TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 318, Number 2, April 1990

ON INFINITE-DIMENSIONAL MANIFOLD TRIPLES

KATSURO SAKAI AND RAYMOND Y. WONG

ABSTRACT. Let Q denote the Hilbert cube $[-1, 1]^{\omega}$, $s = (-1, 1)^{\omega}$ the pseudo-interior of Q, $\Sigma = \{(x_i) \in s \mid \sup|x_i| < 1\}$ and $\sigma = \{(x_i) \in s \mid x_i = 0$ except for finitely many $i\}$. A triple (X, M, N) of separable metrizable spaces is called a (Q, Σ, σ)- (or (s, Σ, σ)-)manifold triple if it is locally homeomorphic to (Q, Σ, σ) (or (s, Σ, σ)). In this paper, we study such manifold triples and give some characterizations.

0. INTRODUCTION

Throughout this paper, spaces are separable and metrizable. A manifold modeled on a given space E is called an *E-manifold*. For a given pair (E, F) (or triple (E, F, G)) of spaces, an *(E, f)-manifold pair* (or *(E, F, G)-manifold triple*) is a pair (X, M) (or triple (X, M, N)) of spaces such that there is an open cover $\mathscr{U}$ of X and open embeddings $\phi_U: U \to E$, $U \in \mathscr{U}$, such that $\phi_U(M \cap U) = F \cap \phi_U(U)$ (and respectively $\phi_U(N \cap U) = G \cap \phi_U(U)$). Let Q denote the Hilbert cube $[-1, 1]^{\omega}$, $s = (-1, 1)^{\omega}$ (the pseudo-interior of Q), $\Sigma = \{(x_i) \in s \mid \sup|x_i| < 1\} = \bigcup_{n=1}^{\infty}[-1+n^{-1}, 1-n^{-1}]^{\omega}$ and $\sigma = \{(x_i) \in s \mid x_i = 0$ except for finitely many $i\}$. In this paper we study (Q, Σ, σ)- (or (s, Σ, σ)-) manifold triples.

A closed subset A of a space X is called a *Z-set in X* if for each map $f: Q \to X$ and each $\varepsilon > 0$, there is a map $g: Q \to X \setminus A$ with $D(f, g) < \varepsilon$. A subset M of X is called a *cap* (or *f.d. cap*) *set for X* if $M = \bigcup_{n\in\mathbf{N}} M_n$, where $M_1 \subset M_2 \subset \cdots$ is a tower of (resp. finite-dimensional) compact Z-sets in X satisfying the following conditions:

$(*)$ *For each (finite-dimensional) compact set A in X, each $\varepsilon > 0$ and each $m \in \mathbf{N}$, there is an integer $n \geq m$ and an embedding $h: A \to M_n$ such that $h|A \cap M_m = \mathrm{id}$ and $d(h, \mathrm{id}) < \varepsilon$*,

where d is a metric for X [An_3 and Ch_1]. We call M a *strong (f.d.) cap set* for X if in the above definition $M = \bigcup_{n\in\mathbf{N}} M_n$ satisfies the following stronger condition:

Received by the editors June 10, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57N20.
Key words and phrases. (Q, Σ, σ)-manifold triple, (s, Σ, σ)-manifold triple, (f.d.) cap pair, cap manifold pair.
This work was done while the first author was visiting the University of California at Santa Barbara.

($*$) *For each* (*finite-dimensional*) *compact* Z*-set* A *in* X, *each open cover* $\mathscr{U}$ *of* X *and each* $m \in \mathbf{N}$, *there is an* $n \geq m$ *and a homeomorphism* $h: X \to X$ *such that* $h|M_m = \mathrm{id}$, $h(A) \subset M_n$ *and* h *is* $\mathscr{U}$*-close to* id.

A strong (f.d.) cap set is called a $\mathscr{Z}$*-skeletoid* (or $\mathscr{E}\mathscr{Z}$*-skeletoid*) in [BP_1], where $\mathscr{Z}$ (or $\mathscr{E}\mathscr{Z}$) is the collection of (finite-dimensional) compact Z-sets in X. In case X is complete-metrizable, strong (f.d.) cap sets are topologically unique, that is, if M and N are strong cap sets (or strong f.d. cap sets) for X then there is a homeomorphism $h: X \to X$ such that $h(M) = N$ ([BP_2, Chaper IV, Theorem 2.1]). In case X is a Q- (or s-)manifold, (f.d.) cap sets for X are strong (f.d.) cap sets [BP_2, Chapter IV, Proposition 4.1]. Note that Σ is a cap set and σ is an f.d. cap set for both Q and s. A pair (X, M) is a (Q, Σ)- (or (s, Σ)-)manifold pair if and only if X is a Q- (or s-)manifold and M is a cap set for X. A pair (X, N) is a (Q, σ)- (or $s, \sigma)$-)manifold pair if and only if X is a Q- (or s-)manifold and N is an f.d. cap set for X [Ch_1].

