t_l : v_{t_{l+1}q+r}) | (t_1, \dots, t_{l+1})$ is a proper subsequence of $I\}$.

Lemma 3.3. *For all $I, \Delta(I) = \Delta'(I)$.*

Proof. Induct on q. For $q = 2$, the lemma is true by definition. Assume the lemma for indexing sequences of length less than q. Suppose $I = (i_1, \dots, i_q)$.

$$\begin{aligned}
\Delta(I) &= \gcd\{m(s_a, \dots, s_b : v_{s_{b+1}q+r}) | (s_a, \dots, s_{b+1}) \text{ is a cyclic permutation of a proper subsequence of } (i_1, \dots, i_q)\} \\
&= \gcd\{\overline{\mu}(s_1, \dots, s_{q-1}), \Delta(s_1, \dots, s_{q-1}) | (s_1, \dots, s_{q-1}) \text{ is a cyclic permutation of a subsequence of } (i_1, \dots, i_q)\} \\
&= \gcd\{\overline{\mu}(t_1, \dots, t_{q-1}), \Delta'(t_1, \dots, t_{q-1}) | (t_1, \dots, t_{q-1}) \text{ is a subsequence of } I\} \\
&= \Delta'(I).
\end{aligned}$$

Lemma 3.4. 1. *If $i_1 = i_2 = \cdots = i_q$ then $m(I : [x_j, v_{jq+r}]) = 0$.*

2. *Otherwise,*

$$m(I:[x_j, v_{jq+r}]) = \sum_{\substack{(I_1,I_2,I_3,I_4)=I \\ I_3\neq\varnothing}} m(I_1 : x_j^{-1})m(I_2 : v_{jq+r}^{-1})m(I_3 : x_j)m(I_4 : v_{jq+r}).$$

Proof. Suppose $i_1 = i_2 = \cdots = i_q = i$. As the sum of the exponents of x_i in $[x_j, v_{jq+r}]$ is zero, $m(I:[x_j, v_{jq+r}]) = 0$.

Statement 2 follows from

$$m(I:[x_j, v_{jq+r}]) = \sum_{(I_1,I_2,I_3,I_4)=I} m(I_1 : x_j^{-1})m(I_2 : v_{jq+r}^{-1})m(I_3 : x_j)m(I_4 : v_{jq+r}).$$

The sum of the terms with $I_3 = \varnothing$ is

$$m(I : x_j^{-1}v_{jq+r}^{-1}v_{jq+r}) = m(I : x_j^{-1}) = 0,$$

as the indices are not all identical.

Lemma 3.5. *Let $1 \le k \le q$, and let $(s_1, \dots, s_k)$ be a subsequence of I.*

$$m(s_1, \dots, s_{k-1} : v_{s_k q+r}) \equiv -m(s_1, \dots, s_{k-1} : v_{s_k q+r}^{-1}) \pmod{\Delta(I)}.$$

Proof.

$$\begin{aligned} 0 &= m(s_1, \dots, s_{k-1} : v_{s_k q+r}v_{s_k q+r}^{-1}) \\ &= \sum_{(I_1,I_2)=(s_1,\dots,s_{k-1})} m(I_1 : v_{s_k q+r})m(I_2 : v_{s_k q+r}^{-1}) \\ &= m(s_1, \dots, s_{k-1} : v_{s_k q+r}) + m(s_1, \dots, s_{k-1} : v_{s_k q+r}^{-1}) \pmod{\Delta(I)}. \end{aligned}$$

Lemma 3.6. *If $k > 1$ and $i_k \neq i_q$, then*

$$m(i_{k+1}, \dots, i_q : v_{i_k q+r}) \equiv 0 \pmod{\Lambda(I)}.$$

Proof. Let I_k^q denote the subsequence $(i_k, \dots, i_q)$ of I. Induct on the number n of indices i_t for $k+1 \le t \le q$, such that $i_t = i_k$. If $n = 0$, then by Lemma 3.4(2),

$$m(I_{k+1}^q : v_{i_k q+r}) = m(I_k^q : [x_{i_k}, v_{i_k q+r}]) = 0 \pmod{\Lambda(I)}.$$

In general:

$$\begin{aligned} 0 &\equiv m(I_k^q : [x_{i_k}, v_{i_k q+r}]) \pmod{\Lambda(I)} \\ &= \left[\sum_{\substack{(I_1,I_2,I_3,I_4)=I_k^q \\ I_3\neq\varnothing,\ (I_1,I_2)\neq\varnothing}} m(I_1 : x_{i_k}^{-1})m(I_2 : v_{v_{i_k} q+r}^{-1})m(I_3 : x_{i_k})m(I_4 : v_{i_k q+r}) \right] \\ &\quad + m(i_{k+1}, \dots, i_q : v_{i_k q+r}). \end{aligned}$$

Since $i_q \neq i_k, I_4$, in the above sum, is not empty. Furthermore, $I_3 = (i_\alpha)$, and $i_\alpha = i_k$. Thus by induction,

$$\sum_{\substack{(I_1,I_2,I_3,I_4)=I_k \\ I_3\neq\varnothing,\ (I_1,I_2)\neq\varnothing}} m(I_1 : x_{i_k}^{-1})m(I_2 : v_{i_k q+r}^{-1})m(I_3 : x_{i_k})m(I_4 : v_{i_k q+r}) \equiv 0 \pmod{\Lambda(I)}.$$

Thus

$$m(i_{k+1}, \dots, i_q : v_{i_k q+r}) \equiv 0 \pmod{\Lambda(I)}.$$

Lemma 3.7. 1. *If* $j \neq i_1$, *and* $j \neq i_q$, *then* $m(I : [x_j, v_{jq+r}]) \equiv 0 \pmod{\Lambda(I)}$.

2. $m(I : [x_{i_1}, v_{i_1 q+r}]) \equiv -m(I : [x_{i_q}, v_{i_q q+r}]) \pmod{\Lambda(I)}$.

Proof. Suppose $j \neq i_1$, and $j \neq i_q$. Invoking Lemma 3.4,

$$m(I : [x_j, v_{jq+r}]) = \sum_{\substack{(I_1, I_2, I_3, I_4) = I \\ I_3 \neq \varnothing}} m(I_1 : x_j^{-1}) m(I_2 : v_{jq+r}^{-1}) m(I_3 : x_j) m(I_4 : v_{jq+r}).$$

Since $j \neq i_q, I_3 \neq (i_q)$, and thus $I_4 \neq \varnothing$. Since $j \neq i_1, I_3 \neq (i_1)$, and thus, by Lemma 3.6, $m(I_4 : v_{jq+r}) \equiv 0 \pmod{\Lambda(I)}$. Therefore, $m(I : [x_j, v_{jq+r}]) \equiv 0 \pmod{\Lambda(I)}$.

