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ADDENDUM TO THE PAPER "EXISTENCE OF WEAK SOLUTIONS FOR THE NAVIER-STOKES EQUATIONS WITH INITIAL DATA IN L^p"

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ABSTRACT. This paper considers the existence of global weak solutions for the Navier-Stokes equations in the infinite cylinder $\mathbf{R}^n \times \mathbf{R}_+$ with initial data in L^r, $n \geq 3$, $1 < r < \infty$. An imbedding theorem as well as related initial value problems are also studied, thus completing results in [2].

INTRODUCTION

This paper considers the initial value problem for the Navier-Stokes equations in the infinite cylinder $S_T = R^n \times [0, T)$. Given $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$, satisfying in the distributions sense $\operatorname{div} f = 0, x \in R^n$, we seek a solution vector $u(x,t) = (u_1(x,t), \dots, u_n(x,t))$ and a pressure function $P(x,t)$ such that

$$(0.1)\qquad \begin{gathered} D_t u_i - \sum_{j=1}^n D_{jj} u_i + \sum_{j=1}^n u_j D_j u_i + D_i P = 0, \qquad (x,t) \in S_T, \\ \sum_j D_j u_j = 0, \qquad (x,t) \in S_T, \\ u(x,0) = f(x). \end{gathered}$$

Here, D_j and D_t denote respectively, the distributional derivatives with respect to x_j and t, D_{ij} denotes the second order derivative with respect to x_i, x_j; likewise, $L(u)$ will denote the heat operator applied to u, and $\operatorname{grad} u$, the square matrix $D_j u_i$. The first equation of $(0,1)$ takes the form

$$(0.2)\qquad L(u) + (\operatorname{grad} u)(u) + \operatorname{grad} P = 0.$$

Following [4], I consider the functional spaces $L^{p,q}(S_T)$ consisting of the Lebesgue-measurable functions u, such that

$$(0.3)\qquad \|u\|_{p,q}(T) < \infty.$$

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Let us define $U_p(t)$ as

$$\left|\int_{R^n} |u|^p\,dx\right|^{1/p}, \tag{0.4}$$

Then, the mixed norm $(0,3)$ can be written as

(0.5) $\|U_p\|_q(T)$ (usual L^q-norm over the interval $(0,T)$).

The norms associated with the maximal operator $u^* = \sup_t |u|$ (the supremum is taken over $t > 0$) are

$$\|u^*\|_p(T) = \left(\int \left(\sup_{0<t<T} |u|\right)^p dx\right)^{1/p}. \tag{0.6}$$

In the particular case when we take $T = \infty$, we have

$$\|u^*\|_p(\infty) = \left(\int \left(\sup_{t>0} |u|\right)^p dx\right)^{1/p}. \tag{0.7}$$

In the same fashion we introduce the norms $\| \quad \|^*_{p,q}$ as

$$\|u\|^*_{p,q} = \left(\int_{\mathbf{R}^n} \left(\int_0^\infty |u(x,t)|^p\,dt\right)^{q/p} dx\right)^{1/q}. \tag{0.8}$$

The aim of this paper is to complete the results in [2] concerning solutions of $(0,1)$ for initial data in $L^n(\mathbf{R}^n)$. Likewise, in §IV below, a relation is established between the $L^{p,q}$ classes of existence and uniqueness introduced by Fabes, Jones, Riviere in [4] and the classes of solutions with initial data in $L^r(\mathbf{R})$.

I. Main results

Weak solutions [4]. A function $u(x,t)$ is said to be a weak solution of the Navier-Stokes equations, with initial values f, $\operatorname{div} f = 0$ (in the distributions sense), if for any C^∞, rapidly decreasing vector function $v(x,t) = (v_1(x,t), \dots, v_n(x,t))$, defined on $R^{n+1}(x,t)$, such that $\operatorname{div} v = 0$, $v(x,t) = 0$, $t > T$, we have

(a) $u \in L^{p,q}(S_T)$ with $p, q \geq 2$,

(b) $\int_0^T \int_{R^n} \langle u, L^*(v) + (\operatorname{grad} v)(u)\rangle dx\,dt = -\int_{R^n} \langle f(x), v(x,0)\rangle\,dx$ where L^* is the adjoint heat operator,

(c) $\operatorname{div} u(x,t) = 0$ (in the distributions sense) for a.e. t, such that $0 < t < T$.

