TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 319, Number 1, May 1990

COMPLETE COINDUCTIVE THEORIES. I

A. H. LACHLAN

ABSTRACT. Let T be a complete theory over a relational language which has an axiomatization by $\exists\forall$-sentences. The properties of models of T are studied. It is shown that quantifier-free formulas are stable. This limited stability is used to show that in $\exists\forall$-saturated models the elementary types of tuples are determined by their $\exists$-types and algebraicity is determined by existential formulas. As an application, under the additional assumption that no quantifier-free formula has the FCP, the models $\mathscr{M}$ of T are completely characterized in terms of certain 0-definable equivalence relations on cartesian powers of M. This characterization yields a result similar to that of Schmerl for the case in which T is $\aleph_0$-categorical.

This is the first of two papers about complete first-order theories which have axiomatizations consisting entirely of $\exists\forall$-sentences over a relational language. We call theories with such axiomatizations $\exists\forall$-*theories*. Schmerl has called them *coinductive*.

This paper is a sequel to [7] which characterized complete theories over a relational language which have axiomatizations consisting of $\exists$-sentences and $\forall$-sentences. (From now on the restriction to relational languages which obtains throughout will be tacit.) Such theories are called *primitive*. Following Hodkinson and Macpherson [5] a structure $\mathscr{M}$ is said to be *finitely partitioned* if there exists a partition $M = F\dot\cup X_1\dot\cup\cdots\dot\cup X_n$ of the universe of $\mathscr{M}$ invariant under $\mathrm{Aut}(\mathscr{M})$ such that $|F| < \omega$, $|X_i| \geq \omega$ $(1 \leq i \leq n)$, and the pointwise stabilizer of $M\backslash X_i$ in $\mathrm{Aut}(\mathscr{M})$ induces $\mathrm{Sym}(X_i)$. In [7] it was shown that a complete theory is primitive if and only if all its models are finitely partitioned. Some acquaintance with [7] would be very helpful to readers of the present work.

Schmerl [10] studied $\aleph_0$-categorical $\exists\forall$-theories over a finite relational language and proved that the unique countable model of such a theory is cellular in the following sense. $\mathscr{M}$ is *cellular* if there exist a finite subset X of M and equivalence relations E, F on $M\backslash X$ such that

(1) $(M\backslash X)/E$ is finite.

(2) If C is an E-class and D an F-class, then $|C\cap D| = 1$.

Received by the editors March 14, 1988 and, in revised form, August 29, 1988.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 03C99; Secondary 03C45.

The author acknowledges the support of the Government of Canada through NSERC Grant A3040 and is grateful to John Baldwin for many helpful comments on an earlier draft of the paper.

(3) If $a_i, b_i \in M$ $(i < k)$, and for all $i, j < k$

$$[a_i \in X \vee b_i \in X] \Rightarrow a_i = b_i,$$
$$a_i, b_i \notin X \Rightarrow a_i E b_i,$$
$$a_i F a_j \Leftrightarrow b_i F b_j,$$

then $(a_0, \dots, a_{k-1})$ and $(b_0, \dots, b_{k-1})$ satisfy the same quantifier-free formulas.

A theorem about the $\aleph_0$-categorical case with essentially the same content as Schmerl's, but formulated rather differently, was obtained independently by the author and stated without proof in [7, p. 699].

This paper has two aims: (1) to give an analysis of complete $\exists\forall$-theories in which no quantifier-free formula has the finite cover property (FCP), (2) to lay the groundwork for a study of arbitrary complete $\exists\forall$-theories to be continued in the sequel to this paper.

Let T be a complete theory with a finite language such that no quantifier-free formula has the FCP. The main result of §5, Theorem 5.4, states that T is $\exists\forall$ if and only if every model $\mathscr{M}$ of T satisfies the following three conditions:

Q1. There is an equivalence relation $\approx$ on $M \setminus \mathrm{acl}(\emptyset)$ such that, for $a \in M \setminus \mathrm{acl}(\emptyset)$ and $B \subseteq M \setminus \mathrm{acl}(\emptyset)$, $a \in \mathrm{acl}(B)$ if and only if $a \approx b$ for some $b \in B$.

Q2. For each n, $1 \le n < \omega$, there is a 0-definable finite equivalence relation R_n on M^n such that, if $\overline{a}, \overline{b} \in (M \setminus \mathrm{acl}(\emptyset))^n$, then $\overline{a} R_n \overline{b}$ if and only if $\overline{a}$ and $\overline{b}$ realize the same quantifier-free type over $(M \setminus \mathrm{acl}\{\overline{a}, \overline{b}\}) \cup \mathrm{acl}(\emptyset)$.

Q3. For each n, $1 \le n < \omega$, if an R_n-class C contains infinitely many disjoint n-tuples, then $C \cap (M \setminus \mathrm{acl}(\emptyset))^n$ contains infinitely many disjoint n-tuples $\overline{a}$ which are closed in the sense that $\mathrm{acl}\{\overline{a}\} \subseteq \overline{a} \cup \mathrm{acl}(\emptyset)$.

The result of Schmerl mentioned above, which is somewhat refined in Theorem 5.8 below, tells us that $\aleph_0$-categorical $\exists\forall$-theories are completely understood. One of the consequences of the theorem just stated is that, if T is a complete $\exists\forall$-theory over a finite language in which no quanitifer-free formula has the FCP, then T has a prime model $\mathscr{M}$ which can be expressed as the union of a chain $\mathscr{M}_0 \subseteq \mathscr{M}_1 \subseteq \cdots$ of 0-definable $\aleph_0$-cateogorical substructures such that $\mathrm{Th}(\mathscr{M}_i)$ is $\exists\forall$ $(i < \omega)$. Thus the $\exists\forall$-theories discussed in §5 in which no quantifier-free formula has the FCP are in some sense limits of $\aleph_0$-categorical $\exists\forall$-theories.

In §1 we describe three examples. In §§2–4 we investigate the properties of arbitrary complete $\exists\forall$-theories at the same time as preparing for the special case treated in §5. In §2 we show that quantifier-free formulas are stable. In §3 we study $\exists\forall$-saturated models which play a key role in all our later work. $\mathscr{M}$ is called $\exists\forall$-*saturated* if every finite $\exists\forall$-type over a finite subset of M realizable in some $\forall$-extension of $\mathscr{M}$ is realized in $\mathscr{M}$. In Lemma 3.8 it is shown that two disjoint $\exists\forall$-formulas over a $\exists\forall$-saturated model can be separated by a Boolean combination of existential formulas containing no additional parameters. It follows that, if $\mathscr{M}$ is a $\exists\forall$-saturated model and $A \subseteq M$, then for all $\overline{b} \in M$ the elementary type of $\overline{b}$ over A is determined by the existential type of $\overline{b}$ over A. In §4 we show that, if $b \in \operatorname{acl}\{\overline{a}\}$ in a $\exists\forall$-saturated model $\mathscr{M}$, then there is an existential formula $\xi(x)$ over $\operatorname{Rng}(\overline{a})$ such that $\mathscr{M} \models \xi(b)$ and $\xi(x)$ has only a finite number of solutions.

We now mention briefly the main result which will appear in the sequel to this paper. A striking feature of the examples of complete $\exists\forall$-theories is that they are stable and the relation "$x = y \vee \operatorname{Tp}(x|y)$ forks over $\varnothing$" is an equivalence relation. Thus the models of our examples are all naturally partitioned into components. We will show that all complete $\exists\forall$-theories have this character by showing that they are tree decomposable in the sense of Baldwin and Shelah [3, p. 253]. Call T *monadically unstable* if there is a monadic formula $\phi(x, y; \overline{X})$ such that for every linear ordering $(I, <_I)$ there is a model $\mathscr{M}$ of T and a tuple $\overline{A}$ of subsets of M such that the interpretation of $\phi(x, y, \overline{A})$ in $\mathscr{M}$ is a linear ordering isomorphic to $(I, <_I)$. By the method of [3, §8] in this definition it makes no difference if one replaces x, y by tuples $\overline{x}$, $\overline{y}$ of the same length. It also follows from various results in [3] that tree decomposability is the same as monadic stability. Thus tree-decomposability is a very strong stability condition. This condition can also be expressed in terms of the independence, i.e. nonforking, relation as follows: T is monadically stable if and only if T is stable and for every $\mathscr{M} \models T$, $b \in M$, and A, C, $D \subseteq M$

$$A \downarrow C(D) \Rightarrow [A \downarrow b(C \cup D) \vee C \downarrow b(A \cup D)].$$

Here $X \downarrow Y(Z)$ is to be read "X and Y are independent over Z".

In this paper *types* are those in which the formulas are quantifier-free. Thus $\operatorname{tp}(\overline{a}|A)$ is the set of all quanitifer-free formulas $\phi(\overline{x})$ over A such that $\mathscr{M} \models \phi(\overline{a})$, where $\overline{x} = (x_1, \ldots, x_{l(\overline{a})})$ and $x_1, x_2, \ldots$ are the variables in canonical order. Types of the more familiar kind containing formulas with quantifiers will be called *elementary types*, and the elementary type of $\overline{a}$ over A is denoted $\operatorname{Tp}(\overline{a}|A)$.

To simplify notation we allow $\overline{a}$ to denote not only the tuple $(a_1, \ldots, a_n)$ but also the set $\{a_1, \ldots, a_n\}$. Where ambiguity may occur the tuple interpretation is meant unless we include the phrase "as sets". Thus "$\overline{a} = \overline{b}$" means that $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$ are tuples of the same length and $a_i = b_i$ for all i, $1 \leq i \leq n$. On the other hand "$\overline{a} = \overline{b}$ as sets" means that $\overline{a} = (a_1, \ldots, a_m)$ and $\overline{b} = (b_1, \ldots, b_n)$ are tuples, possibly of different lengths, such that $\{a_1, \ldots, a_m\} = \{b_1, \ldots, b_n\}$. The tuple obtained by concatenating tuples $\overline{a}$ and $\overline{b}$ will be denoted by $\overline{a}\,\overline{b}$.

A useful preliminary observation about complete $\exists\forall$-theories is

0.1. **Lemma.** *Let T^* be a complete $\exists\forall$-theory over the relational language L^*, and T be the restriction of T^* to $L \subseteq L^*$. Then T is $\exists\forall$.*

Proof. We need the following:

Claim. If $\mathcal{M}$ is an L-structure such that for all L-structures $\mathcal{M}'$, $\mathcal{N}$

$$[\mathcal{M} \preceq \mathcal{M}' \& \mathcal{M} \subseteq \mathcal{N} \subset \mathcal{M}'] \to \mathcal{M} \equiv \mathcal{N},$$

then $\mathrm{Th}(\mathcal{M})$ is $\exists\forall$.

To see this let $\mathcal{M}$ be as in the hypothesis, $\mathcal{M}_0 \equiv \mathcal{M}$, and $\mathcal{M}_0 \subseteq_\forall \mathcal{N}_0$. By compactness there exist L-structures $\mathcal{M}_1$, $\mathcal{N}_1$ with $\mathcal{M}_1 \subseteq \mathcal{N}_1$ such that $(\mathcal{N}_0, M_0) \preceq (\mathcal{N}_1, M_1)$ and $\mathcal{M}$ is elementarily embeddable in $\mathcal{M}_1$. Here $(\mathcal{N}_i, M_i)$ means the expansion of $\mathcal{N}_i$ by a unary predicate picking out M_i. Replacing $\mathcal{M}$ by a copy we can assume $\mathcal{M} \preceq \mathcal{M}_1$ in which case $\mathcal{M} \subseteq_\forall \mathcal{N}_1$. By compactness there exists $\mathcal{M}_2 \succeq \mathcal{M}$ such that $\mathcal{M} \subseteq \mathcal{N}_1 \subseteq \mathcal{M}_2$. By our hypothesis about $\mathcal{M}$, $\mathcal{N}_1 \equiv \mathcal{M}$. Since $\mathcal{N}_0 \preceq \mathcal{N}_1$, we also have $\mathcal{N}_0 \equiv \mathcal{M}$. From [7, Lemma 4.1] this is enough.

We now turn to the proof of the lemma. Let $\mathcal{M}^* \models T^*$ and $\mathcal{M} = \mathcal{M}^* \restriction L$. Let $\mathcal{N} \succeq \mathcal{M}$. By compactness there exists $\mathcal{N}^* \succeq \mathcal{M}^*$ such that $\mathcal{N}^* \restriction L \succeq \mathcal{N}$. Every structure between $\mathcal{M}^*$ and $\mathcal{N}^*$ is a model of T^* since T^* is $\exists\forall$, and so any structure between $\mathcal{M}$ and $\mathcal{N}$ is a model of T. From the claim T is $\exists\forall$, which completes the proof.

From this lemma it is clear that understanding complete $\exists\forall$-theories over finite languages should tell us everything we wish to know about the general case.

We shall use the same notion of rank as in [7] with one important difference. In [7] it was enough to assign ranks to formulas $\phi(x, \overline{b})$ and hence to definable subsets of the universe M of the model $\mathcal{M}$ under consideration. Here we proceed in the same way, but we allow formulas with any number of free variables. Working over a finite relational language, for each n, $1 \leq n < \omega$, we choose a finite complete set of atomic formulas Δ_n with distinguished variables $x_1, \ldots, x_n$. If $\overline{x}$ denotes $(x_1, \ldots, x_n)$, then $\mathrm{Rk}(\phi(\overline{x}, \overline{b}))$ means $R^n(\{\phi(\overline{x}, \overline{b})\}, \Delta_n, \aleph_0)$ in the notation of Shelah [11, p. 21]. We shall use this rank even when the language is infinite. In such a case it is tacitly assumed that we first restrict to an appropriate finite sublanguage by means of Lemma 0.1. We need the following consequence of the definition of rank.

0.2. **Lemma.** *Let a finite relational language L be given. For any quantifier-free L-formula $\phi(\overline{x}, \overline{y})$ and any r, $0 \leq r < \omega$, there exists a set $\Gamma(\overline{y})$ of existential formulas such that for any L-structure $\mathcal{M}$ and $\overline{b} \in M$*

$$\mathrm{Rk}(\phi(\overline{x}, \overline{b})) \geq r \Leftrightarrow \mathcal{M} \models \bigwedge \Gamma(\overline{b}).$$

1. Examples

In this section we present some further instances of complete $\exists\forall$-theories. Like those given in [7, §4] these will all follow a common pattern. A finite relational language L is specified as well as a universal L-theory T_1. Let $\mathcal{M}^*$ be a disjoint union of finite models of T_1 in which every finite model occurs infinitely often. In the cases we consider it turns out that $\mathcal{M}^* \preceq \mathcal{M}^* \dot{\cup} \mathcal{N}$ for all $\mathcal{N} \models T_1$. From this it is clear that $T_2 = \mathrm{Th}(\mathcal{M}^*)$ is a $\exists\forall$-theory. There are variations one can play on this theme but we know of no complete $\exists\forall$-theories which are radically different. Verifying the examples, i.e. that indeed $\mathcal{M}^* \preceq \mathcal{M}^* \dot{\cup} \mathcal{N}$ for all $\mathcal{N} \models T_1$, is straightforward using Ehrenfeucht games. We leave this task to the reader since it sheds no light on the general theory which is our main concern.

Example 1. Let L be the language with one binary relation symbol R. Let $\mathcal{M} = (M, R^{\mathcal{M}})$ be a connected graph with no cycles in which each vertex has degree $\aleph_0$. Let $T_0 = \mathrm{Th}(\mathcal{M})$. In T_0 every formula is equivalent to a Boolean combination of $\forall$- and $\exists$-formulas. Let T_1 be the underlying universal theory, i.e. the theory of graphs without cycles. Note that $\mathcal{M}$ is the unique countable existentially closed model of T_1. For a model of T_1 to be existentially closed it is necessary and sufficient that it be a model of T_0 with one component. Let $\mathcal{M}^*$ be a disjoint union of finite models of T_1 in which every finite model of T_1 occurs infinitely often as a summand. Let $T_2 = \mathrm{Th}(\mathcal{M}^*)$. T_2 is a $\exists\forall$-theory: suitable axioms are those of T_1 together with sentences which say that each finite model $\mathcal{N}$ of T_1 can be embedded as a "disjoint part" of each model of T_2. The formulas of T_2 have arbitrary quantifier complexity because any connected graph without cycles can occur as a component of a model of T_2.

Example 2. Let L be the language with one ternary relation symbol R. A 3-*graph* is an L-structure $\mathcal{M}$ determined by a *vertex set* M and an *edge set* $E \subseteq [M]^3$, where

$$R^{\mathcal{M}} = \{(a_0, a_1, a_2): \{a_0, a_1, a_2\} \in E\}.$$

The *degree* of a vertex $b \in M$ is $|\{e \in E: b \in e\}|$. A *cycle* in $\mathcal{M}$ is a sequence $\langle b_i: i \leq n\rangle$ of vertices, distinct except that $b_0 = b_n$, such that $n > 2$, for each $i < n$ there exists $e \in E$ with $\{b_i, b_{i+1}\} \subseteq e$, and for each $i < n$, $\{b_{i-1}, b_i, b_{i+1}\} \notin E$ where b_{-1} means b_{n-1}. Following the same plan as in Example 1 let $\mathcal{M}$ be a connected 3-graph without cycles in which each vertex has degree $\aleph_0$ and in which any two edges have at most one vertex in common. Let $T_0 = \mathrm{Th}(\mathcal{M})$ and T_1 be the underlying universal theory of T_0. Let $\mathcal{M}^*$ be a disjoint union of finite models of T_1 in which every finite model appears infinitely often as a summand. Letting $T_2 = \mathrm{Th}(\mathcal{M}^*)$ gives another $\exists\forall$-theory. An interesting feature of this example is that it permits $a \in (\mathrm{acl}(b, c))\backslash(\mathrm{acl}(b) \cup \mathrm{acl}(c))$.

Example 3. This example is due to Shelah in another context. Let the language L consist of two unary relation symbols U_0, U_1 and a binary relation symbol R. Define an L-structure $\mathcal{M}$ as follows:

$$U_0^{\mathcal{M}} = \{{}^n\omega: 1 \leq n < \omega\}, \qquad U_1^{\mathcal{M}} = {}^\omega\omega,$$
$$M = U_0^{\mathcal{M}} \cup U_1^{\mathcal{M}} \cup \{(f, m, n): f \in {}^\omega\omega,\ m \leq n < \omega\},$$

and $R^{\mathscr{M}}$ is to be the least symmetric relation containing all pairs of the forms

$$(f, (f, 0, n)), \quad ((f, m, n), (f, m+1, n)), \quad ((f, n, n,), f \upharpoonright (n+1)),$$

where $f \in {}^{\omega}\omega$ and $m < n < \omega$. Let $T_0 = \mathrm{Th}(\mathscr{M})$. Then T_0 is model complete and stable but not superstable. The unsuperstability results from the definability in $\mathscr{M}$ of the functions $G_n: {}^{\omega}\omega \to {}^{(n+1)}\omega$ given by $G_n(f) = f \upharpoonright (n+1)$. Let T_1 be the underlying universal theory and T_2 be obtained in the same way as in Examples 1 and 2. Again T_2 is a $\exists\forall$-theory. Since the functions G_n are defined by $\exists$-formulas and $\mathscr{M}$ is a $\forall$-substructure of some model of T_2, T_2 is unsuperstable.