It is well known that s is homeomorphic to ($\cong$) the separable Hilbert space l_2 [An_1]. Let l_2^Q be the linear span of the Hilbert cube $\prod_{i\in\mathbf{N}}[-i^{-1}, i^{-1}]$ in 1_2, i.e., $l_2^Q = \{(x_i) \in l_2 | \sup|ix_i| < \infty\}$, and l_2^f the linear span of the usual orthonormal basis, i.e., $l_2^f = \{(x_i) \in l_2 | x_i = 0$ except for finitely many $i\}$. Then l_2^Q is a cap set and l_2^f is an f.d. cap set for l_2; hence $(l_2, l_2^Q) \cong (s, \Sigma)$ and $(l_2, l_2^f) \cong (s, \sigma)$. It is not clear whether $(l_2, l_2^Q, l_2^f) \cong (s, \Sigma, \sigma)$, or whether any one of the following triples homeomorphic to (Q, Σ, σ) or (s, Σ, σ):

$$(Q, Q \setminus s, Q_f \setminus s), \quad (Q, \Sigma', \sigma), \quad (s, \Sigma', \sigma), \quad (I^\omega, I^\omega \setminus \mathring{I}^\omega, I_f^\omega),$$

where $Q_f = \{(x_i) \in Q | x_i = 0$ except for finitely many $i\}$, $\Sigma' = \{(x_i) \in s | |x_i| < \frac{1}{2}$ except for finitely many $i\}$, $I = [0, 1]$, $\mathring{I} = (0, 1)$ and $I_f^\omega = I^\omega \cap Q_f$. In general, we can ask whether (X, M, N) is a (Q, Σ, σ)- (or (s, Σ, σ)-)manifold triple when X is a Q- (or s-)manifold, M is a cap set and N is an f.d. cap set for X such that $N \subset M$. This question relates to a conjecture on function spaces as follows: The triple

$$(C(X, Y), \quad LIP(X, Y), \quad PL(X, Y))$$

is an (s, Σ, σ)-manifold triple if X and Y are polyhedra in $\mathbb{R}^n$, X is compact, $\dim X \neq 0$ and Y has no isolated point (or Y is open in $\mathbb{R}^n$), where $C(X, Y)$ is the space of maps from X to Y, $LIP(X, Y)$ the subspace of $C(X, Y)$ consisting of all Lipschitz maps and $PL(X, Y)$ the subspace of $C(X, Y)$ consisting of all piecewise linear maps (cf. [Ge], [Sa_3], [SW]).

In §2, we give a characterization of (Q, Σ, σ)- (or (s, Σ, σ))-manifold triples. Using this characterization, we can easily show, among other results, that $(l_2, l_2^Q, l_2^f) \cong (s, \Sigma, \sigma)$, $(Q, Q \setminus s, Q_f \setminus s) \cong (Q, \Sigma, s)$, etc. This section contains our principal results on manifold triples.

In §3, we establish a relationship between (Σ, σ)-manifold pairs and $\Sigma \times s$-manifolds. In fact, we show that $\Sigma \setminus \sigma \cong \Sigma \times s$; for any (Σ, σ)-manifold pair

(M, N), $M\backslash N$ is a $\Sigma\times s$-manifold and each $\Sigma\times s$-manifold is homeomorphic to $M\backslash N$ for some (Σ, g)-manifold pair (M, N). Using the fact $\Sigma\backslash\sigma\cong\Sigma\times s$, we can easily construct a cap set M and an f.d. cap set N for both Q and s so that $N\subset M$ but $(Q, M, N)\not\cong(Q, \Sigma, \sigma)$ and $(s, M, N)\not\cong(s, \Sigma, \sigma)$.

Example. Let $N = \{(x_i)\in s\,|\, x_i = \frac{2}{3}$ except for finitely many $i\}$ and $M = \Sigma'\cup N$. Then N is clearly an f.d. cap set for both Q and s and M is a cap set for both Q and s ([Ch_1, Theorem 6.6]). Hence $M\backslash N=\Sigma'\cong\Sigma\not\cong\Sigma\times s\cong\Sigma\backslash\sigma$ since $\Sigma\times s$ is not σ-compact. Therefore $(M, N)\not\cong(\Sigma, \sigma)$, which implies $(Q, M, N)\not\cong(Q, \Sigma, \sigma)$ and $(s, M, N)\not\cong(s, \Sigma, \sigma)$.

Applying the result of §2, we give in §3 a characterization of (Σ, σ)-manifold pairs and show that (X, M, N) is a (Q, Σ, σ)- (or (s, Σ, σ)-)manifold triple if and only if (X, M) is a (Q, Σ)- (or (s, Σ)-)manifold pair (or (X, N) is a (Q, σ)- (or (s, σ) -)manifold pair) and (M, N) is a (Σ, σ)-manifold pair.

Notations and terminology. A metric for a space X is denoted by d. for $A\subset X$ and $x\in X$, let $d(x, A) = \inf\{d(x, a)|\, a\in A\}$. For maps $f, g\colon Y\to X$, let $d(f, g) = \sup\{d(f(y), g(y))|\, y\in Y\}$. For $A\subset X$ and a collection $\mathscr{B}$ of subsets of X, let $\mathrm{st}(A, \mathscr{B}) = \bigcup\{B\in\mathscr{B}|\, A\cap B\neq 0\}$. Let $\mathscr{A}$ and $\mathscr{B}$ be collections of subsets of X. Define $\mathrm{st}(\mathscr{A}, \mathscr{B}) = \{\mathrm{st}(A, B)|\, A\in\mathscr{A}\}$. For $n\in\mathbf{N}$, let $\mathrm{st}^n(\mathscr{B}) = \mathrm{st}(\mathrm{st}^{n-1}(\mathscr{B}), \mathscr{B})$, where $\mathrm{st}^0(\mathscr{B}) = \mathscr{B}$. We denote $\mathscr{A}<\mathscr{B}$ if each $A\in\mathscr{A}$ is contained is some $B\in\mathscr{B}$. Maps $f, g\colon Y\to X$ are said to be $\mathscr{B}$-close if $\{\{f(y), g(y)\}|\, y\in Y\}<\mathscr{B}$. A homotopy $h\colon Y\times I\to X$ is called a *$\mathscr{B}$-homotopy* if $\{\{h(\{y\}\times I)\}\,|y\in Y\}<\mathscr{B}$. Then we say that $h_0 = h|\, Y\times\{0\}$ is *$\mathscr{B}$-homotopic* to $h_1 = h|\, Y\times\{1\}$ and denote $h_0\overset{\mathscr{B}}{\simeq} h_1$. Let $\mathscr{U}$ be an open cover of X. A map $f\colon Y\to X$ is called a *$\mathscr{U}$-homotopy equivalence* if there is a map $g\colon X\to Y$ called a *$\mathscr{U}$-homotopy inverse* of f such that $fg\overset{\mathscr{U}}{\simeq}\mathrm{id}_X$ and $gf\overset{f^{-1}(\mathscr{U})}{\simeq}\mathrm{id}_y$. A *fine homotopy equivalence* $f\colon Y\to X$ is a *$\mathscr{U}$-homotopy* equivalence for any open cover $\mathscr{U}$ of X. Let A be a closed set in X. An open cover $\mathscr{V}$ of $X\setminus A$ is said to be *normal relative to* A provided any homeomorphism $h\colon X\setminus A\to X\setminus A$ of $X\setminus A$ onto itself which is $\mathscr{V}$-close to id can be extended to a homeomorphism $h\colon X\to X$ of X onto itself by $\tilde{h}|A = \mathrm{id}$.