Theorem A. *Let n be greater than or equal to* 3 *and $f(x)$, the initial data, a vector function, such that $f(x) \in L^r(\mathbf{R}^n)$, $1 < r < \infty$, and $\operatorname{div} f = 0$ in the distributions sense. Let $F(x,t) = W * f$, where W is the fundamental solution of the heat equation* (*see* II *below*); *the convection is taken in the spatial coordinates. Then*

(i) $\|F\|_{p,q}{}^{1} < C_{p,q}\|f\|_r$, $\frac{n}{r} = \frac{n}{p} + \frac{2}{q}$, $q \geq p$,

[1] $\|F\|_{p,q} = \|F\|_{p,q}(S_\infty)$.

(ii) $\|F\|^*_{p,q} < C_{p,q}\|f\|_r$, $\frac{n}{r} = \frac{2}{p} + \frac{n}{q}$, $p \geq q$,

(iii) *If for some* p, q, $1 = \frac{n}{p} + \frac{2}{q}$, $n < p$, *we have*

$$\|F\|_{p,q} < \varepsilon_{p,q}(n),$$

where $\varepsilon_{p,q}(n)$ *is a fixed small quantity depending on* p, q *and* n *only. Then, there exists a unique solution* u *to the problem* $(0, 1)$, *that is global and satisfies the equations* $(0, 1)$ *and the initial data in the weak sense. The uniqueness holds in the class of functions* u *such that*

$$\|u\|_{p,q} < \infty.$$

Likewise, if for some (p, q); $1 = \frac{n}{q} + \frac{2}{p}$; $n < q$, *we have*

$$\|F\|^*_{p,q} < \varepsilon_{p,q}(n)$$

where $\varepsilon_{p,q}(n)$ *is a small quantity depending on* p, q *and* n *only; then, there exists a weak solution* u *of* $(0, 1)$ *for the initial data* f, *that satisfies* $(0, 1)$ *for all time* $t > 0$. *The solution* u *is unique in the class of functions that satisfy* $\|u\|^*_{p,q} < \infty$.

Corollary. *Let the initial data* f *belong to* $L^n(\mathbf{R}^n)$. *If for some* $L^{p,q}$*-norm we have*

$$\|F\|_{p,q} < \varepsilon_{p,q}(n), \qquad 1 = \frac{n}{p} + \frac{2}{q}, \; p < q.$$

Then, there exists a weak, global solution for the problem $(0, 1)$. *The solution is unique in the corresponding class of functions* $L^{p,q}(S_\infty)$.

II. Fundamental solutions, the bilinear form, mixed points

Fabes-Jones-Riviere, [4], extended to dimension n a formula found for the case $n = 3$ by Oseen [10]. The Oseen-Fabes-Jones-Riviere formula gives a divergence free matrix fundamental solution $E_{ij}(x, t)$ for an n-dimensional heat equation. The matrix $E_{ij}(x, t)$ is defined in the following way:

$$E(x, t) = \delta_{ij}W(x, t) - R_iR_jW(x, t), \tag{2.1}$$

where $W(x, t) = (4\pi t)^{-n/2} \exp -|x|^2/(4t)$, and R_j is the jth Riesz transform, namely,

$$R_j(f) = \text{p.v. } c_j \int (x_j - y_j)|x - y|^{-(n+1)} f(y)\, dy. \tag{2.2}$$

For details, see [4 and 13]. $E_{ij}(x, t)$ is symmetric and divergence free, that is,

$$\sum_j D_j E_{ij}(x, t) = 0, \qquad t \geq 0. \tag{2.3}$$

In the above formula we take classical derivatives if $t > 0$, and distributional derivatives for the limit for t tending to 0. An other important property is the following one:

$\sum_j \int E_{ij}(x, t) f_j(y - x)\, dx$ tends to $f_i(y)$ in L^p, $1 < p < \infty$, as t tends to zero, provided that $\operatorname{div} f = 0$.

A very important theorem in [4] asserts that $u(x,t)$ is a weak solution of the problem $(0,1)$ (see definition in §I) over S_T, with $g \in L^r$, $1 \leq r < \infty$, if and only if it is a solution of the following integral equation [4, Theorem 2.1, p. 226]:

$$u + B(u,u) = F(x,t). \tag{2.4}$$

$F(x,t)$ stands for the convolution in the space variables of the initial data $f(x)$ with the fundamental solution $W(x,t)$. $B(u,v)$ is the bilinear form

$$\int_0^t \int_{R^n} \langle(\operatorname{grad} E(x-y,t-s))(v(y,s)),u(y,s)\rangle dy\,ds. \tag{2.5}$$

We have used here a notation consistent with $(0,2)$. In fact, $(\operatorname{grad} E)(v)$ is the matrix $\sum_k D_k E_{ij} v_k$, whose ith row is dotted with u to obtain the integrand of (2.5).