Statement 2 follows from statement 1 and two additional facts.

1. $\langle \sigma(I), \omega_j \rangle$ is the residue of $(-1)^{q+1} m(I : [x_j, v_{jq+r}])$.
2. $\omega_1 + \cdots + \omega_n = 0$.

Proof of Theorem 3.1. We show that $\Delta(I) | \Lambda(I)$. Lemma 3.7 implies that $\Lambda(I) = \gcd\{m(S : [x_{s_k}, v_{s_k q+r}]) | S = (s_1, \dots, s_k)$ is a proper subsequence of I, and $s_1 \neq s_k\}$.

$$\begin{aligned} &m(S : [x_{s_k}, v_{s_k q+r}]) \\ &\quad = \sum_{\substack{(I_1, I_2, I_3, I_4) = S \\ I_3 \neq \varnothing}} [m(I_1 : x_{s_k}^{-1}) m(I_2 : v_{s_k q+r}^{-1}) m(I_3 : x_{s_k}) m(I_4 : v_{s_k q+r})]. \end{aligned}$$

For each nonzero term of the sum, $I_1 = \varnothing$, and $I_2 \neq \varnothing$. Thus, by Lemma 3.5, $m(S : [x_{s_k}, v_{s_k q+r}]) \equiv 0 \pmod{\Delta(I)}$. Thus $\Delta(I)$ divides $\Lambda(I)$.

Proof that $\langle \sigma(I), \omega_{i_q} \rangle$ *projects onto* $(-1)^q \overline{\mu}(I)$. $h\colon H_2(G) - (N \cap F \sharp F)/F \sharp N$ is the Hopf isomorphism, and $h(\omega_j)$ is the class represented by $-w_{jr_j}$. See presentation (3) above. Thus

$$\begin{aligned} (-1)^q \langle \sigma(I), \omega_{i_q} \rangle &= -m(I : [x_{i_q}, v_{i_q q+r}]) \\ &= -\sum_{\substack{(I_1, I_2, I_3, I_4) = I \\ I_3 \neq \varnothing}} m(I_1 : x_{i_q}^{-1}) m(I_2 : v_{i_q q+r}^{-1}) m(I_3 : x_{i_q}) m(I_4 : v_{i_q q+r}). \end{aligned}$$

Since $i_1 \neq i_q$, in each nonzero term of the above sum $I_1 = \varnothing$, and $I_2 \neq \varnothing$. If $I_2 \neq (i_1, \dots, i_{q-1})$, then by Lemma 3.5, $m(I_2 : v_{i_q q+r}^{-1}) \equiv 0 \pmod{\Delta(I)}$. Thus

$$\begin{aligned} (-1)^q \langle \sigma(I), \omega_{i_q} \rangle &\equiv -m(i_1, \dots, i_{q-1} : v_{i_q q+r}^{-1}) \pmod{\Delta(I)} \\ &\equiv m(i_1, \dots, i_{q-1} : v_{i_q q+r}) \pmod{\Delta(I)} \\ &= \overline{\mu}(I). \end{aligned}$$

Proof that $\Delta(I) = \Lambda(I)$ *if the indices are distinct.* Suppose $S = (s_1, \dots, s_k)$ is a subsequence of $I = (i_1, \dots, i_q)$, and for all $\alpha \neq \beta$, $i_\alpha \neq i_\beta$. Then, $m(s_1, \dots, s_{k-1} : v_{s_k q+r}) = -m(s_1, \dots, s_k : [x_{s_k}, v_{s_k q+r}])$, and thus $\Lambda(I) | \Delta(I)$.

The present example shows that the σ invariants may distinguish links which are not distinguished by the $\overline{\mu}$-invariants. Refer to Figures 1 and 2 below. The curves, a and b, in Figures 1 and 2 are $V_1 \cap V_2$ and $V_1 \cap V_3$ respectively. Let $L = K_1 \cup K_2 \cup K_3 \cup K_4$, and let $L' = K_1 \cup K_2 \cup K_3 \cup K_4'$. One shows that

1. The σ and $\overline{\mu}$ invariants, with distinct indices, are zero for each of these links.

2. Both of these links have $\overline{\mu}(1,2,3,1) = 1$, and therefore, $\overline{\mu}(1,2,3,1,4)$ is totally indeterminate.

3. Each of these links has

$$\Lambda(1,2,3,1,4) = 0,$$

and the invariants, $\sigma(1,2,3,1,4)(w_4)$, of L and L' are 1 and 0 respectively.

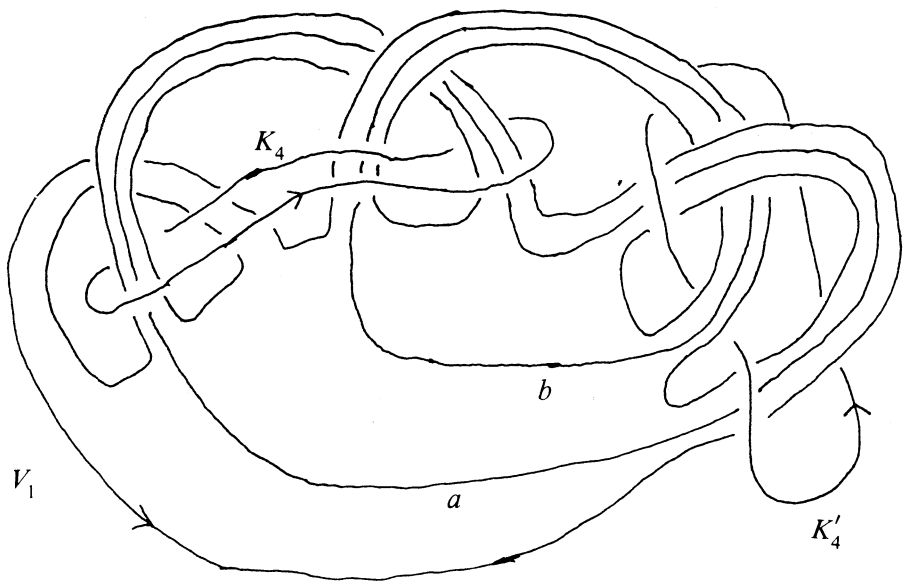
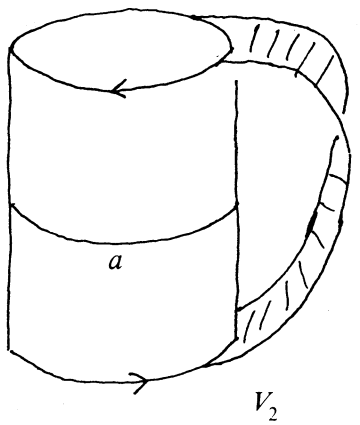


FIGURE 1. $K_1 = \partial V_1$; $a = V_1 \cap V_2$; $b = V_1 \cap V_3$

4. ADDITIONAL INVARIANTS

In this section we show that for two component links certain of the T_V, using the appropriate group, are independent of the basing V. L_V, for these invariants, only depends upon the linking number of the components. If this linking number is zero, $L_V = 0$, and these invariants may be related or equivalent to the Sato-Levine-Cochran invariants [1, 2, 12]. The invariants are defined using the fundamental group of the manifold obtained by surgery on one of the components.