1. Pair versions of (f.d.) cap sets

Let X be a space and $y\subset X$. A strong (f.d.) cap set M for X is called a *strong (f.d.) cap set for* (X, Y) if h in $(*)_s$ above satisfies $h(Y) = Y$. Let $H(X)$ be the group of homeomorphisms of X onto itself and $H(X, Y)$ the subgroup of $H(X)$ consisting of all $h\in H(X)$ satisfying $h(Y) = Y$. A strong (f.d.) cap set for (X, Y) is called an *$H(X, Y)$-$\mathscr{Z}$-skeletoid* (or *$H(X, Y)$-skeletoid*) in [BP_2]. In general $H(X, Y)$ is not closed in $H(X)$ in the topology of uniform convergence with respect to any metric for X. Hence the following lemma does not follow immediately from [BP_2, Chapter IV, Proposition 2.2].

1.1. **Lemma.** *Let X be complete-metrizable and $Y \subset X$. Then for any two strong (f.d.) cap sets M and N for (X, Y) and any open cover $\mathscr{U}$ of X, there is a homeomorphism $h: (X, M, Y) \to (X, N, Y)$ which is $\mathscr{U}$-close to* id.

Proof. A proof can be given following the same procedure as in [Ch_1, Lemma 4.3] and will be omitted.

Let (M, N) be a pair of subsets of a space X. We call (M, N) a *(cap, f.d. cap)-pair for* X if $M = \bigcup_{n=1}^{\infty} M_n$, where $M_1 \subset M_2 \subset \cdots$ is a tower of compact Q-manifolds satisfying $(*)$ above and the following:

$(**)$ *Each M_n is a Z-set for M_{n+1} and each $M_n \cap N$ is an f.d. cap set for M_n.*

Remark. It follows that N is an f.d. cap set for X.

1.2. **Lemma.** *Let (M, N) be a (cap, f.d. cap)-pair for a Q- (or s-)manifold X. Then M is a strong cap set for (X, N).*

Proof. Write $M = \bigcup_{n\in\mathbf{N}} M_n$ as in the definition of (cap, f.d. cap)-pair. Let $\mathscr{U}$ be an open cover of X and A a compact Z-set in X. From $(*)$, there is an $n \geq m$ and an embedding $f: A \to M_n$ such that $f|A \cap M_m = \mathrm{id}$ and f is $\mathscr{U}$-homotopic to id. By [Ch_1, Lemma 5.4, Theorems 6.7 and 6.6],

$$L = (M_{n+1} \cap N) \setminus f(A \cap M_m)$$

and

$$L' = (((M_{n+1} \cap N) \setminus f(A)) \cup f(A \cap N)) \setminus f(A \cap M_m)$$

are f.d. cap sets for $M_{n+1} \setminus f(A \cap M_m)$. Let $\mathscr{V}$ be an open cover of $M_{n+1} \setminus f(A \cap M_m)$ such that $\mathscr{V} < \mathscr{U}$ and $\mathscr{V}$ is normal relative to $f(A \cap M_m)$. By [Ch_1, Theorem 6.2], we have a homeomorphism of pair

$$g': (M_{n+1} \setminus f(A \cap M_m), L') \to (M_{n+1} \setminus f(A \cap M_m), L)$$

which is $\mathscr{V}$-homotopic to id. Then g' extends to a homeomorphism $g: M_{n+1} \to M_{n+1}$ with $g|f(A \cap M_m) = \mathrm{id}$, and g is $\mathscr{U}$-homotopic to id. Observe that $gf(A \cap N) \subset N$ and $gf(A \setminus N) \subset M_{n+1} \setminus N$; that is, $gf(A) \cap N = gf(A \cap N)$. Since $gf: A \to X$ is a Z-embedding which is st $\mathscr{U}$-homotopic to id, gf extends to a homeomorphism $h: X \to X$ which is $\mathrm{st}^2\, \mathscr{U}$-isotopic to id by the Homeomorphism Extension Theorem [AC]. By [Ch_1, Lemma 5.4], $N \setminus h(A)$ and $h(N \setminus A) = h(N) \setminus h(A)$ are f.d. cap set for $X \setminus h(A)$. Similarly, there is a homeomorphism $k: X \to X$ such that $k|h(A) = \mathrm{id}$, $kh(N \setminus A) = N \setminus h(A)$ and k is $\mathscr{U}$-close to id. Since $h(A \cap N) = h(A) \cap N$, we have $kh(N) = N$. Thus we have a homeomorphism $kh: X \to X$ such that $kh(N) = N$, $kh|A \cap M_m = gf|A \cap M_m = \mathrm{id}$, $kh(A) = gf(A) \subset g(M_n) \subset M_{n+1}$ and kh is $\mathrm{st}^3\, \mathscr{U}$-close to id. $\square$

For any countable locally finite simplicial complex K, $|K| \times Q$ is a Q-manifold and $|K| \times s$ is an s-manifold [We]. As easily observed, (Σ, σ) *is* *a (cap, f.d. cap)-pair for both* Q *and* s. It is straightforward to verify the following:

1.3. **Lemma.** *For any countable locally finite simplicial complex* K, $(|K| \times \Sigma, |K| \times \sigma)$ *is a (cap, f.d. cap)-pair for both* $|K| \times Q$ *and* $|K| \times s$.