Estimates for the bilinear operators and fixed-point properties.

In what follows, we are going to consider Banach spaces of Lebesgue measurable functions defined on S_T for which the operator $T(u,v) = B(u,v) + 1(u) + F(x,t)$, ($B(u,v)$ is bilinear and $1(u)$ is assumed to be linear) satisfies an inequality of the type

$$\|T(u,v)\| \leq C_1\|u\|\,\|v\| + C_2\|u\| + \|F\|. \tag{2.6}$$

In (2.9) above, the norm is that of the Banach space in question.

Lemma II. *The quadratic operator $T(u,u)$ maps the ball $\{\|u\| \leq s_1\}$ into itself if s_1 is the smallest root of the equation*

$$C_1 s^2 + (C_2 - 1)s + \|F\| = 0 \tag{2.7}$$

provided that C_1, C_2 and $\|F\|$ satisfy

$$(1-C_2)^2 > 4C_1\|F\|, \qquad C_1 > 0,\ 0 \leq C_2 < 1. \tag{2.8}$$

If $2s_1C_1 + C_2 < 1$, $T(u,u)$ is a contraction mapping in the ball of radius s_1. In particular, $T(u,u)$ is a contraction mapping in the ball of radius s_1 if C_1, C_2 and $\|F\|$ satisfy

$$2C_1\|F\|\{(1-C_2)^2 - 4C_1\|F\|\}^{-1/2} + C_2 < 1. \tag{2.9}$$

For the proof see [2].

III. Estimates for the bilinear form $B(u,u)$

The bilinear form $B(u,v)$ admits the domination

$$|B(u,v)| \leq C\int_{R^n} |x-y|^{-n+1} \int_0^t |x-y|^{-2}(1+|x-y|^{-1}s^{1/2})^{-n-1} \times |u(y,t-s)|\,|v(y,t-s)|\,ds\,dy. \tag{3.1}$$

The above domination is a consequence of the estimate

(3.2) $$|D_k E_{ij}(x,t)| \le C(|x| + t^{1/2})^{-n-1}.$$

Calling $M(u)$ the maximal function of Hardy-Littlewood of $|u|$ on the space variables and u^* the sup on $t > 0$ of $|u|$, we have

(3.3) $$|B(u,u)| \le C_0 \int_{R^n} |x-y|^{-n+1} (M(u^*))^2\, dy.$$

The constant C_0 does not exceed

(3.4) $$C\int_0^\infty (1 + t^{1/2})^{-n-1}\, dt.$$

The right-hand side of (3.14) does not depend on t, hence

(3.5) $$|B(u,u)|^* \le C_0 \int_{R^n} |x-y|^{-n+1} (M(u^*))^2\, dy.$$

We now apply Hardy-Littlewood-Sobolev potential inequality to (3.5) with exponents $\frac{1}{q} = \frac{2}{n} - \frac{1}{n}$ (see [13, pp. 119, 120]) and obtain

(3.6) $$\|B(u,u)^*\|_n(\infty) \le C\{\|M(u^*)\|_n(\infty)\}^2 \le C'\{\|u^*\|_n(\infty)\}^2.$$

We now need estimates on $D_k E_{ij}$ of the type introduced by Benedek-Panzone in [1] (see p. 321, Theorem 1), namely,

(3.5) $$|D_k E_{ij}| \le \frac{C}{|x|^{n-\theta}(t^{1/2})^{1+\theta}}.$$

Here, C is an independent constant, $0 < \theta < 1$, $t > 0$, $x \in \mathbf{R}^n$. A simple adaptation of Theorem 1, p. 321 in [1] (see also [4, Theorem (3.1)]) gives for the operator

$$T(f) = \int_0^t \int_{\mathbf{R}^n} \frac{1}{|x-y|^{n-\theta}(t-\tau)^{\frac{1+\theta}{2}}} f(y,\tau)\, dy\, d\tau$$

the estimates

(3.7) $$\|T(f)\|_{p^*,q^*} \le C_{p,q}\|f\|_{p,q}$$

where

$$\frac{1}{p^*} = \frac{1}{p} - \frac{\theta}{n}\, \frac{1}{q^*} = \frac{1}{q} - \frac{1-\theta}{2}.$$

