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POWER SERIES SPACE REPRESENTATIONS OF NUCLEAR FRÉCHET SPACES

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ABSTRACT. Let E be a nuclear graded Fréchet space such that the norms satisfy inequalities $\| \|_k^2 \leq C_k \| \|_{k-1} \| \|_{k+1}$ for all k, let F be a graded Fréchet space such that the dual (extended real valued) norms satisfy inequalities $\| \|_k^{*2} \leq D_k \| \|_{k-1}^* \| \|_{k+1}^*$ for all k, and let A be a tame (resp. linearly tame) linear map from F to E. Then there exists a tame (resp. linearly tame) factorization of A through a power series space $\Lambda_\infty^2(\alpha)$. In the case of a tame quotient map, E is tamely equivalent to a power series space of infinite type. This applies in particular to the range of a tame (resp. linearly tame) projection in a power series space $\Lambda_\infty^2(\alpha)$. In this case one does not need nuclearity. It also applies to the tame spaces in the sense of the various implicit function theorems. If they are nuclear, they are tamely equivalent to power series spaces $\Lambda_\infty(\alpha)$.

The question whether certain Fréchet spaces have isomorphic representations as Köthe sequence spaces has become more and more important. On one hand, by examples of Mityagin-Zobin [28], Djakov-Mityagin [2] and others, it has become clear that in general, even for the nuclear case, a representation is not possible. On the other hand, the theory of operators in Fréchet spaces has developed tools which are more or less connected to such representations. An important open problem is, for instance, whether complemented subspaces of nuclear Köthe spaces always have a basis, i.e., whether they are again isomorphic to Köthe spaces (see Pełczyński [15]). This is not even clear for the complemented subspaces of the space (s) or, more generally, of a power series space $\Lambda_\infty(\alpha)$ of infinite type (except for special cases; see [4, 5, 10, 23, 27]). Moreover, in this case it is of interest to determine the continuity estimates of isomorphisms, in particular, to determine whether there are tame or linearly tame isomorphisms (see [6]).

In the case of finite type power series spaces any complemented subspace is again a finite type power series space (see Mityagin [12] and Mityagin-Henkin [13]). The consequence of this for questions on the linearization of the basic constructions of complex analysis are presented in Mityagin-Henkin [13]. Extensions and generalizations of the method of Mityagin-Henkin have also been given in Dubinsky [3] and Krone [8, 9].

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In the present note we treat the infinite type case in view of the fact that complemented subspaces of spaces $\Lambda_\infty(\alpha)$ can be characterized by topological invariants (DN) and (Ω). We start with a somewhat more general problem and show by use of interpolation methods, that under certain conditions a continuous linear map A from a Fréchet space F to a Fréchet space E factors through a power series space, and the characteristic of continuity can be estimated. These conditions are satisfied, for instance, if A is a tame (resp. linearly tame) linear map from an (Ω)-space F in standard form, which means that the dual norms for the given fundamental system of seminorms satisfy inequalities $\| \|_k^{*2} \le D_k \| \|_{k-1}^* \| \|_{k+1}^*$ for all k, to a nuclear (DN)-space in standard form, i.e., a space with norms satisfying inequalities $\| \|_k^2 \le C_k \| \|_{k-1} \| \|_{k+1}$ for all k. In this case the factorization is tame (resp. linearly tame) for the standard norms in a power series space $\Lambda_\infty^2(\alpha)$. If in particular A is a tame quotient map, this means that E is tamely equivalent to a power series space of infinite type. This applies in particular to the ranges of tame projections in power series spaces $\Lambda_\infty^2(\alpha)$ (cf. [4, 5, 23]). Here we do not need nuclearity.

The conditions are also satisfied for the identity in a nuclear space E admitting a family of smoothing operators in the sense of Moser [14]. The result shows that these spaces are tamely equivalent to power series spaces $\Lambda_\infty(\alpha)$. In particular the nuclear tame spaces in the sense of Hamilton [6] or Sergeraert [18] are tamely equivalent to power series spaces (cf. [23]).

To prove the basic factorization result we use (in §1) an extension and generalization of the interpolation method used by Mityagin-Henkin [13] for the finite type case. We use it in the form presented in [21, §1] (i.e., using only invariants of (DN)- and (Ω)-type). The generalization gives, by avoiding nuclearity (cf. Mityagin [11]), an improvement, compared with [21, §1], even in the finite type case, which allows it to give a characterization of all subspaces and quotient spaces of a given finite type power series space $\Lambda_0^2(\alpha)$ also in the nonnuclear case (cf. [3, 20, 21, 25, 26]). This will be contained in a forthcoming paper.

Preliminaries

We use common notation concerning Fréchet spaces (see [7, 17]) and tameness [6]. If $a = (a_{n,k})_{n,k\in\mathbf{N}}$ is an infinite matrix with $0 \le a_{n,k} \le a_{n,k+1}$, $\sup_k a_{n,k} > 0$ for all n, k, then we define

$$\lambda^2(a) = \left\{ x = (x_1, x_2, \ldots) : \|x\|_k^2 = \sum_n |x_n|^2 a_{n,k}^2 < +\infty \text{ for all } k \right\}.$$

Equipped with the seminorms $\| \|_k$, $k \in \mathbf{N}$, this is a hilbertizable Fréchet space. Hilbertizable means that there is a fundamental system of seminorms defined by semiscalar products.

A graded Fréchet space is a Fréchet space with a fixed fundamental system $\| \|_1 \le \| \|_2 \le \cdots$ of seminorms. For a given matrix a, $\lambda^2(a)$ is considered a graded Fréchet space with the above seminorms. In particular, for any increasing sequence $\alpha_1 \le \alpha_2 \le \cdots$ tending to infinity, we define $\Lambda^2_\infty(\alpha) := \lambda^2(a)$ with $a_{n,k} = e^{k\alpha_n}$. This is always regarded as a graded Fréchet space (power series space of infinite type).

More generally for any $r \in \mathbf{R} \cup \{+\infty\}$ we set

$$\Lambda^2_r(\alpha) = \left\{ x = (x_1, x_2, \ldots) \colon \|x\|^2_\rho = \sum_n |x_n|^2 e^{2\rho\alpha_n} < +\infty \text{ for all } \rho < r \right\}.$$

Equipped with the norms $\|\ \|_\rho$ this is a hilbertizable Fréchet-Schwartz space. For any sequence $\rho_k \nearrow r$ the norms $\|\ \|_{\rho_k}$, $k = 1, 2, \ldots$, are a fundamental system of seminorms. For $r < +\infty$ (finite type power series space) we do not distinguish a special fundamental system of seminorms.

A linear map $\varphi \colon F \to E$ between graded Fréchet spaces is called linear-tame if there exists a, $b \in \mathbf{N}_0$ and C_k, $k \in \mathbf{N}$, such that $\|\varphi x\|_k \le C_k \|x\|_{ak+b}$ for all k; it is called tame if a can be chosen to be 1. A bijection is called tame equivalence (or isomorphy), if it is tame in both directions. Systems of seminorms on E are called tamely equivalent if identity is a tame equivalence. A sequence $\cdots \xrightarrow{\varphi_1} E \xrightarrow{\varphi_2} \cdots$ is called tame-exact iff $\operatorname{im}\varphi_1 = \ker\varphi_2$ and the quotient seminorms on $\operatorname{im}\varphi_1$ are tamely equivalent to the seminorms induced from E on $\ker\varphi_2$. A map $\varphi \colon F \to E$ is called tame imbedding (tame quotient map) if $0 \to F \xrightarrow{\varphi} E \to \cdots$ (resp. $F \xrightarrow{\varphi} E \to 0$) is tame-exact. All these notations apply in an analogous manner to "linear-tame".