It is also easy to verify that various versions of (Σ, σ) mentioned in the introduction are (cap, f.d. cap)-pairs. The following is rather useful in the proof of our characterization.

1.4. **Lemma.** *Let* (M, N) *be a (cap, f.d. cap)-pair for a* Q*- (or* s*-) manifold* X *and* U *an open set in* X. *Then* $(M \cap U, N \cap U)$ *is a (cap, f.d. cap)-pair for* U.

Proof. Let $M = \bigcup_{n \in \mathbf{N}} M_n$ be as in the definition and for each $n \in \mathbf{N}$, let $A_n = \{x \in U | d(x, X \setminus U) \leq 1/n\}$. Without loss of generality, we may assume that $M_1 \cap A_1 \neq \emptyset$. By [$\mathrm{Ch}_2$], $M_1 \cap U \cong |K| \times Q$ for some countable locally finite simplical complex K. Hence, $|K| \times \sigma$ is an f.d. cap set for $|K| \times Q$. On the other hand, $M_1 \cap N \cap U$ is an f.d. cap set for $M_1 \cap U$ by [Ch_2, Lemma 5.4]. By [Ch_1, Theorem 6.2], we have a homeomorphism

$$h\colon (M_1 \cap U, M_1 \cap N \cap U) \to (|K| \times Q, |K| \times \sigma).$$

Since $M_1 \cap A_1$ is compact, $h(M_1 \cap A_1) \subset |L| \times Q$ for some finite subcomplex L of K. Let $M_1^* = h^{-1}(|L| \times Q)$. Then M_1^* is a compact Q-manifold with $M_1 \cap A_1 \subset M_1^* \subset M_1 \cap U$ and $M_1^* \cap N \cap U = h^{-1}(|L| \times \sigma)$ an f.d. cap set. Using this process inductively, we obtain compact Q-manifolds M_{gn}^*, $n \in \mathbf{N}$, such that $M_{n-1}^* \cup (M_n \cap A_n) \subset M_n^* \subset M_n \cap U$ and $M_n^* \cap N \cap U$ is an f.d. cap set for M_n^*. Then $M \cap U = \bigcup_{n \in \mathbf{N}} M_n^*$ satisfies $(*)$ and $(**)$ relative U and $N \cap U$. $\square$

A subset M of a space X said to be *map-dense in* X if for each compact set A in X and each $\varepsilon > 0$, there is a map $f\colon A \to M$ such that $d(f, \mathrm{id}) < \varepsilon$.

1.5. **Lemma.** *Let* (M, N) *be a pair of subsets of a space* X *such that* M *is an ANR. Then* (M, N) *is a (cap, f.d. cap)-pair for* X *if and only if* M *is map-dense in* X *and* $M = \bigcup_{n \in \mathbf{N}} M_n$, *where* $M_1 \cap M_2 \subset \cdots$ *is a tower of compact* Q*-manifolds satisfying* $(**)$ *and the following*:

$(*)'$ *For each compact set* A *in* M, *each* $\varepsilon > 0$ *and each* $m \in \mathbf{N}$, *there is an* $n \geq m$ *and a map* $f\colon A \to M_n$ *with* $d(f, \mathrm{id}) < \varepsilon$, *and* $f(A \cap M_m) \subset M_m$.

Proof. The "only if" part is trivial. By [Sa_2, Proposition 2.1] (cf. [Sa_1, Lemma 1]), we can verify that $\{M_n\}_{n \in \mathbf{N}}$ satisfies $(*)$. $\square$

Since a cap set for a Q- (or s-)manifold is a Σ-manifold, we have the following.

1.6. **Corollary.** *Let* (M, N) *be a (cap, f.d. cap)-pair for a* Q*- (or* S*-) manifold* X. *Then for any homeomorphism* $f\colon M \to M$ *of* M *onto itself and* $M \subset Y \subset X$, $(M, f(N))$ *is a (cap, f.d. cap)-pair for* Y.

2. Characterization of (Q, Σ, σ)- (or (s, Σ, σ)-)manifold triples

Here we give a characterization of (Q, Σ, σ)- (or (s, Σ, σ)-)manifolds.

2.1. **Theorem.** *Let X be a Q- (or s-) manifold and $N \subset M \subset X$. Then the following are equivalent*:

(i) *(X, M, N) is a (Q, Σ, σ)- (or (s, Σ, σ)-) manifold triple*;

(ii) *(M, N) is a (cap, f.d. cap)-pair for X*;

(iii) *M is a strong cap set for (X, N) and N is an f.d. cap set for X*.

Proof. The implication (ii) $\Rightarrow$ (iii) is Lemma 1.2 and (iii) $\Rightarrow$ (ii) follows from Lemmas 1.3, 1.2 and 1.1. We shall prove the equivalence of (i) and (ii).

(ii) $\Rightarrow$ (i): Each $x \in X$ has an open neighborhood U with an open embedding $g: U \to Q$ (or $g: U \to s$). By Lemma 1.4, $(M \cap U, N \cap U)$ and $(\Sigma \cap g(U), \sigma \cap g(U))$ are (cap, f.d. cap)-pairs for U and $g(U)$, respectively. Then by Lemmas 1.2 and 1.1, there is a homeomorphism

$$\phi: (U, M \cap U, N \cap U) \to (g(U), \Sigma \cap g(U), \sigma \cap g(U)).$$

Therefore (X, M, N) is a (Q, Σ, σ)- (or (s, Σ, σ)-)manifold triple.