Let L be a link in S^3 with components K_1 and K_2. Let W be the complement of K_2 in the manifold obtained by zero-framed surgery on K_1. Let $d = |\mathrm{lk}(K_1, K_2)|$, and let $\mathscr{G} = \pi_1(W, w)$.



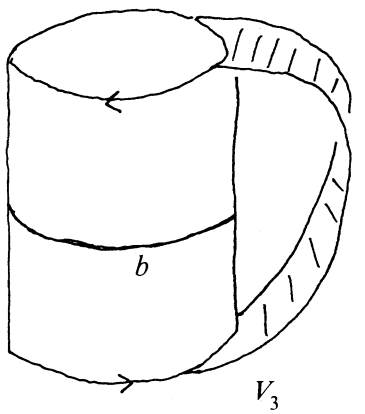


FIGURE 2. $K_2 = \partial V_2$ $K_3 = \partial V_3$

Proposition 4.1. *Let $V = \{v_1, v_2\} \subset \mathscr{G}$ be a collection of meridians of L. Then, V satisfies $M(d)$.*

Proof. Using the presentation (3) of $\pi_1(S^3 - L)$ described above, such that for $i = 1$ or 2, $p(x_{i1}) = v_i$, one sees that $\mathscr{G} \cong \langle x_{ij} | v_{1r_1}, w_{st}; 1 \le i \le 2,\ 1 \le j \le r_i,\ 1 \le s \le 2,\ 1 \le t \le r_i,\ (s,t) \ne (1, r_1)\rangle$, and that v_{1r_1} represents a longitude which has linking number zero with K_1. Let $v_{iq} = n_q(v_{ir_i})$. Milnor's homomorphisms, $n_q : F \to A$, show that

$$\mathscr{G}/\mathscr{G}_q \cong \langle x_1, x_2 | v_{1q}, [x_2, v_{2q}], A_q\rangle .$$

Let $\omega \in A$, then $\omega \in A\sharp_d A \Leftrightarrow \forall 1 \le j \le 2, d | m(j : \omega)$ [4]. As $m(j : v_{1q}) = \mathrm{lk}(K_j, K_1), v_{1q} \in A\sharp_d A$. Thus conditions 1 and 2 are satisfied. We verify 3. Let λ be the algebraic number of crossings of K_1 by itself in the projection of L. One may choose

$$v_{1r_1} = x_{11}^{-\lambda} x_{\alpha_1\beta_1}^{\varepsilon_1} \cdots x_{\alpha_{r_1}\beta_{r_1}}^{\varepsilon_{r_1}} .$$

Let

$$\zeta_1 = v_{1r_1} \prod_{\beta_j \ne 1} (w_{\alpha_j(\beta_j - 1)})^{\varepsilon_j} .$$

$\zeta_1 \in F\sharp_d F$, and $n_q(\zeta_1) \equiv v_{1q} \pmod{A_q}$. Thus V satisfies $M(d)$.

For each $I \in I(2)$ and $C \in (H^1(W))^q$, Theorems 1.7 and 1.11 provide invariants, $\sigma(I) \in H^2(\mathscr{G}, \mathbf{Z}/\Lambda(I))$ and $T_V(C) \in H^2(\mathscr{G}, \mathbf{Z}/L_V(C))$.

From the exact homology sequence of the pair (W, X), $X = S^3 - \mathrm{nbd}(L)$, one sees that $H_2(W; \mathbf{Z}/d) \cong H_2(W, X; \mathbf{Z}/d)$, and thus, $H_2(W; \mathbf{Z}/d) \equiv \mathbf{Z}/d$. Let $\hat{\omega} \in H_2(W; \mathbf{Z}/d)$ be the class represented by $D^2 \cup \Sigma$, where D^2 is the core of the attached handle, and Σ is a mod-d Seifert surface for K_1 in X. Since L has only two components, $\hat{\omega}$ is independent of the choice of the mod-d Seifert surface, and it generates $H_2(W; \mathbf{Z}/d)$. Under the Hopf homomorphisms utilized above, $\hat{\omega}$ corresponds to the class represented by ζ_1. Let ω be the corresponding element of $H_2(\mathscr{G}, \mathbf{Z}/d)$.

Certain of the invariants T_V are independent of V. For each positive integer k let I_k be the sequence

$$(2, \underbrace{1, \dots, 1}_{2k-1 \text{ 1's}}, 2).$$

Theorem 4.2. 1.

$$L_V(I_k) = \gcd\left\{\binom{d}{j} \mid 1 \le j \le \min\{d, 2k\}\right\}.$$

Thus, if the linking number is zero, $L_V(I_k) = 0$, *while if* $d \neq 0$ *and* $2k \geq d$, $L_V(I_k) = 1$, *and* $T_V(I_k)$ *is totally indeterminate.*

2. $T_V(I_k)$ *is independent of* V.

The proof quotes the following lemma.

Lemma 4.3. *Let* E *be the free group* $\langle x_1, \dots, x_m\rangle$. *For each* $w \in E$ *and each positive integer* s *define*

$$\Theta(s, w) = \gcd\{m(\underbrace{1, \dots, 1}_{r \text{ 1's}} : w) \mid 1 \le r \le s\}.$$

Then, $\Theta(s, w) = \Theta(s, w^{-1})$.