2. Stability of quanitifer-free formulas

The foundation for many of the results which follow is an extension of Lemma 4.3 of [7]:

2.1. **Theorem.** *Let T be a complete $\exists\forall$-theory over a relational language. Every quantifier-free formula is stable in T.*

Proof. Towards a contradiction suppose that there are $\mathscr{M} \models T$, a quantifier-free formula $\phi(\overline{x}, \overline{y})$ with parameters $\overline{c}$ from M, and tuples $\overline{a}_i, \overline{b}_i \subseteq M \setminus \mathrm{Rng}(\overline{c})$ $(i < \omega)$ such that $\mathscr{M} \models \phi(\overline{a}_i, \overline{b}_j)$ iff $i \le j$. The formula $\phi(\overline{x}, \overline{y})$ is to be chosen to minimize first $l(\overline{x})$ and then $l(\overline{y})$. Notice that this implies $l(\overline{x}) \le l(\overline{y})$. Using Ramsey's theorem and compactness in the usual way we can choose the $\overline{a}_i$'s and $\overline{b}_j$'s such that, if $n < \omega$ and

$$(*) \qquad i_0 \le j_0 < i_1 \le j_1 < \cdots < i_n \le j_n$$

then

$$(\#) \qquad \mathrm{tp}(\overline{a}_{i_0}, \dots, \overline{a}_{i_n}, \overline{b}_{j_0}, \dots, \overline{b}_{j_n}) = \mathrm{tp}(\overline{a}_0, \dots, \overline{a}_n, \overline{b}_0, \dots, \overline{b}_n).$$

It is technically convenient to have the $\overline{a}_i$'s and $\overline{b}_i$'s defined for $-k \le i < \omega + k$, where k is an integer such $l(\overline{x}) \le l(\overline{y}) \le k$ and the arity of the language is at most $k+1$. Now in $(*)$ we allow $i_0, \dots, j_n$ to range anywhere in the interval $-k \le i < \omega + k$.

Let C, D denote the sets

$$\bigcup\{\overline{a}_i\overline{b}_i : -k \le i < 0 \text{ or } \omega \le i < \omega + k\}, \qquad \bigcup\{\overline{a}_i\overline{b}_i : 0 \le i < \omega\}$$

respectively. Since T is a $\exists\forall$-theory, replacing $\mathscr{M}$ by a suitable $\forall$-extension we can choose $\overline{a}_i, \overline{b}_i$ $(-k \le i < \omega + k)$ such that $\mathscr{N} \models T$, where $\mathscr{N} \subseteq \mathscr{M}$ is defined by $N = M \setminus (C \cup D)$, and such that $(*)$ implies

$$(\#\#) \qquad \mathrm{tp}(\overline{a}_{i_0}, \dots, \overline{a}_{i_n}, \overline{b}_{j_0}, \dots, \overline{b}_{j_n} | N) = \mathrm{tp}(\overline{a}_0, \dots, \overline{a}_n, \overline{b}_0, \dots, \overline{b}_n | N).$$

From the indiscernibility there is a finite set $E \subseteq M$ such that $C \subseteq E$, $E \cap D = \varnothing$, and for any π permuting an $l(\overline{x})$-tuple and any i, $0 \le i < \omega$,

$$\mathrm{tp}(\overline{a}_i | E) = \mathrm{tp}(\pi(\overline{a}_i) | E) \to \mathrm{tp}(\overline{a}_i | M \setminus \overline{a}_i) = \mathrm{tp}(\pi(\overline{a}_i) | M \setminus \overline{a}_i),$$

and similarly for the $\overline{b}_i$'s $(0 \le i < \omega)$. This is where we require the arity of the language to be at most $k+1$.

Claim 1. For any $\overline{a} \subseteq M$, if $\overline{a} \cap E = \emptyset$ and $\mathscr{M} \models \phi(\overline{a}, \overline{b}_\omega) \& \neg\phi(\overline{a}, \overline{b}_{-1})$, then for some i, $\overline{a} = \overline{a}_i$ as sets or $\overline{a} \subseteq \overline{b}_i$.

Consider first the case in which $N \cap \overline{a} \neq \emptyset$. Suppose that $\overline{a} = \overline{a}'\overline{c}'$, where $\overline{a}' \in D$ and $\overline{c}' \in N$. Choose m, $0 \le m < \omega$, such that $\overline{a}' \subseteq \overline{a}_0\overline{b}_0 \cdots \overline{a}_{m-2}\overline{b}_{m-2}$, and for $0 \le i < \omega$ let $\overline{a}'_i$ denote the corresponding subtuple of $\overline{a}_{im}\overline{b}_{im} \cdots \overline{a}_{(i+1)m-2}\overline{b}_{(i+1)m-2}$. Let $\overline{b}'_i$ denote $\overline{b}_{(i+1)m-1}$ $(0 \le i < \omega)$ and $\phi'(\overline{x}', \overline{y})$ denote $\phi(\overline{x}'\overline{c}', \overline{y})$. Then $\overline{a}'_0, \overline{b}'_0, \overline{a}'_1, \overline{b}'_1, \ldots$ witness the unstability of $\phi'(\overline{x}', \overline{y})$. This contradicts the minimality of $l(\overline{x})$. So below we can assume that $\overline{a} \in D$.

From the indiscernibility expressed in (##), if the entries of $\overline{a}$ fall in more than one of the sets $\overline{a}_i$, $\overline{b}_i$ $(0 \le i < \omega)$ we can "spread them out" so that

$$\overline{a} \cap \overline{a}_i \neq \emptyset \to \overline{a} \cap \overline{b}_i\overline{a}_{i+1} = \emptyset$$

and

$$\overline{a} \cap \overline{b}_i \neq \emptyset \to \overline{a} \cap \overline{a}_{i+1}\overline{b}_{i+1} = \emptyset .$$

Fix the least i, $0 \le i < \omega$, such that $\mathscr{M} \models \phi(\overline{a}, \overline{b}_i)$ and $\overline{a} \cap \overline{b}_i = \emptyset$.

Suppose that $\emptyset \neq \overline{a} \cap \overline{a}_i \neq \overline{a}_i$. We can assume that $\overline{a} = \overline{a}'\overline{d}\overline{e}$, where $\emptyset \neq \overline{a}' \subseteq \overline{a}_i$, $\overline{d} \in \overline{a}_0\overline{b}_0 \cdots \overline{a}_{i-2}\overline{b}_{i-2}$, and $\overline{e} \in \overline{b}_{i+1}\overline{a}_{i+2}\overline{b}_{i+2} \cdots$. Let $\overline{e}' \in \bigcup\{\overline{a}_j\overline{b}_j : \omega \le j < \omega + k\}$ be chosen to realize the same type as $\overline{e}$ over $\overline{d}\overline{b}_{i-1}\overline{a}'\overline{b}_i$. Let $\phi'(\overline{x}', \overline{y})$ denote $\phi(\overline{x}'\overline{d}\overline{e}', \overline{y})$ and $\overline{a}'_j$ denote the subtuple of $\overline{a}_{i+j}$ corresponding to the subtuple $\overline{a}'$ of $\overline{a}_i$. Let $\overline{b}'_j$ denote $\overline{b}_{i+j}$. Then $\overline{a}'_0$, $\overline{b}'_0$, $\overline{a}'_1$, $\overline{b}'_1, \ldots$ withness that $\phi(\overline{x}', \overline{y})$ is unstable. Thus we finish unless $\overline{a}_i \cap \overline{a} = \emptyset$.

Since either $\mathscr{M} \models \neg\phi(\overline{a}, \overline{b}_{i-1})$ or $\overline{a} \cap \overline{b}_{i-1} \neq \emptyset$, the latter must be the case from the indiscernibility. We can assume that $\overline{a} = \overline{a}'\overline{d}\overline{e}$, where $\overline{a}' \subseteq \overline{b}_{i-1}$, $\overline{d} \subseteq \overline{a}_0\overline{b}_0 \cdots \overline{b}_{i-3}\overline{a}_{i-2}$, and $\overline{e} \subseteq \overline{a}_{i+1}\overline{b}_{i+1}\overline{a}_{i+2}\overline{b}_{i+2} \cdots$. Let $\overline{e}' \subseteq \bigcup\{\overline{a}_j\overline{b}_j : \omega \le j < \omega + k\}$ realize the same type as $\overline{e}$ over $\overline{d}\overline{b}_{i-2}\overline{a}'\overline{b}_i$. Let $\phi'(\overline{x}', \overline{y})$ denote $\phi(\overline{x}'\overline{d}\overline{e}', \overline{y})$. Let $\overline{a}'_j$ be the subtuple of $\overline{b}_{2j+i-1}$ corresponding to the subtuple $\overline{a}'$ of $\overline{b}_{i-1}$, and $\overline{b}'_j$ denote $\overline{b}_{2j+i}$. Then $\overline{a}'_0, \overline{b}'_0, \overline{a}'_1, \overline{b}'_1, \ldots$ witness the unstability of $\phi'(\overline{x}', \overline{y})$. By the minimality of $l(\overline{x})$, $l(\overline{x}') = l(\overline{x})$ which means that $\overline{a} \subseteq \overline{b}_{i-1}$. This completes the proof of the claim.

Reversing the roles of $\overline{x}$ and $\overline{y}$ we obtain a stronger conclusion because $l(\overline{x}) \le l(\overline{y})$:

Claim 2. If $\overline{b} \subseteq M$ and $\overline{b} \cap E = \emptyset$ and $\mathscr{M} \models \phi(\overline{a}_{-1}, \overline{b}) \& \neg\phi(\overline{a}_\omega, \overline{b})$, then for some i, $\overline{b} = \overline{a}_i$ or $\overline{b} = \overline{b}_i$ as sets.

There are now two cases.

Case 1. There is no $\overline{a} \subseteq \overline{b}_0$ such that $\mathscr{M} \models \phi(\overline{a}, \overline{b}_\omega)\&\neg\phi(\overline{a}, \overline{b}_{-1})$.

Let $A = \bigcup\{\overline{a}_i : 0 \le i < \omega\}$ and E_A be the equivalence relation whose classes are the sets $\overline{a}_i$ $(0 \le i < \omega)$. Let $<_A$ be the natural linear ordering of A/E_A in type ω. From Claim 1 and the case hypothesis there is a symmetric quantifer-free formula $\chi(\overline{x})$ over C such that $\mathscr{M} \models \chi(\overline{a})$ iff $\overline{a}$ is an E_A-class. Let $\psi(\overline{x}_0, \overline{x}_1)$ denote the formula

$$\chi(\overline{x}_0)\&\chi(\overline{x}_1)\& \bigvee_{\pi_0, \pi_1} [\text{tp}(\pi_0(\overline{x}_0)|E) = \text{tp}(\pi_1(\overline{x}_1)|E) = \text{tp}(\overline{a}_0|E)$$
$$\&\exists\overline{y}(\phi(\overline{a}_{-1}, \overline{y})\&\neg\phi(\overline{a}_\omega, \overline{y})\&\phi(\pi_0(\overline{x}_0), \overline{y})$$
$$\&\neg\phi(\pi_1(\overline{x}_1), \overline{y})\&\overline{y} \cap (\overline{x}_0\overline{x}_1 \cup E) = \varnothing)],$$

where (π_0, π_1) varies through all pairs of permutations of an $l(\overline{x})$-tuple.

If $\overline{a}'_0$ and $\overline{a}'_1$ are E_A-classes $\overline{a}_i, \overline{a}_j$ such that $i < j$, then $\mathscr{M} \models \psi(\overline{a}'_0, \overline{a}'_1)$ is witnessed by $\overline{y} = \overline{b}_i$ and the identity permutations. Conversely, suppose that $\mathscr{M} \models \psi(\overline{a}'_0, \overline{a}'_1)$ is witnessed by $\overline{y} = \overline{b}$ and the permutations π_0, π_1. Since $\mathscr{M} \models \chi(\overline{a}'_k)$ $(k < 2)$, there exist i, j such that $0 \le i, j < \omega$, and $\overline{a}'_0 = \overline{a}_i$ and $\overline{a}'_1 = \overline{a}_j$ as sets. Since $\text{tp}(\overline{a}_0|E) = \text{tp}(\pi_0(\overline{a}'_0)|E) = \text{tp}(\pi'_1(\overline{a}'_1)|E)$, without loss of generality we may assume that π_0, π_1 are the identity, and $\overline{a}'_0 = \overline{a}_i$ and $\overline{a}'_1 = \overline{a}_j$ (as tuples). Now we have $\overline{b} \cap (\overline{a}_i\overline{a}_j \cup E) = \varnothing$ and

$$\mathscr{M} \models \phi(\overline{a}_{-1}, \overline{b})\&\phi(\overline{a}_i, \overline{b})\&\neg\phi(\overline{a}_j, \overline{b})\&\neg\phi(\overline{a}_\omega, \overline{b}).$$

From Claim 2 there exists k such that $0 \le k < \omega$ and $\overline{b} = \overline{a}_k$ or $\overline{b} = \overline{b}_k$ as sets. From the indiscernibility, since $-1 < k < \omega$, we have $i < k < j$. Therefore we have $\mathscr{M} \models \psi(\overline{a}'_0, \overline{a}'_1)$ iff $\overline{a}'_0$, $\overline{a}'_1$ are E_A-classes and $\overline{a}'_0 <_A \overline{a}'_1$. Therefore $\mathscr{M}$ has the property Π defined as follows.

The property Π. $\mathscr{M}$ has Π if there exist $m < \omega$, $\overline{e} \in M$, $A \subseteq M$, an equivalence relation E_A on A and a linear ordering $<_A$ of A/E_A satisfying the following four conditions:

P1. $A \cap \overline{e} = \varnothing$ and each E_A-class has size m.

P2. There is a quantifier-free formula $\chi(\overline{x}, \overline{e})$ with $l(\overline{x}) = m$ such that $\mathscr{M} \models \chi(\overline{a}, \overline{e})$ iff $\overline{a}$ is an E_A-class.

P3. $(A/E_a, <_A)$ is a linear ordering with no greatest element.

P4. There is an existential formula $\psi(\overline{x}_0, \overline{x}_1, \overline{e})$ with $l(\overline{x}_0) = l(\overline{x}_1) = m$ such that $\psi(\overline{a}_0, \overline{a}_1, \overline{e})$ is true iff $\overline{a}_0, \overline{a}_1$ enumerate E_A-classes A_0, A_1 with $A_0 <_A A_1$.

Since T is complete every model has property Π. We handle Case 1 by obtaining a contradiction. Let σ be the sentence

$$\exists\overline{x}_0[\chi(\overline{x}_0, \overline{e})\&\forall\overline{x}_1(\neg\chi(\overline{x}_1, \overline{e}) \vee \overline{x}_0 \cap \overline{x}_1 \ne \varnothing \vee \neg\psi(\overline{x}_0, \overline{x}_1, \overline{e}))].$$

Since $(A/E_A, <_A)$ has no greatest element. $\mathscr{M} \models \neg\sigma$. By compactness there exists $\mathscr{M}' \succeq \mathscr{M}$ with an E_A-class $A' \subseteq M'\backslash M$ such that $B <_A A'$ for every E_A-class $B \subseteq M$. Let $\mathscr{N} = \mathscr{M} \cup A'$. Then $\mathscr{M} \subseteq_\forall \mathscr{N}$ and $\mathscr{N} \models \sigma$. Since $\mathscr{M} \subseteq_\forall \mathscr{N}$, $\mathscr{N} \models T$. Therefore T is not complete. This resolves Case 1.

Case 2. There exists $\overline{a} \subseteq \overline{b}_0$ such that $\mathscr{M} \models \phi(\overline{a}, \overline{b}_\omega) \& \neg\phi(\overline{a}, \overline{b}_{-1})$. Let $\mathscr{M}'$ be the model with

$$M' = M \setminus \bigcup\{\overline{a}_i : 0 \leq i < \omega\}.$$

Let $B = \bigcup\{\overline{b}_i : 0 \leq i < \omega\}$, E_B be the equivalence relation whose classes are $\overline{b}_i$ $(0 \leq i < \omega)$, and $<_B$ be the natural linear ordering of the E_B-classes.

From Claim 2 there exists a formula $\chi(\overline{y})$ over E such that $\mathscr{M} \models \chi(\overline{b})$ iff $\overline{b}$ is an E_B-class. Let $\overline{a}'_{-1}$, $\overline{a}'_\omega$ be the subtuples of $\overline{b}_1$, $\overline{b}_\omega$ corresponding to the subtuple $\overline{a}$ of $\overline{b}_0$. Let $\psi(\overline{y}_0, \overline{y}_1)$ be the formula

$$\chi(\overline{y}_0) \& \chi(\overline{y}_1) \& \exists \overline{x} \Big[\mathrm{tp}(\overline{x} \mid E) = \mathrm{tp}(\overline{a} \mid E) \& \overline{x} \subseteq \overline{y}_0 \setminus \overline{y}_1$$

$$\& \bigvee_\pi \{\phi(\overline{x}, \pi(\overline{y}_1)) \& \phi(\overline{a}'_{-1}, \pi(\overline{y}_1)) \& \neg\phi(\overline{a}'_\omega, \pi(\overline{y}_1))\}\Big]$$

where π runs through all permutations of an $l(\overline{y})$-tuple. The formula $\psi(\overline{y}_0, \overline{y}_1)$ linearly orders the E_B-classes in $\mathscr{M}'$ in the same way that $\psi(\overline{x}_0, \overline{x}_1)$ linearly ordered the E_A-classes in Case 1. Hence $\mathscr{M}'$ has property Π and we obtain the same contradiction as before. This completes the proof of the theorem.

In [8] we shall strengthen the above theorem by showing that existential formulas are also stable.

3. ∃∀-SATURATED MODELS

Let T be a $\forall\exists\forall$-theory over the relational language L. Let $\mathscr{M} \models T$ and $A \subseteq M$. A *∃∀-n-type over A* is a set Γ of ∃∀-formulas over A in which at most $x_1, \ldots, x_n$ occur free such that in some model $\mathscr{N}$ of T, with $\mathscr{M} \subseteq_\forall \mathscr{N}$, there is an n-tuple $\overline{b}$ with $\Gamma \subseteq \mathrm{Tp}_{\mathscr{N}}(\overline{b} \mid A)$. Recall from the introduction that "Tp" refers to the full elementary type. Such b is said to *realize* Γ. In this definition we require $\mathscr{M} \subseteq_\forall \mathscr{N}$ because the union of a $\subseteq_\forall$-chain of models of T is again a model of T, while the same is not necessarily true of a $\subseteq$-chain. *∃∀-n-type* without qualification means ∃∀-n-type over $\varnothing$. A *∃∀-n-type of T* is a ∃∀-n-type over $\varnothing$ realized in some model of T. A ∃∀-n-type is called *maximal* if it is maximal with respect to inclusion in the collection of all ∃∀-n-types of T.

If $\overline{b} \in M$, then $\exists\forall\text{-}\mathrm{tp}_{\mathscr{M}}(\overline{b}|A)$ denotes the intersection of $\mathrm{Tp}_{\mathscr{M}}(\overline{b}|A)$ with the set of ∃∀-formulas over A.

$\mathscr{M}$ is *∃∀-closed* if every finite ∃∀-type over M is realized in $\mathscr{M}$.

$\mathscr{M}$ is *∃∀-saturated* if every ∃∀-type over a finite subset of M is realized in $\mathscr{M}$.

3.1. **Lemma.** *Let T be a countable ∀∃∀-theory and $\mathscr{M} \models T$. There exists $\mathscr{N} \models T$ such that $\mathscr{M} \subseteq_\forall \mathscr{N}$, $\mathscr{N}$ is ∃∀-saturated, and $|N| \leq 2^{\aleph_0}|M|$.*

Proof. Let $\overline{a} \in M$ and p be a ∃∀-n-type over $\overline{a}$. By definition of ∃∀-type there exists $\mathscr{M}' \models T$ such that $\mathscr{M} \subseteq_\forall \mathscr{M}'$ and p is realized in $\mathscr{M}'$. Moreover, for any $\overline{b} \in M$ and $\mathscr{M}' \supseteq_\forall \mathscr{M}$, $\exists\forall\text{-}\mathrm{tp}_{\mathscr{M}'}(\overline{b}|A) \supseteq \exists\forall\text{-}\mathrm{tp}_{\mathscr{M}}(\overline{b}|A)$. The number of $\exists\forall$-types over a particular finite set is at most $2^{\aleph_0}$ because L is countable.

From these observations it is clear that the desired model $\mathscr{N}$ can be obtained as the union of a continuous chain of models $\langle \mathscr{M}_\alpha : \alpha < \gamma \rangle$, where $\mathscr{M}_0 = \mathscr{M}$, γ is a limit ordinal, and $\mathscr{M}_\alpha \subseteq_\forall \mathscr{M}_{\alpha+1}$ for all $\alpha < \gamma$.