(i) $\Rightarrow$ (ii): By [Ch_2] $X \cong |K| \times Q$ (or $|K| \times s$) for some countable locally finite simplicial complex K. Since $(|K| \times \Sigma, |K| \times \sigma)$ is a (cap, f.d. cap)-pair for $|K| \times Q$ (or $|K| \times s$) by Lemma 1.3, we have a (cap, f.d. cap)-pair (M', N') for X. From the definition of manifold triples and Lemma 1.4, X has a countable star-finite open cover $\mathscr{U}$ such that for each $U \in \mathscr{U}$, $(M \cap U, N \cap U)$ is a (cap, f.d. cap)-pair for U. By [AHW] there is an ordering $\{U_i\}_{i \in \mathbf{N}}$ of $\mathscr{U}$ and a sequence $n_1 \leq n_2 \leq \cdots$ in $\mathbf{N}$ such that for any $f_i \in H(X)$, $i \in \mathbf{N}$, with $f_i | X \setminus U_i = \mathrm{id}$, $\{f_i \circ \cdots \circ f_1\}_{i \in \mathbf{N}}$ converges pointwise to an $f \in H(X)$ with $f | U_i = f_{n_i} \circ \cdots \circ f_{n_1} | U_i$. Note that $(M' \cap U_1, N' \cap U_1)$ is a (cap, f.d. cap)-pair for U_1 by Lemma 1.4. Using Lemmas 1.2 and 1.1, we have a homeomorphism

$$h_1: (U_1, M' \cap U_1, N' \cap U_1) \to (U_1, M \cap U_1, N \cap U_1)$$

which can be extended to an $f_1 \in H(X)$ by $f_1 | X \setminus U_1 = \mathrm{id}$. Then $f_1(M') \cap U_1 = f_1(M') \cap f_1(U_1) = f_1(M' \cap U_1) = M \cap U_1$. Similarly we have $f_1(N') \cap U_1 = N \cap U_1$. Since $(f_1(M') \cap U_2, f_1(N') \cap U_2)$ is a (cap, f.d. cap)-pair for U_2, we have a homeomorphism

$$h_2: (U_2, f_1(M') \cap U_2, f_1(N') \cap U_2) \to (U_2, M \cap U_2, N \cap U_2)$$

which can be extended to an $f_2 \in H(X)$ by $f_2 | X \setminus U_2 = \mathrm{id}$. Since $f_2 f_1(M') \cap U_2 = M \cap U_2$ and

$$\begin{aligned} f_2 f_1(M') \cap (U_1 \setminus U_2) &= f_2((f_1(M') \cap U_1) \setminus U_2) \\ &= f_2((M \cap U_1) \setminus U_2) = M \cap (U_1 \setminus U_2), \end{aligned}$$

$f_2 f_1(M') \cap (U_1 \cup U_2) = M \cap (U_1 \cup U_2)$. Similarly $f_2 f_1(N') \cap (U_1 \cup U_2) = N \cap (U_1 \cup U_2)$. By induction, we can construct $f_i \in H(X)$, $i \in \mathbf{N}$, such that $f_i | X \setminus U_i = \mathrm{id}$,

$$f_i \circ \cdots \circ f_1(M') \cap (U_1 \cup \cdots \cup U_i) = M \cap (U_1 \cup \cdots \cup U_i)$$

and

$$f_i \circ \cdots \circ f_1(N') \cap (U_1 \cup \cdots \cup U_i) = N \cap (U_1 \cup \cdots \cup U_i).$$

We also note that $f_1 \circ \cdots \circ f_1(U_1 \cup \cdots \cup U_i)U_1 \cup \cdots \cup U_i$. Let $f \in H(X)$ be the limit of $\{f_i \circ \cdots \circ f_1\}_{i \in \mathbf{N}}$. Then $f(M') = M$ and $f(N') = N$. In fact, for any $x \in M' \cap U_i$,

$$f(x) = f_{n_i} \circ \cdots \circ f_{n_1}(x) \in f_{n_1} \circ \cdots \circ f_{n_i}(M') \cap (U_{n_1} \cup \cdots \cup U_{n_i}) \subset M$$

and for $y = f(x) \in M \cap f(U_i)$,

$$\begin{aligned} y &= f_{n_i} \circ \cdots \circ f_{n_1}(x) \in M \cap f_{n_i} \circ \cdots \circ f_{n_1}(U_{n_1} \cup \cdots \cup U_{n_i}) \\ &= M \cap (U_{n_1} \cup \cdots \cup U_{n_i}) \\ &= f_{n_i} \circ \cdots \circ f_{n_1}(M' \cap (U_{n_1} \cup \cdots \cup U_{n_i})) \\ &\subset f(M'). \end{aligned}$$

Hence $f(M') = M$. Similarly $f(N') \subset N$. Therefore

$$(X, M', N') \cong (X, M, N),$$

which implies that (M, N) is a (cap, f.d. cap)-pair for X. □

We can apply Theorem 2.1 to verify that various triples mentioned in the Introduction are homeomorphic to (Q, Σ, σ) or (s, Σ, σ). Combining Theorem 2.1 with Lemma 1.1, we have the following

2.2. **Classification Theorem.** *Let (X, M, N) and (X', M', N') be (Q, Σ, σ)- (or (s, Σ, σ)-)manifold triples. Then $(X, M, N) \cong (X', M', N')$ if and only if $X \cong X'$. Moreover each homeomorphism $f: X \to X'$ is approximated by homeomorphisms $h: X \to X$ such that $h(M) = M'$ and $h(N) = N'$.*

The following follows from the Stability Theorem for Q- (or s-)manifolds [AS] and Theorem 2.2.