Proof. Induct on s. Let $s = 1$. $m(1 : w) = -m(1 : w^{-1})$. Assume the result for $s < q$, and let

$$I = (\underbrace{1, \dots, 1}_{q \text{ 1's}}).$$

$$\begin{aligned} 0 &= m(I : ww^{-1}) \\ &= m(I : w) + m(I : w^{-1}) + \sum_{\substack{(I_1, I_2) = I \\ I_1 \neq \varnothing,\ I_2 \neq \varnothing}} m(I_1 : w) m(I_2 : w^{-1}) \\ &\equiv m(I : w) + m(I : w^{-1}) \pmod{\Theta(q-1, w)}, \end{aligned} \tag{5}$$

$$\begin{aligned} \Theta(q, w) &= \gcd\{\Theta(q-1, w), m(I : w)\} \\ &= \gcd\{\Theta(q-1, w^{-1}), m(I : w^{-1})\} \quad \text{(by induction and (5))} \\ &= \Theta(q, w^{-1}). \end{aligned}$$

Proof of 4.2.1.

$$\begin{aligned}
L_V(I_k) &= \gcd\{m(J : v_{12k+1+r}) | J = (2, \underbrace{1, \dots, 1}_{r \text{ 1's}}), (\underbrace{1, \dots, 1}_{r \text{ 1's}}, 2), \\
&\qquad \text{or } (\underbrace{1, \dots, 1}_{r \text{ 1's}}), \text{ and } 0 \le r \le 2k-1\} \\
&= \Delta(\underbrace{1, \dots, 1}_{2k+1 \text{ 1's}}, 2) \\
&= \gcd\{m(\underbrace{1, \dots, 1}_{r \text{ 1's}} : v_{22k+1+r}) | 1 \le r \le 2k\} \quad \text{(Lemma 3.3)} \\
&= \gcd\{m(\underbrace{1, \dots, 1}_{r \text{ 1's}} : x_1^{\mathrm{lk}(K_1, K_2)}) | 1 \le r \le 2k\} \\
&= \gcd\{m(\underbrace{1, \dots, 1}_{r \text{ 1's}} : x_1^{d}) | 1 \le r \le 2k\} \quad \text{(Lemma 4.3)}.
\end{aligned}$$

The statement follows as

$$m(\underbrace{1, \dots, 1}_{r \text{ 1's}} : x_1^d) = \binom{d}{r}.$$

Definition 4.4. For any basing V of L, $L(I_k) = L_V(I_k)$.

The proof of statement 2 requires a definition and some lemmas.

Definition 4.5. Let $x = \mathscr{G}$, and let $\Upsilon : \mathscr{G} \to \overline{\mathbf{U}}(\mathbf{Z}/L(I_k), 2k+2)$ be a homomorphism. Υ is *internally zero* on x if $\Upsilon_{ij}(x) = 0$ for all $2 \le i < j-1 \le 2k$.

Lemma 4.6. *Let x and $y \in \mathscr{G}$ be meridians of K_1 and K_2 respectively. If Υ, a defining system for I_k, is internally zero on x and semizero on $\{y\}$, then for all $g \in G$, Υ is internally zero on gxg^{-1}.*

Proof. Let A be the free group $\langle x_1, x_2 \rangle$. For each $r > 0$, there is a diagram

$$\begin{tikzcd}
F \arrow[r, "p"] \arrow[d, "n_{2k+2+r}"'] & \mathscr{G} \arrow[d, "a_{2k+2+r}"] \\
A \arrow[r, "p_{2k+2+r}"] & \mathscr{G}/\mathscr{G}_{2k+2+r}
\end{tikzcd} \tag{6}$$

such that if $x' = a_{2k+2+r}(x)$, and $y' = a_{2k+2+r}(y), p_{2k+2+r}(x_1) = x'$, and $p_{2k+2+r}(x_2) = y'$. Choose (i, j) such that $2 \le i < j-1 \le 2k$. Let Υ': $\mathscr{G}/\mathscr{G}_{2k+2+r} \to \overline{\mathbf{U}}(\mathbf{Z}/L(I_k), 2k+2)$ be the induced homomorphism. The argument below hinges upon the fact that, for such (i, j), $\Upsilon_{ij}(x) = \Upsilon_{ij}(y) = 0$, and thus $\Upsilon'_{ij}(x') = \Upsilon'_{ij}(y') = 0$. Let $\tilde{g} \in p^{-1}(g)$, and let $\hat{g} = n_{2k+2+r}(\tilde{g})$. Let

$$\widetilde{\Upsilon}' : A \to \mathbf{U}(\mathbf{Z}/L(I_k), 2k+2)$$

be a homomorphism such that the diagram below commutes.

$$\begin{tikzcd} & & \mathbf{U}(\mathbf{Z}/L(I_k), 2k+2) \arrow[d] \\ A \arrow[rr, "p_{2k+2+r}"] \arrow[urr, "\widetilde{\Upsilon}'"] & & \mathscr{G}/\mathscr{G}_{2k+2+r} \arrow[r, "\Upsilon'"] & \overline{\mathbf{U}}(\mathbf{Z}/L(I_k), 2k+2) \end{tikzcd}$$

$$\begin{aligned}
\Upsilon_{ij}(gxg^{-1}) &= \Upsilon'_{ij}(a_{2k+2+r}(gxg^{-1})) \\
&= \sum_{s=1}^{j-i} \sum_{c_1,\dots,c_s \in S_{ij}} \sum_{I \in I(2)_s} (\widetilde{\Upsilon}'_{ii+c_1} \cdots \widetilde{\Upsilon}'_{j-c_s j})(I) m(I : \hat{g} x_1 \hat{g}^{-1}) \\
&= (-1)^{j-i} m(\underbrace{1, \dots, 1}_{j-i \text{ 1's}} : \hat{g} x_1 \hat{g}^{-1}) \\
&= (-1)^{j-i} m(1, \dots, 1 : x_1) \\
&= 0.
\end{aligned}$$

Lemma 4.7. *Let $V = \{x, y\}$ as above. If Υ, a defining system for I_k, is internally zero on x and zero on $\{y\}$, then $\langle [\Upsilon], \check{\omega} \rangle = \langle T_V(I_k), \check{\omega} \rangle$.*

Proof. Let Υ, Υ', and $\widetilde{\Upsilon}'$ be as defined above, and suppose that Υ is internally zero on x and zero on $\{y\}$.

$$\begin{aligned}
\langle [\Upsilon], \check{\omega} \rangle &= \widetilde{\Upsilon}'_{1\ 2k+2}(v_{1\ 2k+2+r}) \\
&= \sum_{s=1}^{2k+1} \sum_{c_1,\dots,c_s \in S_{1\ 2k+2}} \sum_{I \in I(2)_s} (\widetilde{\Upsilon}'_{1\ 1+c_1} \cdots \widetilde{\Upsilon}'_{2k+2-c_s\ 2k+2})(I) m(I : v_{1\ 2k+2+r}) \\
&= -m(2, \underbrace{1, \dots, 1}_{2k-1 \text{ 1's}}, 2 : v_{1\ 2k+2+r}) \\
&= \langle T_V(I_k), \check{\omega} \rangle.
\end{aligned}$$

