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SURFACES OF E^4 SATISFYING CERTAIN RESTRICTIONS ON THEIR NORMAL BUNDLE

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ABSTRACT. We consider smooth surfaces in E^4 whose normal bundles satisfy certain geometric conditions that entail the vanishing of the normal curvature, and prove that their Gauss curvatures cannot be bounded from above by a negative number. We also give some results towards a classification of flat surfaces with flat normal bundle in E^4.

0. INTRODUCTION

It is an elementary fact that no smooth compact surface M in euclidean space E^3 exists, whose Gauss curvature K satisfies $\max K \leq 0$. Also, according to the theorem of Efimov, no complete M in E^3 exists which has $\sup K < 0$. Furthermore, if $K = 0$ identically, then M is a cylinder according to the theorem of Hartman and Nirenberg.

What can we say along the lines of these three facts in case our surface lies in E^4? The situation seems to be quite different, and results analogous to the above either do not hold, or are not known. Thus, the cartesian product of two closed plane curves is an example of a compact surface in E^4 with $K \equiv 0$. As far as we know, the classification of complete surfaces in E^4 with $K \equiv 0$ is still an unresolved problem. The question posed by Chern many years ago [1, p. 43], namely whether there exist compact surfaces in E^4 with $\max K < 0$, has not been answered yet.

Motivated by these considerations, we investigate here surfaces in E^4 satisfying certain restrictions on their normal bundle, and ask whether they can satisfy at the same time one of the conditions $\sup K < 0$ or $K \equiv 0$. After a short introductory section on definitions, general facts, and tools, we proceed in the second section to the theorems. There we investigate surfaces having a globally defined parallel normal field. Such surfaces are, for example, those which lie on spheres S^3 of E^4. In the third section we investigate surfaces whose tangent planes do not contain the origin O of E^4. Projecting the position vector on the corresponding normal space and taking the unit vector in this direction, we

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consider the special class of surfaces for which this vector is parallel and prove some restrictions concerning their curvature.

Finally in the fourth section, still under certain restrictions on the normal bundle, we give some results towards a classification of surfaces in E^4 with $K \equiv 0$.

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1. General definitions and facts

Surfaces, functions, vectorfields, and the like are all supposed to be C^∞-differentiable. Also, surfaces we deal with are connected and oriented. Their normal bundles then carry an orientation, which is induced by the one of E^4. We shall use orthonormal moving frames $\{e_1, e_2, e_3, e_4\}$; here e_1 and e_2 are tangent, and e_3 and e_4 are normal to the surface M. For the corresponding dual forms ω_i and connection forms ω_{ij}, the equations of Cartan are

$$d\omega_i = \sum_k \omega_{ik} \wedge \omega_k, \qquad i, k = 1, 2, \tag{1.1}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}, \qquad i, j, k = 1, 2, 3, 4. \tag{1.2}$$

The $\omega_{ij} = -\omega_{ji}$ are defined by

$$D_X e_i = \sum_j \omega_{ij}(X) e_j, \qquad X \in TM, \ j = 1, 2, 3, 4. \tag{1.3}$$

D denotes the usual directional (covariant) derivative of E^4. For each normal vectorfield ξ of the surface, the corresponding shape operator A_ξ is defined by

$$A_\xi X = -(D_X \xi)^\top, \qquad X \in TM.$$

We have the orthogonal decomposition

$$D_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{1.4}$$

where $\nabla^\perp$ is the connection on the normal bundle NM of M. The normal vectorfield ξ is called parallel in NM if $\nabla^\perp_X \xi = 0$ for all tangent vectorfields X. The expression

$$B(X, Y) = (D_X Y)^\perp, \qquad X, Y \in TM,$$

where $(\dots)^\perp$ is the normal part of a vector, defines the second fundamental tensor of M. For every normal vectorfield ξ, we have

$$\langle B(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The equation of Gauss is $D_X Y = \nabla_X Y + B(X, Y)$, where ∇ is the covariant derivative of M.

In terms of the connection forms ω_{ij}, we have:

$$A_3 = A_{e_3} = \omega_{13} \otimes e_1 + \omega_{23} \otimes e_2,$$
$$A_4 = A_{e_4} = \omega_{14} \otimes e_1 + \omega_{24} \otimes e_2.$$

The equations $d\omega_{12} = -K\omega_1 \wedge \omega_2$ and $d\omega_{34} = -K_n\omega_1 \wedge \omega_2$ define respectively the Gauss curvature and the normal curvature of the surface. We have the identity

$$K_n = -\langle (A_3 \circ A_4 - A_4 \circ A_3)e_1, e_2\rangle;$$

from this, it follows that $K_n = 0$ if and only if A_3 and A_4 commute; this is equivalent to the existence of an orthonormal basis $\{E_1, E_2\}$ of tangent vectors which diagonalize simultaneously all the operators A_ξ. Replacing e_1, e_2 with these, we get a new frame, still denoted by $\{e_1, e_2, e_3, e_4\}$, for which

$$(1.5) \qquad A_3 = k_1\omega_1 \otimes e_1 + k_2\omega_2 \otimes e_2, \qquad A_4 = \lambda_1\omega_1 \otimes e_1 + \lambda_2\omega_2 \otimes e_2,$$

where k_i and λ_i are respectively the eigenvalues of A_3 and A_4. In this frame,

$$(1.6) \qquad \omega_{13} = k_1\omega_1, \quad \omega_{23} = k_2\omega_2, \quad \omega_{14} = \lambda_1\omega_1, \quad \omega_{24} = \lambda_2\omega_2.$$

From Cartan's equations we get

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} + \omega_{14} \wedge \omega_{42} = -(k_1k_2 + \lambda_1\lambda_2)\omega_1 \wedge \omega_2.$$

Therefore,

$$(1.7) \qquad K = \det A_3 + \det A_4.$$

The mean curvature vector H is defined by the equation

$$(1.8) \qquad \langle \xi, H\rangle = \tfrac{1}{2}\operatorname{trace} A_\xi \quad \text{for all normals } \xi,$$

or

$$(1.9) \qquad H = \tfrac{1}{2}(k_1 + k_2)e_3 + \tfrac{1}{2}(\lambda_1 + \lambda_2)e_4.$$

$h = |H|$ is the mean curvature of M.

The vector space generated by the set $\{B(X, Y) | X, Y \in T_pM\}$ is called the first normal space at p and is denoted by N_1.

2. Surfaces which have a parallel normal vectorfield

If a surface M of E^4 is contained in some sphere S^3 of E^4, then the sphere normal is also a normal of M in E^4 and parallel in the normal bundle of M. Here, more generally, we investigate surfaces which have a globally defined unit normal vectorfield e, which is parallel in the normal bundle. We call such an e *nonsingular* if the corresponding shape operator A_e has everywhere $\det A_e \neq 0$, and *singular* if $\det A_e = 0$ identically. The central question here is whether there exist such surfaces having $\sup K < 0$. We prove the following theorem.