If E is a linear space, U, V are absolutely convex sets, and $F \subset E$ is a linear subspace, then we have

$$d(U, V; F) = \inf\{d > 0 \colon U \subset dV + F\}$$

which is in $[0, +\infty]$. We have

$$d_n(U, V) = \inf\{d(U, V; F) \colon \dim F \le n\}.$$

This is called the nth Kolmogorov-diameter of U with respect to V. If $U \subset V$ are Hilbert balls and U in V is precompact, then the $d_n(U, V)$ are equal to the characteristic numbers of the imbedding of the associated Hilbert spaces.

For any continuous seminorm $\|\ \|$ on a locally convex space E, $\|y\|^* := \sup\{|y(x)| \colon \|x\| \le 1\}$ denotes the extended real valued dual norm.

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This section contains the basic construction and the main result. The formulation is somewhat technical. It will be adjusted later to the most interesting special cases.

Let F, H, E be Fréchet spaces with fixed increasing fundamental systems of seminorms $\|\ \|_0 \le \|\ \|_1 \le \cdots$. We put $U_k = \{x \colon \|x\|_k \le 1\}$ in F, H, E. We assume that F is separable and that the seminorms in H are defined by semiscalar products $\langle\ ,\ \rangle_k$, i.e., $\|x\|^2_k = \langle x, x\rangle_k$ for all $x \in H$. We call F_k (resp. H_k, F_k) the Banach spaces defined by $\|\ \|_k$. H_k is a Hilbert space for all k.

Let $A_1: F \to H$, $A_2: H \to E$ be continuous linear maps such that $\|A_j x\|_k \leq \|x\|_k$ for $j = 1, 2$ and all k. We put $A = A_2 \circ A_1$. We assume that there is a q_0 such that the map $F_{q_0} \to H_0$ induced by A_1 is compact, or equivalently that $A_1 U_{q_0}$ is U_0-precompact. This is satisfied for instance if F or H is a Schwartz space.

It does not mean an additional restriction on A to assume, as we will do from now, that A_2 is injective. Moreover we make the following assumptions:

(α) For all k and $K \gg k$ there exists $\mu > 0$ and $M > 0$ such that for all $x \in E$

$$\|x\|_k^{\mu+1} \leq M\|x\|_0^{\mu}\|x\|_K.$$

(ω) There exists q_0 such that for all Q we have $\nu > 0$ and $D > 0$ such that for all $y \in F'$

$$\|y\|_{q_0}^{*\nu+1} \leq D\|y\|_0^{*\nu}\|y\|_Q^{*}.$$

We may assume without loss of generality that the q_0 in (ω) and the q_0 mentioned above is the same.

For all k, $K \gg k$ and $q \geq q_0$, Q we put $\mu_0(K, k) =$ sup of all μ such that with some M the inequality in (α) holds for all $x \in E$, and $\nu_0(Q, q) =$ inf of all $\nu > 0$ such that with some D the inequality in (ω) holds for all $y \in F'$ (with q instead of q_0).

Then $\mu_0(K, k)$ and $\nu_0(Q, q)$ are increasing in the first variable and decreasing in the second variable. We make the following crucial assumption:

(τ) For every k there exist $q = q(k)$ such that

$$\limsup_{Q\to+\infty} \frac{\nu_0(Q+2, q)+1}{\mu_0(Q, k)+1} < 1.$$

Obviously the condition does not change if we replace $Q+2$ by $Q+1$. However we have to use $q(k)$ as defined in (τ).

Well-known arguments (cf. [19, 24]) show that the inequalities in (α) and (ω) can be written as

(α) $\| \ \|_k \leq r\| \ \|_0 + M'\| \ \|_K/r^{\mu}$ for $r > 0$,

(ω) $U_q \subset r^{\nu} U_Q + D' U_0/r$ for $r > 0$,

with modified constants M', D'. Such inequalities hold for $K \gg k$ and $\mu < \mu_0(K, k)$, and for $q \geq q_0$ and $\nu > \nu_0(Q, q)$.

By applying (α) (resp. (ω)) to μ' with $\mu < \mu' < \mu_0(K, k)$ (resp. ν' with $\nu > \nu' > \nu_0(Q, q)$) and cancelling M' with $r^{\mu'-\mu}$ (resp. replacing r by $r^{\nu/\nu'}$ and cancelling D' with $r^{\nu/\nu'-1}$) for large r we obtain constants $R(K, k)$ depending on the choice of μ and ν such that

(α') $\| \ \|_k \leq r\| \ \|_0 + \| \ \|_K/r^{\mu}$ for $r \geq R(K, k)$,

(ω') $U_q \subset r^{\nu} U_Q + U_0/r$ for $r \geq R(Q, q)$.

We shall use a generalization of an interpolation argument of Mityagin and Henkin [13] (cf. Introduction). Basic for this is Lemma 1.3, which provides us with the "dead-end space" together with the information needed in the interpolation. It needs a preparation, which uses only that $A_1 U_{q_0}$ is U_0-precompact in H.

We assume, without loss of generality, that $U_0 \supset U_1 \supset \cdots$ is a *basis* of neighborhoods of zero.

1.1. **Lemma.** *For any sequence $(A(k))_{k\geq q_0}$ of positive numbers and any sequences $(\varphi_k)_{k\geq q_0}$, $(\psi_k)_{k\geq q_0}$ of positive real valued functions on $\mathbf{R}_+$ there exist a strictly increasing sequence $(C(k))_{k\geq q_0}$ with $C(K)\geq A(K)$ and $C(K+1)\geq \psi_K(C(K))$ for all $K\geq q_0$ and a total (= linear span dense) null-sequence $(x_n)_n$ in F such that with $B_0=\{x_n : n\in\mathbf{N}\}$:*

(1) $A_1 U_k \subset A_1 B_0 + U_0/\varphi_K(C(K))$,
(2) $B_0 \subset C(K) U_K$

for all $K\geq q_0$.

Proof. Since F is separable we have a countable total subset $\{\xi_n : n\in\mathbf{N}_0\}$. We may assume $\xi_n\in U_{q_0+n}$ for all n. We determine inductively finite sets $N_q\subset U_q$ in F and constants $C(q)\geq 1$, $q\geq q_0$. We start with $C(q_0)$ according to the (then noninductive) definition below.

Assume $N_{q_0},\dots,N_{K-1}$, $C(q_0),\dots,C(K-1)$ are chosen. Put

$$C(K)=\max\{\|x\|_K,\ x\in N_q,\ q_0\leq q\leq K-1;\ \psi_{K-1}(C(K-1));\ C(K-1)+1;\ A(K);\ 1\}.$$

Then there exists a finite set $N_K\subset U_K$ in F with $\xi_{K-q_0}\in N_K$ and

(3) $A_1 U_K \subset A_1 N_K + U_0/\varphi_K(C(K))$.