For $\mathscr{N} \vDash T$ and $\overline{b}$ an m-tuple in N the notion of $\exists\forall$-n-type over $\overline{b}$ was defined with respect to $\mathscr{N}$. In fact, this notion depends only on T and $\exists\forall\text{-}\mathrm{tp}_{\mathscr{N}}(\overline{b})$. To see this let Γ be a set of $\exists\forall$-formulas over $\overline{b}$ in which at most $x_1, \ldots, x_n$ occur free and Ψ be the $\forall$-diagram of $\mathscr{N}$. Then Γ is a $\exists\forall$-n-type over $\overline{b}$ iff $\Gamma \cup \Psi \cup T$ is satisfiable. By compactness $\Gamma \cup \Psi \cup T$ is satisfiable if and only if $\Gamma \cup \Psi' \cup T$ is satisfiable, where $\Psi' = \exists\forall\text{-}\mathrm{tp}_{\mathscr{N}}(\varnothing \mid \overline{b})$. Clearly, $\exists\forall\text{-}\mathrm{tp}_{\mathscr{N}}(\varnothing \mid \overline{b})$ is determined by $\exists\forall\text{-}\mathrm{tp}_{\mathscr{N}}(\overline{b})$.

Another crucial point is the following. There is a natural one-one correspondence $p \mapsto p'$ between $\exists\forall$-n-types over $\overline{b}$ and $\exists\forall$-$(m+n)$-types (over $\varnothing$) which include $\exists\forall\text{-}\mathrm{tp}_{\mathscr{N}}(\overline{b})$, such that for any n-tuple $\overline{c} \subseteq N$

$$\exists\forall\text{-}\mathrm{tp}_{\mathscr{N}}(\overline{c} \mid \overline{b}) \supseteq p \Leftrightarrow \exists\forall\text{-}\mathrm{tp}_{\mathscr{N}}(\overline{b}\overline{c}) \supseteq p'.$$

3.2. **Lemma.** *Let T be a $\forall\exists\forall$-theory and $\mathscr{M}$, $\mathscr{N} \vDash T$ be $\exists\forall$-saturated.*

(1) *For all $\overline{b} \subseteq M$, $\exists\forall\text{-}\mathrm{tp}_{\mathscr{M}}(\overline{b})$ is a maximal $\exists\forall$-type of T.*

(2) *If $\overline{b} \subseteq M$, $\overline{c} \subseteq N$, and $\exists\forall\text{-}\mathrm{tp}_{\mathscr{M}}(\overline{b}) = \exists\forall\text{-}\mathrm{tp}_{\mathscr{N}}(\overline{c})$, then $\mathrm{Tp}_{\mathscr{M}}(\overline{b}) = \mathrm{Tp}_{\mathscr{N}}(\overline{c})$.*

(3) *The set of elementary types realized in $\mathscr{M}$ is determined by $\exists\forall\text{-}\mathrm{tp}_{\mathscr{M}}(\varnothing)$.*

(4) *$\mathscr{M}$ is $\aleph_0$-homogeneous.*

Note. The hypothesis that T is a $\exists\forall\exists$-theory sets the context and is not used directly.

Proof. (1) Consider an m-tuple $\overline{b}$ in the $\exists\forall$-saturated model $\mathscr{M}$. Let p be a $\exists\forall$-m-type of T such that $p \supseteq \exists\forall\text{-}\mathrm{tp}_{\mathscr{M}}(\overline{b})$. Let $p(\overline{b})$ denote $\{\phi(\overline{b}) : \phi(\overline{x}) \in p\}$. Since p is a $\exists\forall$-m-type of T, $p(\overline{b}) \cup T$ is consistent. Since $p \supseteq_\forall \exists\forall\text{-}\mathrm{tp}_{\mathscr{M}}(\overline{b})$, $p(\overline{b}) \cup T \cup \forall\text{-}\mathrm{Diag}(\mathscr{M})$ is consistent. Hence $p(\overline{b})$ is a $\exists\forall$-0-type over $\overline{b}$ relative to $\mathscr{M}$. Since $\mathscr{M}$ is $\exists\forall$-saturated the 0-type $p(\overline{b})$ is realized in $\mathscr{M}$, i.e. $p \subseteq \exists\forall\text{-}\mathrm{tp}_{\mathscr{M}}(\overline{b})$, which means that $\exists\forall\text{-}\mathrm{tp}_{\mathscr{M}}(\overline{b})$ is maximal.

Since $\mathscr{M}$ is $\exists\forall$-saturated, the set of $\exists\forall$-types over $\varnothing$ realized in $\mathscr{M}$ is determined by $\exists\forall\text{-}\mathrm{tp}_{\mathscr{M}}(\varnothing)$. Further, if p' is any $\exists\forall$-$(m+n)$-type of T extending $\exists\forall\text{-}\mathrm{tp}_{\mathscr{M}}(\overline{b})$, then there exists $\overline{c} \subseteq M$ such that $\overline{b}\,\overline{c}$ realizes p' because the corresponding $\exists\forall$-n-type p over $\overline{b}$ is realized in $\mathscr{M}$. Next by induction on the length of the formula $\phi(\overline{x})$ we can show that for all formulas $\phi(\overline{x})$ and all $\overline{c} \subseteq M$ with $l(\overline{c}) = l(\overline{x})$, the truth-value of $\phi(\overline{c})$ is determined by $\exists\forall\text{-}\mathrm{tp}_{\mathscr{M}}(\overline{c})$ independently of $\mathscr{M}$. This establishes (2), and once we know that the type of a tuple in $\mathscr{M}$ is determined by its $\exists\forall$-type, (3) and (4) follow from the remarks above.

3.3. **Lemma.** *Let $\mathscr{M}$ be a $\exists\forall$-closed model of a $\exists\forall$-theory T. Let $\phi(\overline{x}, \overline{y}, \overline{z})$ be a quantifier-free formula and $\overline{a} \subseteq M$ such that $\mathscr{M} \vDash \forall\overline{y}\exists\overline{z}\phi(\overline{a}, \overline{y}, \overline{z})$. There* *exists a finite subset* $C = C_{\overline{a}}$ *of* M *such that*

$$\mathscr{M} \models \forall\overline{y}(\exists\overline{z} \subseteq C \cup \overline{y})\phi(\overline{a}, \overline{y}, \overline{z}).$$

Proof. If the conclusion fails, by compactness there exist $\mathscr{N} \succeq \mathscr{M}$ and $\overline{b} \subseteq N$ such that $\mathscr{N} \models \neg\phi(\overline{a}, \overline{b}, \overline{c})$ for every $\overline{c} \subseteq M \cup \overline{b}$. Let $\mathscr{M}'$ be the substructure of $\mathscr{N}$ such that $M' = M \cup \overline{b}$. Since T is $\exists\forall$, $\mathscr{M}' \models T$. Also, $\mathscr{M} \subseteq_\forall \mathscr{M}'$ since $\mathscr{M} \subseteq_\forall \mathscr{N}$. Clearly, $\mathscr{M}' \models \exists\overline{y}\forall\overline{z}\neg\phi(\overline{a}, \overline{y}, \overline{z})$. This contradicts the assumption that $\mathscr{M}$ is $\exists\forall$-closed.

It is relatively easy to analyze a complete $\exists\forall$-theory in which every formula is equivalent to a $\exists\forall$-formula. There is a familiar property which guarantees this.

3.4. **Lemma.** *Let* T *be a complete* $\exists\forall$*-theory over a relational language in which no quantifier-free formula has the FCP. Every formula of* T *is equivalent to a* $\exists\forall$*-formula.*

Proof. Consider an arbitrary $\forall\exists$-formula $\forall\overline{y}\exists\overline{z}\phi(\overline{x}, \overline{y}, \overline{z})$, where $\phi(\overline{x}, \overline{y}, \overline{z})$ is quantifier-free. Let $\mathscr{M} \models T$ be $\exists\forall$-closed, $\overline{a} \subseteq M$, and $\mathscr{M} \models \forall\overline{y}\exists\overline{z}\phi(\overline{a}, \overline{y}, \overline{z})$. From Lemma 3.3 there is a finite $C \subseteq M$ such that

$$\mathscr{M} \models \forall\overline{y}(\exists\overline{z} \subseteq C \cup \overline{y})\phi(\overline{a}, \overline{y}, \overline{c}).$$

Since $\phi(\overline{x}, \overline{y}, \overline{z})$ (see as $\psi(\overline{y}; \overline{xz})$) does not have the FCP, there exists $n < \omega$ depending only on ϕ such that for all $\overline{a}$ satisfying $\mathscr{M} \models \forall\overline{y}\exists\overline{z}\phi(\overline{a}, \overline{y}, \overline{z})$. C can be found with $|C| \leq n$. So in $\mathscr{M}$ the formula $\forall\overline{y}\exists\overline{z}\phi(\overline{x}, \overline{y}, \overline{z})$ is equivalent to

$$\exists z_1 \cdots \exists z_n \forall\overline{y} \bigvee\{\phi(\overline{x}, \overline{y}, \overline{z}^i) : i \in I\},$$

where $\overline{z}^i$ runs through all $l(\overline{z})$-tuples from $\{z_1, \ldots, z_n\} \cup \overline{y}$. Since T is complete the equivalence of the two formulas is a theorem of T.

3.5. **Corollary.** *With the hypothesis of the lemma, if* $\mathscr{M} \models T$ *and* $\mathscr{M} \subseteq_\forall \mathscr{M}'$, *then* $\mathscr{M} \preceq \mathscr{M}'$.

Note that the conclusions of the last lemma apply to $\aleph_0$-categorical theories.

3.6. **Lemma.** *Let* T *be an* $\aleph_0$*-categorical* $\exists\forall$*-theory over a relational language. No quantifier-free formula of* T *has the FCP.*

Proof. In [11, II, Theorem 4.4] it is shown that when T is stable the FCP is equivalent to a number of other properties some of which are obviously incompatible with $\aleph_0$-categoricity. Looking at the part of the proof which shows $(2)_m$ implies $(5)_m$ in the notation of [11, p. 62], one easily sees that in an $\aleph_0$-categorical theory, whose quantifier-free formulas are stable, no quantifier-free formula has the FCP.

If one is interested only in $\aleph_0$-categorical theories, one can avoid all mention of the FCP. Where the lack of the FCP is used in the proof of Lemma 3.4, $\aleph_0$-categoricity will do just as well.

Using the stability of quantifier-free formulas we can refine the conclusion of Lemma 3.3 as follows.

3.7. **Lemma.** *Let $\mathscr{M}$ be a $\exists\forall$-saturated model of a complete $\exists\forall$-theory T, $\phi(\overline{x}, \overline{y}, \overline{z})$ be a quanifier-free formula and $\overline{a} \subseteq M$ s.t. $\mathscr{M} \models \forall\overline{y}\exists\overline{z}\phi(\overline{a}, \overline{y}, \overline{z})$. There exists a quantifier-free formula $\psi(\overline{x}, \overline{w})$ s.t. the formulas $\exists\overline{w}\psi(\overline{a}, \overline{w})$ and*

$$\psi(\overline{a}, \overline{w}) \to \forall\overline{y}(\exists\overline{z} \subseteq \cdot w\overline{y})\phi(\overline{a}, \overline{y}, \overline{z})$$

are both valid in $\mathscr{M}$.

Proof. Using Lemma 0.1 we can suppose that the language is finite. The rank we use below is that based on quantifier-free formulas discussed in [7, §3] and also mentioned in the Introduction.

We study finite sequences $\mathbf{c} = \langle\overline{c}_1, \dots, \overline{c}_k\rangle$ such that $l(\overline{c}_i) = l(\overline{z})$. The set of all such sequences is denoted $\mathbf{C}$. To establish notation let

$$C_i = \{\overline{b} \in M : \mathscr{M} \models (\exists\overline{z} \subseteq \overline{b}\overline{c}_i)\phi(\overline{a}, \overline{b}, \overline{z})\},$$
$$r_i(\mathbf{c}) = \mathrm{Rk}(C_i \backslash (C_1 \cup \cdots \cup C_{i-1})),$$
$$\nu(\mathbf{c}) = (r_1(\mathbf{c}), \dots, r_k(\mathbf{c})).$$

Note that C_i is definable by a quantifier-free formula over $\overline{a}_i\overline{c}_i$. The sequence $\mathbf{c}$ is called *good* if $r_i(\mathbf{c}) > -1$ for all i, $1 \le i \le k$. The codomain of ν restricted to good sequences, i.e. the set of nonempty finite sequences of natural numbers, is ordered lexicographically.

Claim. Amongst the good sequences $\mathbf{c}$ is one for which $\nu(\mathbf{c})$ is maximum.

Since the rank of the universe is finite the claim can fail only if there are arbitrarly long good sequences. Suppose the claim fails. Since the rank function is monotonic with respect to inclusion, for any good $\mathbf{c} \in \mathbf{C}$ by deleting some components and reordering the rest we can obtain a good $\mathbf{d} \in \mathbf{C}$ such that $\nu(\mathbf{d})$ is monotonic decreasing and $\nu(\mathbf{c}) \le \nu(\mathbf{d})$. From the failure of the claim there exist natural numbers $j, r, r_1, r_2, \dots, r_j$ such that $r < r_j \le r_{j-1} \le \cdots \le r_2 \le r_1$, every tuple $(r_1, r_2, \dots, r_j, r, \dots, r)$ is in $\mathrm{rng}(\nu)$, and for every good $\mathbf{c}$

$$\nu(\mathbf{c}) \le (r_1, r_2, \dots, r_j, r, \dots, r)$$

provided that the length of the sequence on the right is at least $l(\mathbf{c})$. Let $r_i = r$ for all $i > j$. Let $\overline{e}_1, \overline{e}_2, \overline{e}_3, \dots$ be disjoint sequences of constants with $l(\overline{e}_i) = l(\overline{z})$ and let E_i denote the set of $l(\overline{y})$-tuples which stands in the same relation to $\overline{e}_i$ as C_i to $\overline{c}_i$. Let $\widehat{E}_i$ denote $E_i \backslash (E_1 \cup \cdots \cup E_{i-1})$. From Lemma 0.2 there are quantifier-free formulas $\psi_{i,m}(\overline{x}, \overline{u}_1, \dots, \overline{u}_i, \overline{v}_{i,m})$ $(1 \le i < \omega$, $m < \omega)$ such that for any L-structure $\mathscr{N}$ interpreting $\overline{a}$ and $\overline{e}_1, \dots, \overline{e}_i$

$$\mathrm{Rk}(\widehat{E}_i) \ge r_i \Leftrightarrow \forall m[\mathscr{N} \models \exists\overline{v}_{i,m}\psi_{i,m}(\overline{a}, \overline{e}_1, \dots, \overline{e}_i, \overline{v}_{i,m})].$$

Let

$$\Gamma = \{\exists\overline{v}_{i,m}\psi_{i,m}(\overline{a}, \overline{e}_1, \dots, \overline{e}_i, \overline{v}_{i,m}) : 1 \le i < \omega, m < \omega\}.$$

Any finite subset of Γ can be satisfied in $\mathscr{M}$. Thus there exists $\mathscr{N} \succeq \mathscr{M}$ in which Γ is satisfiable, and $\mathscr{N}$ has a $\forall$-extension $\mathscr{M}'$ which is $\exists\forall$-saturated.

Clearly, Γ is satisfiable in $\mathscr{M}'$. From Lemma 3.2, $\mathscr{M}$ and $\mathscr{M}'$ are L_{∞_ω}-equivalent and so Γ is actually satisfiable in $\mathscr{M}$. Choose particular interpretations of $\overline{e}_i$ $(1 \le i < \omega)$ in $\mathscr{M}$ which satisfy Γ. Let $\mathbf{c}'$ be the interpretation of $\langle \overline{e}_1, \dots, \overline{e}_j \rangle$ and $\langle C'_1, \dots, C'_j \rangle$ be the corresponding sequence of subsets of $M^{l(\overline{z})}$, the satisfaction of Γ yields $r_i(\mathbf{c}') \ge r_i$ $(1 \le i \le j)$ and $\mathrm{Rk}(\neg C'_1 \wedge \cdots \wedge \neg C'_j) > r$. From Lemma 3.3 $M^{l(\overline{y})}$ is covered by a finite number of sets of the form

$$C' = \{\overline{b} \in M : \mathscr{M} \models (\exists \overline{z} \subseteq \overline{b}\,\overline{c})\phi(\overline{a}, \overline{b}, \overline{z})\} \qquad (\overline{c} \subseteq M, \, l(\overline{c}) = l(\overline{z})).$$

From the properties of rank we can fix $\overline{c}$ such that

$$\mathrm{Rk}(C' \wedge \neg C'_1 \wedge \cdots \wedge \neg C'_j) > r.$$

Now $\mathbf{c}'^{\wedge}\langle \overline{c} \rangle$ is a good sequence contradicting the choice of $j, r, r_1, \dots, r_j$ made above. This completes the proof of the claim.

Fix a good sequence $\mathbf{d} = \langle \overline{d}_1, \dots, \overline{d}_k \rangle$ such that $\nu(\mathbf{d})$ is maximum. Henceforth let r_i denote $r_i(\mathbf{d})$. Let $\overline{u}$ be a sequence of variables with $l(\overline{u}) = k \cdot l(\overline{z})$, and $\{\psi_i(\overline{x}, \overline{u}, \overline{v}_i) : i < \omega\}$ be a set of quantifier-free formulas such that, for $\mathbf{c} \in \mathbf{C}$ with $l(\mathbf{c}) = k$

$$\mathscr{M} \models \bigwedge \{\exists \overline{v}_i \psi_i(\overline{a}, \mathbf{c}, \overline{v}_i) : i < \omega\}$$

iff $r_i(\mathbf{c}) \ge r_i$ $(1 \le i \le k)$. Since C_i is definable without quantifiers, there is a quantifier-free formula $\chi(\overline{x}, \overline{y}, \overline{u})$ such that $\mathscr{M} \models \forall \overline{y} \chi(\overline{a}, \overline{y}, \mathbf{c})$ iff $\bigcup \{C_i : 1 \le i \le k\} = M^{l(\overline{y})}$. Since $\nu(\mathbf{d})$ is maximum, for $\mathbf{c} \in \mathbf{C}$ with $l(\mathbf{c}) = k$, the conditions $r_i(\mathbf{c}) \ge r_i$ $(1 \le i \le k)$ imply $\bigcup \{C_i : 1 \le i \le k\} = M^{l(\overline{y})}$. Thus

$$\bigwedge \{\exists \overline{v}_i \psi_i(\overline{a}, \overline{u}, \overline{v}_i) : i < \omega\} \rightarrow \forall \overline{y} \chi(\overline{a}, \overline{y}, \overline{u})$$

is valid in $\mathscr{M}$. Since $\mathscr{M}$ is $\exists\forall$-saturated, by compactness there exists $j < \omega$ such that

$$\bigwedge \{\exists \overline{v}_i \psi_i(\overline{a}, \overline{u}, \overline{v}_i) : i < j\} \rightarrow \forall \overline{y} \chi(\overline{a}, \overline{y}, \overline{u})$$

is valid in $\mathscr{M}$. Without loss of generality the sequences $\overline{v}_i$ of variables are disjoint, and so the formula $\bigwedge \{\psi_i(\overline{x}, \overline{u}, \overline{v}_i) : i < j\}$ satisfies the conclusion of the lemma.

3.8. **Lemma.** *Let T be a complete $\exists\forall$-theory over a finite relational language. Let $\mathscr{M}$ be a $\exists\forall$-saturated model of T and $\phi(\overline{a}, \overline{x}, \overline{y}, \overline{z})$, $\theta(\overline{a}, \overline{x}, \overline{y}, \overline{z})$ be quantifier-free formulas over M such that $\exists \overline{y} \forall \overline{z} \phi(\overline{a}, \overline{x}, \overline{y}, \overline{z})$ and $\exists \overline{y} \forall \overline{z}$ $\theta(\overline{a}, \overline{x}, \overline{y}, \overline{z})$ are disjoint in $\mathscr{M}$. Then there is a Boolean combination of existential formulas over $\overline{a}$ which separates $\exists \overline{y} \forall \overline{z} \phi(\overline{a}, \overline{x}, \overline{y}, \overline{z})$ from $\exists \overline{y} \forall \overline{z} \theta(\overline{a}, \overline{x}, \overline{y}, \overline{z})$.*

Proof. Let $\phi(\overline{a}, \overline{x}, \overline{y}, \overline{z})$ and $\theta(\overline{a}, \overline{x}, \overline{y}, \overline{z})$ be quantifier-free formulas satisfying the hypothesis. We will show that there is a Boolean combination of existential formulas over $\overline{a}$ separating the two $\exists\forall$-formulas.