Theorem 1. *Let M be a complete surface of E^4 with bounded mean curvature and a global parallel nonsingular normal e, such that* $\det A_e > 0$. *Then* $\sup K \geq 0$.

Theorem 2. *Let M be a complete surface of E^4 with bounded mean curvature and a global parallel nonsingular normal e, such that* $\sup \det A_e < 0$. *Then* $\sup K \geq 0$.

Corollary 1. *A complete surface of S^3, with bounded mean curvature, has always* $\sup K \geq 0$.

Corollary 2. *A compact surface of E^4 having a global, parallel, and nonsingular normal e, has* $\max K > 0$, *unless* $K \equiv 0$.

Corollary 3. *A compact surface of S^3 with $K \leq 0$ must satisfy $K \equiv 0$.*

Corollary 4. *A compact surface of E^4 having a global, parallel and nonsingular normal e and $K \leq 0$, must satisfy $K \equiv 0$.*

To prove these propositions, we apply a method already used in [2] by two of the authors. More precisely, we change the metric of M using the nonsingular operator A_e, and consider the curvature restrictions for the new metric.

Lemma 1. *Let M be a surface of E^4 admitting a parallel and nonsingular normal e. Define the new metric on M by*

$$(X, Y)_e = \langle A_e X, A_e Y \rangle .$$

The curvature K_e of this metric is given in terms of the curvature K of M by $K_e = K/\det A_e$.

Proof. We use the well-known relation between a metric and the corresponding covariant derivative

$$(\nabla'_Y X, Z)_e = \tfrac{1}{2}(X(Z, Y)_e + Y(Z, X)_e - Z(X, Y)_e - ([X, Z], Y)_e - ([Y, Z], X)_e - ([X, Y], Z)_e),$$

and the Codazzi equation $(\nabla_X A_e)Y = (\nabla_Y A_e)X$ which holds because e is parallel in the normal bundle. A direct calculation now gives

$$\nabla'_Y X = A_e^{-1}(\nabla_Y(A_e X)),$$

where ∇ is the covariant derivative of the original metric of M. For the curvature tensors of the connections ∇ and ∇' we obtain the expression

$$R'(X, Y)Z = A_e^{-1}(R(X, Y)A_e Z).$$

If e_1 and e_2 are eigenvectors of A_e with corresponding eigenvalues k_1 and k_2, then we get

$$K_e = (R'(e_1, e_2)e_2, e_1)_e/((e_1, e_1)_e(e_2, e_2)_e - (e_1, e_2)_e^2) = K/\det A_e .$$

Lemma 2. *Let M be a complete surface of E^4 with bounded mean curvature, admitting a global, parallel and nonsingular normal e, such that* $\inf|\det A_e| > 0$. *Then M is complete with respect to the new metric* $(\ldots,\ldots)_e$.

The assertion follows directly if we show that $\inf k_i^2 > 0$, where the k_i are the eigenvalues of A_e. This, however, follows easily from our hypotheses

$$\inf|\det A_e| > 0 \quad \text{and} \quad |\operatorname{trace} A_e|^2 \leq 4|H|^2 \leq c^2.$$

Proof of Theorem 1. Let $h_0 = \sup|H|$. For every unit normal vectorfield ξ we have

$$\det A_\xi \leq (\operatorname{trace} A_\xi)^2/2 \leq (4h_0^2)/2 = 2h_0^2.$$

Hence, according to our hypothesis, we have $0 < \det A_e \leq 2h_0^2$.

Suppose now that $\sup K = -\alpha < 0$ and ξ is a normal unit vectorfield orthogonal to e. For the curvature of M, we have

$$K = \det A_e + \det A_\xi \leq -\alpha < 0,$$

and so $\det A_\xi < -\alpha$.

According to Lemma 2, the metric $(\ldots,\ldots)_\xi$ will be complete with corresponding curvature

$$K_\xi = K/\det A_\xi = K/(K - \det A_e).$$

However

$$K_\xi = K/(K - \det A_e) \geq \alpha/(\alpha + 2h_0^2).$$

In fact, the real function $f(x) = x/(x - \det A_e)$ is decreasing and continuous on $(-\infty, -\alpha]$. Thus M is complete with respect to $(\ldots,\ldots)_\xi$, with corresponding curvature $K_\xi \geq \alpha/(\alpha + 2h_0^2)$. Hence M is compact and, besides, its genus is 0. But then, by the Gauss-Bonnet theorem, it is impossible to have everywhere $K \leq -\alpha < 0$. So $\sup K \geq 0$.

Proof of Theorem 2. Let us suppose that $\sup K = -\alpha < 0$, $\sup|H| = h_0$, and $\sup \det A_e = -c < 0$. According to Lemma 2, the metric $(\ldots,\ldots)_e$ is complete with corresponding curvature $K_e = K/\det A_e$. Denoting again by ξ a unit normal vectorfield of M orthogonal to e, we get

$$K_e = K/\det A_e = (\det A_e + \det A_\xi)/\det A_e = 1 + (\det A_\xi/\det A_e).$$

Obviously, at points of M where $\det A_\xi \leq 0$, we have $K_e \geq 1$, whereas at points where $\det A_\xi > 0$ we have

$$K_e = K/\det A_e = K/(K - \det A_\xi).$$

As in the preceding proof, since $\det A_\xi \leq 2h_0^2$, we will have again

$$K_e \geq \alpha/(\alpha + 2h_0^2).$$

Thus, we obtain

$$K_e \geq \min(1, \alpha/(\alpha + 2h^2)) = \alpha/(\alpha + 2h_0^2) > 0.$$

Consequently, M will have genus 0, which is incompatible with the restriction $K \leq -\alpha < 0$.

Proofs of the corollaries. (1) An immediate consequence of the fact that the sphere normal e is a parallel normal vectorfield of M with $\det A_e = 1$.

(2) Because of the compactness of M, all hypotheses of Theorem 1 are valid. Hence $\max K \geq 0$. Let $\max K = 0$. Then $K \leq 0$ everywhere on M and the genus $g(M) \geq 1$ or $K \equiv 0$. There are two cases:

(i) $\det A_e > 0$. From the inequality $K = \det A_e + \det A_\xi \leq 0$ we get as before that $\det A_\xi < 0$, and the metric $(\dots, \dots)_\xi$ is complete with corresponding curvature $K_\xi = K/\det A_\xi \geq 0$. Hence M has genus 0; a contradiction, unless $K \equiv 0$.

(ii) $\det A_e < 0$. Then $(\dots, \dots)_e$ is a complete metric with corresponding curvature $K_e = K/\det A_e \geq 0$, and again the genus must be 0; a contradiction, unless $K \equiv 0$.

Thus we have proved that $\max K > 0$, unless $K \equiv 0$.

Corollaries 3 and 4 are immediate consequences of the preceding corollaries.