We put $B_0=\bigcup_{q\geq q_0} N_q$. Then $B_0=\{x_n : n\in\mathbf{N}\}$ where $(x_n)_n$ is a null-sequence. Inclusion (1) follows immediately from (3). For (2) we prove $N_q\subset C(K)U_K$. For $q=q_0,\dots,K-1$ we have $N_q\subset C(K)U_K$ by definition of $C(K)$, and for $q\geq K$ we have $N_q\subset U_q\subset U_K$ and the inclusion follows from $C(K)\geq 1$.

The next lemma is true for any hilbertizable Fréchet space H.

1.2. **Lemma.** *Let $(b_n)_n$ be a null-sequence in H, $\sup_n\|b_n\|_K\leq C(K)$ for all $K\geq q_0$, and $\sum_{K=q_0}^{\infty}\varepsilon_K\leq 1$. Then there exists a compact Hilbert ball $B\subset H$ such that $\{b_n : n\in\mathbf{N}\}\subset B\subset C(K)U_K/\varepsilon_K$ for all $K\geq q_0$.*

Proof. We may assume without restriction of generality that $\{b_n : n\in\mathbb{N}\}$ is total. We fix K and a sequence $(\tilde b_n)_n=1,2,\dots$ such that $\{\tilde b_n : n=1,2,\dots\}=\{b_n : n\in\mathbf{N},\ \|b_n\|_K>0\}$ and $d_n=\|\tilde b_n\|_K$ is decreasing. We apply the Gram-Schmidt orthogonalization method to the sequence $\tilde b_1,\ \tilde b_2,\dots$ and obtain an orthonormal system $h_1, h_2, \ldots$ such that

$$\tilde{b}_n = \sum_{k=1}^{n} \langle \tilde{b}_n, h_k \rangle_K h_k,$$

hence

$$d_n^2 = \sum_{k=1}^{n} |\langle \tilde{b}_n, h_k \rangle_K|^2$$

for all n. We put

$$(x, y)_K := \sum_k \frac{1}{d_k^2} \langle x, h_k \rangle_K \langle h_k, y \rangle_K$$

and, using $|x|_K^2 = (x, x)_K$, we obtain

$$|\tilde{b}_n|_K^2 = \sum_{k=1}^{n} \frac{1}{d_k^2} |\langle \tilde{b}_n, h_k \rangle_K|^2 \leq 1$$

for all n, hence $|b_n|_K \leq 1$ for all n. Certainly $\{x\colon |x|_K \leq 1\}$ is compact in H_K and $\| \, \|_K \leq d_1 | \, |_K \leq C(K) | \, |_K$.

We put

$$\langle x, y \rangle_B = \sum_{K=q_0}^{\infty} \varepsilon_K (x, y)_K, \qquad \|x\|_B^2 = \langle x, x \rangle_B.$$

Then $B = \{x\colon \|x\|_B \leq 1\}$ is compact in H and satisfies the assertion of the lemma.

We now choose functions $\mu(K, k) < \mu_0(K, k)$ and $\nu(Q, q) > \nu_0(Q, q)$ increasing in the first variable and decreasing in the second variable such that with suitable $\delta(k) > 0$ we have

(τ') $\nu(Q+2, q(k)) + 1 < (1 - \delta(k))(\mu(Q, k) + 1)$ for large Q.

1.3. **Lemma.** *There exists a compact Hilbert ball $B \subset H$ and an increasing integer valued function $Q(r)$ such that:*

(1) $Q(e^r) \leq Q(r) + 1$ *for large* r.

(2) *With* $h_k(r) = r^{\mu(Q(r), k)+1}$ *and suitable constants* C_k *we have*

$$\|A_2 x\|_k \leq C_k \left(r\|x\|_0 + \frac{r}{h_k(r)} \|x\|_B \right)$$

for all $k \geq q_0$ *and* $r > 0$.

(3) *With* $g_q(r) = r^{\nu(Q(r)+1, q)+1}$ *and suitable* C_q *we have*

$$A_1 U_q \subset C_q \left(\frac{1}{r} U_0 + \frac{g_q(r)}{r} B \cap \operatorname{im} A_1 \right)$$

for all $q \geq q_0$ *and* $r > 0$.

(4) $A_1^{-1} H_B$ *is dense in* F.

(5) $H_B \cap \operatorname{im} A_1$ *is dense in the Hilbert space* H_B.

Proof. We choose a decreasing sequence $\varepsilon(Q)$ such that for any k the inequality $\mu(Q, k)+\varepsilon(q) < \mu_0(Q, k)$ holds for large Q. We determine $R(Q, k)$ such that (α') and (ω') hold for $\mu = \mu(Q, k) + \varepsilon(Q)$ and $\nu = \nu(Q, k)$.

We chose increasing sequences $S(l) > 0$ and $\alpha(Q) > 0$ tending to infinity such that

$$\lim_{l\to+\infty} \frac{R(l+1, q)}{S(l)} = \lim_{Q\to+\infty} \frac{\nu(Q+1, q)}{\alpha(Q)} = 0 \tag{1}$$

for all q.

We apply Lemma 1.1 with $\varphi_K(x) = x^{2\alpha(K)/\varepsilon(K)}$, $\psi_K(x) = e^{\varepsilon(K+1)x^{2/\varepsilon(K)}/2}$, and $A(K) = \max\{2^{K+1}; S(K)^{\varepsilon(K)/2}\}$ and obtain $B_0 \subset F$ according to 1.1. By application of Lemma 1.2 with $\varepsilon_K = 2^{-K}$ to $A_1 B_0$ we obtain a compact Hilbert ball $B \subset H$ with $A_1 B_0 \subset B$ (hence assertion (4)) and

$$A_1 U_k \subset B_1 + C(K)^{-2\alpha(K)/\varepsilon(K)} U_0, \tag{2}$$

$$\|x\|_K \le C(K)^2 \|x\|_B \tag{3}$$

for all $K \ge q_0$, where $B_1 = B \cap \operatorname{im} A_1$.

For large r there exists one and only one $Q = Q(r)$ with

$$C(Q)^{2/\varepsilon(Q)} \le r < C(Q+1)^{2/\varepsilon(Q+1)}. \tag{4}$$

This implies $S(Q) \le r$. Since $Q(r)$ is increasing and $\lim_{r\to+\infty} Q(r) = +\infty$, we have from (1) that $R(Q(r)+1, q) \le r$ for large r. Then (ω'), (2), and (4) give us for large r with $Q = Q(r)$:

$$\begin{aligned} A_1 U_q &\subset r^{\nu(Q+1, q)} B_1 + \left(r^{\nu(Q+1, q)} C(Q+1)^{-2\alpha(Q+1)/\varepsilon(Q+1)} + \frac{1}{r}\right) U_0 \\ &\subset r^{\nu(Q+1, q)} B_1 + \left(r^{\nu(Q+1, q)-\alpha(Q+1)} + \frac{1}{r}\right) U_0 \\ &\subset r^{\nu(Q+1, q)} B_1 + \frac{2}{r} U_0. \end{aligned} \tag{5}$$

From (α'), (3), and (4) we obtain for large r with $Q = Q(r)$:

$$\begin{aligned} \|A_2 x\|_k &\le r\|x\|_0 + \frac{C(Q)^2}{r^{\mu(Q, k)+\varepsilon(Q)}} \|x\|_B \\ &\le r\|x\|_0 + \frac{1}{r^{\mu(Q, k)}} \|x\|_B. \end{aligned} \tag{6}$$

This proves (2) and (3) of the assertion for large r, hence for all $r > 0$ with suitable constants. Assertion (1) follows from

$$e^{C(Q)^{2/\varepsilon(Q)}} \le C(Q+1)^{2/\varepsilon(Q+1)}$$

if one applies it to $Q = Q(e^r) - 1$ and uses (4).