This argument uses the fact that

$$\widetilde{\Upsilon}'_{1\ 1+c_1} \cdots \widetilde{\Upsilon}'_{2k+2-c_s\ 2k+2})(I) m(I : v_{1\ 2k+2+r}) \equiv 0 \pmod{L(I_k)}$$

unless $s = 2k+1$, $(c_1, \dots, c_{2k+1}) = (1, \dots, 1)$, and $I = (2, 1, \dots, 1, 2)$, which we explain. Suppose $s = 1$. $m(1 : v_{1\ 2k+2+r}) = 0$ as the self-linking number is zero, and $m(2 : v_{1\ 2k+2+r}) = \mathrm{lk}(K_1, K_2) = d$. Suppose

$$(\widetilde{\Upsilon}'_{1\ 1+c_1} \cdots \widetilde{\Upsilon}'_{2k+2-c_s\ 2k+2})(I) m(I : v_{1\ 2k+2+r}) \not\equiv 0.$$

Each of the numbers, $m(1, \dots, 1 : v_{1\ 2k+2+r}), m(2, 1, \dots, 1 : v_{1\ 2k+2+r})$, and $m(1, \dots, 1, 2 : v_{1\ 2k+2+r})$, is congruent to zero $\bmod L(I_k)$. Since $\widetilde{\Upsilon}'$ is zero on $\{x_2\}$, $c_1 = 1$, and $c_s = 1$. For $2 \le i < j - 1 < 2k+1$, $\widetilde{\Upsilon}'_{ij}(x_1) = 0$. Thus, for $1 < t < s$, $c_t = 1$. As $c_1 + c_2 + \cdots + c_s = 2k+2$, the claim follows.

Lemma 4.8. *Let $\{m_1, m_2\}$ be a collection of meridians of L in $\mathscr{G}$, and let Υ be a defining system for I_k which is zero on $\{m_1, m_2\}$ as above. Let $m_2^+ = m_1 m_2 m_1^{-1}$, and let $m_2^- = m_1^{-1} m_2 m_1$. Let Υ_+ and Υ_- be defining systems for*

I_k which are zero on $\{m_1, m_2^+\}$, and $\{m_1, m_2^-\}$ respectively. Then $\langle[\Upsilon_+], \omega\rangle = \langle[\Upsilon], \omega\rangle = \langle[\Upsilon_-], \omega\rangle$.

Proof.

$$\Upsilon'_+ : \mathscr{G}/\mathscr{G}_{2k+2+r} \to \overline{\mathbf{U}}(\mathbf{Z}/L(I_k), 2k+2),$$

and

$$\Upsilon'_- : \mathscr{G}/\mathscr{G}_{2k+2+r} \to \overline{\mathbf{U}}(\mathbf{Z}/L(I_k), 2k+2),$$

are the induced defining systems. Let $p_{2k+2+r} : A \to \mathscr{G}/\mathscr{G}_{2k+2-r}$ be as defined above, and let $\widetilde{\Upsilon}'_+$ and $\widetilde{\Upsilon}'_-$ be homomorphisms such that the following diagrams commute.

$$\begin{tikzcd}
 & & \mathbf{U}(\mathbf{Z}/L(I_k), 2k+2) \arrow[d, "\iota"] \\
A \arrow[r, "p_{2k+2+r}"] \arrow[urr, "\widetilde{\Upsilon}'_\pm"] & \mathscr{G}/\mathscr{G}_{2k+2+r} \arrow[r, "\Upsilon'_\pm"] & \overline{\mathbf{U}}(\mathbf{Z}/L(I_k), 2k+2)
\end{tikzcd}$$

Claim. 1. $\widetilde{\Upsilon}'(x_1) = \widetilde{\Upsilon}'_+(x_1) = \widetilde{\Upsilon}'_-(x_1)$.

2. For $2 \le i < j - 1 < 2k+1$, $\widetilde{\Upsilon}'_{\pm ij}(x_2) = 0$.

3. For $3 \le j \le 2k+1$,

$$\widetilde{\Upsilon}'_{+1j}(x_2) = \begin{cases} 1, & \text{if } j = 3, \\ 0, & \text{otherwise}, \end{cases} \qquad \widetilde{\Upsilon}'_{-1j}(x_2) = -1.$$

4. For $2 \le l \le 2k$,

$$\widetilde{\Upsilon}'_{+l2k+2}(x_2) = -1, \qquad \widetilde{\Upsilon}'_{-l2k+2}(x_2) = \begin{cases} 1, & \text{if } l = 2k, \\ 0, & \text{otherwise}. \end{cases}$$

Proof of 1. For $g \in \mathscr{G}$, let $\overline{g} = a_{2k+2+r}(g)$. Statement 1 follows from the fact that Υ', Υ'_+, and Υ'_- are zero on $\overline{m_1}$.

Proof of 2. Let $\Phi_{2\ 2k+1} : \overline{\mathbf{U}}(\mathbf{Z}/L(I_k), 2k+2) \to \mathbf{U}(\mathbf{Z}/L(I_k), 2k)$ be the homomorphism defined in §1. $\overline{m_2} = \overline{m_1}^{-1}\overline{m_2^+}\,\overline{m_1}$, and $\overline{m_2} = \overline{m_1}\,\overline{m_2^-}\,\overline{m_1}^{-1}$. We state the argument for $\widetilde{\Upsilon}'_+$, and the argument for $\widetilde{\Upsilon}'_-$ is similar. Let ID be the identity matrix.

$$\begin{aligned}
\Phi_{2\ 2k+1} \circ \iota \circ \widetilde{\Upsilon}'_+(x_2) &= \Phi_{2\ 2k+1} \circ \Upsilon'_+(\overline{m_2}) \\
&= \Phi_{2\ 2k+1} \circ \Upsilon'_+(\overline{m_1}^{-1}\overline{m_2^+}\,\overline{m_1}) \\
&= \Phi_{2\ 2k+1} \circ \Upsilon'_+(\overline{m_1}^{-1}) \cdot \Phi_{2\ 2k+1} \circ \Upsilon'_+(\overline{m_2^+}) \cdot \Phi_{2\ 2k+1} \circ \Upsilon'_+(\overline{m_1}) \\
&= \Phi_{2\ 2k+1} \circ \Upsilon'_+(\overline{m_1}^{-1}) \cdot \mathrm{ID} \cdot \Phi_{2\ 2k+1} \circ \Upsilon'_+(\overline{m_1}) \\
&= \mathrm{ID}.
\end{aligned}$$