3. Surfaces having a canonical parallel normal vectorfield

We assume here that the complete surface M of E^4 has tangent planes T_pM none of which contains the origin O of E^4. We then decompose the position vector $x = x^\top + x^\perp$ in tangential $x^\top$ and normal $x^\perp$ components to M. By hypothesis $x^\perp$ is everywhere nonzero, hence $e_3 = x^\perp/|x^\perp|$ is a normal unit vectorfield which we call the *canonical normal vectorfield* of M. Let e_4 be the unit normal vectorfield which, together with e_3, builds a basis of the oriented normal space of M. Let $f = |x^\perp|$. Then we have $x = x^\top + fe_3$.

Differentiating this we get

$$\begin{aligned} X = D_X x &= \nabla_X x^\top + B(X, x^\top) + fD_X e_3 + (Xf)e_3 \\ &= \nabla_X x^\top + B(X, x^\top) + f(-A_3X + \omega_{34}(X)e_4) + (Xf)e_3, \end{aligned} \tag{3.1}$$

where ∇ is the covariant derivative of the metric of M. Taking tangential and normal components of (3.1), we get respectively

$$X = \nabla_X x^\top - fA_3X, \quad B(X, x^\top) + f\omega_{34}(X)e_4 + (Xf)e_3 = 0. \tag{3.2}$$

Multiplying the last equation by e_3, we obtain

$$\langle A_3X, x^\top\rangle + \langle X, \operatorname{grad} f\rangle = 0 \quad \text{for every } X \in TM.$$

Hence

$$A_3x^\top = -\operatorname{grad} f. \tag{3.3}$$

Multiplying (3.2) by e_4, we obtain also

$$\omega_{34}(X) = -\tfrac{1}{f}\langle A_4x^\top, X\rangle. \tag{3.4}$$

These are the fundamental relations of the kind of surface we are considering.

Now we proceed to the investigation of the central projections of our surface on the unit hypersphere S^3 of E^4. This projection is given analytically by the formula $y = x/r$, where $r = |x|$. If (u, v) are coordinates of M, then

$$y_u = (x_u/r) - (r_u/r^2)x, \qquad y_v = (x_v/r) - (r_v/r^2)x.$$

Since O is not contained in any tangent plane of M, the vectors x_u, x_v, and x are linearly independent and the projection M' of M is a regular surface of the unit sphere S^3. The coefficients of the corresponding first fundamental form are easily computed:

$$E' = (E - r_u^2)/r^2, \quad F' = (F - r_u r_v)/r^2, \quad G' = (G - r_v^2)/r^2. \tag{3.5}$$

From this we get

$$E'G' - F'^2 = (EG - F^2)(1 - |\operatorname{grad} r|^2)/r^4. \tag{3.6}$$

By parallel translation of e_4 along the radii, we define a vectorfield e_4' of M', which is orthogonal to y_u and y_v, and so coincides with the unit normal of M' in S^3. Let A' be the corresponding shape operator of M'. For the curvature of M' we have

$$K' = 1 + \det A'. \tag{3.7}$$

The coefficients of the second fundamental form of M' and the fundamental form $\langle A_4 X, Y\rangle$ of M are related by the equations

$$\begin{aligned} L' &= \langle y_{uu}, e_4'\rangle \\ &= \langle (x_{uu}/r) - (r_u/r^2)x_u - ((r_{uu}x + r_u x_u)/r^2) + (2r_u^2/r^3)x, e_4\rangle \\ &= L/r. \end{aligned}$$

Analogously,

$$\begin{gathered} M' = M/r \quad \kappa\alpha\iota \quad N' = N/r, \\ L'N' - M'^2 = (LN - M^2)/r^2. \end{gathered}$$

This and (3.6) imply

$$\det A' = (r^2/(1 - |\operatorname{grad} r|^2)) \det A_4$$

which, together with (3.7) gives

$$K' = 1 + (r^2/(1 - |\operatorname{grad} r|^2)) \det A_4. \tag{3.8}$$

We are now ready to prove our next theorem.

Theorem 3. *Let M be a compact surface of E^4 whose set of tangent planes does not contain the origin O. If the canonical normal e_3 is parallel in the normal bundle of M, then*

(a) $\max K \geq (1/R^2)$, *unless M lies in a hypersphere whose center is O.*

(b) $\min K \le (1/r^2)$, *unless* M *lies in a hypersphere whose center is* O.

(c) *If* $K \le 0$ *on* M, *then* $K \equiv 0$ *and* M *lies in a hypersphere whose center is* O.

Here $R = \max d(O, M)$ *and* $r = \min d(O, M)$ *are respectively the maximal and minimal distances of* O *from the surface.*

Proof. Since e_3 and e_4 are assumed to be parallel, we have $\omega_{34} = 0$. This, in view of (3.4), gives $A_4 x^{\top} = 0$ on M.

Let $M_1 = \{p \in M | x^{\top}(p) = 0\}$ and $M_2 = M - M_1$. It is clear that the connected components of $\operatorname{Int}(M_1)$ are in hyperspheres with center at O. We denote by U_α a connected component which is in the hypersphere of radius α. We have also $\det A_4 = 0$ on M_2.

Consider now the radial projection M' of M on the unit hypersphere. The image U'_α of U_α under this projection has curvature (because of (3.5) and $r_u = r_v = 0$)

$$K' = \alpha^2 K. \tag{3.9}$$

Besides, at points of M'_2, the projection of M_2, because of (3.8) and (3.9) we have

$$K' = 1 \quad \text{on } M'_2. \tag{3.10}$$

Obviously we have also $r^2 \le \alpha^2 \le R^2$. With these preparations we pass to the

Proof of (a) *and* (b). Let $\max K < 1/R^2$; hence $KR^2 < 1$ everywhere on M, and so $K\alpha^2 < 1$ for all our α's. This, in view of (3.9) and (3.10) and by continuity, means that we cannot have simultaneously $\operatorname{Int} M_1 \ne \varnothing$ and $\operatorname{Int} M_2 \ne \varnothing$. If $M_2 = \varnothing$, then $M_1 = M$ and $x^{\top} = 0$ everywhere on M. Hence M is contained in a hypersphere with center at O. If $\operatorname{Int} M_1 = \varnothing$, then everywhere on M' we shall have $K' = 1$, so M' will be a great sphere S^2 of S^3. But M will then be contained in some three-dimensional subspace E^3 of E^4, and for the point of M furthest from O we will have $K \ge 1/R^2$, which contradicts $\max K < 1/R^2$. Thus if $\max K < 1/R^2$, then M must lie in some hypersphere with center O. (b) is proved along the same lines.

Proof of (c). If $K \le 0$, then, by continuity of K', either (3.9) or (3.10) will be true. If (3.10) is true, then M' is a great sphere of S^3, hence M is contained in some three-dimensional subspace E^3 of E^4. We know then that there exists a point of M, where $K > 0$. So only (3.9) can be true. Then M lies in some hypersphere of radius r_0 and center O (since $x^{\top} = 0$ on M). In this case, from Corollary 3, we get $K \equiv 0$.