Finally we replace B by $\overline{B \cap \operatorname{im} A_1}^{H_B}$ if necessary and so fulfill (5) in the assertion, without destroying (1)–(4).

We shall use the following modification of the fundamental lemma of interpolation theory (cf. [1, p. 45]).

1.4. **Lemma.** *Let G, L be linear spaces, $\|\ \|_0 \le \|\ \|_1 \le \|\ \|_2$ seminorms on G (resp. on L) and $A: G \to L$ a linear map with $\|Ax\|_j \le C_j\|x\|_j$, $j = 0, 2$. Let φ and ψ be increasing positive real-valued functions on $\mathbf{R}_0$ such that $s \le \varphi(s) \le \psi(sh(s))$, where h is a function with $\sum_{n=1}^{\infty} h(2^n) \le C < +\infty$. We assume that*

$$\|x\|_1 \le C\left(r\|x\|_0 + \frac{r}{\psi(r)}\|x\|_2\right) \quad \textit{for all } r > 0,\ x \in L,$$

$$U_1 \subset C\left(\frac{1}{r}U_0 + \frac{\varphi(r)}{r}U_2\right) \quad \textit{for all } r > 0,$$

where $U_j = \{x \in G: \|x\|_j \le 1\}$, $j = 0, 1, 2$, C a constant, $C \ge \varphi(1)$. Then with $D = 4C^3(C_0 + C_2)$ we have $\|Ax\|_1 \le D\|x\|_1$ for all $x \in G$.

Proof. Let $x \in U_1$. For every $n \in \mathbf{N}_0$ we have $x_n \in C2^{-n}\varphi(2^n)U_2$, $y_n \in C2^{-n}U_0$ with $x = x_n - y_n$. Since $x_n - x_{n+1} = y_n - y_{n+1}$ we obtain

$$\begin{aligned}
\|Ax_n - Ax_{n+1}\|_1 &\le C\left(rC_0 2C2^{-n} + \frac{r}{\psi(r)}C_2 2C2^{-n}\varphi(2^{n+1})\right) \\
&= C^2 2^{-n+1} r\left(C_0 + C_2\frac{\varphi(2^{n+1})}{\psi(r)}\right) \\
&\le 4C^2(C_0 + C_2)h(2^{n+1})
\end{aligned}$$

with $r = 2^{n+1}h(2^{n+1})$. Since $C \ge \varphi(1) \ge 1$ we may assume $x_0 = 0$ and obtain

$$\sup_n \|Ax_n\|_1 \le 4C^3(C_0 + C_2) = D.$$

Using $\|y_n\|_2 \le C2^{-n}\varphi(2^n) + \|x\|_2$ we get

$$\begin{aligned}
\|Ay_n\|_1 &\le C\left(rC_0C2^{-n} + \frac{r}{\psi(r)}C_2C2^{-n}\varphi(2^n) + \frac{r}{\psi(r)}C_2\|x\|_2\right) \\
&\le C(C_0C + C_2C + C_2\|x\|_2)h(2^n)
\end{aligned}$$

with $r = 2^n h(2^n)$, hence $\|Ax\|_1 \le D + \text{const}\cdot h(2^n)$, which yields the result.

Remark. If $(2^{-n}\varphi(2^n))_n$ is unbounded the last estimate is unnecessary, since we may assume $y_n = 0$ for some n, i.e., $x_n = x$.

We come back to the functions $g_q(r)$ and $h_k(r)$ defined in Lemma 1.3. Using (τ') we obtain

$$\begin{aligned}
g_q(r) = r^{\nu(Q(r)+1,q)+1} &\le r^{(1-\delta)(\mu(Q(r)-1,k)+1)} \\
&\le r^{(1-\delta)(\mu(Q(r^{1-\delta}),k)+1)} = h_k(r^{1-\delta})
\end{aligned}$$

for large r with $\delta = \delta(k)$ and $q = q(k)$.

Hence we can find an increasing function $\beta_k(r) > 1$ and $\varepsilon = \varepsilon(k)$ such that with $\gamma_k(r) = r^{\beta_k(r)}$ we have $g_q(r) \le \gamma_k(r^{1-\varepsilon})$ and $\gamma_k(r) \le h_k(r^{1-\varepsilon})$ for large r.

Let a_n be the nth characteristic number (cf. [16, 8.3.2]) of the imbedding $H_B \to H_0$. We put $a_{n,k} = \gamma_k^{-1}(1/a_n)$, where for an increasing function γ we define $s = \gamma^{-1}(t)$ by $\gamma(s-) \le t \le \gamma(s+)$, and obtain with $a = (a_{n,k})$:

1.5. **Theorem.** *There are linear maps $S\colon F \to \lambda^2(a)$ and $T\colon \lambda^2(a) \to E$ such that $A = T \circ S$, $\operatorname{im} S$ is dense in $\lambda^2(a)$, T is injective on $\operatorname{im} S$, and*

$$\|Sx\|_k \le C_k \|x\|_{q(k)}, \qquad \|Tx\|_k \le C_k \|x\|_k$$

for $k \ge q_0$ with suitable constants C_k.

Proof. We apply the spectral theorem (see [16, 8.3.1]) to the compact imbedding $H_B \to H_0$ and obtain orthonormal systems (b_n) in H_B and (f_n) in H_0 such that

$$x = \sum_{n=0}^{\infty} a_n \langle x, b_n \rangle_B f_n$$

for every $x \in H_B$, where the series converges in H_0. (b_n) is complete.

For $x \in H_B$ we put $x_n = \langle x, f_n \rangle_0$ and

$$|x|_k^2 = \sum_{n=0}^{\infty} a_{n,k}^2 |x_n|^2.$$

The series converges, since $a_{n,k} \le 1/a_n$ and

$$\sum_{n=0}^{\infty} \frac{1}{a_n^2} |x_n|^2 = \|x\|_B^2 < +\infty$$

for all $x \in H_B$. $|\ |_k$ is a norm since (b_n) is complete in H_B.

For these norms we have

$$\begin{aligned} |x|_k^2 &\le \gamma_k^{-1}\left(\frac{1}{a_m}\right)^2 \sum_{n=0}^{m} |x_n|^2 + a_{m+1}^2 \gamma_k^{-1}\left(\frac{1}{a_{m+1}}\right)^2 \sum_{n=m+1}^{\infty} \frac{1}{a_n^2}|x_n|^2 \\ &\le r^2 \|x\|_0^2 + \left(\frac{r}{\gamma_k(r)}\right)^2 \|x\|_B^2 \end{aligned}$$

for large r and choosing m such that $1/a_m \le \gamma_k(r) \le 1/a_{m+1}$. We have used that $\gamma^{-1}(r)/r$ is decreasing. Using suitable C_k' we obtain

$$|x|_k \le C_k' \left(r\|x\|_0 + \frac{r}{\gamma_k(r)} \|x\|_B \right)$$

for all $x \in H_B$ and $r > 0$.