Proof of 3. Let $x_2^+ = x_1 x_2 x_1^{-1}$ and $x_2^- = x_1^{-1} x_2 x_1$. A is generated by both $\{x_1, x_2^+\}$ and $\{x_1, x_2^-\}$. Since $x_2 = x_1^{-1} x_2^+ x_1$, and $x_2 = x_1 x_2^- x_1^{-1}$,

$$\begin{aligned}
\widetilde{\Upsilon}'_{+1j}(x_2) &= (-1)^{j-1} m(2, \underbrace{1, \dots, 1}_{j-2\ 1\text{'s}} : x_1^{-1} x_2^+ x_1) \quad \text{(Lemma 1.5)} \\
&= \begin{cases} 1, & \text{if } j = 3, \\ 0, & \text{otherwise}, \end{cases} \\
\widetilde{\Upsilon}'_{-1j}(x_2) &= (-1)^{j-1} m(2, \underbrace{1, \dots, 1}_{j-2\ 1\text{'s}} : x_1 x_2^- x_1^{-1}) \\
&= (-1)^{j-1}(-1)^{j-2} \\
&= -1 .
\end{aligned}$$

Proof of 4. For $2 \le l \le 2k$,

$$\begin{aligned}
\widetilde{\Upsilon}'_{+l\ 2k+2}(x_2) &= (-1)^l m(\underbrace{1, \dots, 1}_{2k+1-l\ 1\text{'s}}, 2 : x_1^{-1} x_2^+ x_1) \\
&= (-1)^l (-1)^{l-1} \\
&= -1, \\
\widetilde{\Upsilon}'_{-l\ 2k+2}(x_2) &= (-1)^l m(\underbrace{1, \dots, 1}_{2k+1-l\ 1\text{'s}}, 2 : x_1 x_2^- x_1^{-1}) \\
&= \begin{cases} 1, & \text{if } l = 2k, \\ 0, & \text{otherwise}. \end{cases}
\end{aligned}$$

Proof of the lemma.

$$\begin{aligned}
\langle [\Upsilon_-], \omega \rangle &= \widetilde{\Upsilon}'_{-1\ 2k+2D}(v_{1\ 2k+2+r}) \\
&= \widetilde{\Upsilon}'_{-1\ 2} \widetilde{\Upsilon}'_{-2\ 3} \cdots \widetilde{\Upsilon}'_{-2k+1\ 2k+2}(I_k) m(I_k : v_{1\ 2k+2+r}) \\
&\quad + \sum_{j=3}^{2k+1} [[\widetilde{\Upsilon}'_{-1j} \widetilde{\Upsilon}'_{-j\ j+1} \cdots \widetilde{\Upsilon}'_{-2k+1\ 2k+2}(2, \underbrace{1, \dots, 1}_{2k+1-j\ 1\text{'s}}, 2) \\
&\qquad\qquad \times m(2, \underbrace{1, \dots, 1}_{2k+1-j\ 1\text{'s}}, 2 : v_{1\ 2k+2+r})] \\
&\quad + [\widetilde{\Upsilon}'_{-1\ j-1} \widetilde{\Upsilon}'_{-j-1\ j} \cdots \widetilde{\Upsilon}'_{-2k-1\ 2k} \widetilde{\Upsilon}'_{-2k\ 2k+2}(2, \underbrace{1, \dots, 1}_{2k+1-j\ 1\text{'s}}, 2) \\
&\qquad\qquad \times m(2, \underbrace{1, \dots, 1}_{2k+1-j\ 1\text{'s}}, 2 : v_{1\ 2k+2+r})]] \\
&\quad + \widetilde{\Upsilon}'_{-1\ 2k+2}(1) m(1 : v_{1\ 2k+2+r}) + \widetilde{\Upsilon}'_{-1\ 2k+2}(2) m(2 : v_{1\ 2k+2+r}) .
\end{aligned}$$

$m(j : v_{1\ 2k+2+r}) \equiv 0 \pmod{L(I_k)}$ for $j = 1, 2$, and

$$\begin{aligned}(-1)^{2k+3-j} &= \widetilde{\Upsilon}'_{-1j}\widetilde{\Upsilon}'_{-j\ j+1}\cdots\widetilde{\Upsilon}'_{-2k+1\ 2k+2}(2, \underbrace{1, \dots, 1}_{2k+1-j\ 1\text{'s}}, 2)\\ &= -\widetilde{\Upsilon}'_{-1\ j-1}\widetilde{\Upsilon}'_{-j-1\ j}\cdots\widetilde{\Upsilon}'_{-2k-1\ 2k}\widetilde{\Upsilon}'_{-2k\ 2k+2}(2, \underbrace{1, \dots, 1}_{2k+1-j\ 1\text{'s}}, 2).\end{aligned}$$

Thus

$$\begin{aligned}\langle[\Upsilon_-], \omega\rangle &\equiv \widetilde{\Upsilon}'_{-1\ 2}\widetilde{\Upsilon}'_{-2\ 3}\cdots\widetilde{\Upsilon}'_{-2k+1\ 2k+2}(I_k)m(I_k : v_{1\ 2k+2+r})\\ &= (-1)^{2k+1}m(I_k : v_{1\ 2k+2+r})\\ &= \langle[\Upsilon], \omega\rangle.\end{aligned}$$

A similar argument establishes the corresponding result for $\langle[\Upsilon_+], \omega\rangle$.

Definition 4.9. Let $V = \{m_1, m_2\}$ and $\check{V} = \{\check{m}_1, \check{m}_2\}$ be basings of the link L. $V \sim \check{V}$ if $T_V(I_k) = T_{\check{V}}(I_k)$.

Lemma 4.10. *Suppose g is a loop in W, and* $\mathrm{lk}(g, K_1) = 0$. *If Υ is a defining system for I_k which is zero on $\{m_1, m_2\}$, then Υ is zero on $\{m_1, gm_2g^{-1}\}$.*

Proof. Let $w_g \in F$ be a word such that $p(w_g) = g$, and let $w'_g = n_{2k+2}(w_g)$. Suppose $1 \le i < j \le 2k+2$, and $(i, j) \ne (1, 2k+2)$. For $I = (l_1, \dots, l_q)$ and $1 \le s \le t \le q$, let $I_s^t = (l_s, \dots, l_t)$.

$$\begin{aligned}\Upsilon_{ij}(gm_2g^{-1}) &= \widetilde{\Upsilon}'_{ij}(w'_g x_2 w'^{-1}_g)\\ &= (-1)^{j-i}m(I_i^{j-1} : w'_g x_2 w'^{-1}_g)\\ &= \sum_{\substack{(I_1, I_2, I_3) = I_i^{j-1}\\ I_2 = 2}} m(I_1 : w'_g)m(I_2 : x_2)m(I_3 : w'^{-1}_g).\end{aligned}$$