Remark. The surfaces in E^4 for which the canonical normal e_3 is parallel in the normal bundle are locally either surfaces in hyperspheres with center O, or surfaces in some subspace E^3 of E^4, or generalized moulding surfaces of Monge. To show this, note that, by (3.4) and (1.3), $D_X e_3 = 0$ is equivalent to

$A_4 x^\top = 0$. Now if $x^\top = 0$ on some open set of M, then this set lies on some hypersphere with center O. So suppose $x^\top \neq 0$ on any open set. If $A_4 = 0$ on an open set, then, from the Weingarten formula (1.4), we obtain $e_4 = \text{const}$, therefore our M lies in some E^3 perpendicular to e_4. Since $x^\top$ lies in this E^3 and $x = x^\top + |x^\perp| e_3$, we obtain $\langle x, e_4\rangle = 0$, hence x is always in this E^3, and so is O.

Assume finally $A_4 \neq 0$ on an open set where $x^\top \neq 0$. Since A_4 has one eigenvalue zero, the other must be different from zero, and so A_4 has two well-defined principal directions, which are also principal directions of A_3. Thus, we must have $A_3 x^\top = k \cdot x^\top$, so from (3.3) we obtain $\operatorname{grad} f = -k \cdot x^\top$. Using this relation, it is easy to establish that, for any point P on M, the plane spanned by O and $x^\top(P)$ intersects M orthogonally along a line of curvature; this property is characteristic of the classical moulding surfaces in E^3.

4. Isometric immersions of E^2 in E^4

We shall begin with the local analysis of the structure of a flat surface M, immersed substantially in E^4, and with flat normal bundle. So we assume $K = K_n = 0$ and $\dim N_1 = 2$. Let (e_1, e_1, ξ_1, ξ_2) be a local orthonormal frame field adapted to M, with (e_1, e_2) a common eigenbasis of all the A_ξ, and (ξ_1, ξ_2) an Otsuki-frame for the normal bundle; that is, if

$$A_{\xi_1} \sim \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \qquad A_{\xi_2} \sim \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

and $\xi = \alpha_1 \xi_1 + \alpha_2 \xi_2$, $|\xi| = 1$, so that

$$\det A_\xi = k_1 k_2 \alpha_1^2 + (k_1 \lambda_2 + k_2 \lambda_1)\alpha_1 \alpha_2 + \lambda_1 \lambda_2 \alpha_2^2, \tag{4.1}$$

then,

$$k_1 \lambda_2 + k_2 \lambda_1 = 0 \tag{4.2}$$

because (ξ_1, ξ_2) is an Otsuki-frame,

$$k_1 k_2 + \lambda_1 \lambda_2 = 0 \tag{4.3}$$

because $K = 0$, and

$$k_1 \lambda_2 - k_2 \lambda_1 \neq 0 \tag{4.4}$$

because $\dim N_1 = 2$; this last inequality follows from the fact that $B(e_1, e_1) = k_1 \xi_1 + \lambda_1 \xi_2$ and $B(e_2, e_2) = k_2 \xi_1 + \lambda_2 \xi_2$ are linearly independent. Because of (4.2) and (4.4), we have $k_1 k_2 \neq 0$ and $\lambda_1 \lambda_2 \neq 0$; using this together with (4.2) and (4.3), we obtain $k_1^2 = \lambda_1^2$. We may assume, possibly by replacing ξ_1 by $-\xi_1$, that $k_1 = \lambda_1$, so that $k_2 = -\lambda_2$. Thus,

$$A_{\xi_1} \sim \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \qquad A_{\xi_2} \sim \begin{pmatrix} k_1 & 0 \\ 0 & -k_2 \end{pmatrix},$$

and (4.1) becomes

$$\det A_\xi = k_1 k_2(\alpha_1^2 - \alpha_2^2), \tag{4.5}$$

where $|k_1 k_2| = |\det A_{\xi_1}| = \max_{|\xi|=1}(\det A_\xi)$.

We replace (ξ_1, ξ_2) by an orthonormal frame (e_3, e_4) of NM in the asymptotic directions of the quadratic form (4.5):

$$e_3 = \frac{1}{\sqrt{2}}(\xi_1 + \xi_2), \qquad e_4 = -\frac{1}{\sqrt{2}}(\xi_1 - \xi_2).$$

Now we have

$$A_{e_3} = A_3 \sim \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_{e_4} = A_4 \sim \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}, \tag{4.6}$$

where $k = \sqrt{2}k_1 \neq 0$ and $\lambda = -\sqrt{2}k_2 \neq 0$. Note that the pair k, λ is, up to sign, an invariant of the immersion since

$$k^2 + \lambda^2 = 4|H|^2, \qquad |k\lambda| = 2\max_{|\xi|=1}(\det A_\xi). \tag{4.7}$$

The pair of line elements determined by e_1, e_2 is well defined everywhere, because A_3 has distinct eigenvalues. The pair of line elements determined by e_3, e_4 is also well defined since it gives the asymptotic directions of the invariant indefinite quadratic form (4.5). In the sequel we shall work with the frame (e_1, e_2, e_3, e_4). The corresponding connection forms are

$$\omega_{13} = k\omega_1, \quad \omega_{23} = 0, \quad \omega_{14} = 0, \quad \omega_{24} = \lambda\omega_2.$$

Writing down Cartan's second structure equations, we obtain immediately from these $d\omega_{13} = 0$, $d\omega_{24} = 0$, so that there exist locally functions u_1, u_2 with the property

$$du_1 = \omega_{13} = k\omega_1, \qquad du_2 = \omega_{24} = \lambda\omega_2. \tag{4.8}$$

Since we also have $d\omega_{12} = 0$ because $K = 0$ and $d\omega_{34} = 0$ because $K_n = 0$, there exist locally functions φ_1, φ_2 so that $d\varphi_1 = \omega_{12}$ and $d\varphi_2 = \omega_{34}$.