For $x \in V_q = \{x \in H_B : |x|_q \le 1\}$, large r, and m such that $1/a_m \le \gamma_k(r) \le 1/a_{m+1}$, we have

$$\xi = \sum_{k=0}^{m} x_n f_n = \sum_{k=0}^{m} \frac{1}{a_n} x_n b_n \in H_B, \qquad \eta = x - \xi \in H_B,$$

and obtain

$$\|\xi\|_B = \left(\sum_{k=0}^{m} \frac{1}{a_n^2}|x_n|^2\right)^{1/2} \leq \frac{1}{a_m\gamma_q^{-1}(1/a_m)} \leq \frac{\gamma_q(r)}{r},$$

$$\|\eta\|_0 = \left(\sum_{k=m+1}^{\infty} |x_n|^2\right)^{1/2} \leq \frac{1}{\gamma_q^{-1}(1/a_{m+1})} \leq \frac{1}{r}.$$

Again we used that $\gamma_q^{-1}(r)/r$ is decreasing. We proved for all $r > 0$ with suitable C_q':

$$V_q \subset C_q'\left(\frac{1}{r}V_0 + \frac{\gamma_q(r)}{r}B\right).$$

From Lemmas 1.4 (set $h(r) = r^{-\varepsilon}$) and 1.3 we obtain

$$\|A_2 x\|_k \leq D_k|x|_k$$

for all $x \in H_B$ and $k \geq q_0$, and with suitable constants D_k. Also, with $q = q(k)$, (then $q \geq q_0$)

$$|x|_k \leq D_k\|x\|_{A_1 U_q}$$

for all $x \in H_B \cap \operatorname{im} A_1$, hence

$$|A_1 x|_k \leq D_k\|x\|_q$$

for all $x \in A_1^{-1}(H_B)$ which is dense in F (by 1.3(4)).

We denote H_B equipped with the norms $|\ |_k$ by G and its completion by $\widehat{G}$. $\widehat{G}$ can be identified with $\{x \in H_0 \colon (x_n)_n \in \lambda^2(a)\}$. Since A_1 maps $A_1^{-1}(H_B)$ continuously into G, it maps F into $H \cap \widehat{G}$. A_2 can be extended into a continuous linear map $\widehat{A}_2 \colon \widehat{G} \to E$. Clearly $A = \widehat{A}_2 \circ A_1$ and $\widehat{A}_2$ is injective on $H \cap \widehat{G} \supset \operatorname{im} A_1$. Since $H_B \cap \operatorname{im} A_1$ is dense in H_B, hence in G, $\operatorname{im} A_1$ is dense in $\widehat{G}$.

By $x \sim (x_n)_n$ we identify $\widehat{G}$ with $\lambda^2(a)$ and obtain from A_1 (resp. $\widehat{A}_2$) maps S (resp. T) with the desired properties.

Remark. If we put $e_n = A_2(f_n) = T((\delta_{j,n})_j)$ and $\varphi_n(x) = \langle A_1 x, f_n\rangle_0$, i.e., $Sx = (\varphi_n(x))_n$, then

$$Ax = \sum_n \varphi_n(x)e_n \quad \text{for all } x \in F.$$

Hence every $\xi \in \operatorname{im} A$ can be expanded into a series $\xi = \sum_n \xi_n e_n$. If there is an estimate $\|x\|_0 \leq C_0\|A_2 x\|_{k_0}$ for some k_0, then the coefficients are uniquely determined and therefore $\operatorname{im} A$ has a basis.

In the most important cases either A_2 or A_1 is identity (after our standardization of a quotient map), i.e., E or F is properly hilbertizable. We formulate the above remark for the first case.

1.6. **Corollary.** *If A_2 is an isomorphism, then* $\operatorname{im} A$ *has a basis.*

If A is surjective, then certainly A_2 is an isomorphism, and we obtain

1.7. **Theorem.** *If A is surjective and $\varepsilon_k U_{\tau(k)} \subset AU_k$ for suitable $\varepsilon_k > 0$ and all k, then $T\colon \lambda^2(a) \to E$ is an isomorphism with $\|Tx\|_k \le C_k|x|_k$ and $|T^{-1}x|_k \le C_k\|x\|_{q(\tau(k))}/\varepsilon_k$ for $\tau(k) \ge q_0$, where the D_k are the constants from Theorem 1.5.*

Proof. The inclusion $\imath\colon G \hookrightarrow H$ can be written as $A_2^{-1} \circ (A_2|_G)$ and is therefore continuous. Hence $\widehat{G} \subset H$ and, since $A_2|_{\widehat{G}}$ is also surjective, even $\widehat{G} = H$. Therefore T is a topological isomorphism. The rest follows from 1.5.

Finally we want to look at special forms of g_k and h_k and the isomorphic type of $\lambda^2(a)$. In many cases we have $\mu(K, k) + 1 = \mu_0(K)/\mu(k)$ for certain increasing functions $\mu_0(\cdot)$ and $\mu(\cdot)$. Then for large r with $\delta = \delta(k)$, assuming $\mu(k) \ge 1$, we obtain

$$\begin{aligned} g_k(r) &= r^{\nu(Q(r)+1,q)+1} \le r^{(1-\delta)(\mu(Q(r)-1,k)+1)} \\ &\le r^{(1-\delta)(\mu(Q(r^{(1-\delta)/\mu(k)}),k)+1)} = (r^{(1-\delta)/\mu(k)})^{\mu_0(Q(r^{(1-\delta)/\mu(k)})} \\ &\le r^{(1-\delta)(\mu_0(Q(r^{1-\delta}))/\mu(k))} = h_k(r^{1-\delta}). \end{aligned}$$

Therefore, with $\gamma(r) = r^{\mu_0(Q(r))}$ and $\rho(k) = 2\mu(k)/(2-\delta(k))$ we can choose $\gamma_k(r) = \gamma(r^{1/\rho(k)})$ for large r.

Similarly we get the same form of $\gamma_k(r)$ with $\gamma(r) = r^{\nu_0(Q(r)+2)}$ and $\rho(k) = (1-\delta(k)/2)\nu(q(k))$ if we assume $\nu(Q, q)+1 = \nu_0(Q)/\nu(q)$ for some increasing functions $\nu_0(\cdot)$ and $\nu(\cdot)$.

In both cases we get $\lambda^2(a) = \Lambda_\rho^2(\alpha)$ with $\alpha_n = \log\gamma^{-1}(1/a_n)$ and $\rho = \sup_k \rho(k)$, and for the canonical norms $|\ |_t$ in $\Lambda_\rho^2(\alpha)$ we have

$$|Sx|_{\rho(k)} \le C_k\|x\|_{q(k)}, \qquad \|Tx\|_k \le C_k|x|_{\rho(k)}$$

for the maps S, T of Theorem 1.5.