Let $\psi_i(\overline{x}, \overline{w}_i)$ $(i < \omega)$ be an enumeration of all quantifier-free formulas over $\overline{a}$. For brevity $\overline{a}$ has been omitted in writing ψ_i and will be omitted from all formulas below. Let Γ denote the set of all formulas:

$$\neg\exists\overline{w}_i\psi_i(\overline{x}, \overline{w}_i). \vee .\exists\overline{w}_i[\psi_i(\overline{x}, \overline{w}_i)\&\exists\overline{y}\neg(\exists\overline{z} \subseteq \overline{w}_i\overline{y})\neg\phi(\overline{x}, \overline{y}, \overline{z})$$
$$\&\exists\overline{w}_i[\psi(\overline{n}, \overline{w}_i)\&\exists\overline{y}\neg(\exists\overline{z} \subseteq \overline{w}_i\overline{y})\neg\theta(\overline{x}, \overline{y}, \overline{z})] \quad (i < \omega).$$

Let $\Gamma(\overline{a}')$ denote the set of formulas obtained by substituting $\overline{a}'$ for $\overline{x}$ in Γ. Then $\bigwedge\Gamma(\overline{a}')$ says that the conclusion of Lemma 3.7 fails both for the input formula $\neg\phi(\overline{a}', \overline{y}, \overline{z})$ and the input formula $\neg\theta(\overline{a}', \overline{y}, \overline{z})$. Since $\exists\overline{y}\forall\overline{z}\phi(\overline{x}, \overline{y}, \overline{z})$ and $\exists\overline{y}\forall\overline{z}\theta(\overline{x}, \overline{y}, \overline{z})$ are disjoint, the formula

$$\forall\overline{y}\exists\overline{z}\neg\phi(\overline{x}, \overline{y}, \overline{z}). \vee .\forall\overline{y}\exists\overline{z}\neg\theta(\overline{x}, \overline{y}, \overline{z})$$

is valid in $\mathscr{M}$. From Lemma 3.7, Γ is not satisfiable in $\mathscr{M}$. Let $\chi_i(\overline{x})$ denote the formula

$$\forall\overline{w}_i[\psi_i(\overline{x}, \overline{w}_i) \to \forall\overline{y}(\exists\overline{z} \in \mathrm{Rng}(\overline{y}\,\overline{w}_i))\neg\phi(\overline{x}, \overline{y}, \overline{z})]$$

and $\pi_i(\overline{x})$ denote the corresponding formula with θ instead of ϕ. Since $\mathscr{M}$ is $\exists\forall$-saturated, by compactness there exists $j < \omega$ such that the set consisting of the first j formulas of Γ is not satisfiable in $\mathscr{M}$. Hence

$$\bigvee_{i<j}[\exists\overline{w}_i\psi_i(\overline{x}, \overline{w}_i).\&.\chi_i(\overline{x}) \vee \pi_i(\overline{x})]$$

is valid in $\mathscr{M}$. Notice that

$$[\exists\overline{w}_i\psi_i(\overline{x}, \overline{w}_i).\&.\chi_i(\overline{x})] \to \neg\exists\overline{y}\forall\overline{z}\phi(\overline{x}, \overline{y}, \overline{z})$$

and

$$[\exists\overline{w}_i\psi_i(\overline{x}, \overline{w}_i).\&.\pi_i(\overline{x})] \to \neg\exists\overline{y}\forall\overline{z}\theta(\overline{x}, \overline{y}, \overline{z})$$

are both valid in $\mathscr{M}$. Thus the formula

$$\bigvee_{i<j}[\exists\overline{w}_i\psi_i(\overline{x}, \overline{w}_i).\&.\pi_i(\overline{x})]$$

separates $\exists\overline{y}\forall\overline{z}\phi(\overline{x}, \overline{y}, \overline{z})$ from $\exists\overline{y}\forall\overline{z}\theta(\overline{x}, \overline{y}, \overline{z})$ in $\mathscr{M}$. Since π_i is clearly equivalent to a universal formula the proof of the lemma is complete.

If $A \cup \overline{b} \subseteq M$, then $\exists\text{-}\mathrm{tp}_{\mathscr{M}}(\overline{b}|A)$ denotes the intersection of $\mathrm{Tp}_{\mathscr{M}}(\overline{b}|A)$ with the set of $\exists$-formulas over A.

3.9. **Corollary.** *Let T be a complete $\exists\forall$-theory over a relational language. Let $\mathscr{M}$ be a $\exists\forall$-saturated model of T and $A \subseteq M$. For all $\overline{b} \subseteq M$, $\mathrm{Tp}_{\mathscr{M}}(\overline{b} \mid A)$ is determined by $\exists\text{-}\mathrm{tp}_{\mathscr{M}}(\overline{b}|A)$.*

Proof. Drop the subscript $\mathscr{M}$, and suppose that $\mathrm{Tp}(\overline{b}_0|A) \neq \mathrm{Tp}(\overline{b}_1|A)$, where $l(\overline{b}_0) = l(\overline{b}_1)$. For some $\overline{a} \subseteq A$, $\mathrm{Tp}(\overline{a}\overline{b}_0) \neq \mathrm{Tp}(\overline{a}\overline{b}_1)$. From Lemma 3.2(2), $\exists\forall\text{-}\mathrm{tp}(\overline{a}\overline{b}_0) \neq \exists\forall\text{-}\mathrm{tp}(\overline{a}\,\overline{b}_1)$. By Lemma 3.2(1) there are $\exists\forall$-formulas $\psi_0(\overline{x}, \overline{y})$, $\psi_1(\overline{x}, \overline{y})$ disjoint in T such that $\mathscr{M} \models \psi_i(\overline{a}, \overline{b}_i)$ $(i < 2)$. From Lemma 3.8 there is an existential formula $\chi(\overline{x}, \overline{y})$ such that for either $i = 0$ or $i = 1$

we have $\mathcal{M} \models \chi(\overline{a}, \overline{b}_i) \& \neg\chi(\overline{a}, \overline{b}_{1-i})$. Hence $\exists$-$\mathrm{tp}(\overline{b}_0|A) \neq \exists$-$\mathrm{tp}(\overline{b}_1|A)$, which completes the proof.

The following theorem is an immediate consequence of Lemmas 3.4 and 3.8.

3.10. **Theorem.** *Let T be a complete $\exists\forall$-theory over a finite relational language in which no quantifier-free formula has the FCP. Every formula of T is equivalent to a Boolean combination of existential formulas.*

Below we shall often appeal to the following consequence of Theorem 3.10. If $\mathcal{M}$ is a model of such a theory and $\mathcal{M} \subseteq_\forall \mathcal{N}$, then $\mathcal{M} \subseteq \mathcal{M}' \subseteq \mathcal{N}$ implies that $\mathcal{M}$ is an elementary submodel of $\mathcal{M}'$.

4. Algebraic closure

In the known examples of complete $\exists\forall$-theories the algebraic closure relation plays a key role. Some information may be gleaned from the argument [7, Lemma 4.3] which shows that the Δ-rank of such a theory is finite: unless the theory is finitely partitioned, for any model $\mathcal{M}$ and finite $A \subseteq M$ there is a quantifier-free formula $\phi(x, \overline{y})$ and $\overline{b} \in M$ such that $\phi(x, \overline{b})$ has only finitely many solutions and at least one $\notin A \cup \{\overline{b}\}$. Thus, except in a trivial case, the quantifier-free formulas give nontrivial algebraic structure involving an infinite number of elements of the model.

In general the formulas giving the algebraic closure of a subset of a model are of arbitrary quantifier complexity as may be seen from Example 2 of §1. We shall now show that in a $\exists\forall$-saturated model algebraic elements are captured by existential formulas.

4.1. **Lemma.** *Let T be a complete $\exists\forall$-theory over a relational language. Let $\mathcal{M}$ be a $\exists\forall$-saturated model of T and $b \in \mathrm{acl}(\overline{a})$ in $\mathcal{M}$. Then there is an existential formula $\xi(x)$ over $\overline{a}$ such that $\mathcal{M} \models \xi(b)$ and $\xi(x)$ has only a finite number of solutions in $\mathcal{M}$.*

Proof. Let $\overline{a}$, b, and $\mathcal{M}$ satisfy the hypothesis. Let B be the set of solutions of $\mathrm{tp}(b \mid \overline{a})$ in M and $n = |B|$.

From Lemma 3.9 $\mathrm{Tp}(b|\overline{a})$ is determined by $\exists$-$\mathrm{tp}\,(b|\overline{a})$. Hence there is a set Ψ of universal and existential formulas over $\overline{a}$, in which at most x occurs free, such that B is the solution set in $\mathcal{M}$ of $\bigwedge\Psi$. Since $\mathcal{M}$ is $\exists\forall$-saturated there exists a finite subset Φ of Ψ such that B is the solution set of $\bigwedge\Phi$. It follows easily that there are quantifier-free formulas $\phi(x, \overline{y})$ and $\psi(x, \overline{z})$ over $\overline{a}$ such that B is the solution set of $\exists\overline{y}\phi(x, \overline{y}).\&.\forall\overline{z}\psi(x, \overline{z})$. We study finite sequences $\mathbf{c} = \langle a, \overline{b}, \overline{c}_1, \dots, \overline{c}_k\rangle$ such that $\mathcal{M} \models \phi(a, \overline{b})$ and $l(\overline{c}_i) = l(\overline{z})$ $(1 \leq i \leq k)$. The set of all such sequences is denoted $\mathbf{C}$. To establish notation let

$$C_i = \{e\overline{f} \in M : \mathcal{M} \models \phi(e, \overline{f}) \& (\forall\overline{z} \subseteq a\overline{b}\,\overline{c}_i)\psi(a, \overline{z}) \& (\exists\overline{z} \subseteq e\overline{f}\,\overline{c}_i)\neg\psi(e, \overline{z})\},$$
$$r_i(\mathbf{c}) = \mathrm{Rk}(C_i \backslash (C_1 \cup \cdots \cup C_{i-1})),$$
$$\nu(\mathbf{c}) = (r_1(\mathbf{c}), \dots, r_k(\mathbf{c})).$$

Let F denote the solution set of $\phi(x, \bar{y})$ in $\mathscr{M}$. Then $\{C_1, \ldots, C_k\}$ is a partial, possibly total, covering of $D = F \cap ((M \backslash B) \times M^{l(\bar{y})})$. If $e\bar{f} \in F$, then $\bar{c}$ is a *witness* for $e\bar{f}$ if $\mathscr{M} \models (\exists \bar{z} \subseteq e\bar{f}\bar{c})\neg\psi(e, \bar{z})$. If $e\bar{f} \in F$, then $e\bar{f} \in D$ if and only if there is a witness for $e\bar{f}$. Now C_i is empty if $\bar{c}_i$ is a witness for $a\bar{b}$, and is the set of all $e\bar{f}$ for which $\bar{c}_i$ is a witness otherwise. Thus $a\bar{b} \notin C_i$ $(1 \leq i \leq k)$ and $\{C_1, \ldots, C_k\}$ can be a covering of D only if $a \in B$.

The sequence $\mathbf{c}$ is called good if $r_i(\mathbf{c}) > -1$ $(1 \leq i \leq k)$. The codomain of ν restricted to good sequences, i.e. the set of nonempty finite sequences of natural numbers, is ordered lexicographically. Using the same idea as in Lemma 3.7 we can prove:

Claim. For each tuple $a\bar{b}$ such that $\mathscr{M} \models \phi(a, \bar{b})$, amongst the good sequences $\mathbf{c} = \langle a, \bar{b}, \ldots \rangle$ is one for which $\nu(\mathbf{c})$ is maximum.

Let $\nu(a, \bar{b})$ be the maximum value of $\nu(\mathbf{c})$ referred to in the claim. The following formula is valid in $\mathscr{M}$:

$$\phi(x_0, \bar{y}_0) \& \phi(x_1, \bar{y}_1). \rightarrow .(x_0, x_1 \in B \rightarrow \nu(x_0, \bar{y}_0) = \nu(x_1, \bar{y}_1)) \& \\ (x_0 \notin B \& x_1 \in B. \rightarrow \nu(x_0, \bar{y}_0) < \nu(x_1, \bar{y}_1)).$$

The reason is as follows. If $a\bar{b} \in F$ and $a \in B$, then $a\bar{b}$ imposes no restriction on a sequence $\bar{c}_1, \bar{c}_2, \ldots$ such that $\mathbf{c} = \langle a, \bar{b}, \bar{c}_1, \bar{c}_2, \ldots \rangle$ is good. Thus $\bar{c}_1, \bar{c}_2, \ldots$ can be chosen so that $C_1, C_2, \ldots$ cover D. Indeed, for $\nu(\mathbf{c})$ to be maximal $C_1, C_2, \ldots$ must cover D. However, if $a\bar{b} \in F$, $a \notin B$, and $\mathbf{c}$ is good, then no $\bar{c}_i$ is a witness for $a\bar{b}$. Thus none of the C_i's contains $a\bar{b}$, and $a\bar{b} \in D$. This prevents $\nu(\mathbf{c})$ from attaining the same maximum value which can be achieved in the other case.

Fix a good sequence $\mathbf{d} = \langle a_d, \bar{b}_d, \bar{d}_1, \ldots, \bar{d}_k \rangle \in \mathbf{C}$ such that $a_d \in B$ and $\nu(\mathbf{d})$ is maximum. Let r_i denote $r_i(\mathbf{d})$. Let $\bar{z}_i$ $(1 \leq i \leq k)$ be disjoint sequences of variables with $l(\bar{z}_i) = l(\bar{z})$, and let $\Gamma(x, \bar{y}, \bar{z}_1, \ldots, \bar{z}_k)$ be a set of existential formulas over $\bar{a}$ such that for any sequence $\mathbf{c} = (a, \bar{b}, \bar{c}_1, \ldots, \bar{c}_k)$, $\mathscr{M} \models \Gamma(a, \bar{b}, \bar{c}_1, \ldots, \bar{c}_k)$ iff $r_i(\mathbf{c}) \geq r_i$ $(1 \leq i \leq k)$. Note that the formula

$$\phi(x, \bar{y}) \& \bigwedge \Gamma(x, \bar{y}, \bar{z}_1, \ldots, \bar{z}_k). \rightarrow .x \in B$$

is valid in $\mathscr{M}$. Since $\mathscr{M}$ is $\exists\forall$-saturated, by compactness there is a finite subset Γ' of Γ such that the last formula is still valid when Γ is replaced by Γ'. On the other hand by choice of Γ

$$x \in B \rightarrow \exists \bar{y} \exists \bar{z}_1 \cdots \exists \bar{z}_k \left[\phi(x, \bar{y}) \& \bigwedge \Gamma(x, \bar{y}, \bar{z}_1, \ldots, \bar{z}_k)\right]$$

is valid in $\mathscr{M}$. Thus for the formula $\xi(x)$ we can take

$$\exists \bar{y} \exists \bar{z}_1 \cdots \exists \bar{z}_k \left[\phi(x, \bar{y}) \& \bigwedge \Gamma'(x, \bar{y}, \bar{z}_1, \ldots, \bar{z}_k)\right].$$

From the previous lemma, Lemma 3.1, and Corollary 3.5 we have

4.2. **Corollary.** *Let T be a complete $\exists\forall$-theory over a relational language in which no quantifier-free formula has the FCP. Then the conclusion of the previous lemma holds in every model $\mathcal{M}$ of T.*

As it turns out a great deal more can be said about algebraic closure in this special case:

4.3. **Lemma.** *Let T be a complete $\exists\forall$-theory over a relational language in which no quantifier-free formula has the FCP. Let $\mathcal{M} \models T$ and $a, b, c, \overline{d} \subseteq M$.*

(1) *If $a \in \operatorname{acl}(b\overline{d})$, then $b \in \operatorname{acl}(a\overline{d})$ or $a \in \operatorname{acl}(\overline{d})$.*

(2) *If $a \in \operatorname{acl}(bc\overline{d})$, then $a \in \operatorname{acl}(b\overline{d})$ or $a \in \operatorname{acl}(c\overline{d})$.*

Proof. (1) Since it does not affect the argument we assume that $\overline{d} = \varnothing$. Towards a contradiction suppose $a \in \operatorname{acl}(b)$, $b \notin \operatorname{acl}(a)$, $a \notin \operatorname{acl}(\varnothing)$. From the last corollary a is the solution of an existential formula over b having only a finite number of solutions. Thus there exist $n < \omega$ and a quantifier-free formula $\phi(x, y, \overline{z})$ such that $\mathcal{M} \models \psi(a, b)$, where $\psi(x, y)$ denotes

$$\exists\overline{z}\phi(x, y, \overline{z}).\&.\exists^{\leq n}x\exists\overline{z}\phi(x, y, \overline{z}).$$

Since $b \notin \operatorname{acl}(a)$, $\psi(a, y)$ has infinitely many solutions for y in $\mathcal{M}$. Permuting the variables $\overline{z}$ if necessary we can find the following configuration $\mathcal{C}$ in $\mathcal{M}$: $a, \overline{d}, \langle b_i\overline{e}_i : i < \omega\rangle$ such that $l(\overline{d}) + l(\overline{e}_i) = l(\overline{z})$,

$$\mathcal{M} \models \psi(a, b_i)\&\phi(a, b_i, \overline{d}\,\overline{e}_i),$$

and $b_i\overline{e}_i \cap b_j\overline{e}_j = \varnothing$ $(i < j < \omega)$.

By compactness there exists $\mathcal{M}' \succeq \mathcal{M}$ such that in $\mathcal{M}'$ there are two configurations $a', \overline{d}', \langle b_i'\overline{e}_i' : i < \omega\rangle$ and $a'', \overline{d}'', \langle b_i''\overline{e}_i'' : i < \omega\rangle$ of the same kind as $\mathcal{C}$ satisfying the following two additional conditions:

(i) $\operatorname{tp}_{\mathcal{M}'}(a'\overline{d}'|M) = \operatorname{tp}_{\mathcal{M}'}(a''\overline{d}''|M)$.

(ii) $a'' \notin \operatorname{acl}_{\mathcal{M}'}(b_i')$ $(i < \omega)$.

For $k < \omega$ let $\mathcal{M}_k$ denote the substructure of $\mathcal{M}'$ with universe

$$M \cup a'a''\overline{d}'\overline{d}'' \cup \bigcup\{b_i'\overline{e}_i' : i < k\}.$$

Let $m = l(\overline{e}_i) + 1$ and consider the following subsets of $(M_k)^m$:

$$A = \{b\overline{e} \subseteq M_k : \mathcal{M}_k \models \neg\phi(a', b, \overline{d}'\overline{e})\},$$
$$B = \{b\overline{e} \subseteq M_k : \mathcal{M}_k \models \phi(a'', b, \overline{d}''\overline{e})\},$$
$$C(j, c) = \{\overline{f} \subseteq M_k : l(\overline{f}) = m, f_j = c\} \qquad (1 \leq j \leq m, c \in M_k \backslash M).$$

From (i) every tuple in $(M_k)^m \backslash (A \cup B)$ has at least one entry in $M_k \backslash M$. Thus the sets displayed cover $(M_k)^m$. Since $\psi(a', b_i')$ is true in $\mathcal{M}'$ so is $\exists^{\leq n}x\exists\overline{z}\phi(x, b_i', \overline{z})$. From (ii) it follows that $b_i'\overline{e}_i'$ is not in B and hence not in $A \cup B$. Since the tuples $b_i'\overline{e}_i'$ are pairwise disjoint, to cover $M_k \backslash (A \cup B)$

we need at least k of the sets $C(j, c)$. This is sufficient to prove that some quantifier-free formula of T has the FCP.