Now $du_1 \wedge du_2 = k\lambda\omega_1 \wedge \omega_2 \neq 0$, so we may introduce (u_1, u_2) locally as coordinates, and we compute

$$\begin{aligned} 0 = d\omega_{23} &= \omega_{21} \wedge \omega_{13} + \omega_{24} \wedge \omega_{43} \\ &= \left(-\frac{\partial\varphi_1}{\partial u_2} du_2\right) \wedge du_1 + du_2 \wedge \left(-\frac{\partial\varphi_2}{\partial u_1} du_1\right), \end{aligned}$$

or

$$\frac{\partial\varphi_1}{\partial u_2} + \frac{\partial\varphi_2}{\partial u_1} = 0, \tag{4.9}$$

and similarly, from $d\omega_{14} = 0$,

$$\frac{\partial\varphi_1}{\partial u_1} + \frac{\partial\varphi_2}{\partial u_2} = 0. \tag{4.10}$$

Using (4.9) and (4.10), we write

$$\omega_{12} = -\frac{\partial \varphi_2}{\partial u_2} du_1 - \frac{\partial \varphi_2}{\partial u_1} du_2,$$

and so

$$0 = d\omega_{12} = -\left(\frac{\partial^2 \varphi_2}{\partial u_1^2} - \frac{\partial^2 \varphi_2}{\partial u_2^2}\right) du_1 \wedge du_2.$$

The general solution of the equation $(\partial^2 \varphi_2/\partial u_1^2) - (\partial^2 \varphi_2/\partial u_2^2) = 0$ is

$$\varphi_2 = \overline{f}(u_1 + u_2) + \overline{g}(u_1 - u_2), \tag{4.11}$$

where $\overline{f}$ and $\overline{g}$ are arbitrary functions of one real variable. Similarly, using again (4.9) and (4.10), we write

$$\omega_{34} = -\frac{\partial \varphi_1}{\partial u_2} du_1 - \frac{\partial \varphi_1}{\partial u_1} du_2,$$

and we compute

$$0 = d\omega_{34} = \left(-\frac{\partial^2 \varphi_1}{\partial u_1^2} + \frac{\partial^2 \varphi_1}{\partial u_2^2}\right) du_1 \wedge du_2,$$

so that

$$\varphi_1 = f(u_1 + u_2) + g(u_1 - u_2), \tag{4.12}$$

where f and g are arbitrary functions of one real variable. If we replace φ_1, φ_2 in (4.9) and (4.10) by their expressions (4.11) and (4.12), we get $f' - g' + \overline{f}' + \overline{g}' = 0$ and $f' + g' + \overline{f}' - \overline{g}' = 0$, and so $f' + \overline{f}' = 0$ and $g' - \overline{g}' = 0$. Thus, $\overline{f} = -f + \text{const}$ and $\overline{g} = g + \text{const}$.

From (4.8), we obtain the expression of the first fundamental form of our surface in the coordinate system (u_1, u_2):

$$I = \omega_1^2 + \omega_2^2 = \frac{du_1^2}{k^2} + \frac{du_2^2}{\lambda^2}. \tag{4.13}$$

The functions k and λ are not independent. Indeed, the equation $d\omega_1 = \omega_{12} \wedge \omega_2$ assumes now the form $d(du_1/k) = d\varphi_1 \wedge du_2/\lambda$, which yields

$$\frac{\partial \varphi_1}{\partial u_1} = -\lambda \frac{\partial}{\partial u_2}\left(\frac{1}{k}\right), \tag{4.14}$$

and similarly $d\omega_2 = \omega_{21} \wedge \omega_2$ assumes now the form $d(du_2/\lambda) = -d\varphi_1 \wedge du_1/k$, which yields

$$\frac{\partial \varphi_1}{\partial u_2} = k \frac{\partial}{\partial u_1}\left(\frac{1}{\lambda}\right). \tag{4.15}$$

The integrability condition for the system (4.14) and (4.15) is

$$\frac{\partial}{\partial u_2}\left[-\lambda \frac{\partial}{\partial u_2}\left(\frac{1}{k}\right)\right] = \frac{\partial}{\partial u_1}\left[k \frac{\partial}{\partial u_1}\left(\frac{1}{\lambda}\right)\right],$$

which is simply the condition $K = 0$ for the metric (4.13). Further, since φ_1 is of the form (4.12), we have

$$\frac{\partial^2 \varphi_1}{\partial u_2^2} = f'' + g'' = \frac{\partial^2 \varphi_1}{\partial u_1^2},$$

and so we obtain from (4.14) and (4.15) a second relation between k and λ:

$$\frac{\partial}{\partial u_1}\left[-\lambda \frac{\partial}{\partial u_2}\left(\frac{1}{k}\right)\right] = \frac{\partial}{\partial u_2}\left[k \frac{\partial}{\partial u_1}\left(\frac{1}{\lambda}\right)\right]. \tag{4.16}$$

So far we have proven the existence of a distinguished coordinate system (u_1, u_2) in the vicinity of any point on M, so that the metric assumes the form (4.13), where k and λ are certain essentially invariant functions (cf. (4.7)) which, in addition, satisfy the relation (4.16). These coordinate systems characterize our class of immersed flat surfaces in the sense made precise by the following.

Theorem 4. *Let (u_1, u_2) be the cartesian coordinates of E^2. Suppose we are given near the origin a flat metric of the form*

$$I = E\,du_1^2 + G\,du_2^2, \tag{4.17}$$

where E and G also satisfy the relation

$$\frac{\partial}{\partial u_1}\left(-\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial u_2}\right) = \frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial u_1}\right). \tag{4.18}$$

For a sufficiently small disc D around the origin, we can construct an immersion $x: D \to E^4$ with the properties:

(i) *I is the first fundamental form of x in the system (u_1, u_2).*
(ii) *$K_n = 0$ and $\dim N_1 = 2$.*
(iii) *$4|H^2| = (1/E) + (1/G)$ and $2\max_{|\xi|=1}(\det A_\xi) = 1/\sqrt{EG}$.*

Conversely, given a flat surface in E^4 which satisfies (ii), *there exists locally a coordinate system (u_1, u_2) so that* (4.17), (4.18), *and* (iii) *hold.*

Remark. Any given flat metric of the form (4.17) on the disc D is isometric to the standard flat metric $I_0 = du_1^2 + du_2^2$. Let $(D, I) \xrightarrow{\varphi} (D, I_0)$ be this isometry. We can realize I_0 as the Clifford torus $x: D \to S^3 \subset E^4$; this imbedding satisfies conditions (ii) and (iii). Therefore, $x \circ \varphi: D \to S^3$ is a realization of I which again satisfies (ii), but now (iii) need not hold.

Proof. We have already shown the second half of the theorem, with $E = 1/k^2$ and $G = 1/\lambda^2$.