2

In this section we assume that E is a nuclear (DN)-space in standard form, i.e., E is nuclear and a fundamental system of norms $\|\ \|_0 \le \|\ \|_1 \le \cdots$ is given such that, with suitable constants C_k,

$$\|\ \|_k^2 \le C_k\|\ \|_{k-1}\|\ \|_{k+1}$$

for all $k \in \mathbf{N}$. We may assume $C_k = 1$ for all k.

2.1. **Lemma.** *There exist p and a system of scalar products $\langle\ ,\ \rangle_k$ such that, with $|x|_k^2 = \langle x, x\rangle_k$ and certain constants D_k, we have*

$$\|\ \|_k \le |\ |_k \le D_k\|\ \|_{k+p} \quad \textit{for all } k \in \mathbf{N}.$$

Proof. The assumption can be written as

$$\| \ \|_k \leq \tfrac{1}{2}(r\| \ \|_{k-1} + \tfrac{1}{r}\| \ \|_{k+1}) \quad \text{for all } r > 0,$$

which is equivalent to

$$B_k \subset \tfrac{1}{2}(rB_{k-1} + \tfrac{1}{r}B_{k+1}) \quad \text{for all } r > 0,$$

with $B_k = \{y \in E' : \|y\|_k^* \leq 1\}$.

This follows from the bipolar theorem (see [17, p. 126]) since for $x \in E$ the norms on both sides of the inequality are the suprema of x, considered as a function on E', over the sets on the respective sides of the set inclusion.

For $F \subset E'$ and $d > d(B_{k-1}, B_{k+1}; F)$ we obtain

$$B_k \subset \tfrac{1}{2}(rd + \tfrac{1}{r})B_{k+1} + F \quad \text{for all } r > 0,$$

hence taking the infima over the d's and r's

$$\begin{aligned} d(B_k, B_{k+1}; F)^2 &\leq d(B_{k-1}, B_{k+1}; F) \\ &\leq d(B_{k-1}, B_k; F) d(B_k, B_{k+1}; F). \end{aligned}$$

Cancelling equal terms on both sides and taking $\inf$ over all subspaces F with $\dim F \leq n$ we have

$$d_n(B_k, B_{k+1}) \leq d_n(B_{k-1}, B_k) \quad \text{for all } k \text{ and } n.$$

We choose p such that $d_n(B_0, B_p) = O(n^{-3})$ and apply the previous construction to the norm system $(\| \ \|_{lp})_l$. We obtain that $d_n(B_{lp}, B_{(l+1)p}) = O(n^{-3})$ for all l, hence $d_n(B_k, B_{k+2p}) \leq d_n(B_{lp}, B_{(l+1)p}) = O(n^{-3})$ for all k (where $l = [k/p] + 1$).

By [23] there is a Hilbert ball $\widetilde{B}_k$ and a constant C_k for each k such that $B_k \subset \widetilde{B}_k \subset C_k B_{k+2p}$. Setting $| \ |_k = \sup\{|y(x)| ; y \in \widetilde{B}_k\}$ gives the result (with $2p$ instead of p).

We assume that F is an (Ω)-space in standard form, i.e., there is given a fundamental system of seminorms such that, with suitable constants D_k,

$$\| \ \|_k^{*2} \leq D_k \| \ \|_{k-1}^* \| \ \|_{k+1}^* \quad \text{for all } k \in \mathbf{N}.$$

Let $A: F \to E$ be a continuous linear map with $\|Ax\|_k \leq C_k \|x\|_{\sigma(k)}$ for all k. Then we can apply the theory of §1 to E, $H = E$, equipped with the norms $| \ |_k$ of Lemma 2.1, and to F equipped with the seminorms $C_k' \| \ \|_{\sigma(k+p)}$, C_k' a suitable constant, p as in Lemma 2.1.

We have

$$\mu(K, k) + 1 = \frac{K}{k}, \qquad \nu(Q, q) + 1 = \frac{\sigma(Q+p) - \sigma(p)}{\sigma(q+p) - \sigma(p)}$$

and it can easily be seen that condition (τ) means: $\lim_Q \sup \sigma(Q)/Q < +\infty$. Therefore we can restrict our attention to maps A with $\sigma(k) = ak + b$, i.e., to linear tame maps. In this case $\nu(Q, p) + 1 = Q/q$ and we obtain condition (τ) with $q(k) = k + 1$.

From Corollary 1.6 we immediately obtain

2.2. **Theorem.** *If E is a nuclear (DN)-space in standard form, and F a separable (Ω)-space in standard form, then the range of any linear tame map has a basis.*

This applies in particular to linear-tame operators in nuclear power series spaces of infinite type. In this case a direct proof is given in [5].

More detailed information can easily be obtained by evaluations the data in Theorem 1.5. We do this in the case where A is even surjective. In this case separability is not a restriction on F, since we may replace F by $F/\ker A$ with the quotient norms, which is separable since it is isomorphic to E.

The function $\rho(k)$ at the end of §1 can be chosen such that $k \le \rho(k) \le k+1$ for all k. Hence we obtain, with the canonical norms $|\ |_t$ on $\Lambda^2_\infty(\alpha)$,

$$|Sx|_k \le C_k \|x\|_{a(k+p+1)+b}, \qquad \|Tx\|_k \le C_k |x|_{k+1}$$

and, using the notation of Theorem 1.7, we have

$$|T^{-1}x|_k \le \frac{C_k}{\varepsilon_k} \|x\|_{a(\tau(k)+p+1)+b}.$$

Assuming $\tau(k) = a'k + b$, we obtain

2.3. **Theorem.** *Let E be a nuclear (DN)-space in standard form, and F an (Ω)-space in standard form, where $F \xrightarrow{A} E \to 0$ is tame (resp. linear-tame) exact. Then E is tamely (resp. linearly-tamely) equivalent to a power series space $\Lambda_\infty(\alpha)$.*

Again this applies to tame (resp. linear-tame) operators in a power series space of infinite type (cf. [5] and, for a projection which includes tameness estimates, [23]). In particular the range of any tame (resp. linear-tame) projection in a space $\Lambda_\infty(\alpha)$ is tamely (resp. linear-tamely) equivalent to a power series space.

An interesting special case of 2.3. is

2.4. **Corollary.** *Every nuclear E which is tame in the sense of Hamilton* [6] *or belongs to the category $\mathscr{L}$ of Sergeraert* [18] *is tamely equivalent to a power series space $\Lambda_\infty(\alpha)$.*

Proof. In the Sergeraert case we may take $q = \mathrm{id}$, since E is a (DN)-space in standard form and an (Ω)-space in standard form. For the (DN) case see [18, Theorem 2.2.2], and for the (Ω) case use the same argument, applied to S^τ_ϑ.

In the Hamilton case E is tamely equivalent to a subspace of a space $\Sigma(B)$ (see [6]), hence to a (DN)-space in standard form E_0. Moreover we have a tame exact sequence $\Sigma(B) \xrightarrow{q} E_0 \to 0$.