2.3. **Stability Theorem.** *Let (X, M, N) be a (Q, Σ, σ)- (or (s, Σ, σ)-)manifold triple. Then the projection $p: X \times Q \to X$ (or $p: X \times s \to X$) is approximated by homeomorphisms $h: X \times Q \to X$ (or $h: X \times s \to X$) such that $h(M \times \Sigma) = M$ and $h(N \times \sigma) = N$.*

By Lemma 1.3, Theorems 2.1 and 2.2, the Triangulation Theorem for Q- (or s-)manifolds [Ch_2] yields the following.

2.4. **Triangulation Theorem.** *For any (Q, Σ, σ)- (or (s, Σ, σ)-)manifold triple (X, M, N), there is a countable locally finite simplical complex K such that*

$$(X, M, N) \cong (|K| \times Q, |K| \times \Sigma, |K| \times \sigma) \quad (or\ (|K| \times s, |K| \times \Sigma, |K| \times \sigma)).$$

Lemma 1.4, Theorems 2.1 and 2.2 and the Open Embedding Theorem for s-manifolds [He] yield the following.

2.5. **Open Embedding Theorem.** *For any (s, Σ, σ)-manifold (X, M, N), there is an open embedding $g: X \to s$ such that $g(M) = g(X) \cap \Sigma$ and $g(N) = g(X) \cap \sigma$.*

In the (s, Σ, σ)-manifold case, we have the following version of Theorem 2.2 since two s-manifolds having the same homotopy type are homeomorphic [He].

2.6. **Classification Theorem II.** *Let (X, M, N) and (X', M', N') be (s, Σ, σ)-manifold triple. Then $(X, M, N) \cong (X', M', N')$ if and only if $X \cong X'$ or $M \cong M'$ of $N \cong N'$.*

3. (Σ, σ)-MANIFOLD PAIRS AND $\Sigma \times s$-MANIFOLDS

In [BM], Bestvina and Mogilski gave a characterization of $\Sigma \times s$-manifolds and established the Stability, Triangulation, Open Embedding Theorems, etc. In particular, they proved that $\Sigma \times s \cong s_f^\omega$, where $s_f^\omega = \{(x_i) \in s^\omega | x_i = 0$ except for finitely many $i\}$. Note that

$$\Sigma \times s \cong \sigma \times Q \times s \cong \sigma \times s.$$

The following can be shown by using the characterization of $\Sigma \times s$ [BM, Corollary 6.3], but we give here a different proof using Theorems 2.1 and 2.2.

3.1. **Theorem.** $\Sigma \setminus \sigma \cong \Sigma \times s$.

Proof. Note $(I^\omega, I_f^\omega) \cong (Q, \sigma)$. It is easy to see that $(I_f^\omega \times Q, I_f^\omega \times \sigma)$ is a (cap, f.d. cap)-pair for $I^\omega \times Q$. By Theorems 2.1 and 2.2,

$$(I^\omega \times Q, I_f^\omega \times Q, I_f^\omega \times \sigma) \cong (Q, \Sigma, \sigma).$$

Since $Q \setminus \sigma \cong s$ [An_3, Theorem 5.3] (cf. [Ch_1, Corollary 8.3]), we have

$$\Sigma \setminus \sigma \cong (I_f^\omega \times Q) \setminus (I_f^\omega \times \sigma) = I_f^\omega \times (Q \setminus \sigma) \cong \sigma \times s \cong \Sigma \times s. \quad \square$$

We have the following relationship between (Σ, σ)-manifolds and $\Sigma \times s$-manifolds.

3.2. **Corollary.** (1) *For any (Σ, σ)-manifold pair (M, N), $M \setminus N$ is a $\Sigma \times s$-manifold.* (2) *Each $\Sigma \times s$-manifold is homeomorphic to $M \setminus N$ for some (Σ, σ)-manifold pair (M, N).*

Proof. Clearly Theorem 3.1 implies (1), and (2) follows from the Triangulation Theorem for $\Sigma \times s$-manifolds [BM, Theorem 3.1 and Lemma 1.3]. $\square$

We next establish the following lemma which is needed in the proof of Theorem 3.4.

3.3. **Lemma.** *Let (M, N) be a (Σ, σ)-manifold pair and (M', n') a (cap, f.d. cap)-pair for an s-manifold. If $M \cong M'$ then $(M, N) \cong (M', N')$.*

Proof. We may assume $M = M'$. There is a countable star-finite open cover $\mathscr{U} = \{U_i\}_{i \in \mathbf{N}}$ of M and open embeddings $\phi_i: U_i \to \Sigma$, $i \in \mathbf{N}$, such that

$\phi_i(U_i \cap N) = \phi_i(U_i) \cap \sigma$. By [AHW, Theorem 2] there is a sequence $n_1 \leq n_2 \leq \ldots$ in $\mathbf{N}$ such that $f_i \in H(M)$, satisfying $f_1|M \setminus U_i = \mathrm{id}$; then $\{f_i \circ \cdots \circ f_1\}_{i \in \mathbf{N}}$ converges pointwise to $f \in H(M)$ with $f|U_i = f_{n_i} \circ \cdots \circ f_{n_1}|U_i$.