Suppose now that, in our fixed coordinate system, we are given the flat metric (4.17) which satisfies (4.18). The system of differential equations for the unknown function φ_1,

$$-\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial u_2} = \frac{\partial \varphi_1}{\partial u_1}, \qquad \frac{1}{\sqrt{EG}}\frac{\partial G}{\partial u_1} = \frac{\partial \varphi_1}{\partial u_2}, \tag{4.19}$$

is solvable on the disc B where E and G are defined, and has a unique solution up to constants, because the integrability condition for this system is just the Theorema Egregium for the metric (4.17):

$$K = -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial u_2}\right) + \frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial u_1}\right)\right] = 0.$$

In terms of the solution φ_1, the condition (4.18) reads $\partial^2\varphi_1/\partial u_1^2 - \partial^2\varphi_1/\partial u_2^2 = 0$, so that the function φ_1 must have the form

$$\varphi_1(u_1, u_2) = f(u_1 + u_2) + g(u_1 - u_2).$$

Define now

$$\varphi_2(u_1, u_2) = -f(u_1 + u_2) + g(u_1 - u_2),$$

and define the matrix of 1-forms on the disc B:

$$(\omega_{ij}) = \begin{bmatrix} 0 & \omega_{12} = d\varphi_1 & \omega_{13} = du_1 & \omega_{14} = 0 \\ -d\varphi_1 & 0 & \omega_{23} = 0 & \omega_{24} = du_2 \\ -du_1 & 0 & 0 & \omega_{34} = d\varphi_2 \\ 0 & -du_2 & -d\varphi_2 & 0 \end{bmatrix}.$$

The orthonormal frame along the coordinate lines of the system (u_1, u_2) is $(1/\sqrt{E})\partial/\partial u_1$, $(1/\sqrt{G})\partial/\partial u_2$; so we also define $\omega_1 = \sqrt{E}\,du_1$ and $\omega_2 = \sqrt{G}\,du_2$. The differential 1-forms ω_i and ω_{ij} satisfy all Cartan's equations for E^4. Namely, one verifies immediately that the equation $d\omega_1 = \omega_{12} \wedge \omega_2$ and $d\omega_2 = \omega_{21} \wedge \omega_1$ are just the system (4.19). The verification of $d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$ is also straightforward. For example,

$$\begin{aligned}\omega_{12} \wedge \omega_{24} + \omega_{13} \wedge \omega_{34} &= d\varphi_1 \wedge du_2 + du_1 \wedge d\varphi_2 \\ &= [(f' + g')\,du_1 + (f' - g')\,du_2] \wedge du_2 \\ &\quad + du_1 \wedge [(-f' + g')\,du_1 + (-f' - g')\,du_2] \\ &= (f' + g')\,du_1 \wedge du_2 - (f' + g')\,du_1 \wedge du_2 \\ &= 0 = d\omega_{14}.\end{aligned}$$

Next we realize our metric (4.17) as a surface in E^4 following a procedure that goes back to E. Cartan. We describe briefly the various steps. First consider the system of differential equations

$$\frac{\partial e_i}{\partial u_j} = \sum_{k=1}^{4} \omega_{ik}\left(\frac{\partial}{\partial u_j}\right) e_k, \qquad i = 1, 2, 3, 4, \; j = 1, 2,$$

for the vectors $e_i(u_1, u_2)\colon B \to E^4$. According to the Frobenius Theorem, this system has a unique local solution, provided the integrability conditions that result from

$$\frac{\partial^2 e_i}{\partial u_k \partial u_j} = \frac{\partial^2 e_i}{\partial u_j \partial u_k}$$

are satisfied, and initial data $e_i(0, 0)$ are given. One verifies that the integrability conditions are simply Cartan's second structure equations, which are satisfied by construction. We assume that our initial data form an orthonormal basis of E^4. Then, the solution vectors $e_i(u_1, u_2)$, $1 \leq i \leq 4$, form an orthonormal basis of E^4 for all (u_1, u_2) in a neighborhood of the origin. Indeed, if we set $\langle e_i, e_j\rangle = g_{ij}$, and we differentiate, we obtain the system of differential equations for the g_{ij}:

$$\frac{\partial g_{ij}}{\partial u_k} = \sum_r \left[\omega_{ir}\left(\frac{\partial}{\partial u_k}\right) g_{rj} + \omega_{jr}\left(\frac{\partial}{\partial u_k}\right) g_{ir}\right].$$

The integrability conditions for this system are again Cartan's second structure equations. Therefore, according to the Frobenius Theorem again, the system is uniquely solvable if we demand $g_{ij}(0, 0) = \delta_{ij}$. Now the constant δ_{ij} is clearly a solution of the system, and so $g_{ij} = \delta_{ij}$ identically.

As a final step, and using the e_1, e_2 just constructed, we solve the system

$$\frac{\partial x}{\partial u_j} = \sum_{i=1}^{2} \omega_i\left(\frac{\partial}{\partial u_j}\right) e_i, \qquad j = 1, 2,$$

for the vector-valued function $x: D \to E^4$. We verify that the integrability conditions for this system are simply Cartan's first structure equations. Given $x(0, 0)$, we can therefore solve this system locally by virtue of the Frobenius Theorem, and we obtain a unique solution $x(u_1, u_2)$. Observe that $\partial x/\partial u_1 = \sqrt{E} e_1$ and $\partial x/\partial u_2 = \sqrt{G} e_2$, so that $(dx)^2 = I$ as desired.

Further, since $d\omega_{34} = d(d\varphi_2) = 0$, we have $K_n = 0$. The shape operators that correspond to e_3 and e_4 are, in matrix form,

$$A_3 \sim (\omega_{i3}(e_j)) = \begin{pmatrix} 1/\sqrt{E} & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_4 \sim (\omega_{i4}(e_j)) = \begin{pmatrix} 0 & 0 \\ 0 & 1/\sqrt{G} \end{pmatrix}.$$

From these expressions we conclude that $\dim N_1 = 2$, and that our assertion (iii) is true. This concludes the proof.

As an example, consider a plane curve C_1 of curvature $k(s) \neq 0$ and a plane curve C_2 of curvature $\lambda(s) \neq 0$. Set $E(u_1) = 1/k^2(u_1)$ and $G(u_2) = 1/\lambda^2(u_2)$. The metric $I = E\,du_1^2 + G\,du_2^2$ is flat and satisfies (4.18). The realization of I according to Theorem 4 is the cartesian product $C_1 \times C_2 \subset E^2 \times E^2 = E^4$. One verifies easily that the vectorfields e_3 and e_4 are, in the present case, singular and parallel in the normal bundle; the converse is also true.

Theorem 5. *A surface M in E^4 is the cartesian product of two plane curves with nowhere zero curvatures if and only if $K = 0$, $\dim N_1 = 2$, and there exists a unit normal vectorfield of M which is singular and parallel in the normal bundle.*

Proof. We need only prove the sufficiency of the conditions. Let e_3 be the unit normal vectorfield of M which is singular, that is $\det A_3 = 0$, and parallel in the normal bundle, that is $\nabla^{\perp}_X e_3 = 0$ for all tangent vectors X. Choose e_1, e_2, e_4 so that (e_1, e_2, e_3, e_4) is an adapted local orthonormal frame. We have $\omega_{34}(X) = \langle \nabla^{\perp}_X e_3, e_4 \rangle = 0$, so that $\omega_{34} = 0$, hence $K_n = 0$. We may assume that e_1, e_2 are the common eigenvectors of all the operators A_ξ. Let k_1, k_2 be the eigenvalues of A_3, and λ_1, λ_2 the eigenvalues of A_4. We have $k_1 k_2 = 0$ by assumption. Since $0 = K = \det A_3 + \det A_4$, we also have $\lambda_1 \lambda_2 = 0$. At no point can we have $k_1 = k_2 = 0$, because in that case $\dim N_1 \leq 1$. Let $k_1 \neq 0$ near a point; then $k_2 = 0$ and $\lambda_1 = 0$ there, because $\lambda_2 = 0$ would invalidate again the assumption $\dim N_1 = 2$. We set $k = k_1 \neq 0$ and $\lambda = \lambda_2 \neq 0$, and we obtain

$$\omega_{13} = k\omega_1, \quad \omega_{23} = 0, \quad \omega_{14} = 0, \quad \omega_{24} = \lambda\omega_2.$$