3

In this section we continue and extend the last example. Let E be a nuclear graded Fréchet space admitting a family $(S_\vartheta)_{\vartheta>0}$ of smoothing operators, i.e., continuous linear operators such that, with suitable constants $C_{k,m}$,

$$\|S_\vartheta x\|_k \le C_{k,m}\vartheta^{k+p-m}\|x\|_m \quad \text{for all } m < k+p,\ \vartheta > 0,\ x \in E,$$
$$\|x - S_\vartheta x\|_k \le C_{k,m}\vartheta^{k+p-m}\|x\|_m \quad \text{for all } m \ge k+p,\ \vartheta > 0,\ x \in E.$$

This is a family of smoothing operators in the sense of [14]. It is easily seen that this class of graded Fréchet spaces includes both classes considered in Corollary 2.4.

We introduce the norms $\|x\|_k^\sim = D_k\|x\|_{k^2p}$ on E. Then we have

$$\|S_\vartheta x\|_k^\sim \le C_k\vartheta^{2kp}\|x\|_{k-1}^\sim,$$
$$\|\operatorname{id} - S_\vartheta x\|_k^\sim \le C_k\vartheta^{-2kp}\|x\|_{k+1}^\sim$$

for all k and $\vartheta \ge 1$. Therefore

$$\|x\|_k^\sim \le C_k'(r\|x\|_{k-1}^\sim + \tfrac{1}{r}\|x\|_{k+1}^\sim)$$

for all $r \ge 1$, and even (by proper choice of C_k') for all $r > 0$. Hence E, equipped with the norms $\|\ \|_k^\sim$, is a (DN)-space in standard form.

According to Lemma 2.1, assuming p chosen large enough and by proper choice of D_k, there are scalar products $\langle\ ,\ \rangle_k$ on E such that with $|x|_k^2 = \langle x, x\rangle_k$ we have

$$\|x\|_k^\sim \le |x|_k \le \|x\|_{k+1}^\sim.$$

We apply §1 to the identity maps between $F = (E, \|\ \|_{k+1}^\sim)$, $H = (E, |\ |_k)$, and $E = (E, \|\ \|_k^\sim)$. Conditions (α) and (ω) are satisfied with

$$\mu(K, k) + 1 = \frac{K^2}{k^2+1},$$
$$\nu(Q, q) + 1 = \frac{(Q+1)^2p + p - (q+1)^2p}{(q+1)^2p - 2p} + 1 = \frac{Q^2 + 2Q}{q^2 + 2q - 1}.$$

Hence (τ) is fulfilled with $q = q(k)$ such that $k^2 + 1 < q^2 + 2q - 1$. We may choose $q(k) = k$ for $k = 2, 3, \ldots$.

The $\rho(k)$ at the end of §1 can be chosen such that $k^2 + 1 < \rho(k) < k^2 + 2$. We choose (different from §1) $\alpha_n = p^{-1}\log\gamma^{-1}(1/a_n)$. From Theorem 1.5 we obtain linear maps $S\colon E \to \Lambda_\infty^2(\alpha)$ and $T\colon \Lambda_\infty^2(\alpha) \to E$ such that

$$|Sx|_{(k^2+1)p} \le C_k\|x\|_{(k+1)^2p},$$
$$\|Tx\|_{k^2p} \le C_k|x|_{(k^2+2)p},$$

where $|\ |_t$ denotes the canonical norms in $\Lambda_\infty(\alpha)$ and $\|\ \|_k$ denotes the original norms in E.

We denote Lemma 1.4 on interpolation first to $S\colon E \to \Lambda_\infty^2(\alpha)$ with respect to the norms $|\ |_p \le |\ |_k \le |\ |_{(K^2+1)p}$ (for $p < k$ and large K) on $\Lambda_\infty^2(\alpha)$ and $\|\ \|_p \le \|\ \|_q \le \|\ \|_{(K+1)^2p}$ on E, with $\psi(r) = r^{\mu+1}$, $\varphi(r) = r^{\nu+1}$, $\mu + 1 =$

$((K^2+1)p-p)/(k-p)$, $\nu+1=(K+1)^2p/(q-2p)$, and $q=k+p+1$. It yields a constant D_k such that $|Sx|_k \le D_k\|x\|_{k+p+1}$ for all $x \in E$.

Next we apply Lemma 1.4 to $T\colon \Lambda^2_\infty(\alpha) \to E$ with respect to the norms $\|\ \|_0 \le \|\ \|_k \le \|\ \|_{K^2p}$ (for $0<k$ and large K) on E and $|\ |_p \le |\ |_q \le |\ |_{(K^2+2)p}$ on $\Lambda^2_\infty(\alpha)$ with ψ, φ as above, $\mu+1=K^2p/(k+p)$, $\nu+1=((K^2+2)p-p)/(q-p)$, and $q=k+2p+1$.

We obtain

3.1. **Theorem.** *A nuclear graded Fréchet space admitting a family of smoothing operators (as defined in the beginning of this section) is tamely isomorphic to a power series space of infinite type.*

This clearly contains the result of Corollary 2.4. It should also be remarked that we needed only that

$$\|x\|_k \le C(\vartheta^{k+p-m}\|x\|_m + \vartheta^{k+p-K}\|x\|_K),$$
$$U_q \subset C(\vartheta^{m+p-q}U_m + \vartheta^{Q+p-q}U_Q)$$

for $m<k+p<K$ (resp. $m<q-p<Q$), $\vartheta>0$, and $C=C_{m,k,K}$ (resp. $C=C_{m,q,Q}$).

If the norms $\|\ \|_k$ are already given by scalar products, we can take $E=H=F$ and $A_1=A_2=\mathrm{id}$, and have (α) with $\mu(K,k)+1=K/(k+p)$, (ω) with $\mu(Q,q)+1=Q/(q-p)$, and (τ) with $q(k)=q+2p+1$. In this case we only need that E is a Schwartz space.

4

We show now that the results of §1 also include cases which are connected with finite type power series spaces. In particular they contain and improve [21, Satz 1.1], which is crucial for the structure theory of nuclear power series spaces of finite type. Therefore they also contain the results of Mityagin-Henkin [13, Theorem 1.1] (cf. Mityagin [12]) and Mityagin [11].

The following properties $(\underline{DN})$ and $(\overline{\Omega})$ characterize in the (strongly) nuclear case the subspaces and the quotient spaces of power series spaces of finite type (see [4, 20, 21, 25, 26]). E denotes a Fréchet space with a fundamental system $\|\ \|_0 \le \|\ \|_1 \le \cdots$ of seminorms.

$(\underline{DN})$ There exists p such that for every k there are K, $\mu>0$ and $M>0$ such that for all $x \in E$

$$\|x\|_k^{\mu+1} \le M\|x\|_p^{\mu}\|x\|_K.$$

$(\overline{\Omega})$ For every p there exists q such that for all Q we have $D>0$ such that for all $y \in E'$

$$\|y\|_q^{*2} \le D\|y\|_p^*\|y\|_Q^*.$$

It is easy to see that both properties are independent of the fundamental system of seminorms and that $(\overline{\Omega})$ is equivalent to the following conditions:

$$\left.\begin{array}{l}\forall p\,,\ \nu > 0\ \exists q\ \forall Q\ \exists D > 0\ \forall y \in E' \\ \exists \nu > 0\ \forall p\ \exists q\ \forall Q\ \exists D > 0\ \forall y \in E'\end{array}\right\} \|y\|_q^{*1+\nu} \leq D\|y\|_p^{*\nu}\|y\|_Q^{*}.$$

E is called hilbertizable if it admits a fundamental system of seminorms defined by semiscalar products.