(2) Again let $\overline{d} = \varnothing$ because it makes no difference to the argument. Let $a \in \operatorname{acl}(bc)$, and towards a contradiction suppose that $a \notin \operatorname{acl}(b) \cup \operatorname{acl}(c)$. From (1) $b \in \operatorname{acl}(ac)$ and $c \in \operatorname{acl}(ab)$. From Corollary 4.2 there exist $n < \omega$ and a quantifier-free formula $\phi(x, y, z, \overline{u})$ such that $\mathscr{M} \models \psi(a, b, c)$, where $\psi(x, y, z)$ is

$$\exists\overline{u}\phi(x, y, z, \overline{u})\&\exists^{\leq n}x\exists\overline{u}\phi(x, y, z, \overline{u})\&\exists^{\leq n}y\exists\overline{u}\phi(x, y, z, \overline{u})\& \exists^{\leq n}z\exists\overline{u}\phi(x, y, z, \overline{u}).$$

Since $a \notin \operatorname{acl}(b) \cup \operatorname{acl}(c)$, from (1) $b \notin \operatorname{acl}(a)$. Therefore $\exists^{\omega}y\exists z\psi(a, y, z)$ is true in $\mathscr{M}$. Permuting the variables $\overline{z}$ if necessary, we can find the following configuration in $\mathscr{M}$: $a, \overline{d}, \langle b_i c_i \overline{e}_i : i < \omega\rangle$ such that $l(\overline{d}) + l(\overline{e}_i) = l(\overline{u})$,

$$\mathscr{M} \models \psi(a, b_i, c_i)\&\phi(a, b_i, c_i, \overline{d}\,\overline{e}_i).$$

and $b_i c_i \overline{e}_i \cap b_j c_j \overline{e}_j = \varnothing \quad (i < j < \omega)$.

Now we finish the proof in the same way as for (1).

4.4. **Corollary.** *Let T be as in the previous lemma and $\mathscr{M} \models T$. If $a, a_1, \dots, a_n \in M$ and $a \in \operatorname{acl}\{a_1, \dots, a_n\}\setminus \operatorname{acl}(\varnothing)$, then for some i, $1 \leq i \leq n$, $a \in \operatorname{acl}(a_i)$ and $a_i \in \operatorname{acl}(a)$.*

5. Complete $\exists\forall$-theories in which no quantifier-free formula has the FCP

In this section theories in which no quantifier-free formula has the FCP will be called *NFCP theories*. We have already seen that theories of this kind have very nice properties: all formulas are equivalent to Boolean combinations of existential formulas, and in any model the relation $a \in \operatorname{acl}(b)$ is an equivalence relation on the set of nonalgebraic elements. Here we show that the models of such theories can be even more narrowly circumscribed *provided the language is finite.* Of course, since every reduct of a complete $\exists\forall$-theory is also $\exists\forall$, our results apply indirectly to the case of an infinite relational language.

We need some more terminology. When $a \in \operatorname{acl}(b)$ is an equivalence relation on the set of nonalgebraic elements, the equivalence classes are called *components.*

Let $\mathscr{M}$ be a structure, and fix $n < \omega$. Following [1, Definition 2.6.3] a set $A \subseteq M^n$ is called *unpseudofinite* if there are $\overline{a}_i \in A$ $(i < \omega)$ such that all the entries of $\overline{a}_i$ are distinct and $\overline{a}_i \cap \overline{a}_j = \varnothing$ $(i < j < \omega)$. The set $A \subseteq M^n$ is called *pseudofinite* if it is not unpseudofinite. Note that A is pseudofinite if and only if there exists finite B such that $\overline{a} \cap B \neq \varnothing$ for all $\overline{a} \in A$.

For every formula ψ let ψ^0 denote ψ, and ψ^1 denote $\neg\psi$. Consider $A \subseteq M^n$ and atomic formulas $\phi_i(\overline{x})$ $(i < j)$ over M, where $\overline{x} = (x_1, \dots, x_n)$. For $\sigma \in {}^j2$ let $\theta_\sigma(\overline{x})$ denote $\bigwedge\{\phi_i(\overline{x})^{\sigma(i)} : i < j\}$. Ignoring any unsatisfiable formulas which turn up, the intersections with A of the solution sets in M^n of the formulas θ_σ partition A. Such a partition is called a *partition of A by atomic formulas.* The elements of the partition are called *pieces.*

$A \subseteq M^n$ is called *broad* if for all $m < \omega$ there is a partition of A by atomic formulas in which at least m of the pieces are unpseudofinite. If $A \subseteq M^n$ is not broad, call it *narrow.* A formula $\phi(\overline{x})$ is called *broad* or *narrow* according as its solution set is broad or narrow.

5.1. **Lemma.** *Let* $\mathrm{Th}(\mathscr{M})$ *be a complete NFCP ∃∀-theory over a finite relational language. For each* n, $1 \leq n < \omega$, M^n *is narrow.*

Proof. Towards a contradiction suppose that M^n is broad. Since atomic formulas are stable there exists a broad quantifier-free formula $\phi(\overline{x})$ over M such that, for every atomic formula $\psi(\overline{x})$ over M, one of $\phi(\overline{x})\&\psi(\overline{x})$ and $\phi(\overline{x})\&\neg\psi(\overline{x})$ is narrow. Thus, we can find basic formulas $\psi_i(\overline{x})$ $(i < \omega)$ over M such that for each i, $\phi(\overline{x})\&\psi_i(\overline{x})$ is narrow but

$$\phi(\overline{x})\&\psi_i(\overline{x})\&\neg\psi_0(\overline{x})\&\cdots\&\neg\psi_{i-1}(\overline{x})$$

is unpseudofinite. Since the language is finite we can choose $\psi_i(\overline{x})$ of the form $\psi(\overline{x}, \overline{b}_i)$.

Let $\alpha \in \mathrm{On}$, $\alpha > \omega$. By compactness we can choose $\mathscr{M}' \succeq \mathscr{M}$ in which the sequence $\psi(\overline{x}, \overline{b}_i)$ $(i < \omega)$ can be extended to length α such that, for all $i < \alpha$,

$$\phi(\overline{x})\&\psi(\overline{x}, \overline{b}_i)\&\bigwedge\{\neg\psi(\overline{x}, \overline{b}_j): j < i\}$$

is unpseudofinite. Choosing α large enough we obtain $\overline{b}, \overline{b}' \in M'$ such that $\mathrm{tp}(\overline{b}|M) = \mathrm{tp}(\overline{b}'|M)$ and

$$\chi(\overline{x}, \overline{c}) =_{\mathrm{dfn}} \phi(\overline{x})\&\psi(\overline{x}, \overline{b}')\&\neg\psi(\overline{x}, \overline{b})$$

is unpseudofinite.

Choose $m < \omega$. We can find $\mathscr{N}$, with $\mathscr{M} \subseteq \mathscr{N} \subseteq \mathscr{M}'$ and $N\backslash M$ finite such that $\overline{b}, \overline{b}' \subseteq N$ and $\chi(\overline{x}, \overline{c})$ has at least m pairwise disjoint solutions for $\overline{x}$ in N^n in which no entry is repeated. Since $\chi(\overline{x}, \overline{c})$ has no solutions in M^n and $N\backslash M$ is finite $\chi(\overline{x}, \overline{c})$ is pseudofinite in $\mathscr{N}$. Notice that N^n is covered by $\neg\chi(\overline{x}, \overline{c})$ together with a finite number of formulas of the form $x_i = c$, where $1 \leq i \leq n$ and $c \in N\backslash M$. Any such covering has at least $m+1$ formulas. Since m was arbitrary and the existence of such a covering is expressible by a first-order formula, some quantifier-free formula has the FCP. This completes the proof of the lemma.

For $P \subseteq M$ let E_n^P be the binary relation on M^n consisting of all pairs $(\overline{a}, \overline{b})$ such that for all $\mathscr{N} \succeq \mathscr{M}$, $\overline{a}$ and $\overline{b}$ realize the same n-type over $(N\backslash \mathrm{acl}(\overline{a}\,\overline{b})) \cup P$ in $\mathscr{N}$. We write E_n for $E_n^{\varnothing}$.

5.2. **Lemma.** *With the same hypothesis as before, if* $P \subseteq M$ *is finite, then* E_n^P *is a finite equivalence relation on* M^n *which is* 0*-definable in* $(\mathscr{M}, P)$ *by a universal formula.*

Note. $(\mathcal{M}, P)$ denotes the expansion of $\mathcal{M}$ by a new unary relaiton symbol whose interpretation is P.

Proof. Without loss of generality suppose $\mathcal{M}$ is $\aleph_0$-saturated. This will save us the trouble of taking elementary extensions. We give the proof for the case $P = \emptyset$ since this eases notation and the adustments needed for general case are straightforward.

We first check that E_n is an equivalence relation. Suppose $\overline{a}E_n\overline{b}E_n\overline{c}$ and towards a contradiction that $\mathcal{M} \models \phi(\overline{a}, \overline{d})\&\neg\phi(\overline{c}, \overline{d})$, where $\phi(\overline{x}, \overline{y})$ is a basic formula and $\overline{d} \cap \operatorname{acl}(\overline{a}\,\overline{c}) = \emptyset$. We need:

5.2.1. **Sublemma.** *For any structure $\mathcal{N}$, formula $\psi(\overline{x}, \overline{y})$, and $\overline{e} \subseteq N$, if $\psi(\overline{x}, \overline{e})$ is pseudofinite, then there exists finite $A \subseteq \operatorname{acl}(\overline{e})$ such that*

$$\mathcal{N} \models \forall\overline{x}(\psi(\overline{x}, \overline{e}) \to A \cap \overline{x} \neq \emptyset).$$

From the sublemma and the $\aleph_0$-saturation of $\mathcal{M}$ we can choose $\overline{d}$ above disjoint from $\operatorname{acl}(\overline{b})$. Since

$$\mathcal{M} \models \phi(\overline{a}, \overline{d})\&\neg\phi(\overline{b}, \overline{d}). \vee .\phi(\overline{b}, \overline{d})\&\neg\phi(\overline{c}, \overline{d}),$$

one of $\overline{a}E_n\overline{b}$ and $\overline{b}E_n\overline{c}$ is contradicted. So E_n is an equivalence relation.

From the definition of E_n and the sublemma, $\overline{a}E_n\overline{b}$ fails if and only if there are disjoint k-tuples $\overline{c}_i$ $(i < \omega)$ such that $\operatorname{tp}(\overline{a} \mid \overline{c}_i) \neq \operatorname{tp}(\overline{b} \mid \overline{c}_i)$ $(i < \omega)$. As $(\overline{a}, \overline{b})$ runs through E_n there is a finite bound, m say, on the maximum length of a sequence $\langle \overline{c}_j : j < i\rangle$ of disjoint k-tuples such that

$$\operatorname{tp}(\overline{a} \mid \overline{c}_j) \neq \operatorname{tp}(\overline{b} \mid \overline{c}_j) \qquad (j < i).$$

Otherwise, some quantifier-free formula has the FCP. Let $\overline{z}_i$ $(i \le m)$ be disjoint k-tuples of variables. Then $\overline{x}E_n\overline{y}$ is equivalent to

$$\forall\overline{z}_0 \cdots \forall\overline{z}_m \left[\left(\bigvee_{i<j\le m} \overline{z}_i \cap \overline{z}_j \neq \emptyset\right) \vee \left(\bigvee_{i\le m} \operatorname{tp}(\overline{x} \mid \overline{z}_i) = \operatorname{tp}(\overline{y} \mid \overline{z}_i)\right)\right]$$

which is a universal formula.

Finally, we must show that E_n has only finitely many equivalence classes. Towards a contradiction suppose that E_n has infinitely many classes. Then there are representatives $\overline{a}_i = \overline{b}\,\overline{c}_i$ $(i < \omega)$ of distinct E_n-classes such that $\overline{c}_i \cap \overline{c}_j = \emptyset$ whenever $i \neq j$. By compactness and $\aleph_0$-saturation we may suppose that $\overline{c}_i \cap \operatorname{acl}(\overline{b}) = \emptyset$. Let $m = l(\overline{c}_i)$. For each pair (i, j), $i < j < \omega$, there exists $\overline{d}_{ij}$ such that $\overline{d}_{ij} \cap \operatorname{acl}(\overline{a}_i\overline{a}_j) = \emptyset$ and $\operatorname{tp}(\overline{a}_i \mid \overline{d}_{ij}) \neq \operatorname{tp}(\overline{a}_j \mid \overline{d}_{ij})$. Using the sublemma we can move the $\overline{d}_{ij}$'s so that $\overline{d}_{ij} \cap \operatorname{acl}(\overline{a}_k) = \emptyset$ for all $k < \omega$. Now fix $h < \omega$ and let

$$D = \overline{b} \cup \bigcup\{\overline{d}_{ij} : i < j < h\}.$$

Then the $\overline{c}_i$ $(i < h)$ all realize different types over D and by Corollary 4.4 $\overline{c}_i \cap \operatorname{acl}(D) = \emptyset$. By the sublemma for each $i < h$ there are infinitely many pairwise disjoint m-tuples each realizing the same type over D as $\bar{c}_i$. Thus D witnesses that there is a partition of M^m by atomic formulas in which at least h of the pieces are unpseudofinite. Since h is arbitrary this contradicts the previous lemma. This completes the proof.

We continue in the same context. Call a tuple $\bar{b} \subseteq M$ *closed* if $\mathrm{acl}(\bar{b}) \subseteq \bar{b} \cup \mathrm{acl}(\emptyset)$.

5.3. **Lemma.** *Let* $\mathrm{Th}(\mathscr{M})$ *be a complete NFCP* $\exists\forall$*-theory over a finite relational language.*

(1) $E_n^{\mathrm{acl}(\emptyset)} \restriction (M \setminus \mathrm{acl}(\emptyset))^n$ *is a finite equivalence relation and each of its classes contains infinitely many pairwise disjoint closed* n*-tuples.*

(2) *There is a* 0*-definable finite equivalence relation on* M^n *which agrees with* $E_n^{\mathrm{acl}(\emptyset)}$ *on* $(M \setminus \mathrm{acl}(\emptyset))^n$.

(3) *If* $\bar{a}, \bar{b} \subseteq M \setminus \mathrm{acl}(\emptyset)$ *are closed disjoint* n*-tuples and* $\bar{a} E_n^{\mathrm{acl}(\emptyset)} \bar{b}$, *then the permutation of* M *with support* $\bar{a}\bar{b}$, *which switches* $\bar{a}$ *and* $\bar{b}$, *is an automorphism of* $\mathscr{M}$.

(4) *If* $\bar{a}, \bar{b} \subseteq M \setminus \mathrm{acl}(\emptyset)$, $l(\bar{a}) = m$, *and* $l(\bar{b}) = n$, *then the* $E_{m+n}^{\mathrm{acl}(\emptyset)}$*-class of* $\bar{a}\bar{b}$ *determines the* $E_m^{\mathrm{acl}(\emptyset)}$*-class of* $\bar{a}$.

(5) *The predicate "*$\bar{x}$ *is a closed* n*-tuple in* $M \setminus \mathrm{acl}(\emptyset)$*" is* 0*-definable.*

Proof. (1), (2) Note that $E_n^{\mathrm{acl}(\emptyset)}$ is an equivalence relation since it is the intersection of the set $\{E_n^P : P \subseteq \mathrm{acl}(\emptyset),\ |P| < \aleph_0\}$. Towards a contradiction suppose that $E_n^{\mathrm{acl}(\emptyset)}$ partitions $(M \setminus \mathrm{acl}(\emptyset))^n$ into infinitely many classes. Using compactness, Corollary 4.4, and Ramsey's theorem we can find $\mathscr{M}' \succeq \mathscr{M}$, $\bar{a}_q \subseteq (M' \setminus \mathrm{acl}(\emptyset))^n$ $(q \in \mathbf{Q})$ and $\bar{b}_{qr} = \bar{c}\bar{d}_{qr} \subseteq M'$ $(q, r \in \mathbf{Q})$ such that:

(i) If $\pi \in \mathrm{Aut}(\mathbf{Q}, <)$, then there is an elementary map F with $\mathrm{dom}(F)$, $\mathrm{rng}(F) \subseteq M'$, such that $F(\bar{a}_q) = \bar{a}_{\pi(q)}$, and $F(\bar{b}_{qr}) = \bar{b}_{\pi(q)\pi(r)}$ $(q, r \in \mathbf{Q})$.

(ii) $\bar{c} \subseteq \mathrm{acl}(\emptyset)$ and $\bar{d}_{qr} \subseteq M \setminus \mathrm{acl}(\bar{a}_q \bar{a}_r)$.

(iii) $\bar{a}_q$ and $\bar{a}_r$ realize different types over $\bar{b}_{qr}$.

It follows that $E_n^{\bar{c}}$ has infinitely many equivalence classes, which contradicts Lemma 5.2. This shows that $E_n^{\mathrm{acl}(\emptyset)}$ has only a finite number of equivalence classes on $(M \setminus \mathrm{acl}(\emptyset))^n$.

Fix a 0-definable finite subset P of $\mathrm{acl}(\emptyset)$ such that E_n^P partitions $(M \setminus \mathrm{acl}(\emptyset))^n$ into the same number of classes as $E_n^{\mathrm{acl}(\emptyset)}$. Then E_n^P agrees with $E_n^{\mathrm{acl}(\emptyset)}$ on $(M \setminus \mathrm{acl}(\emptyset))^n$. This proves (2). Let $\bar{c} \subseteq M \setminus \mathrm{acl}(\emptyset)$ be an n-tuple, and C, C^P denote $\bar{c}/E_n^{\mathrm{acl}(\emptyset)}$, $\bar{c}/E_n^P$ respectively. Applying the sublemma to $\mathscr{M}^*$ obtained from $\mathscr{M}$ by adjoining the imaginary points corresponding to the classes of E_n^P we see that there are infinitely many disjoint n-tuples in C^P, and hence infinitely many in C. Note that $M \cap \mathrm{acl}(\emptyset)$ is the same in $\mathscr{M}^{\mathrm{eq}}$ as in $\mathscr{M}$. By compactness there exists $\mathscr{M}' \succeq \mathscr{M}$ with $\bar{a} \subseteq (M' \setminus M)^n \cap C$. Let $\mathscr{M}''$ denote the submodel of $\mathscr{M}'$ with universe $M \cup \bar{a}$. From Theorem 3.10 $\mathscr{M} \preceq \mathscr{M}''$. Since M is algebraically closed in $\mathscr{M}''$, $\bar{a}$ is closed in $\mathscr{M}''$

by Corollary 4.4. Since $\mathcal{M}'' \subseteq \mathcal{M}'$ and E_n^P is defined by a universal formula, $\mathcal{M}'' \models \overline{a}E_n^P\overline{c}$ and so $\overline{a} \subseteq C$ in $\mathcal{M}''$. Thus we can add a closed tuple to C which means that in any $\aleph_0$-saturated model all the $E_n^{\mathrm{acl}(\varnothing)}$-classes contain infinitely many pairwise disjoint closed tuples. Once (5) is proved it will follow that the same is true of every model.

(3) Without loss of generality $\mathcal{M}$ is $\aleph_0$-saturated. Let $\overline{a}, \overline{b}$ be disjoint $E_n^{\mathrm{acl}(\varnothing)}$-equivalent closed n-tuples $\subseteq M\setminus \mathrm{acl}(\varnothing)$. Towards a contradiction let $\phi(\overline{x}, \overline{y}, \overline{z})$ be a basic formula and $\overline{d} \subseteq M\setminus(\overline{a}, \overline{b})$ such that

(#) $$\mathcal{M} \models \phi(\overline{a}, \overline{b}, \overline{d})\&\neg\phi(\overline{b}, \overline{a}, \overline{d}).$$

From the saturation there exists a closed n-tuple $\overline{c} \subseteq M\setminus(\mathrm{acl}(\varnothing) \cup \overline{abd})$ which is in the same $E_n^{\mathrm{acl}(\varnothing)}$-class as $\overline{a}$ and $\overline{b}$. From the definition of $E_n^{\mathrm{acl}(\varnothing)}$, $\overline{a}$ and $\overline{b}$ realize the same type over $M\setminus\overline{a}\,\overline{b}$, and likewise for the pairs $(\overline{b}, \overline{c})$ and $(\overline{c}, \overline{a})$. Using $(\overline{c}, \overline{a})$, from $\mathcal{M} \models \phi(\overline{a}, \overline{b}, \overline{d})$ we have $\mathcal{M} \models \phi(\overline{c}, \overline{b}, \overline{d})$. Using $(\overline{a}, \overline{b})$, we deduce that $\mathcal{M} \models \phi(\overline{c}, \overline{a}, \overline{d})$; and then, using $(\overline{b}, \overline{c})$, we get $\mathcal{M} \models \phi(\overline{b}, \overline{a}, \overline{d})$. This contradicts (#).