If $i \ne 1$, and $j \ne 2k+2$, then

$$I_i^{j-1} = (\underbrace{1, \dots, 1}_{j-i\ 1\text{'s}}),$$

and thus $m(I_i^{j-1} : w'_g x_2 w'^{-1}_g) = 0$. If $i = 1$, and $j \ne 2k+2$, then

$$I_1^{j-1} = (2, \underbrace{1, \dots, 1}_{j-2\ 1\text{'s}}),$$

and thus

$$\begin{aligned}m(I_1^{j-1} : w'_g x_2 w'^{-1}_g) &= m(\underbrace{1, \dots, 1}_{j-2\ 1\text{'s}} : w'^{-1}_g)\\ &= m(\underbrace{1, \dots, 1}_{j-2\ 1\text{'s}} : x_1^{-\mathrm{lk}(g, k_1)})\\ &= 0.\end{aligned}$$

If $i \neq 1$, and $j = 2k+2$, then

$$I_i^{2k+1} = (\underbrace{1, \dots, 1}_{2k-i \text{ 1's}}, 2)$$

and thus

$$\begin{aligned} m(I_i^{2k+1} : w_g' x_2 w_g'^{-1}) &= m(\underbrace{1, \dots, 1}_{2k-i \text{ 1's}} : w_g') \\ &= m(\underbrace{1, \dots, 1}_{2k-i \text{ 1's}} : x_1^{\mathrm{lk}(g, k_1)}) \\ &= 0. \end{aligned}$$

Lemma 4.11. *Let $V = \{m_1, m_2\}$ be a basing of L, and for $n \in \mathbf{Z}$, let $V_n = \{m_1, m_1^n m_2 m_1^{-n}\}$. $V \sim V_n$.*

Proof. Induct on n. Lemma 4.8 covers the cases $n = \pm 1$. Let Υ_n be a defining system for I_k which is zero on V_n. Then, for $n > 0$,

$$\begin{aligned} \langle [\Upsilon_n], \omega \rangle &= \langle [\Upsilon_{n-1}], \omega \rangle \quad \text{(by Lemma 4.8)} \\ &= \langle [\Upsilon_0], \omega \rangle \quad \text{(by induction)}. \end{aligned}$$

Thus $V \sim V_n$. The case $n < 0$ follows similarly.

Proof of 4.2.2. Let $V = \{m_1, m_2\}$ and $\check{V} = \{\check{m}_1, \check{m}_2\}$ be basings of L. For some $g \in \mathcal{G}$, $\check{m}_2 = g m_2 g^{-1}$. Suppose g is represented by the loop Γ, and let $n = \mathrm{lk}(\Gamma, K_1)$. Of course, n depends only upon the homology class represented by Γ.

$$\begin{aligned} \{m_1, m_2\} &\sim \{\check{m}_1, m_2\} \quad \text{(by Lemmas 4.6 and 4.7)} \\ &\sim \{\check{m}_1, \check{m}_1^n m_2 \check{m}_1^{-n}\} \quad \text{(by Lemma 4.11)} \\ &\sim \{\check{m}_1, \check{m}_2\} \quad \text{(by Lemma 4.10)}. \end{aligned}$$

The invariants in the surgered manifold are compared with the invariants in the complement of the link. For $k > 0$, let I_k be as above, and let

$$J_k = (2, \underbrace{1, \dots, 1}_{2k-1 \text{ 1's}}, 2, 1).$$

Let $\omega_1 \in H_2(G)$, $\omega \in H_2(\mathcal{G})$, and $V = \{m_1, m_2\}$ be as defined above. We have

$$T_V(J_k) \in H^2(G, \mathbf{Z}/L_V(J_k)), \quad \sigma(J_k) \in H^2(G, \mathbf{Z}/\Lambda(J_k)),$$

and

$$T(I_k) \in H^2(\mathcal{G}, \mathbf{Z}/L(I_k)).$$

Theorem 4.12. 1. *If $\mathrm{lk}(K_1, K_2) = 0$, then $L(I_k) = 0 = L_V(J_k)$, and $T(I_k)(\omega) = T_V(J_k)(\omega_1)$. In general $L(I_k) | L_V(J_k)$ and*

$$T(I_k)(\omega) \equiv -T_V(J_k)(\omega_1) \pmod{L(I_k)}.$$

2. $\Lambda(J_k)|L(I_k)$ *and*

$$\sigma(J_k)(\omega_1) \equiv -T(I_k)(\omega) \pmod{\Lambda(J_k)}.$$

Proof of 4.12.1.

$$L_V(J_k) = \gcd\{m(2, \underbrace{1, \dots, 1}_{l \text{ 1's}} : [x_1, v_{12k+2+r}]),$$
$$m(\underbrace{1, \dots, 1}_{l \text{ 1's}}, 2 : [x_1, v_{1\ 2k+2+r}]) \mid 0 \le l \le 2k-1\}.$$

Results analogous to Lemmas 3.4 and 3.5 and the proof of Theorem 4.2.1 show that $L(I_k)|L_V(J_k)$. If $\mathrm{lk}(K_1, K_2) = 0, L(I_k) = 0$, and thus, $L_V(J_k) = 0$.

$$\begin{aligned}
T_V(J_k)(\omega_1) &= m(2, \underbrace{1, \dots, 1}_{2k-1 \text{ 1's}}, 2, 1 : v_{1\ 2k+2+r} x_1 v_{1\ 2k+2+r}^{-1} x_1^{-1}) \\
&= \sum_{\substack{(I_1, I_2, I_3, I_4) = J_k \\ I_2 \neq \varnothing}} m(I_1 : v_{1\ 2k+2+r}) m(I_2 : x_1) m(I_3 : v_{1\ 2k+2+r}^{-1}) m(I_4 : x_1^{-1}) \\
&\equiv m(2, \underbrace{1, \dots, 1}_{2k-1 \text{ 1's}}, 2 : v_{1\ 2k+2+r}) \pmod{L(I_k)} \\
&= -T(I_k)(\omega).
\end{aligned}$$

The proof of 2 utilizes the lemma below.

Lemma 4.13. *For each* $0 \le s \le 2k-1$,

$$m(2, \underbrace{1, \dots, 1}_{s \text{ 1's}} : v_{1\ 2k+2+r}) \equiv 0 \pmod{\Lambda(J_k)}.$$

Proof.

$$\begin{aligned}
\Lambda(J_k) = \gcd\{&m(2, \underbrace{1, \dots, 1}_{r \text{ 1's}} : v_{1\ 2k+2+r} x_1 v_{1\ 2k+2+r}^{-1} x_1^{-1}), \\
&m(2, \underbrace{1, \dots, 1}_{l \text{ 1's}}, 2, 1 : v_{1\ 2k+2+r} x_1 v_{1\ 2k+2+r}^{-1} x_1^{-1}), \\
&m(\underbrace{1, \dots, 1}_{p \text{ 1's}}, 2 : v_{1\ 2k+2+r} x_1 v_{1\ 2k+2+r}^{-1} x_1^{-1}) \mid \\
&0 \le r \le 2k,\ 0 \le l \le 2k-2,\ 0 \le p \le 2k-1\}.
\end{aligned}$$