4.1. **Theorem.** *If E is a hilbertizable Schwartz space with properties $(\underline{DN})$ and $(\overline{\Omega})$ then $E \cong \Lambda_0^2(\alpha)$ for suitable α.*

Proof. Since $(\underline{DN})$ and $(\overline{\Omega})$ are independent of the fundamental system of seminorms, we may assume that $\| \; \|_k$ is defined by a semiscalar product. Moreover, we may assume that $p = 0$ in the definition of $(\underline{DN})$. We apply §1 to $E = H = F$. Then $(\underline{DN})$ gives (α) and $(\overline{\Omega})$ gives (ω). The first of the equivalent forms of $(\overline{\Omega})$ implies that $\lim_q \sup_Q \nu_0(Q, q) = 0$. This shows (τ). Hence Theorem 1.7 tells us that $E \cong \lambda^2(a)$ and Wagner [26] tells us that $\lambda^2(a) \cong \Lambda_0^2(\alpha)$.

The latter also follows by proper choice of γ_k. Assuming E is not Banach, setting $\beta_q = (1+q)(\sup_Q \nu_0(Q, q) + 1)/q$, and choosing $q = q(k)$ such that $\beta_q < k(\mu_0(Q, k) + 1)/(1+k)$, $Q = Q(k)$ large enough, we may put $\gamma_k(r) = r^{\beta_{q(k)}}$ and obtain $a_{n,k} = (1/a_n)^{1/\beta_{q(k)}}$. Hence with $\rho_k = 1/\beta_{q(k)} \nearrow 1$ and $\alpha_n = -\log a_n$ we have $E \cong \Lambda_1^2(\alpha) \cong \Lambda_0^2(\alpha)$.

The previous theorem generalizes Theorem 1.1 in [21] and leads to a structure theory of power series spaces of finite type in nonnuclear cases (in analogy to [21, §2]). This will be carried out elsewhere. It should also be remarked that for hilbertizable E or F, one of them a Schwartz space such that E has property (DN) and F has property $(\tilde{\Omega})$ (see [22]), $q(k)$ in (τ) can be chosen independent of k. Hence in this case every continuous linear operator from F to E is compact. However, the method of [22, §6] gives a better result.

5

We close this paper by briefly showing the consequences of §1 to endomorphisms of power series spaces of infinite type. The results should be compared with those in [5, 23]. The advantage is that we do not need nuclearity.

Let A be a linear map in $\Lambda_\infty^2(\alpha)$ with $\|Ax\|_k \leq C_k\|x\|_{ak+b}$. Then we may argue as in the proof of Theorem 2.2 and obtain the following theorem from Corollary 1.6, Theorem 1.7, and the open mapping theorem (cf. [5]).

5.1. **Theorem.** *The range of A has a basis. If the range is closed, then it is isomorphic to a space $\Lambda_\infty^2(\beta)$.*

In Theorem 2.3 we may replace the assumptions on E by the assumption that it is closed subspace of $\Lambda_\infty^2(\alpha)$. This leads in the case of a projection to (cf. [5, 23])

5.2. **Theorem.** *The range of a tame (resp. linear tame) projection in* $\Lambda^2_\infty(\alpha)$ *is tamely (resp. linear tamely) equivalent to a power space* $\Lambda^2_\infty(\beta)$.

It should be noticed that in the nuclear case $\Lambda^2_\infty(\alpha)$ is tamely equivalent to $\Lambda_\infty(\alpha)$.

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References

1. J. Bergh and J. Löfström, *Interpolation spaces*, Springer, 1976.
2. P. B. Djakov and B. S. Mityagin, *Modified construction of a nuclear Fréchet space without basis*, J. Funct. Anal. **23** (1976), 415–423.
3. E. Dubinsky, *The structure of nuclear Fréchet spaces*, Lecture Notes in Math., vol. 720, Springer, 1979.
4. E. Dubinsky and D. Vogt, *Bases in complemented subspaces of power series spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **34** (1986), 65–67.
5. ——, *Complemented subspaces in tame power series spaces*, Studia Math. **93** (1989), 71–85.
6. R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), 65–222.
7. G. Köthe, *Topologische lineare Räume*, Springer, 1960.
8. J. Krone, *Bases in the range of operators between Köthe spaces*, Doga Math. Ser. **10** (1986), 162–166.
9. ——, *Basisprobleme in Fréchetraümen*, Dissertation, Wuppertal, 1986.
10. ——, *On projections in power series spaces and the existence of bases*, Proc. Amer. Math. Soc. **105** (1989), 350–355.
11. B. S. Mityagin, *Sur l'équivalences des bases inconditionelles dans les échelles de Hilbert*, C. R. Acad. Sci. Paris **269** (1969), 426–428.
12. ——, *The equivalence of bases in Hilbert scales*, Studia Math. **37** (1970), 111–137. (Russian)
13. B. S. Mityagin and G. R. Henkin, *Linear problems of complex analysis*, Russian Math. Surveys **26** (1971), 99–164.
14. J. Moser, *A new technique for the construction of solutions of nonlinear differential equations*, Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 1824–1831.
15. A. Pełczyński, Problem 37, Studia Math. **38** (1970), 476.
16. A. Pietsch, *Nuclear locally convex spaces*, Ergebnisse der Math., no. 66, Springer, 1972.
17. H. H. Schaefer, *Topological vector spaces*, Macmillan, 1966.
18. F. Sergeraert, *Un théorème de fonctions implicites sur certains espaces de Fréchet et quelques applications*, Ann. Sci. École Norm. Sup. (4) **5** (1972), 599–660.
19. D. Vogt, *Charakterisierung der Unterräume von* s, Math. Z. **155** (1977), 109–117.
20. ——, *Charakterisierung der Unterräume eines nuklearen stabilen Potenzreihenraumes von endlichem Typ*, Studia Math. **71** (1982), 251–270.
21. ——, *Eine Charakterisierung der Pontenzreihenräume von endlichem Typ und ihre Folgerungen*, Manuscripta Math. **37** (1982), 269–301.
22. ——, *Fréchetträume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. Reine Angew. Math. **345** (1983), 182–200.
23. ——, *Tame spaces and power series spaces*, Math. Z. **196** (1987), 523–536.
24. D. Vogt and M. J. Wagner, *Charakterisierung der Quotientenräume von* s *und eine Vermutung von Martineau*, Studia Math. **67** (1980), 225–240.

25. M. J. Wagner, *Unterräume und Quotienten von Potenzreihenräumen*, Dissertation, Wuppertal, 1977.

26. ——, *Quotientenräume von stabilen Potenzreihenräumen endlichen Typs*, Manuscripta Math. **31** (1980), 97–109.

27. ——, *Stable complemented subspaces of* (s) *have a basis*, Seminar Lecture, AG Funktionanalysis Düsseldorf/Wuppertal, 1985.

28. N. M. Zobin and B. S. Mityagin, *Examples of nuclear linear metric spaces without a basis*, Funct. Anal. Appl. **8** (1975), 304–313.

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