Similar results hold for the norms $\| \ \|^*_{p,q}$, namely,

(3.8) $$\|T(f)\|^*_{p^*,q^*} \le C_{p,q}\|f\|^*_{p,q}.$$

Here,

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1-\theta}{2}\, \frac{1}{q^*} = \frac{1}{q} - \frac{\theta}{n}.$$

The estimates (3.5), (3.7), (3.8) and (3.6) lead to the following inequalities for $B(u,u)$:

(3.9) $$\|B(u,u)\|_{\frac{n}{\theta},\frac{2}{1-\theta}} \le C_\theta(\|u\|_{\frac{n}{\theta},\frac{2}{1-\theta}})^2,$$

$0 < \theta < 1$, which is a consequence of

$$|B(u,u)| \leq CT(|u|^2). \tag{3.10}$$

T as defined in (3.6) above. Likewise we get for the $\| \|^*_{p,q}$ norms the estimate

$$\|B(u,u)\|^*_{\frac{2}{1-\theta},\frac{n}{\theta}} \leq C_\theta(\|u\|^*_{\frac{2}{1-\theta},\frac{n}{\theta}})^2, \tag{3.11}$$

$0 < \theta < 1$. In the above expressions we may replace $\frac{n}{\theta}$ and $\frac{2}{1-\theta}$ by p and q respectively, satisfying

$$1 = \frac{n}{p} + \frac{2}{q}; \qquad p > n.$$

IV. Imbedding the initial data

As we have seen in (2.4), $F(x,t)$ is the term arising from the initial data:

$$F(x,t) = W * f. \tag{4.1}$$

The above convolution is on the spatial variables only. The "a priori estimate" (3.86) [2] and Lemma A_3 in [2] give the following result as a trivial consequence:

Lemma IV. *Let* $1 \leq n < \infty$, $f \in L^p(\mathbf{R}^n)$, $1 < p \leq \infty$. *Then, the function* $F(x,t)$ *as defined above, satisfies*

(i) $\|F\|_{\frac{n+2}{n}p,\frac{n+2}{n}p} \leq C_p(n)\|f\|_p$,

(ii) $\|F\|^*_{\frac{n+2}{n}p,\frac{n+2}{n}p} \leq C_p(n)\|f\|_p$.

$C_p(n)$ *depends only on* p *and* n.

Theorem B. *The function* $F(x,t)$ *defined above satisfies*

(i) $\|F\|_{s,v} \leq C_r\|f\|_r$,

(ii) $\|F\|^*_{v,s} \leq C_r\|f\|_r$, $1 < r < \infty$, $v > s$, $\frac{1}{r} = \frac{1}{s} + \frac{2}{nv}$.

C_r *depends only on* r, v, s *and* n.

Proof. We shall consider the norms $\| \|_{p,q}$ only, since $\| \|^*_{p,q}$ can be dealt with in a similar manner ($\tilde{u}(t,x) = u(x,t)$, thus $\|\tilde{u}\|_{p,q} = \|u\|^*_{p,q}$). On one hand we have

$$\|F\|_{p,\infty} < C_p\|f\|_p, \qquad 1 < p \leq \infty,$$

and from Lemma IV

$$\|F\|_{\frac{n+2}{n}q,\frac{n+2}{n}q} < C_q\|f\|_q, \qquad 1 < q \leq \infty.$$

The Benedek-Panzone interpolation theorem for mixed norms (see [1, Theorems 1, 2]); gives the desired result for

$$\begin{aligned} \frac{1}{r} &= \frac{t}{p} + \frac{1-t}{q}, \\ \frac{1}{s} &= \frac{t}{p} + \frac{1-t}{q}\frac{n}{n+2}, \\ \frac{1}{v} &= \frac{1-t}{q}\frac{n}{n+2}, \qquad 0 < t < 1, \end{aligned} \tag{4.2}$$

or

$$\frac{1}{r} = \frac{1}{s} - \frac{1}{v} + \frac{n+2}{n}\frac{1}{v}, \qquad v > s,$$

hence,

$$\frac{1}{r} = \frac{1}{s} + \frac{2}{nv}. \tag{4.3}$$

Concerning the norms $\| \|^*_{p,q}$, one should notice that

$$\|F\|^*_{\infty,p} < C_p \|f\|_p, \qquad 1 < p \leq \infty,$$

is a consequence of the maximal theorem associated with the Weierstrass kernel. This finishes the proof.

V. Proof of Theorem A

Parts (i) and (ii) follow from the imbedding results of §IV. Part (iii) follows using the estimates (3.9) and (3.11) and Lemma II. This concludes the proof.

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