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THE METHOD OF NEGATIVE CURVATURE: THE KOBAYASHI METRIC ON $\mathbf{P}_2$ MINUS 4 LINES

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ABSTRACT. Bloch, and later H. Cartan, showed that if $H_1, \ldots, H_{n+2}$ are $n+2$ hyperplanes in general position in complex projective space $\mathbf{P}_n$, then $\mathbf{P}_n - H_1 \cup \cdots \cup H_{n+2}$ is (in current terminology) hyperbolic modulo Δ, where Δ is the union of the hyperplanes $(H_1 \cap \cdots \cap H_k) \oplus (H_{k+1} \cap \cdots \cap H_{n+2})$ for $2 \leq k \leq n$ and all permutations of the H_i. Their results were purely qualitative. For $n = 1$, the thrice-punctured sphere, it is possible to estimate the Kobayashi metric, but no estimates were known for $n \geq 2$. Using the method of negative curvature, we give an explicit model for the Kobayashi metric when $n = 2$.

Pour arriver à des resultats d'ordre quantatif,
il faudrait construire... la fonction modulaire.
Henri Cartan 1928

0. INTRODUCTION

For a compact complex manifold M, Brody [5] has given a simple criterion for hyperbolicity: M is hyperbolic if and only if every holomorphic map $f: \mathbf{C} \to M$ is constant. For noncompact manifolds, hyperbolicity is much more difficult to establish, as Brody's Theorem is false (Green's example, see [14]). The situation is even more complicated for hyperbolicity modulo a subset, the most general case.

For example, let D be a divisor on $\mathbf{P}_n$, let $X = \mathbf{P}_n - D$, and let Y be a proper subvariety of $\mathbf{P}_n$. We say that X is *hyperbolic modulo* Y, if the Kobayashi distance d_X satisfies $d_X(x, y) > 0$ for all x, y distinct and not both in Y. If every nonconstant holomorphic map $f: \mathbf{C} \to X$ lies in the subvariety Y, then X is called *Brody hyperbolic modulo* Y [14]. It is unknown if Brody hyperbolicity modulo Y implies hyperbolicity modulo Y when $X = \mathbf{P}_n - D$; a modification of Green's example shows this is false if $\mathbf{P}_n$ is replaced by an arbitrary variety. Furthermore, the results so far for hyperbolicity modulo Y are mostly qualitative, giving little insight into the quantitative behavior of the Kobayashi metric, which approaches 0 near Y in the tangent directions to Y.

In this paper we give an explicit model for the Kobayashi metric on a particular X, namely $\mathbf{P}_2$ minus 4 lines. We prove that the model does indeed

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approximate the Kobayashi metric by using the method of negative curvature, an approach initiated by Ahlfors [1] in the late 1930's to study the Poincaré metric on Riemann surfaces. This is a very powerful method, but difficult to apply to manifolds, since a metric which models the Kobayashi metric need not necessarily have negative curvature; its curvature must be played off against the curvature of metrics involving higher order jets. Thus, in addition to constructing models for the Kobayashi metric, we must also construct jet pseudo-metrics, so that the entire family of metrics—including the approximating metric we actually want—is negatively curved.

The difficulty with this approach is that any modification to a particular jet metric in order to achieve negative curvature of the family may require corresponding changes to all the other jet metrics. This is a process which may never converge to yield negative curvature of the family. Nevertheless, when the process does give negative curvature, as in our case, we can then extend Ahlfors' method to make an end run around Cartan's dictum [6]; we can achieve explicit quantitative results without uniformizing an n-dimensional manifold.

The Kobayashi metric on $\mathbf{P}_n$ minus hyperplanes is subject with a history going back to the turn of the century—well before there was a Kobayashi metric. It begins with E. Borel's generalization of the Little Picard Theorem [4]:

Theorem 0.1 (Borel). *Let $f\colon \mathbf{C} \to \mathbf{P}_n - H_1 \cup \cdots \cup H_{n+2}$ be holomorphic and nonconstant, where the H_i are hyperplanes in general position. Then f lies in a hyperplane.*

Remark 0.2. Inductive use of Borel's Theorem shows that f must lie in one of a finite number of hyperplanes: the hyperplanes $(H_1 \cap \cdots \cap H_k) \oplus (H_{k+1} \cap \cdots \cap H_{n+2})$ for $2 \leq k \leq n$ and permutations thereof, where we view the H_i as being in $\mathbf{C}^{n+1}$, take $\oplus$ there, then projectivize. The union of these hyperplanes is Δ. For example, when $n = 2$, Δ consists of three lines, the diagonals of the quadrilateral formed by $H_1, \ldots, H_4$.

The history continues with the results of Bloch [3] and then H. Cartan [6], which generalize the Schottky-Landau Theorem:

Theorem 0.3 (Bloch, H. Cartan). *Let $f\colon \mathbf{D}_r \to \mathbf{P}_n - H_1 \cup \cdots \cup H_{n+2}$ be holomorphic. Then r is bounded in terms of $f(0)$ and $f'(0)$, as long as $f(0)$ is not in the exceptional set Δ.*

Kiernan and Kobayashi [11] present the results–but not the proofs—of Bloch and Cartan in modern terms:

Theorem 0.4 (H. Cartan, via Kiernan and Kobayashi). *$\mathbf{P}_n - H_1 \cup \cdots \cup H_{n+2}$ is hyperbolic modulo Δ.*

These results are difficult to generalize, since the methods of Bloch and Cartan are very technical and more than a little obscure (see [14, VIII, §4]). Furthermore, their results are purely qualitative. They have no explicit bounds for the radius r, that is, there are no estimates for the Kobayashi metric, other than that it is positive off Δ. Our results are also technical; that is, after all, the nature of the subject. Nonetheless, our primary goal is to illustrate a method—negative curvature for families of jet metrics—that can be made to yield quantitative results about the Kobayashi metric.