Then, $d\omega_{13} = \omega_{12} \wedge \omega_{23} + \omega_{14} \wedge \omega_{43} = 0$ and $d\omega_{24} = \omega_{21} \wedge \omega_{14} + \omega_{23} \wedge \omega_{34} = 0$. So there exist locally functions u_1, u_2 with

$$du_1 = \omega_{13} = k\omega_1, \qquad du_2 = \omega_{24} = \lambda\omega_2. \tag{4.20}$$

Since $du_1 \wedge du_2 \neq 0$, we may introduce u_1, u_2 as local coordinates, and our frame assumes the form $e_1 = k(\partial/\partial u_1)$, $e_2 = \lambda(\partial/\partial u_2)$. We compute further:

$$0 = d\omega_{14} = \omega_{12} \wedge \omega_{24} + \omega_{13} \wedge \omega_{34} = \lambda\omega_{12} \wedge \omega_2,$$
$$0 = d\omega_{23} = \omega_{21} \wedge \omega_{13} + \omega_{24} \wedge \omega_{43} = -k\omega_{12} \wedge \omega_1,$$

and so $\omega_{12} = 0$, which implies that e_1 and e_2 are parallel vectorfields on M. Now we take the exterior derivative of the two equations (4.20):

$$0 = dk \wedge \omega_1 + k\omega_{12} \wedge \omega_2 = dk \wedge \omega_1 = dk \wedge (du_1/k),$$
$$0 = d\lambda \wedge \omega_2 + \lambda\omega_{21} \wedge \omega_1 = d\lambda \wedge \omega_2 = d\lambda \wedge (du_2/\lambda),$$

which mean that k depends only on u_1, and λ only on u_2.

We shall show finally that the integral curves of the vectorfield e_1 are plane congruent curves. We have by construction $B(e_1, e_2) = 0$ and $A_4 e_1 = 0$. Furthermore, because e_2 and e_4 are parallel vectorfields in M and NM respectively,

$$D_{e_1} e_2 = \nabla_{e_1} e_2 + B(e_1, e_2) = 0,$$
$$D_{e_1} e_4 = -A_4 e_1 + \nabla^{\perp}_{e_1} e_4 = 0.$$

Therefore, e_2 and e_4 are parallel in E^4 along the e_1-curves. This means that every single e_1-curve is perpendicular to a constant plane Π, the one spanned by e_2 and e_4 along this curve, so this e_1-curve lies in an E^2 perpendicular to Π. The curvature of each of these plane curves is

$$|D_{e_1} e_1| = |B(e_1, e_1)| = |k|,$$

which is constant along the e_2-curves, since k depends only on u_1. It follows that any two e_1-curves are congruent. By a similar reasoning, we establish that the e_2-curves are congruent plane curves of curvature $|\lambda|$. This completes the proof.

What can be said of compact orientable flat surfaces with $K_n = 0$ and $\dim N_1 = 2$? They must of course all be topologically tori, according to the Gauss-Bonnet Theorem, and the cartesian product of two closed plane curves is an example, but there are others which are not products, e.g. the so-called Hopf tori. The next theorem gives a characterization of the standard flat torus in E^4. We call a normal vector e at a point of M an *umbilical direction* if A_e is a scalar multiple of the identity at that point.

Theorem 6. *Let M be an oriented complete flat surface in E^4. If $K_n = 0$, $|H|$ is bounded and H is an umbilical direction everywhere, then M is either a plane or the cartesian product of two circles of the same radius.*

Proof. Suppose $H(P) \neq 0$. Set $e_3' = H/|H|$ and pick a unit normal vector e_4' with $\langle e_3', e_4' \rangle = 0$. Let (e_1, e_2) be the common eigenbasis of all the shape operators. By our assumption on H and our choice of e_3', we have in the frame (e_1, e_2, e_3', e_4')

$$A_{e_3'} \sim \begin{pmatrix} |H| & 0 \\ 0 & |H| \end{pmatrix}.$$

Since $\operatorname{trace} A_{e_4'} = 0$, we find from $K = 0$ that

$$A_{e_4'} \sim \begin{pmatrix} \pm|H| & 0 \\ 0 & \mp|H| \end{pmatrix}.$$

Therefore, $B(e_1, e_1) = |H|(e_3' \pm e_4')$, $B(e_2, e_4) = |H|(e_3' \mp e_4')$, and $B(e_1, e_2) = 0$. We conclude that $\dim N_1 = 2$ at P.

Let U be the open set of points where $H \neq 0$. We may apply to U the local analysis preceding Theorem 4. With respect to the adapted frame (e_1, e_2, e_3, e_4) used there, we have $2H = ke_3 + \lambda e_4$, and we obtain from the above representation of $A_{e_3'}$

$$k^2 = \lambda^2 = 2|H|^2.$$

According to Theorem 4 again, there exists locally in U a coordinate system (u_1, u_2) so that $I = E(du_1^2 + du_2^2)$, $E = 1/2|H|^2$. The Theorema Egregium in these coordinates gives locally $\Delta(\log E) = 0$, Δ the laplacian on M. Therefore,

$$\Delta[\log(|H|^2)] = 0 \quad \text{in } U.$$

Since we have identically

$$\Delta[\log(|H|^2)] = \frac{\Delta(|H|^2)}{|H|^2} - \frac{|\operatorname{grad}(|H|^2)|^2}{|H|^4},$$

we deduce $\Delta(|H|^2) \geq 0$ in U.

On the set $V = M - U$ we have $H = 0$. If V has nonempty interior, then $\Delta(|H|^2) = 0$ there. Since $\Delta(|H|^2)$ is a continuous function on M, we have everywhere

$$\Delta(|H|^2) \geq 0.$$

If M is compact, we infer that $|H| = \text{const}$. If M is complete but not compact, then its universal covering space is E^2, since $K = 0$. Therefore, M is parabolic. Now $|H|^2$ is subharmonic, and bounded by assumption, so again $|H| = \text{const}$ by the maximum principle.