(4) Without loss of generality $\mathcal{M}$ is $\aleph_0$-saturated. Let $\overline{a}_0, \overline{a}_1, \overline{b}_0, \overline{b}_1 \subseteq M\setminus \mathrm{acl}(\varnothing)$, where $l(\overline{a}_0) = l(\overline{a}_1) = m$, $l(\overline{b}_0) = l(\overline{b}_1) = n$, and $\overline{a}_0\overline{b}_0E_{m+n}^{\mathrm{acl}(\varnothing)}\overline{a}_1\overline{b}_1$. Towards a contradiction suppose that there exists $\overline{c} \subseteq (M\setminus \mathrm{acl}(\overline{a}_0\overline{a}_1)) \cup \mathrm{acl}(\varnothing)$ such that $\overline{a}_0, \overline{a}_1$ realize different types over $\overline{c}$. Write $\overline{c} = \overline{d}\,\overline{e}$, where $d \subseteq \mathrm{acl}(\varnothing)$ and $\overline{e} \subseteq M\setminus \mathrm{acl}(\varnothing)$. From (1) there exists a closed $l(\overline{e})$-tuple $\overline{e}' \subseteq M\setminus \mathrm{acl}(\overline{a}_0\overline{a}_1\overline{b}_0\overline{b}_1)$ such that $\overline{e}'E_{l(\overline{e})}^{\mathrm{acl}(\varnothing)}\overline{e}$. Since $\overline{e}$ and $\overline{e}'$ realize the same type over $\overline{a}_0\overline{a}_1 \cup \mathrm{acl}(\varnothing)$, $\overline{a}_0$ and $\overline{a}_1$ realize different types over $\overline{c}'$, where $\overline{c}' = \overline{d}\,\overline{e}'$. This contradicts $\overline{a}_0\overline{b}_0E_{m+n}^{\mathrm{acl}(\varnothing)}\overline{a}_1\overline{b}_1$.

(5) Let $l(\overline{x}) = l(\overline{y}) = n$ and $\phi(\overline{x}, \overline{y})$ be a formula such that $\mathcal{M} \models \phi(\overline{a}, \overline{b})$ if and only if $\overline{a}\cap\overline{b} = \varnothing$ and $\overline{a}\,\overline{b}$, $\overline{b}\overline{a}$ realize the same type over $M\setminus\overline{a}\overline{b}$. Note that $\mathcal{M} \models \phi(\overline{a}, \overline{b})$ if and only if there is an automorphism of $\mathcal{M}$ which switches $\overline{a}$ and $\overline{b}$ and fixes $M\setminus\overline{a}\,\overline{b}$ pointwise.

Claim. There exists $k < \omega$ such that for all $\overline{a}$, if $\{\overline{b}_1, \dots, \overline{b}_m\}$ is a maximal set of pairwise disjoint solutions of $\phi(\overline{a}, \overline{y})$ in $\mathcal{M}$, then $m < k$.

This is a special case of Lemma 4 of Baldwin and Kueker [2].

Let $\psi(\overline{x})$ be a formula such that $\mathcal{M} \models \psi(\overline{a})$ if and only if there exist pairwise disjoint solutions $\overline{b}_1, \dots, \overline{b}_k$ of $\phi(\overline{a}, \overline{y})$. Suppose that $\mathcal{M} \models \psi(\overline{a})$. By choice of k it follows that $\phi(\overline{a}, \overline{y})$ has infinitely many pairwise disjoint solutions. Let $\overline{b}$ be one of them. Then $\overline{b}\cap\overline{a} = \varnothing$, and $\overline{b}$ has infinitely many pairwise disjoint conjugates over $\overline{a}$ since any solution $\overline{c}$ of $\phi(\overline{a}, \overline{y})$ can be switched with $\overline{a}$ keeping $M\setminus\overline{a}\,\overline{c}$ fixed pointwise. Let $d \in \mathrm{acl}(\overline{a})\setminus\overline{a}$. Switching $\overline{a}$ and $\overline{b}$ we see that $d \in \mathrm{acl}(\overline{b})$. If $d \notin \mathrm{acl}(\varnothing)$, by Corollary 4.4 there exists $e \in \overline{b}$ such that $e \in \mathrm{acl}(d)$. Hence $\overline{b} \cap \mathrm{acl}(\overline{a}) \neq \varnothing$, which is impossible since $\overline{b}$ has infinitely many pairwise disjoint conjugates over $\overline{a}$. Thus $d \in \mathrm{acl}(\varnothing)$ and so $\overline{a}$ is closed. Since the solutions of $\phi(\overline{a}, \overline{y})$ are all conjugates of $\overline{a}$ over $\varnothing$, $\overline{a}\cap\mathrm{acl}(\varnothing) = \varnothing$. Thus, if $\mathcal{M} \models \psi(\overline{a})$, then $\overline{a}$ is a closed tuple $\subseteq M\setminus \mathrm{acl}(\varnothing)$. On the other hand, from as much of (1) as we have already proved, if $\overline{a} \in (M \setminus \text{acl}(\varnothing))^n$ is closed, then in any $\aleph_0$-saturated $\mathcal{M}' \succeq \mathcal{M}$ there exist pairwise disjoint closed n-tuples $\overline{b}_1, \ldots, \overline{b}_k$ in the same $E_n^{\text{acl}(\varnothing)}$-class as $\overline{a}$. Hence by (3) $\mathcal{M}' \models \psi(\overline{a})$, which means that $\mathcal{M} \models \psi(\overline{a})$. This completes the proof.

We are now able to characterize the theories being considered in this section in terms of certain properties of their models. The properties we need of a model $\mathcal{M}$ are:

Q1. There is an equivalence relation $\approx$ on $M \setminus \text{acl}(\varnothing)$ such that, for $a \in M \setminus \text{acl}(\varnothing)$ and $B \subseteq M \setminus \text{acl}(\varnothing)$, $a \in \text{acl}(B)$ if and only if $a \approx b$ for some $b \in B$.

Q2. For each n, $1 \leq n < \omega$, there is a 0-definable finite equivalence relation R_n on M^n such that, if $\overline{a}, \overline{b} \in (M \setminus \text{acl}(\varnothing))^n$, then $\overline{a} R_n \overline{b}$ if and only if $\overline{a}$ and $\overline{b}$ realize the same type over $(M \setminus \text{acl}(\overline{a}\,\overline{b})) \cup \text{acl}(\varnothing)$.

Q3. For each n, $1 \leq n < \omega$, if the R_n-class C is unpseudofinite, then $C \cap (M \setminus \text{acl}(\varnothing))^n$ contains infinitely many pairwise disjoint closed n-tuples.

Note that $a \approx b$ if and only if $a, b \in M \setminus \text{acl}(\varnothing)$ and $a \in \text{acl}(b)$. Recall from above that $\approx$-classes are called *components*. Note that Q3 guarantees the existence of many finite components.

5.4. **Theorem.** *Let T be a complete theory over a finite relational language. T is an NFCP $\exists\forall$-theory if and only if every model of T has the properties* Q1–Q3.

Proof. The "only if" part is clear from the work above. For the rest we assume that T is a complete theory over a finite relational language all of whose models satisfy Q1–Q3.

Our first aim is to show that every formula is equivalent to a $\exists\forall$-formula. Consider an arbitrary model $\mathcal{M}$ of T, a quantifier free formula $\phi(\overline{x}, \overline{y}, \overline{z})$ and $\overline{a} \in M$ such that $\mathcal{M} \models \forall\overline{y}\exists\overline{z}\phi(\overline{a}, \overline{y}, \overline{z})$.

Claim 1. There exists a finite subset C of M such that

$$\mathcal{M} \models \forall\overline{y}(\exists\overline{z} \in C \cup \overline{y})\phi(\overline{a}, \overline{y}, \overline{z}).$$

To see this we use induction on $l(\overline{y}) = n$. Without loss of generality $\phi(\overline{x}, \overline{y}, \overline{z})$ can be replaced by the formula

$$\overline{z} \cap \overline{xy} = \varnothing \& \bigvee\{\phi(\overline{x}, \overline{y}, \overline{u}) : \overline{u} \subseteq \overline{x}\,\overline{y}\,\overline{z}\}.$$

Thus we may assume that $\phi(\overline{x}, \overline{y}, \overline{z})$ implies $\overline{z} \cap \overline{xy} = \varnothing$. Let B be an R_n-class. Since R_n is a finite equivalence relation it is sufficient to find a finite subset C of M such that

$$\mathcal{M} \models (\forall\overline{y} \in B)(\exists\overline{z} \in C \cup \overline{y})\phi(\overline{a}, \overline{y}, \overline{z}).$$

If B is a pseudofinite, there is a finite subset of M which meets every member of B and so we are done by the induction hypothesis. So we may also assume that B is unpseudofinite.

By Q3 choose closed $\overline{b} \in B$ such that $\overline{b} \cap \overline{a} = \varnothing$. Choose $\overline{c}$ such that $\mathcal{M} \models \phi(\overline{a}, \overline{b}, \overline{c})$. Then $\overline{a}\overline{b} \cap \overline{c} = \varnothing$ by choice of ϕ. Permuting the variables $\overline{z}$

we can write $\bar{c} = \bar{d}\,\bar{e}$, where $\bar{d} \subseteq \text{acl}(\bar{a})$ and $\bar{e} \subseteq M\backslash \text{acl}(\bar{a})$. Since $\bar{b}$ is closed and $\bar{e} \cap \bar{b} = \varnothing$, by Q1 $\bar{e} \cap \text{acl}(\bar{a}\,\bar{b}) = \varnothing$. By Q3 there exist $\bar{e}_i \subseteq M\backslash(\bar{a}\,\bar{b} \cup \text{acl}(\varnothing))$ $(i \leq n)$ pairwise disjoint closed tuples in the same $R_{l(\bar{e})}$-class as $\bar{e}$. By Q1 $\bar{e}_i \cap \bar{d} = \varnothing$ since $\bar{e}_i$ is closed, $\bar{e}_i \cap (\bar{a} \cup \text{acl}(\varnothing)) = \varnothing$, and $\bar{d} \subseteq \text{acl}(\bar{a})$. From Q1 we see that

$$\bar{a}\,\bar{b}\,\bar{d} \subseteq (M\backslash \text{acl}(\bar{e}\,\bar{e}_i)) \cup \text{acl}(\varnothing) \qquad (i \leq n).$$

Applying Q2 to $\bar{e}$ and $\bar{e}_i$ we get $\mathscr{M} \models \phi(\bar{a}, \bar{b}, \bar{d}\bar{e}_i)$ $(i \leq n)$. Again by Q1, since $\bar{b}$ is closed,

$$\bar{a}\,\bar{d}\bar{e}_i \subseteq (M\backslash \text{acl}(\bar{b})) \cup \text{acl}(\varnothing) \qquad (i \leq n).$$

Consider $\bar{b}' \subseteq M\backslash \text{acl}(\varnothing)$ such that $\bar{b}' R_n \bar{b}$ and $\mathscr{M} \models \neg\phi(\bar{a}, \bar{b}', \bar{d}\,\bar{e}_i)$ $(i \leq n)$. Now fix j such that $\bar{e}_j \cap \bar{b}' = \varnothing$. This is possible since $l(\bar{b}') = n$ and the $\bar{e}_i$'s are pairwise disjoint. If $\bar{a}\,\bar{d}\,\bar{e}_j \subseteq (M\backslash \text{acl}(\bar{b}')) \cup \text{acl}(\varnothing)$, then by Q1 we have $\bar{a}\,\bar{d}\bar{e}_j \subseteq (M\backslash \text{acl}(\bar{b}\,\bar{b}')) \cup \text{acl}(\varnothing)$. By Q2 applied to $\bar{b}$ and $\bar{b}'$ we get $\mathscr{M} \models \phi(\bar{a}, \bar{b}, \bar{d}\,\bar{e}_j) \leftrightarrow \phi(\bar{a}, \bar{b}', \bar{d}\bar{e}_j)$, contradiction. It follows that $\bar{a}\,\bar{d}\,\bar{e}_j \cap (\text{acl}(\bar{b}')\backslash \text{acl}(\varnothing)) \neq \varnothing$. By Q1 this implies $\bar{b}' \cap \text{acl}(\bar{a}) \neq \varnothing$ since $\bar{e}_j \cap \bar{b}' = \varnothing$ and $\bar{d} \subseteq \text{acl}(\bar{a})$. We conclude that

$$\left[\bar{y} R_n \bar{b} \,\&\, \bigwedge\{\neg\phi(\bar{a}, \bar{y}, \bar{d}\,\bar{e}_i) \colon i \leq n\}\right] \to \bar{y} \cap \text{acl}(\bar{a}) \neq \varnothing$$

is valid in every elementary extension of $\mathscr{M}$. By compactness there exists finite $A \subseteq \text{acl}(\bar{a})$ of M such that

$$\left[\bar{y} R_n \bar{b} \,\&\, \bigwedge\{\neg\phi(\bar{a}, \bar{y}, \bar{d}\,\bar{e}_i) \colon i \geq n\}\right] \to \bar{y} \cap A \neq \varnothing.$$

The claim now follows by the induction hypothesis.

By compactness, once $\phi(\bar{x}, \bar{y}, \bar{z})$ is fixed, there is a bound on the size of C in Claim 1. Thus in T every formula is equivalent to a $\exists\forall$-formula.

Let $\mathscr{M}' \succeq \mathscr{M}$, $a_1, a_2, \ldots \in M'\backslash M$, and $\bar{a}_n$ denote $a_1, \ldots, a_n$. Let $\mathscr{M}_n$ be the structure $\mathscr{M} \cup \{a_1, \ldots, a_n\}$, and $\mathscr{M}_\omega$ be $\mathscr{M} \cup \{a_1, a_2, \ldots\}$. Notice that, if $\bar{b} \subseteq M\backslash \text{acl}(\varnothing)$ is a closed n-tuple in the same R_n-class as $\bar{a}_n$, then $\bar{a}_n\bar{b}$ and $\bar{b}\bar{a}_n$ realize the same type over $M\backslash\bar{a}_n\bar{b}$. This follows by the argument used to prove (3) of Lemma 5.3. Since there are infinitely many pairwise disjoint possibilities for $\bar{b}$ which are permutable over the rest of the model, $\mathscr{M}_n \cong \mathscr{M}$ and the isomorphism may be chosen to fix any finite subset of M. Let $\phi(\bar{x}, \bar{y}, \bar{z})$ be a quantifier-free formula.

Claim 2. If $\forall\bar{x}\exists\bar{y}\forall\bar{z}\phi(\bar{x}, \bar{y}, \bar{z})$ is true in $\mathscr{M}$, then it is true in $\mathscr{M}_\omega$.

Note that without loss of generality we can assume that $\bar{x}\,\bar{y} \cap \bar{z} \neq \varnothing$ implies $\phi(\bar{x}, \bar{y}, \bar{z})$. Let $\bar{a} \subseteq M_k$ where $l(\bar{a}) = l(\bar{x})$ and $1 \leq k < \omega$. It is sufficient to show that $\mathscr{M}_\omega \models \exists\bar{y}\forall\bar{z}\phi(\bar{a}, \bar{y}, \bar{z})$. For each n, $k \leq n < \omega$, choose $\bar{b}_n$ such that $\mathscr{M}_n \models \forall\bar{z}\phi(\bar{a}, \bar{b}_n, \bar{z})$. Permuting the variables $\bar{y}$ if necessary, we can find an infinite set $I \subseteq \{k, k+1, \ldots\}$, $\bar{c} \subseteq M_\omega$, and $\bar{d}_n \subseteq M_n$ $(n \in I)$

such that $\overline{b}_n = \overline{c}\,\overline{d}_n$ and $\overline{d}_m \cap \overline{d}_n\overline{c} = \emptyset$ $(m, n \in I, m \neq n)$. Let $l(\overline{d}_n) = j$. By thinning I we can suppose that each $\overline{d}_n$ is in the same R_j-class D. Let $\overline{d} \subseteq M\backslash(\overline{a}\,\overline{c} \cup \mathrm{acl}(\emptyset))$ be a closed j-tuple $\subseteq D$ and $\overline{e} \subseteq M_\omega$ be arbitrary with $l(\overline{e}) = l(\overline{z})$. Since there are infinitely many pairwise disjoint $\overline{d}_n$'s $\subseteq D$ such that $\mathscr{M}' \models \phi(\overline{a}, \overline{c}\overline{d}_n, \overline{e})$, for some $\mathscr{M}'' \geq \mathscr{M}'$ and $\overline{d}' \subseteq M''\backslash M'$ we have $\mathscr{M}'' \models \phi(\overline{a}, \overline{c}\,\overline{d}', \overline{e})$ and $\overline{d}' \subseteq D$. Since $\overline{d}$ is also closed in $\mathscr{M}''$ we have $\mathscr{M}'' \models \phi(\overline{a}, \overline{c}\,\overline{d}, \overline{e})$ unless $\overline{a}\,\overline{c}\,\overline{e} \cap \overline{d} \neq \emptyset$. The latter is possible only if $\overline{d} \cap \overline{e} \neq \emptyset$, in which case $\mathscr{M}'' \models \phi(\overline{a}, \overline{c}, \overline{d}\,\overline{e})$ by choice of $\phi(\overline{x}, \overline{y}, \overline{z})$. Since $\mathscr{M}_\omega \models \phi(\overline{a}, \overline{c}\overline{d}, \overline{e})$ for all $\overline{e} \in (M_\omega)^{l(\overline{z})}$ the claim is proved.

Recall that in $\mathrm{Th}(\mathscr{M}) = T$ every formula is equivalent to a $\exists\forall$-formula, which is the same as saying that every $\forall\exists$-formula is equivalent to a $\exists\forall$-formula. The equivalence of a $\exists\forall$-formula and a $\exists\forall$-formula is expressed by a $\forall\exists\forall$-sentence. From Claim 2, if the formula ψ is equivalent to the $\exists\forall$-formula θ in T, then the same is true in $\mathrm{Th}(\mathscr{M}_\omega)$. Since the same $\forall\exists\forall$-sentences are true in $\mathscr{M}$ and $\mathscr{M}_\omega$, $\mathscr{M}_\omega$ is a model of T. It follows by [7, Lemma 4.1] that T is a $\exists\forall$-theory.

Finally we wish to show that no quantifier-free formula has the FCP. Towards a contradiction suppose that the quantifier-free formula $\phi(\overline{x}, \overline{y})$ has the FCP. Let $l(\overline{x}) = m$, $l(\overline{y}) = m$, and $\mathscr{M} \models T$. A set $\{\overline{b}_i : i \leq k\} \subseteq M^n$ is a (minimal) *ϕ-covering of* M^m if $\bigvee\{\phi(\overline{x}, \overline{b}_i) : i \leq k\}$ is valid in $\mathscr{M}$ and for each $j \leq k$, $\bigvee\{\phi(\overline{x}, \overline{b}_i) : i \leq k, i \neq j\}$ is not. Since $\phi(\overline{x}, \overline{y})$ has the FCP there are arbitrarily large finite ϕ-coverings. Given a particular ϕ-covering $B = \{\overline{b}_i : i \leq k\}$ we proceed as follows. For each $j \leq k$ we choose $\overline{a}_j \subseteq M^n$ such that

$$\mathscr{M} \models \phi(\overline{a}_j, \overline{b}_j) \& \bigwedge\{\neg\phi(\overline{a}_j, \overline{b}_i) : i \leq k, i \neq j\}.$$

We permute $\overline{x}$ and choose $\overline{c} \subseteq M^{<m}$, $k' < \omega$, and $\overline{c}_i \in M^{m-l(\overline{c})}$ $(i \leq k')$ such that for all $i, j \leq k'$

$$\begin{gathered} \overline{c}\,\overline{c}_i \in \{\overline{a}_h : h \leq k\}, \\ i \neq j \rightarrow \overline{c}_i \cap \overline{c}\,\overline{c}_j = \emptyset, \\ \overline{c}_i R_{m-l(\overline{c})} \overline{c}_j. \end{gathered}$$

The choices are made to maximize k'. Next choose $\overline{c}'$ a closed tuple in the same $R_{m-l(\overline{c})}$-classes as $\overline{c}_i$ such that $\overline{c}' \cap \overline{b}_j\overline{c} = \emptyset$ $(j \leq k)$. Finally, choose $\overline{b} \in B$ such that $\mathscr{M} \models \phi(\overline{c}\,\overline{c}', \overline{b})$.