Induct on s. Consider the case $s = 0$. Let $r = 1$.

$$\begin{aligned}
0 &\equiv m(2, 1 : v_{1\ 2k+2+r} x_1 v_{1\ 2k+2+r}^{-1} x_1^{-1}) \pmod{\Lambda(J_k)} \\
&= m(2 : v_{1\ 2k+2+r}).
\end{aligned}$$

Assume the lemma is true for $0 \le s < q \le 2k-1$. Let $r = q+1$, and let

$$I = (2, \underbrace{1, \dots, 1}_{r \text{ 1's}}).$$

$$\begin{aligned}
0 &\equiv m(I : v_{1\ 2k+2+r} x_1 v_{1\ 2k+2+r}^{-1} x_1^{-1}) \\
&= \sum_{\substack{(I_1, I_2, I_3, I_4) = I \\ I_2 \neq \varnothing}} m(I_1 : v_{1\ 2k+2+r}) m(I_2 : x_1) m(I_3 : v_{1\ 2k+2+r})^{-1} m(I_4 : x_1^{-1}) \\
&\equiv m(2, \underbrace{1, \ldots, 1}_{q\ 1\text{'s}} : v_{1\ 2k+2+r}) \quad (\operatorname{mod} \Lambda(J_k)) \quad (\text{by induction}).
\end{aligned}$$

Proof of 4.12.2.

$$\begin{aligned}
L(I_k) &= \Delta(2, \underbrace{1, \ldots, 1}_{2k+1\ 1\text{'s}}) \\
&= \gcd\{m(2, \underbrace{1, \ldots, 1}_{l\ 1\text{'s}} : v_{1\ 2k+2+r}) \mid 0 \leq l \leq 2k-1\} \quad (\text{Lemma } 3.3) \\
&\equiv 0 \quad (\operatorname{mod} \Lambda(J_k)) \quad (\text{Lemma } 4.13).
\end{aligned}$$

$$\begin{aligned}
\sigma(J_k)(\omega_1) &\equiv m(2, \underbrace{1, \ldots, 1}_{2k-1\ 1\text{'s}}, 2, 1 : v_{1\ 2k+2+r} x_1 v_{1\ 2k+2+r}^{-1} x_1^{-1}) \\
&\equiv m(2, \underbrace{1, \ldots, 1}_{2k-1\ 1\text{'s}}, 2 : v_{1\ 2k+2+r}) \quad (\operatorname{mod} \Lambda(J_k)) \quad (\text{Lemmas } 3.4.2 \text{ and } 4.13) \\
&\equiv -T(I_k)(\omega).
\end{aligned}$$

Using results of [1], one can construct links such that $\Lambda(J_k)$ is a proper divisor of $L(I_k)$, or if the linking number of the components is not zero, $L(I_k)$ is a proper divisor of $L_V(J_k)$. For $1 \leq l \leq 2k-2$,

$$m(2, \underbrace{1, \ldots, 1}_{l\ 1\text{'s}}, 2, 1 : v_{1\ 2k+2+r} x_1 v_{1\ 2k+2+r}^{-1} x_1^{-1})$$

enters into $\Lambda(J_k)$, but the corresponding term,

$$m(2, \underbrace{1, \ldots, 1}_{l\ 1\text{'s}}, 2 : v_{1\ 2k+2+r}),$$

does not contribute to $L(I_k)$. Furthermore,

$$m(2, \underbrace{1, \ldots, 1}_{2k-1\ 1\text{'s}} : v_{1\ 2k+2+r})$$

and

$$m(\underbrace{1, \ldots, 1}_{2k-1\ 1\text{'s}}, 2 : v_{1\ 2k+2+r})$$

contribute to $L(I_k)$, but the corresponding terms,

$$m(2, \underbrace{1, \ldots, 1}_{2k\ 1\text{'s}} : v_{1\ 2k+2+r} x_1 v_{1\ 2k+2+r}^{-1} x_1^{-1})$$

and

$$m(\underbrace{1, \ldots, 1}_{2k\ 1\text{'s}}, 2 : v_{1\ 2k+2+r} x_1 v_{1\ 2k+2+r}^{-1} x_1^{-1}),$$

do not contribute to $L_V(J_k)$.

REFERENCES

1. T. D. Cochran, *Geometric invariants of link cobordism*, Comment. Math. Helv. **60** (1985), 291–311.
2. ——, *Derivatives of links: Milnor's concordance invariants and Massey's products*, preprint.
3. W. G. Dwyer, *Homology, Massey products, and maps between groups*, J. Pure and Appl. Algebra **6** (1975), 177–190.
4. R. Fenn, *Techniques of geometric topology*, London Math. Soc. Stud. Text, No. 57, Cambridge Univ. Press, Cambridge, 1983.
5. D. Kraines, *Massey higher products*, Trans. Amer. Math. Soc. **124** (1966), 431–449.
6. W. S. Massey, *Higher order linking numbers*, Conf. on Algebraic Topology (Univ. of Illinois at Chicago Circle, 1968), Univ. of Chicago at Chicago Circle, Chicago, Ill., pp. 174–205.
7. ——, *Some higher order cohomology operations*, Symposium International de Topologia Algebraica, 1958, pp. 145–154.
8. J. Milnor, *Link groups*, Ann. of Math. (2) **2** (1954), 177–195.
9. ——, *Isotopy of links*, Algebraic Geometry and Topology: A Symposium in Honor of Solomon Lefshetz, Princeton Univ. Press, Princeton, N.J., 1957, pp. 280–306.
10. R. Porter, *Milnor's $\bar{\mu}$-invariants and Massey products*, Trans. Amer. Math. Soc. **257** (1980), 39–71.
11. D. Rolfsen, *Knots and links*, Publish or Perish, Berkeley, California, 1976.
12. N. A. Sato, *Cobordism of semi-boundary links*, Topology Appl. **18** (1984).
13. J. Stallings, *Homology and central series of groups*, J. Algebra **2** (1965), 170–181.
14. D. W. Stein, *Massey products in the cohomology of groups with applications to link theory*, Ph.D. Thesis, Brandeis Univ., 1986.
15. ——, *Computing Massey product invariants of links*, Proc. 1987 Georgia Topology Conference, Topology Appl. (to appear).
16. V. G. Turaev, *The Milnor invariants and Massey products*, Studies in Topology. II, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Kogo oi Stet. Acad. Nauk USSR **66** (1976) (translated in J. Soviet Math.).

DEPARTMENT OF MATHEMATICS, TEXAS A& M UNIVERSITY, COLLEGE STATION, TEXAS 77843

Current address: Naval Ocean Systems Center, Code 632 (B), San Diego, California 92152-5000