Let $\mathscr{V}$ be an open cover of U_1 such that $\mathrm{st}^6 \mathscr{V}$ is normal relative to $M \setminus U_1$. For each $V \in \mathscr{V}$, let V^X and V^s be open sets in X and s respectively such that $V^X \cap M = V$ and $V^s \cap \Sigma = \phi_1(V)$. Let $\mathscr{V}^X = \{V^X | V \in \mathscr{V}\}$, $\mathscr{V}^s = \{V^s | V \in \mathscr{V}\}$, $U_1^X = \bigcup \mathscr{V}^X$ and $U_1^s = \bigcup \mathscr{V}^s$. Then $U_1^X \cap M = U_1$, $U_1^X \cap N' = U_1 \cap N'$, $U_1^s \cap \Sigma = \phi_1(U_1)$ and $U_1^s \cap \sigma = \phi_1(U_1 \cap N)$. By Lemma 1.4, $(U_1, U_1 \cap N')$ and $(\phi_1(U_1), \phi_1(U_1 \cap N))$ are (cap, f.d. cap)-pair for U_1^X and U_1^s respectively. Since the inclusions $i\colon U_1 \subset U_1^X$ and $j\colon \phi_1(U_1) \subset U_1^s$ are fine homotopy equivalences, there are maps $f\colon U_1^X \to U_1$ and $g\colon U_1^s \to \phi_1(U_1)$ such that $fi \overset{\mathscr{V}}{\simeq} \mathrm{id}$, $if \overset{\mathscr{V}^X}{\simeq} \mathrm{id}$, $gj \overset{\phi_1(\mathscr{V})}{\simeq} \mathrm{id}$ and $jg \overset{\mathscr{V}^s}{\simeq} \mathrm{id}$. Then

$$(i \circ \phi_1^{-1} \circ g)(j \circ \phi_1 \circ f) \overset{\mathscr{V}}{\simeq} i \circ f \overset{\mathscr{V}^X}{\simeq} \mathrm{id}$$

and

$$(j \circ \phi_1 \circ f)(i \circ \phi_1^{-1} \circ g) \overset{\phi_1(\mathscr{V})}{\simeq} j \circ g \overset{\mathscr{V}^s}{\simeq} \mathrm{id}.$$

Note that $(j \circ \phi_1 \circ f)^{-1}(\mathscr{V}^s) = f^{-1}(\mathscr{V})$ and $\mathscr{V}^X < \mathrm{st}\, f^{-1}(\mathscr{V})$. Hence $\mathrm{st}\, \mathscr{V}^X < \mathrm{st}^2 (j \circ \phi_1 \circ f)^{-1}(\mathscr{V}^s) = (j \circ \phi_1 \circ f)^{-1}(\mathrm{st}^2 \mathscr{V}^s)$. Thus $j \circ \phi_1 \circ f\colon U_1^X \to U_1^s$ is a $\mathrm{st}^2 \mathscr{V}^s$-homotopy equivalence, which is $\mathrm{st}^4 \mathscr{V}^s$-close to a homeomorphism by [Fe, Theorem 3.4]. By Theorems 1.2 and 2.2, we have a homeomorphism

$$h_1\colon (U_1^X, U_1, U_1 \cap N') \to (U_1^s, \phi_1(U_1), \phi_1(U_1 \cap N))$$

which is $\mathrm{st}^5 \mathscr{V}^s$-close to $j \circ \phi_1 \circ f$. Since $h_1|U_1$ is $\mathrm{st}^5 \phi_1(\mathscr{V})$-close to $\phi_1 f i$,

$$\phi_1^{-1} h_1 | U_1\colon (U_1, U_1 \cap N') \to (U_1, U_1 \cap N)$$

is $\mathrm{st}^5 \mathscr{V}$-close to fi, hence $\mathrm{st}^5 \mathscr{V}$-close to id. Since $\mathrm{st}^6 \mathscr{V}$ is normal relative to $M \setminus U_1$, $\phi_1^{-1} h_1 | U_1$ extends to a $f_1 \in H(M)$ with $f_1 | M \setminus U_1 = \mathrm{id}$ and $f_1(N') \cap U_1 = N \cap U_1$. By Corollary 1.6 and Theorem 1.2, $(X, M, f_1(N))$ is an (s, Σ, σ)-manifold triple. By the same procedure, we have a $f_2 \in H(M)$ with $f_2 | M \setminus U_2 = \mathrm{id}$ and $f_2 f_1(N) \cap U_2 = N' \cap U_2$. Similar to the implication (i) $\Rightarrow$ (ii) in Thoerem 2.1, $f_2 f_1(N') \cap (U_1 \cup U_2) = N \cap (U_1 \cup U_2)$. By induction we obtain, for each $i \in \mathbf{N}$, an $f_i \in H(M)$, such that $f_i | M \setminus U_i = \mathrm{id}$ and

$$f_i \circ \cdots \circ f_1(N') \cap (U_1 \cup \cdots \cup U_i) = N \cap (U_1 \cup \cdots \cup U_i).$$

Let $f \in H(M)$ be the limit of $\{f_i \circ \cdots \circ f_1\}_{i \in \mathbf{N}}$. Then $f(N') = N$. Therefore $(M', N'') \cong (M, N)$. $\square$

Since each Σ-manifold M is homeomorphic to $|K| \times \Sigma$ for some countable locally finite simplicial complex K, we have the theorem which followed from Lemmas 3.1 (using Theorem 2.1) and Lemma 3.3.

3.4. **Theorem.** *Each* (Σ, σ)*-manifold pair can be embedded as a* (*cap, f.d. cap*)*-pair for an* s*-manifold and also for a* Q*-manifold.*

Thus we have the (Σ, σ)-manifold pair versions for Theorems 2.2–2.6. We now give a characterization of (Σ, σ)-manifold pairs:

3.5. **Theorem.** *Let* (M, N) *be a pair of spaces.* (M, N) *is a* (Σ, σ)*-manifold pair if and only if* M *is an* ANR *and* $M = \bigcup_{n\in\mathbf{N}} M_n$, *where* $M_1 \subset M_2 \subset \cdots$ *is a tower of compact* Q*-manifolds satisfying* $(*)'$ *and* $(**)$.