There are two points to note about this construction. Firstly, with some fixed permutation of $\overline{x}$ and some fixed value for $l(\overline{c})$ we can find ϕ-coverings for which k' is arbitrarily large. Secondly, $\phi(\overline{c}\,\overline{c}_i, \overline{b})$ is true in $\mathscr{M}$ for at most one $i \leq k'$. By compactness in some model $\mathscr{N}$ of T we can find $\overline{c}, \overline{c}', \overline{c}''$ and $\overline{b}$ such that

$$(\#) \qquad \mathscr{N} \models \phi(\overline{c}\,\overline{c}', \overline{b}) \& \neg\phi(\overline{c}\,\overline{c}'', \overline{b}),$$

$\bar{c}'$ is closed, $\bar{c}' R_{m-l(\bar{c})} \bar{c}''$ and

$$\bar{c}' \cap \bar{b}\bar{c} = \varnothing = \bar{c}'' \cap \mathrm{acl}(\bar{b}\bar{c}).$$

From Q1 it follows that

$$\bar{b}\bar{c} \cap (\mathrm{acl}(\bar{c}'\bar{c}'') \setminus \mathrm{acl}(\varnothing)) = \varnothing$$

whence $\bar{c}'$ and $\bar{c}''$ realize the same type over $\bar{b}\bar{c}$. This contradicts (#) and completes the proof of the theorem.

5.5. **Lemma.** *Let T be a complete NFCP $\exists\forall$-theory over a finite relational language. Let $\mathcal{M} \models T$ and R_n $(1 \le n < \omega)$ be equivalence relations satisfying conditions* Q2 *and* Q3 *listed above.*

(1) *n-tuples $\bar{a}, \bar{b} \subseteq M \setminus \mathrm{acl}(\varnothing)$ are R_n-inequivalent iff there exist $\bar{c} \subseteq \mathrm{acl}(\varnothing)$ and pairwise disjoint tuples $\bar{d}_i$ $(i < \omega)$ of equal length such that* $\mathrm{tp}(\bar{a} \mid \bar{c}\bar{d}_i) \ne \mathrm{tp}(\bar{b} \mid \bar{c}\bar{d}_i)$ $(i < \omega)$.

(2) *If $\mathcal{N}$ is obtained from $\mathcal{M}$ by deleting one component, or even all infinite ones, then $\mathcal{N} \preceq \mathcal{M}$.*

Proof. (1) Suppose that the specified $\bar{c}$ and $\bar{d}_i$ $(i < \omega)$ exist. In some $\mathcal{M}' \succeq \mathcal{M}$ there exists $\bar{d} \in M' \setminus M$ such that $\mathrm{tp}(\bar{a} \mid \bar{c}\bar{d}) \ne \mathrm{tp}(\bar{b} \mid \bar{c}\bar{d})$ and $\bar{d} \cap \mathrm{acl}(\bar{a}\bar{b}) = \varnothing$. Thus $\bar{a} R_n \bar{b}$ fails in $\mathcal{M}'$ and hence also in $\mathcal{M}$. For the other direction, suppose $\bar{a} R_n \bar{b}$ fails. There exists $\bar{c} \subseteq \mathrm{acl}\,\varnothing$ and $\bar{d} \subseteq M \setminus \mathrm{acl}(\bar{a}\bar{b})$ such that $\mathrm{tp}(\bar{a} \mid \bar{c}\bar{d}) \ne \mathrm{tp}(\bar{b} \mid \bar{c}\bar{d})$. In the last inequality $\bar{d}$ can be replaced by any closed tuple $\bar{d}'$ such that $\bar{d}' R_{l(\bar{d})} \bar{d}$ and $\bar{d}' \cap \bar{a}\bar{b} = \varnothing$.

(2) Let C be a component of $\mathcal{M}$. Suppose $\bar{a} \subseteq M \setminus C$, $\bar{b} \subseteq C$, and $\mathcal{M} \models \forall \bar{z}\psi(\bar{a}, \bar{b}, \bar{z})$, where $\psi(\bar{x}, \bar{y}, \bar{z})$ is quantifer-free. Let $\bar{b}' \subseteq M \setminus (\mathrm{acl}(\varnothing) \cup \bar{a}\bar{b})$ be a closed tuple such that $\bar{b}' R_{l(\bar{b})} \bar{b}$. From Q2 $\bar{b}$ and $\bar{b}'$ realize the same type over $M \setminus (C \cup \bar{b}')$. Therefore $\mathcal{M} \setminus C \models \forall \bar{z}\psi(\bar{a}, \bar{b}', \bar{z})$. Hence any $\exists\forall$-sentence over $\mathcal{M} \setminus C$ true in $\mathcal{M}$ is also true in $\mathcal{M} \setminus C$. Since T is $\exists\forall$, $\mathcal{M} \setminus C \models T$ and so $\mathcal{M} \setminus C \preceq \mathcal{M}$ from Theorem 3.10. It is clear that the same argument works if we delete all infinite components from $\mathcal{M}$ instead of just C.

The relations R_n $(1 \le n < \omega)$ will continue to have the same meaning acquired from conditions Q2 and Q3 for the rest of this section.

5.6. **Theorem.** *Let T be a complete NFCP $\exists\forall$-theory over a relational language.*

(1) *T has a prime model.*

(2) *A countable model of T is prime iff all its components are finite.*

(3) *If $\mathcal{M} \models T$, then the number of nonprincipal elementary* 1*-types over M is at most $2^{\aleph_0}$.*

Note. In fact we can easily see that prime models exist over arbitrary subsets of models. An example which shows that T may not be $\aleph_0$-stable can be constructed as follows. Let $L = \{R, U\}$, where R is a binary relation symbol and U is a unary relation symbol. Let T_1 be the L-theory such that $\mathcal{M} \models T_1$

if and only if $\mathscr{M} \restriction R$ is a graph in which each vertex has degree ≤ 2. Form $\mathscr{M}^*$ from T_1 as in the examples of §1. Then $\mathrm{Th}(\mathscr{M}^*)$ is an NFCP $\exists\forall$-theory which is not $\aleph_0$-stable.

Proof. (1), (2) Let $\mathscr{M} \models T$ and $\mathscr{N} \subseteq \mathscr{M}$ be obtained by deleting the infinite components of $\mathscr{M}$. From the last lemma $\mathscr{N} \preceq \mathscr{M}$.

Now consider two countable models $\mathscr{N}$, $\mathscr{M}$ such that $\mathscr{N} \preceq \mathscr{M}$ and all components of $\mathscr{M}$ are finite. We claim that $\mathscr{N} \cong \mathscr{M}$. Fix $n \geq 1$ and an $E_n^{\mathrm{acl}(\varnothing)}$-class C. Let $\langle \overline{a}_i \colon i < \omega \rangle$ be an enumeration of the closed tuples in $M^n \cap C$ and $\langle \overline{b}_i \colon i < \omega \rangle$ be a corresponding enumeration for $\mathscr{N}$. Let A, B denote $\bigcup\{\overline{a}_i \colon i < \omega\}$, $\bigcup\{\overline{b}_i \colon i < \omega\}$. Let $F_C \colon A \to B$ be the map which takes $\overline{a}_i$ to $\overline{b}_i$ for each $i < \omega$. From Lemma 5.3(3) $F_C \cup \mathrm{id}_{M \setminus A}$ is an isomorphism from $\mathscr{M}$ into $\mathscr{M}$. Taking the union of $\mathrm{id}_{\mathrm{acl}(\varnothing)}$ together with a collection of the F_C's, which is maximal subject to the domains not overlapping, we obtain an isomorphism from $\mathscr{M}$ onto $\mathscr{N}$.

Let $\mathscr{N}_0$, $\mathscr{N}_1$ be two countable models neither of which has an infinite component. Let $\mathscr{M}$ be a countable model which embeds both $\mathscr{N}_0$ and $\mathscr{N}_1$ elementarily. By deleting any infinite components we can suppose that $\mathscr{M}$ has only finite ones. From the last paragraph $\mathscr{M} \cong \mathscr{N}_i$ $(i = 0, 1)$. Hence $\mathscr{N}_0 \cong \mathscr{N}_1$. This completes the proof of (1) and (2).

(3) Let $\mathscr{M} \models T$ be of arbitrary size, $\mathscr{M} \preceq \mathscr{M}'$, and $a \in M' \setminus M$. Let $A \subseteq M' \setminus M$ be the component of $\mathscr{M}'$ in which a lies. The proof of Lemma 5.5(2) shows that $\mathscr{M} \cup A \preceq \mathscr{M}'$. ($\mathscr{M} \cup A$ denotes the substructure of $\mathscr{M}'$ with universe $M \cup A$.) Hence $\mathrm{Tp}_{\mathscr{M}'}(a|M) = \mathrm{Tp}_{\mathscr{M} \cup A}(a|M)$. Let $a = a_1, a_2, \ldots$ be an enumeration of A. For $1 \leq n < \omega$ let C_n be the R_n-class of $(a_1, \ldots, a_n)$. From Q2, C_n fixes $\mathrm{tp}(a_1, \ldots, a_n|M)$. Hence $\langle C_i \colon 1 \leq i < \omega \rangle$ fixes the isomorphism type of $\mathscr{M} \cup A$ over M, and hence fixes $\mathrm{Tp}_{\mathscr{M} \cup A}(a|M)$. Since there are only a finite number of choices for each C_i the proof is complete.

Let $\mathscr{M}$ be a structure and $\phi(x, y)$ be a formula. For any $A \subseteq M$, the *ϕ-closure* of A, denoted $\phi\text{-}\mathrm{cl}\,A$, is the least B, $A \subseteq B \subseteq M$, such that

$$[c \in B \& \mathscr{M} \models \phi(c, d) \vee \phi(d, c)] \to d \in B.$$

5.7. **Theorem.** *Let T be a complete NFCP $\exists\forall$-theory over a finite relational language. There is an existential formula $\phi(x, y)$, algebraic in both x and y, such that in any model $\mathscr{M}$ of T, $\phi\text{-}\mathrm{cl}(a) = \mathrm{acl}(a) \setminus \mathrm{acl}(\varnothing)$ $(a \in M \setminus \mathrm{acl}(\varnothing))$.*

Proof. Let $\mathscr{M} \models T$, where T satisfies the hypothesis. Let C be a component of $\mathscr{M}$ and $\mathscr{M} \setminus C \subseteq \mathscr{M}' \subseteq \mathscr{M}$. From Lemma 5.5(2), $\mathscr{M} \setminus C \preceq \mathscr{M}$. Since T is $\exists\forall$, $\mathscr{M}' \models T$. From Theorem 3.10, $\mathscr{M} \setminus C \preceq \mathscr{M}'$ also. Thus we have:

Claim 1. If C is a component of $\mathscr{M}$ and $\mathscr{M} \setminus C \subseteq \mathscr{M}' \subseteq \mathscr{M}$, then $\mathrm{acl}_{\mathscr{M}'}(\varnothing) = \mathrm{acl}_{\mathscr{M}}(\varnothing)$ and every component of $\mathscr{M}$ other than C is a component of $\mathscr{M}'$.

Next we shall establish:

Claim 2. In $\mathscr{M}$, if $b \in \mathrm{acl}(a) \setminus \mathrm{acl}(\varnothing)$, then there exists $\overline{c} \subseteq \mathrm{acl}(a) \setminus \mathrm{acl}(\varnothing)$ such that for any $a'b'\overline{c}' \subseteq M \setminus \mathrm{acl}\,\varnothing$ in the same $R_{l(\overline{c})+2}$-class as $ab\overline{c}$, $b' \in \mathrm{acl}(a')$.

From Corollary 4.2 there is a quantifier-free formula $\psi(x, y, \overline{u})$ such that $\exists\overline{u}\psi(a, y, \overline{u})$ has only a finite number of solutions one of which is b. Permuting the variables $\overline{u}$ if necessary, we can find $\overline{c} \subseteq \operatorname{acl}(a)\setminus\operatorname{acl}(\varnothing)$, $\overline{d} \subseteq M\setminus\operatorname{acl}(a)$, and $\overline{e} \subseteq \operatorname{acl}(\varnothing)$ such that $\mathscr{M} \models \psi(a, b, \overline{c}\,\overline{d}\,\overline{e})$. Let a', b', and $\overline{c}'$ be given in $M\setminus\operatorname{acl}(\varnothing)$ such that $ab\overline{c}R_{l(\overline{c})+2}a'b'\overline{c}'$. Without loss of generality we can suppose that $\overline{d}$ is a closed tuple disjoint from $a'b'\overline{c}'$. Temporarily we assume that $a'b'\overline{c}'$ is disjoint from $\operatorname{acl}(ab\overline{c})$. Let $\mathscr{N} \subseteq \mathscr{M}$ be given by

$$N = (M\setminus\operatorname{acl}(aa'b'\overline{c}')) \cup ab\overline{c}a'b'\overline{c}' \cup \operatorname{acl}(\varnothing).$$

We can pass from $\mathscr{M}$ to $\mathscr{N}$ in a finite number of steps in each of which some elements are deleted from a single component of $\mathscr{M}$. Claim 1 tells us that two elements in the same component of $\mathscr{N}$ are in the same component of $\mathscr{M}$. Since $\mathscr{N} \models \psi(a, b, \overline{c}\,\overline{d}\,\overline{e})$, $b \in \operatorname{acl}_{\mathscr{N}}(a)$. since $ab\overline{c}R_{l(\overline{c})+2}a'b'\overline{c}'$ and $ab\overline{c}$, $a'b'\overline{c}'$ are closed tuples in $\mathscr{N}$, there is an automorphism of $\mathscr{N}$ switching $ab\overline{c}$ and $a'b'\overline{c}'$. Hence $b' \in \operatorname{acl}_{\mathscr{N}}(a')$ which implies $b' \in \operatorname{acl}_{\mathscr{M}}(a')$. We can draw the same conclusion when $a'b'\overline{c} \cap \operatorname{acl}(ab\overline{c}) \neq \varnothing$, by interpolating a closed tuple $a''b''\overline{c}''$ disjoint from $\operatorname{acl}(ab\overline{c}a'b'\overline{c}')$. This completes the proof of Claim 2.

If a, $b \in M\setminus\operatorname{acl}(\varnothing)$ and $\overline{c}$ satisfies the conclusion of Claim 2 with $l(\overline{c}) = i$, then we say that a and b are *i-adjacent*. It is clear that i-adjacency is a reflexive relation.

Recall that R_{i+2} agrees with $E_{i+2}^{\operatorname{acl}(\varnothing)}$ on tuples $\subseteq M\setminus\operatorname{acl}(\varnothing)$. Thus there is an obvious correspondence between $E_{i+2}^{\operatorname{acl}(\varnothing)}$-classes and R_{i+2}-classes which meet $(M\setminus\operatorname{acl}(\varnothing))^{i+2}$. From the definition of $E_{i+2}^{\operatorname{acl}(\varnothing)}$, if A is an $E_{i+2}^{\operatorname{acl}(\varnothing)}$-class and $B = \{b'a'\overline{c}' : a'b'\overline{c}' \in A\}$, then B is an $E_{i+2}^{\operatorname{acl}(\varnothing)}$-class. From this observation, the correspondence between $E_{i+2}^{\operatorname{acl}(\varnothing)}$-classes and R_{i+2}-classes, and the symmetry of the relation $x \in \operatorname{acl}(y)$ on $M\setminus\operatorname{acl}(\varnothing)$, it follows that i-adjacency is a symmetric relation.

The *i-closure* of $A \subseteq M$, denoted $i\text{-}\operatorname{cl}(A)$, is the closure of A under i-adjacency. Clearly, $i\text{-}\operatorname{cl}(A) \subseteq \operatorname{acl}(A)$. Let k be the arity of the language.

Claim 3. If $A \subseteq M\setminus\operatorname{acl}(\varnothing)$ is k-closed, then A is closed in the sense that $A = \operatorname{acl}(A)\setminus\operatorname{acl}(\varnothing)$.

If not, we have a, $b \in M$ such that $b \in \operatorname{acl}(a)\setminus(k\text{-}\operatorname{cl}(a) \cup \operatorname{acl}(\varnothing))$. Let $\overline{c} \subseteq \operatorname{acl}(a)\setminus\operatorname{acl}(\varnothing)$ witness the conclusion of Claim 2. Let $a'b'\overline{c}' \subseteq M\setminus\operatorname{acl}(\varnothing)$ be a closed tuple such that $ab\overline{c}R_{l(\overline{c})+2}a'b'\overline{c}'$. From Lemma 5.3(4), $b' \notin k\text{-}\operatorname{cl}(a')$ since $b \notin k\text{-}\operatorname{cl}(a)$. Let $\overline{d}$ enumerate $k\text{-}\operatorname{cl}(a')$ and $\overline{e}$ enumerate $a'b'\overline{c}'\setminus\overline{d}$. Then $\{\overline{d}, \overline{e}\}$ is a partition of $a'b'\overline{c}'$ into k-closed sets. Permuting entries of the various tuples we can suppose that $\overline{d}\,\overline{e}$ is $a'\overline{c}'b'$.

Let $\overline{d}'$, $\overline{e}'$ denote closed tuples such that $\overline{d}R_{l(\overline{d})}\overline{d}'$ and $\overline{e}R_{l(\overline{e})}\overline{e}'$. Let $\overline{d}_0$, $\overline{d}_0'$ be corresponding subtuples of $\overline{d}$, $\overline{d}'$ respectively, and $\overline{e}_0$, $\overline{e}_0'$ be corresponding subtuples of $\overline{e}$, $\overline{e}'$ respectively such that $l(\overline{d}_0)+l(\overline{e}_0) = k$. Let $\overline{d}_1\overline{e}_1$ be a closed tuple in $M\setminus\operatorname{acl}(\varnothing)$ such that $\overline{d}_1\overline{e}_1R_k\overline{d}_0\overline{e}_0$. Towards a contradiction suppose

$\bar{d}_1$ is not a closed tuple. Then $\mathrm{acl}(\bar{d}_1) \cap \bar{e}_1 \neq \varnothing$. We can easily deduce that $k\text{-}\mathrm{cl}(\bar{d}_0) \cap \bar{e}_0 \neq \varnothing$, contradiction. Therefore $\bar{d}_1$ is closed, and hence $\bar{e}_1$ also. Without loss of generality $\bar{d}'\bar{e}'$ and $\bar{d}_1\bar{e}_1$ are disjoint tuples. By Lemma 5.3(3) $\bar{d}'_0$ and $\bar{d}_1$ can be switched, and then $\bar{e}'_0$ and $\bar{e}_1$. This gives $\bar{d}'_0\bar{e}'_0 R_k \bar{d}_1\bar{e}_1$, whence $\bar{d}'_0\bar{e}'_0 R_k \bar{d}_0\bar{e}_0$.

Since $\bar{d}_0\bar{e}_0$ is an arbitrary k-set from $\bar{d}\bar{e}$, it follows from the definition of the relation $E_n^{\mathrm{acl}(\varnothing)}$ (with which R_n agrees on $(M \setminus \mathrm{acl}(\varnothing))^n$) that $\bar{d}'\bar{e}' R_{l(\bar{d}\bar{e})} \bar{d}\bar{e}$. Let $\bar{d}'\bar{e}'$ be $a''\bar{c}''b''$. Then $a''b''\bar{c}'' R_{l(\bar{c})+2} ab\bar{c}$. Since $\bar{c}$ witnesses $b \in \mathrm{acl}(a)$, $b'' \in \mathrm{acl}(a'')$. This contradicts the choice of $\bar{d}'$ and $\bar{e}'$ and so completes the proof of Claim 3.