If $|H| = 0$, then all the shape operators are zero, so M is a plane. Suppose now $|H| \neq 0$; this means $k = \pm\lambda = \text{const} \neq 0$. We may suppose $k = \lambda$. Using (4.8) we compute

$$0 = d(k\omega_1) = dk \wedge \omega_1 + kd\omega_1 = k\omega_{12} \wedge \omega_2,$$
$$0 = d(k\omega_2) = dk \wedge \omega_2 + kd\omega_2 = k\omega_{21} \wedge \omega_2;$$

therefore $\omega_{12} = 0$ and e_1, e_2 are parallel vectorfields on M. Since $\omega_{13} = k\omega_1$, $\omega_{23} = \omega_{14} = 0$ and $\omega_{24} = k\omega_2$, we have further:

$$0 = d\omega_{23} = \omega_{21} \wedge \omega_{13} + \omega_{24} \wedge \omega_{43} = k\omega_2 \wedge \omega_{43},$$
$$0 = d\omega_{14} = \omega_{12} \wedge \omega_{24} + \omega_{13} \wedge \omega_{34} = k\omega_1 \wedge \omega_{34},$$

and so $\omega_{34} = 0$. The matrix of the connection forms now becomes

$$(\omega_{ij}) = \begin{bmatrix} 0 & 0 & k\omega_1 & 0 \\ 0 & 0 & 0 & -k\omega_2 \\ -k\omega_1 & 0 & 0 & 0 \\ 0 & k\omega_2 & 0 & 0 \end{bmatrix},$$

k a nonzero constant, whence we deduce that M is the cartesian product of two circles of radius $1/|k|$.

We turn now to surfaces with $K = K_n = 0$ and $\dim N_1 = 1$. The cylinders over curves in E^3 are examples of such surfaces. We prove a global converse of this fact.

Theorem 7. *A complete oriented surface M of E^4 with $K = 0$ and $\dim N_1 = 1$ is a cylinder $(s, t) \to (\alpha(s), t)$, $-\infty < s, t < +\infty$, where $\alpha(s)$ is a complete curve with everywhere nonzero curvature in E^3.*

Proof. Suppose $K = 0$ and $\dim N_1 = 1$ on our surface M. Take a unit normal vector e_3 generating everywhere N_1. This e_3 is globally defined, as is an e_4 chosen so that (e_3, e_4) is an oriented basis of NM. Since $B(X, Y)$ is parallel to e_3 for all X, Y, we have $\langle A_4X, Y\rangle = \langle B(X, Y), e_4\rangle = 0$, so $A_4 = 0$ and therefore $K_n = 0$. We now have $0 = K = \det A_3$. Let k be the nonzero eigenvalue of A_3; it is globally defined, as is the corresponding unit eigenvector e_1. We complete e_1 to the global oriented basis (e_1, e_2) of TM. In our global frame, the connection forms are:

$$\omega_{13} = k\omega_1, \quad \omega_{23} = 0, \quad \omega_{14} = 0, \quad \omega_{24} = 0. \tag{4.21}$$

We now compute

$$0 = d\omega_{23} = \omega_{21} \wedge \omega_{13} + \omega_{24} \wedge \omega_{43} = k\omega_{21} \wedge \omega_1 ,$$
$$0 = d\omega_{14} = \omega_{12} \wedge \omega_{24} + \omega_{13} \wedge \omega_{34} = k\omega_1 \wedge \omega_{34} .$$

Therefore,

$$\omega_{12} = \rho\omega_1 , \qquad \omega_{34} = \tau\omega_1 . \tag{4.22}$$

Clearly, the functions ρ and τ are globally defined. We shall prove that $\rho = 0$ on M, whereas $|k|$ and τ are, respectively, the curvature and torsion of the curve in E^3 which defines our cylinder. From (4.21) and (4.22), we get first

$$D_{e_2} e_2 = \nabla_{e_2} e_2 + B(e_2 , e_2) = 0 .$$

This means that the e_2-curves are straight lines of E^4. We consider $\rho = \rho(t)$ along one of these straight lines; here t is the arclength, and $-\infty < t < +\infty$ because M is complete. We have

$$\begin{aligned} 0 = d\omega_{12} &= d\rho \wedge \omega_1 + \rho d\omega_1 = d\rho \wedge \omega_1 + \rho\omega_{12} \wedge \omega_2 \\ &= (-e_2(\rho) + \rho^2)a\omega_1 \wedge \omega_2 , \end{aligned}$$

or

$$\frac{d\rho}{dt} - \rho^2 = 0 . \tag{4.23}$$

Suppose there exists a point $t = 0$ where $\rho(0) \neq 0$. Let (a , b) be the maximal interval containing 0, where $\rho \neq 0$. In (a , b), (4.23) has the solution

$$\rho(t) = \left(\frac{1}{\rho(0)} - t \right)^{-1} . \tag{4.24}$$

By continuity of ρ and maximality of (a , b), if $b < +\infty$, we would have

$$0 = \rho(b^-) = \left(\frac{1}{\rho(0)} - b \right)^{-1} \neq 0 ,$$

which is absurd. A similar contradiction is obtained for $\rho(\alpha^+)$ if we were to assume $\alpha > -\infty$. Therefore, $(a , b) = R$, and $\rho \neq 0$ for all t which is impossible since ρ in (4.24) is not defined for $t = 1/\rho(0)$. It follows that $\rho = 0$ along the straight line, as we asserted; hence,

$$\omega_{12} = 0 . \tag{4.25}$$

From this and Cartan's first equations, we infer the local existence of functions u_1 and u_2 with the property $\omega_1 = du_1$ and $\omega_2 = du_2$. We may introduce (u_1 , u_2) as local coordinates on M. Now from (4.21), (4.22) and (4.25), we have

$$dk \wedge \omega_1 = d\omega_{13} = \omega_{12} \wedge \omega_{23} + \omega_{14} \wedge \omega_{43} = 0 ,$$
$$d\tau \wedge \omega_1 = d\omega_{34} = \omega_{31} \wedge \omega_{14} + \omega_{32} \wedge \omega_{24} = 0 ,$$

which means that the functions k and τ depend only on u_1.

In addition to $D_{e_2}e_2 = 0$, we have because of (4.21) and (4.25):

$$D_{e_1}e_2 = \nabla_{e_1}e_2 + B(e_1, e_2) = 0,$$

which says that e_2 is parallel in E^4 along M. Thus, all e_2-curves, which are straight lines of E^4, are parallel to a fixed vector e of E^4. Besides, the e_1-curves are contained in affine 3-subspaces which are orthogonal to e. In fact, the equations

$$D_{e_1}e_1 = ke_3,$$
$$D_{e_1}e_3 = -A_3e_1 + \nabla^{\perp}_{e_1}e_3 = -ke_1 + \omega_{34}(e_1)e_4 = -ke_1 + \tau e_4$$

show that (e_1, e_3, e_4, k, τ) is the Frenet apparatus in E^3 of the e_1-curves. This completes the proof.

Remark. This theorem can be generalized almost verbatim for surfaces of E^n. Thus, the only complete surfaces of E^n with $K = 0$ and $\dim N_1 = 1$ are cylinders over curves in E^{n-1} with nonzero first curvature.

Remark. We learned in the course of writing this paper that a more general form of Theorem 7 is found in [3]; the proof there, however, uses much heavier machinery.

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