Proof. The "only if" part is a direct consequence of Theorem 3.4 and Lemma 1.5. We need to show the "if" part. By [CDM, Proposition 2.2] and [Mo], M is a Σ-manifold. Since M can be embedded as a cap set for an s-manifold [Ch_1], (M, N) is a (Σ, σ)-manifold pair by Lemma 1.5 and Theorem 2.1. □

By Lemma 1.5, Theorems 2.1 and 3.5, we state another characterization of (Q, Σ, σ)- (or (s, Σ, σ)-)manifold triples:

3.6. **Theorem.** *Let* X *be a* Q- (*or* s-)*manifold and* $N \subset M \subset X$. *Then* (X, M, N) *is a* (Q, Σ, σ)- (*or* (s, Σ, σ)-)*manifold triple if and only if* (M, N) *is a* (Σ, σ)*-manifold pair and* M *or* N *is map-dense in* X.

In other words, this theorem asserts that (X, M, N) is a (Q, Σ, σ)- (or (s, Σ, σ)-manifold triple if and only if (X, M) is a (Q, Σ)- (or (s, gE)-)manifold pair (or (X, N) is a (Q, σ)- (or (s, σ)-) manifold pair) and (M, N) is a (Σ, σ)-manifold pair.

3.7. **Problem.** (i) If M is a Σ-manifold and N is a σ-manifold such that $N \subset M$ and $M \setminus N$ is a $\Sigma \times s$-manifold, then is (M, N) a (Σ, σ)-manifold pair? (ii) If M is a cap set and N is an f.d. cap set for a Q- (or (s-)manifold X such that $N \subset M$ and $M \setminus N$ is a $\Sigma \times s$-manifold, then is (X, M, N) a (Q, Σ, σ)- (or (s, Σ, σ)-)manifold triple?

References

[An_1] R. D. Anderson, *Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **72** (1966), 515–519.

[An_2] ——, *Strongly negligible sets in Fréchet manifolds*, Bull. Amer. Math. Soc. **75** (1969), 64–67.

[An_3] ——, *On sigma-compact subsets of infinite-dimensional spaces*, unpublished manuscript.

[AC] R. D. Anderson and T. A. Chapman, *Extending homeomorphisms to Hilbert cube manifolds*, Pacific J. Math. **38** (1971), 281–293.

[AHW] R. D. Anderson, D. W. Henderson, and J. E. West, *Negligible subsets of infinite-dimensional manifolds*, Compositio Math. **21** (1969), 143–150.

[AM] R. D. Anderson and J. D. McCharen, *On extending homeomorphisms to Fréchet manifolds*, Proc. Amer. Math. Soc. **25** (1970), 283–289.

[AS] R. D. Anderson and R. M. Schori, *Factors of infinite-dimensional manifolds*, Trans. Amer. Math. Soc. **142** (1969), 315–330.

[BP_1] C. Bessaga, and A. Pelczyński, *The estimated extension theorem, homogeneous collections and skeletons, and their application to the topological classification of linear metric spaces and convex sets*, Fund. Math. **69** (1970), 153–190.

[BP_2] ——, *Selected topics in infinite-dimensional topology*, Monogr. Mat. 58, PWN, Warsaw, 1970.

[BM] M. Bestvina, and J. Mogilski, *Characterizing certain incomplete infinite-dimensional absolute retracts*, Michigan Math. J. **33** (1986), 291–313.

[Ch_1] T. A. Chapman, *Dense sigma-compact subsets of infinite-dimensional manifolds*, Trans. Amer. Math. Soc. **154** (1971), 399-426.

[Ch_2] ——, *Lectures on Hilbert cube manifolds*, CBMS Regional Conf. Ser. in Math., no. 28, Amer. Math. Soc., Providence, R.I., 1976.

[Cu] D. W. Curtis, *Boundary sets in Hilbert cube*, Topology Appl. **20** (1985), 201–221.

[CDM] D. Curtis, T. Dobrowolski, and J. Mogilski, *Some applications of the topological characterizations of the sigma-compact spaces* l_f^2 *and* Σ, Trans. Amer. Math. Soc. **284** (1984), 837–846.

[En] R. Engelking, *General topology*, Monogr. Mat., 60, PWN, Warsaw, 1977.

[Fe] S. Ferry, *The homeomorphism group of a subcompact Hilbert cube manifold is an* ANR, Ann. of Math. (2) **106** (1977), 101–119.

[Ge] R. Geoghegan, *On spaces of homeomorphisms, embeddings, and functions*. II. *The piecewise linear case*, Proc. London Math. Soc. (3) **27** (1973), 463–483.

[He] D. W. Henderson, *Infinite dimensional manifolds are open subsets of Hilbert space*, Topology **9** (1970), 25–33.

[Mo] J. Mogilski, *Characterizing the topology of infinite-dimensional* σ*-compact manifolds*, Proc. Amer. Math. Soc. **92** (1984), 111–118.

[Sa_1] K. Sakai, *On topologies of triangulated infinite-dimensional manifolds*, J. Math. Soc. Japan **39** (1987), 287–300.

[Sa_2] ——, *A Q-manifold local-compactification of metric combinatorial* ∞*-manifold*, Proc. Amer. Math. Soc. **100** (1987).

[Sa_3] ——, *The space of Lipschitz maps from a compactum to a polyhedron*, in preparation.

[SW] K. Sakai and R. Y. Wong, *The space of Lipschitz maps from a compactum to a locally convex set*, Topology Appl. (submitted).

[We] J. E. West, *Infinite products which are Hilbert cubes*, Trans. Amer. Math. Soc. **150** (1970), 1–25.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, SAKURA-MURA, IBARAKI, 305 JAPAN

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CALIFORNIA 93106