We return to the proof of the theorem. From Claim 3 it is sufficient to find a formula $\psi(x, y)$ algebraic in y such that, if $a, b \in M \setminus \mathrm{acl}(\varnothing)$ are k-adjacent, then $\mathscr{M} \models \psi(a, b)$. Since k-adjacency is symmetric, for $\phi(x, y)$ we can take $\psi(x, y) \& \psi(y, x)$.

Let $a, b \in M \setminus \mathrm{acl}(\varnothing)$ be k-adjacent and $\bar{c}$ be a tuple satisfying the conclusion of Claim 2 with $l(\bar{c}) = k$. Let C be the R_{k+2}-class of $ab\bar{c}$. Then a, b are said to be *k-adjacent via C*. Since there are only a finite number of possible C, it is sufficient to find an existential formula $\psi_C(x, y)$, depending only on C and algebraic in y, such that $\mathscr{M} \models \psi_C(a, b)$. A disjunction of formulas of the form $\psi_C(x, y)$ will serve as the $\psi(x, y)$ of the last paragraph.

Let $n < \omega$ be the greatest number for which there exist $a' \in M \setminus \mathrm{acl}(\varnothing)$, distinct $b_i \in M \setminus \mathrm{acl}(\varnothing)$ $(i < n)$, and $\bar{c}_i \subseteq M \setminus \mathrm{acl}(\varnothing)$ $(i < n)$ such that $a'b_i\bar{c}_i \subseteq C$ $(i < n)$. We call n the *bound* of C. The *multiplicity* of n (as the bound of a class) is the number of $R_{l(\bar{c})+2}$-classes having n as bound. Notice that, once $l(\bar{c})$ is fixed, the multiplicity of a particular bound is fixed by T.

Let $\mathscr{N} \subseteq \mathscr{M}$ be the substructure with universe

$\mathrm{acl}(\varnothing) \cup ab\bar{c} \cup \bigcup\{D : D$ is a finite component of $\mathscr{M}$ disjoint from $ab\bar{c}\}$.

Our next task is to establish:

Claim 4. There exists $h < \omega$ such that for any model $\mathscr{M}'$ of T and any embeddings $F_i : \mathscr{N} \to \mathscr{M}'$ $(i < h)$, with $F_i(a) = F_j(a)$ $(i, j < h)$, there exist $i, j < h$ such that $F_i(b) = F_j(b)$ and $i \neq j$.

A key point for the proof of the claim is as follows. If $\mathscr{M}_0, \mathscr{M}_1 \models T$, $\mathscr{M}_0 \subseteq \mathscr{M}_1$, $1 \leq m < \omega$, and $\bar{e}, \bar{e}' \in (M_0 \setminus \mathrm{acl}(\varnothing))^m$, then $\bar{e} R_m \bar{e}'$ in $\mathscr{M}_0$ iff $\bar{e} R_m \bar{e}'$ in $\mathscr{M}_1$. To see this, note that $\mathrm{acl}(\varnothing)$ is the same in $\mathscr{M}_1$ as in $\mathscr{M}_0$ by Lemma 4.1. Also, from Lemma 5.5(1), if $\bar{e} R_m \bar{e}'$ fails in $\mathscr{M}_0$, then it fails in $\mathscr{M}_1$. Since R_m is a finite equivalence relation, this justifies the assertion that, in $\mathscr{M}_0$, R_m is its restriction from $\mathscr{M}_1$ at least as regards tuples without algebraic entries. Thus for every R_m-class C_0 which meets $(M_0 \setminus \mathrm{acl}(\varnothing))^m$ there is an R_m-class C_1 of $\mathscr{M}_1$ such that

$$C_0 \cap (M_0 \setminus \mathrm{acl}(\varnothing))^m = C_1 \cap (M_0 \setminus \mathrm{acl}(\varnothing))^m .$$

C_1 is said to *correspond* to C_0.

Let C_0 be an R_m-class in $\mathscr{M}_0$ with bound n_0 and C_1 be the corresponding R_m-class of $\mathscr{M}_1$, with bound n_1 say. From the definition of bound, $n_0 \leq n_1$. Since the possible bounds and their multiplicities are fixed by T, we have $n_0 = n_1$.

Let $h = n^{\cdot} \mid (M \setminus \operatorname{acl}(\varnothing))/R_{l(\overline{c})+2}| + 1$, and $\mathscr{M}'$ and the $F_i : \mathscr{N} \to \mathscr{M}'$ $(i < h)$ satisfy the hypothesis of Claim 4. By the pigeon-hole principle there is an $R_{l(\overline{c})+2}$-class C' of $\mathscr{M}'$ such that $F_i(ab\overline{c}) \in C'$ for at least $n + 1$ values of i, say for all $i \leq n$. Since $(ab\overline{c})/R_{l(\overline{c})+2}$ $(= C$ in $\mathscr{M})$ has bound n in $\mathscr{M}$, it has the same bound in $\mathscr{N}$. Clearly, $C' = F_0((ab\overline{c})/R_{l(\overline{c})+2})$ also has bound n. Let $a' = F_i(a)$, $b_i = F_i(b)$, and $\overline{c}_i = F_i(\overline{c})$ $(i \leq n)$. Since $a'b_i\overline{c}_i \in C'$ $(i \leq n)$, there exist i, j such that $i < j \leq n$ and $b_i = b_j$. This completes the proof of the claim.

Let $\overline{u}$ be an ω-sequence of variables and $\Phi(x, y, \overline{u})$ be a set of basic formulas such that for some enumeration $\overline{d}$ of $N \setminus \{a, b\}$, $\Phi(a, b, \overline{d})$ is the set of all basic sentences over N which are true in $\mathscr{N}$. Let $\overline{u}_i$ $(1 \leq i \leq h)$ be nonoverlapping ω-sequences of variables. From Claim 4

$$\vdash_T \left[\bigwedge \Phi(x, y_1, \overline{u}_1) \& \cdots \& \bigwedge \Phi(x, y_h, \overline{u}_h)\right] \to \bigvee \{y_i = y_j : 1 \leq i < j \leq h\}.$$

By compactness there exists a finite $\Psi(x, y, \overline{u}) \subseteq \Phi(x, y, \overline{u})$ such that, if $\psi_C(x, y)$ is $\exists \overline{u} \Psi(x, y, \overline{u})$, then

$$\vdash_T [\psi_C(x, y_1) \& \cdots \& \psi_C(x, y_h)] \to \bigvee \{y_i = y_j : 1 \leq i < j \leq h\}.$$

Clearly $\psi_C(x, y)$ is algebraic in y, and $\mathscr{M} \models \psi_C(a, b)$. This completes the proof of the theorem.

We close this section with a theorem which is more or less equivalent to the theorem of Schmerl [10].

5.8. **Theorem.** *Let T be a complete theory over a relational language. T is an $\aleph_0$-categorical $\exists\forall$-theory if and only if for every $\mathscr{M} \models T$:*

(i) *there exists a 0-definable equivalence relation E on M with finite classes such that $\mathscr{M}/E$ is finitely partitioned, and*

(ii) *for all $a \in M \setminus \operatorname{acl}\varnothing$, $\mathscr{M} \cong \mathscr{M} \setminus \{a\}$.*

Terminology. By $\mathscr{M}/E$ we mean the permutation structure $(M/E, G)$, where G is the subgroup of $\operatorname{Sym}(M/E)$ induced by $\operatorname{Aut}(\mathscr{M})$. (For a brief discussion of permutation structures see [4, §1].) A permutation structure (X, H) is *finitely partitioned* if there is an H-invariant partition

$$X = F \dot{\cup} X_1 \dot{\cup} \cdots \dot{\cup} X_n$$

of X such that $|F| < \omega$, $|X_i| \geq \omega$ $(1 \leq i \leq n)$, and the pointwise stabilizer of $X \setminus X_i$ in H induces $\operatorname{Sym}(X_i)$.

Proof. Suppose that T is an $\aleph_0$-categorical $\exists\forall$-theory and for the moment suppose that the language is finite. From Lemma 3.6 no quantifier-free formula of T has the FCP. Let $\mathscr{M} \models T$ and $a \in M$. Let E be the equivalence relation on M defined by

$$xEy \Leftrightarrow [(x, y \in \operatorname{acl}(\emptyset) \& x = y) \vee (x \in \operatorname{acl}(y) \setminus \operatorname{acl}(\emptyset))].$$

By Corollary 4.4 for all $b \in M$ and $B \subseteq M$, $b \in \operatorname{acl}(B)$ iff $b \in \operatorname{acl}(\emptyset)$ or $(b/E) \cap B \neq \emptyset$. Since T is $\aleph_0$-categorical the E-classes are finite, $\operatorname{acl}(\emptyset)$ is finite, and E is 0-definable. Of course, the E-classes contained in $M \setminus \operatorname{acl}(\emptyset)$ are just the components of $\mathscr{M}$. Call nonalgebraic E-classes A_0, A_1 *equivalent*, written $A_0 \approx A_1$, if $|A_0| = |A_1|$ and there exist $\overline{a}_0, \overline{a}_1$ such that $A_i = \overline{a}_i$ $(i < 2)$ and $\overline{a}_0 E_n^{\operatorname{acl}\emptyset} \overline{a}_1$, where $n = l(\overline{a}_0) = l(\overline{a}_1)$. Clearly $\approx$ is invariant under $\operatorname{Aut}(\mathscr{M})$. From Lemma 5.3(1), $\approx$ partitions the components of $\mathscr{M}$ into finitely many classes. From (3) of the same lemma, if $A_0 \approx A_1$, then there is an automorphism of $\mathscr{M}$ fixing $M \setminus (A_0 \cup A_1)$ pointwise which switches A_0 and A_1. M/E can be written as $F \dot{\cup} X_1 \cup \cdots \dot{\cup} X_m$, where $F = \{\{a\};\ a \in \operatorname{acl}(\emptyset)\}$ and the X_i are the necessarily infinite classes into which $\approx$ partitions $(M \setminus \operatorname{acl}(\emptyset))/E$. Let $G \leq \operatorname{Sym}(M/E)$ be the group induced by $\operatorname{Aut}(M/E)$. We have remarked above that, if a, $b \in X_i$, then the transposition (ab) is in G. Hence in $\mathscr{M}^{\mathrm{eq}}$ the sets X_i are mutually indiscernible over F. Since $M \subseteq \operatorname{acl}(M/E)$ in $\mathscr{M}^{\mathrm{eq}}$, any elementary map of M/E into itself can be extended to an elementary map of $\mathscr{M} \dot{\cup} (\mathscr{M}/E)$ into itself. In particular, any automorphism of $\mathscr{M}/E$ is induced by some automorphism of $\mathscr{M}$. Thus $\mathscr{M}/E$ is finitely partitioned.

Let C be a component of $\mathscr{M}$ and $a \in C$. We have to show that $\mathscr{M} \setminus \{a\} \cong \mathscr{M}$. From Lemma 5.5(2), $\mathscr{M} \setminus C \preceq \mathscr{M}$ and so $\mathscr{M} \setminus \{a\} \equiv \mathscr{M}$. If $\mathscr{M}$ were countable, this would suffice. In general, note that every component of $\mathscr{M} \setminus C$ is a component in $\mathscr{M} \setminus \{a\}$, while $C \setminus \{a\}$ is partitioned into a number of components in $\mathscr{M} \setminus \{a\}$, To simplify notation suppose $C \setminus \{a\}$ is a single component D in $\mathscr{M} \setminus \{a\}$. Let $\overline{c}_i$ $(i < \omega)$ enumerate distinct components of $\mathscr{M}$ such that $C = \overline{c}_0$ and for each $i > 0$ there is an automorphism of $\mathscr{M}$ switching $\overline{c}_0$ and $\overline{c}_i$ and fixing $M \setminus \overline{c}_0 \overline{c}_i$ pointwise. Let $\overline{d}_i$ $(i < \omega)$ enumerate distinct components of $\mathscr{M} \setminus \{a\}$ such that $D = \overline{d}_0$ and for each $i > 0$ there is an automorphism of $\mathscr{M} \setminus \{a\}$ switching $\overline{d}_0$ and $\overline{d}_i$ and fixing $M \setminus a \overline{d}_0 \overline{d}_1$. Let $F: M \to M \setminus A$ be the bijection such that $F(\overline{c}_i) = \overline{c}_{i+1}$ and $F(\overline{d}_{i+1}) = \overline{d}_i$ for all $i < \omega$. Then F witnesses that $\mathscr{M} \cong \mathscr{M} \setminus \{a\}$.

This completes the proof of the "only if" part of the theorem in the case when the language is finite. Suppose the language $L(T)$ of T is infinite and consider $\mathscr{M} \models T$. For each finite sublanguage L of $L(T)$ we get a 0-definable equivalence relation E_L such that $\mathscr{M}_L/E_L$ is finitely partitioned, where $\mathscr{M}_L = \mathscr{M} \restriction L$. Also, since $\approx$ partitions the components of $\mathscr{M}$ into finitely many classes, there are only finitely many nonprincipal elementary 1-types over any model. Hence $\operatorname{Th}(\mathscr{M})$ is ω-stable. From [6, Theorem 2.1], T has a finite language in the sense that for some finite $L \subseteq L(T)$ every formula is equivalent to an L-formula. Since properties (i) and (ii) hold for $\mathcal{M} \restriction L$, they also hold for $\mathcal{M}$.

For the other direction suppose that a countable structure $\mathcal{M}$ is given satisfying (i) and (ii) above. We shall show that $T = \mathrm{Th}(\mathcal{M})$ is $\aleph_0$-categorical and $\exists\forall$. The $\aleph_0$-categoricity of T is clear from (i). It also follows from (i) that T is $\aleph_0$-stable. To see that T is $\exists\forall$ we need:

Claim 1. Let C be a component of $\mathcal{M}$. If $\mathcal{M}\backslash C \subseteq \mathcal{M}' \subseteq \mathcal{M}$, then $\mathcal{M}\backslash C \preceq \mathcal{M}'$ and the components of $\mathcal{M}$ other than C are all components of $\mathcal{M}'$.

To see this let R be the finest 0-definable finite equivalence relation on M which is refined by E and, let D be the R-class of C. There exist $\mathcal{M}' \succeq \mathcal{M}$ and $\bar{c}_i \in M'\backslash M$ $(i < \omega)$ such that $C_i = \bar{c}_i$ is a component of $\mathcal{M}'$ R-equivalent to C and $\{\bar{c}_i : i < \omega\}$ is indiscernible over M. Since any two components in the R-class of C can be switched by an automorphism of $\mathcal{M}$ which fixes the rest of $\mathcal{M}/E$ pointwise, $(\mathcal{M}\backslash D) \cup \bigcup\{\bar{c}_i : i < \omega\} \cong \mathcal{M}$. So let us replace $\mathcal{M}$ by $(\mathcal{M}\backslash D) \cup \bigcup\{\bar{c}_i : i < \omega\}$ and C by C_0. Clearly, every permutation of the $\bar{c}_i$'s is induced by some automorphism of $\mathcal{M}$ which fixes $M\backslash D$ pointwise. The other nonalgebraic R-classes of $\mathcal{M}$ must have the same property, i.e. their components can be permuted arbitrarily while the rest of the model is fixed pointwise.

Since C is finite, $\mathcal{M}\backslash C \preceq \mathcal{M}$. From (ii), $\mathcal{M}' \vDash T$. Using the automorphisms of $\mathcal{M}$ which fix C pointwise, we see that $\mathrm{acl}_{\mathcal{M}'}(\varnothing)$ does not intersect any component of $\mathcal{M}\backslash C$. Also, if D^* is any R-class of $\mathcal{M}\backslash C$, in which the components of $\mathcal{M}\backslash C$ are also components of $\mathcal{M}'$, then D^* is included in some R-class of $\mathcal{M}'$. Let B be a component of $\mathcal{M}$ other than C, and B' be a component of $\mathcal{M}'$ such that $B \cap B' \neq \varnothing$. We have $B' \subseteq B$; otherwise there is an evident contradiction. Towards a contradiction suppose that B is a component of $\mathcal{M}\backslash C$, of maximum size, which is actually partitioned by $E_{\mathcal{M}'}$. Every component of $\mathcal{M}\backslash C$ which is R-equivalent to B is also partitioned by $E_{\mathcal{M}'}$ and so $\mathcal{M}'$ will have too few R-classes with components of size $|B|$. Thus the components of $\mathcal{M}\backslash C$ are all components of $\mathcal{M}'$, and $R_{\mathcal{M}\backslash C}$ and $R_{\mathcal{M}'}$ agree on $M\backslash C$.

We need to see that $\mathrm{acl}_{\mathcal{M}'}(\varnothing) = \mathrm{acl}_{\mathcal{M}}(\varnothing)$. To this end let $C' = C \cap M'$, and C_i' be the image of C' under the bijection which takes $\bar{c}_0$ to $\bar{c}_i$. We have already seen that $\mathrm{acl}_{\mathcal{M}'}(\varnothing) \subseteq C' \cup \mathrm{acl}_{\mathcal{M}}(\varnothing)$. let $m = |\mathrm{acl}_{\mathcal{M}}(\varnothing)|$ and $\mathcal{N} \subseteq \mathcal{M}$ be defined by $N = M\backslash\bigcup\{C_i\backslash C_i' : i \leq m\}$. If $C_i' \cap \mathrm{acl}_{\mathcal{N}}(\varnothing) \neq \varnothing$ for some i, $1 \leq i \leq m$, then $C_i' \cap \mathrm{acl}_{\mathcal{N}}(\varnothing) \neq \varnothing$ for every such i, contradicting the cardinality of $\mathrm{acl}_{\mathcal{N}}(\varnothing)$. Thus $\mathrm{acl}_{\mathcal{N}}(\varnothing) = \mathrm{acl}_{\mathcal{M}}(\varnothing)$. However, we can obtain $\mathcal{N}$ from $\mathcal{M}$ by deleting the sets $C_i\backslash C_i'$ $(i = 0, 1, \ldots, m)$ in turn. If at any step an element is displaced from $\mathrm{acl}(\varnothing)$, then it cannot return to $\mathrm{acl}(\varnothing)$ at a subsequent step. Hence $\mathrm{acl}(\varnothing)$ is the same at every step which shows that $\mathrm{acl}_{\mathcal{M}'}(\varnothing) = \mathrm{acl}_{\mathcal{M}}(\varnothing)$. It follows that C' is the union of a finite number of components of $\mathcal{M}'$. We have already seen that deleting a component yields an elementary substructure. Therefore $\mathcal{M}\backslash C \preceq \mathcal{M}'$ which completes the proof of the claim.

Claim 2. Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be countable models of T with $\mathcal{M}_0 \preceq \mathcal{M}_1$, and let $\mathcal{M}_0 \subseteq \mathcal{N} \subseteq \mathcal{M}_1$. Then $\mathcal{N} \models T$.

If the claim is granted, it follows easily that the same is true without restriction on the cardinalities of $\mathcal{M}_0$ and $\mathcal{M}_1$. From [7, Theorem 4.1] we infer that T is $\exists\forall$ which completes the proof of the theorem.

It remains to prove the claim. Let C_i $(i < \omega)$ be all the components of $\mathcal{M}_1$ which meet $N\backslash M_0$. We define an increasing chain of structures $\langle \mathcal{N}_i : i < \omega \rangle$ by $N_i = M_0 \cup \bigcup\{N \cap C_j : j < i\}$. Fix i and suppose $\mathcal{M}_0 \preceq \mathcal{N}_i$. From the discussion of the components R-equivalent to C in the proof of Claim 1, C_i is indistinguishable from infinitely many components of $\mathcal{M}_0$ when we fix the rest of M_1 pointwise. Thus $\mathcal{N}_i \preceq \mathcal{N}_i \cup C_i$, and C_i is a component of $\mathcal{N}_i \cup C_i$ since components of $\mathcal{M}_0$ are components of $\mathcal{N}_i$. From Claim 1 $\mathcal{N}_i \preceq \mathcal{N}_{i+1}$. It follows by induction that $\langle \mathcal{N}_i : i < \omega \rangle$ is an elementary chain, whence $\mathcal{N} \models T$. This completes the proof of Claim 2 and of the theorem.

Remark. In the presence of $\aleph_0$-categoricity condition (i) of Theorem 5.8 is equivalent to T being strongly decomposable in the sense of [3, Definition 3.2.2].

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Department of Mathematics and Statistics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6