The plan of this paper is as follows:

1. *Negative curvature and the Kobayashi metric*: the Ricci formalism; Ahlfors' Lemma and the method of negative curvature; estimating the infinitesimal Kobayashi metric using negatively curved families of metrics.

2. *Motivation for the model*: using the Kobayashi metric on $\mathbf{P}_1$ minus 3 points to motivate the model of the metric on $\mathbf{P}_2$ minus 4 hyperplanes.

3. *Distance functions and calculation of Ricci forms*: notation; geometry of holomorphic curves in $\mathbf{P}_n$; Ricci forms for distance to hyperplanes.

4. *The model for the Kobayashi metric*: modifications of the $\mathbf{P}_1$ metric which lead to the model for the metric on $\mathbf{P}_2$. The main result is stated in Theorem 4.4.

5. *Construction of a negatively curved family*: modification of a negatively curved 2-jet metric so that the model becomes part of a negatively curved family of metrics.

6. $\mathbf{P}_n$ *minus hyperplanes*: why $\mathbf{P}_n$ is harder than $\mathbf{P}_2$.

General references are [12–14].

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1. Negative curvature and the Kobayashi metric

Let M be a complex manifold and let $\mathbf{D}_r$ denote the disc of radius r in $\mathbf{C}$. As in [15], we define the *infinitesimal Kobayashi metric* F_M on the holomorphic tangent bundle T_M as follows: if $x \in M$ and $\xi \in T_x$, then

$$F_M(x, \xi) = \inf r^{-2}, \tag{1.0.1}$$

where the infimum is taken over the set of all r for which there is a holomorphic map $f: \mathbf{D}_r \to M$ with $f(0) = x$, $f'(0) = \xi$. Note that F_M satisfies the defining property for a *differential metric*:

$$F_M(x, \lambda\xi) = |\lambda|^2 F_M(x, \xi) \quad \text{for all } \lambda \in \mathbf{C}. \tag{1.0.2}$$

In addition, the Kobayashi metric is *distance decreasing*, that is, if $g: M \to N$ is a holomorphic map, and $g^* F_N$ is the differential metric induced on M by $g^* F_N(x, \xi) = F_N(g(x), g_* \xi)$, then

$$g^* F_N \leq F_M. \tag{1.0.3}$$

We say M is *hyperbolic* if M has a Hermitian metric such that $F_M(x, \xi) \geq \|\xi\|$ for all $x \in M$ and $\xi \in T_x$; this corresponds to M being a metric space in the Kobayashi distance d_M, the integrated version of F_M.

Note that by Schwarz's Lemma, the Kobayashi metric on $\mathbf{D}_r$ equals the Poincaré metric ds_r^2, where

$$ds_r^2 = \frac{r^2|dz|^2}{(r^2 - |z|^2)^2}. \tag{1.0.4}$$

The following trivial lemma, describing how to verify a lower bound on the Kobayashi metric, lies behind the method of negative curvature.

Lemma 1.1. *Let $F(x, \xi)$ be any differential metric on M. Then $F \leq F_M$ on T_M if and only if $f^*F \leq ds_r^2$ for all $f\colon \mathbf{D}_r \to M$ and all $r > 0$.*

Thus a metric gives a lower bound on the Kobayashi metric if and only if the pulled-back metric is always less than the Poincaré metric. Ahlfors gave us a method for checking this condition; the Ricci formalism, which we describe now, is due to Griffiths.

Let $ds^2 = h(z)|dz|^2$ be a C^2 Hermitian pseudo-metric on $\mathbf{D}_r$, that is, $h(z) \geq 0$ and $h(z) = 0$ only at isolated points. Let

$$\omega = \tfrac{i}{\pi} h(z)\, dz\, d\bar{z} \tag{1.1.1}$$

be the *associated* $(1, 1)$-*form* on $\mathbf{D}_r$. Define the *Ricci form* $\operatorname{Ric}\omega$ by

$$\operatorname{Ric}\omega = dd^C \log h, \tag{1.1.2}$$

where $d^c = \frac{i}{4\pi}(\overline{\partial} - \partial)$, so that $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$. Note that the Gaussian curvature $K = -\{\Delta \log h\}/\{2h\}$ of the Hermitian metric ds^2 can be expressed in terms of the Ricci form by

$$\operatorname{Ric}\omega = -(K/4)\omega. \tag{1.1.3}$$

Thus, in keeping with the Gaussian curvature interpretation, we say that ω is *negatively curved* if $\operatorname{Ric}\omega \geq \omega$. In particular, if

$$\omega_r = \tfrac{i}{\pi} r^2(r^2 - |z|^2)^{-2}\, dz\, d\bar{z} \tag{1.1.4}$$

denotes the associated form for the Poincaré metric, then $\operatorname{Ric}\omega_r = \omega_r$. That is, the Gaussian curvature of the normalized Poincaré metric is -4.

Ahlfors' Lemma [1, 12] gives the principal criterion for obtaining lower bounds on the Kobayashi metric.

Lemma 1.2 (Ahlfors). *Let $ds^2 = h(z)|dz|^2$ be a Hermitian pseudo-metric on $\mathbf{D}_r$, $h \in C^2(\mathbf{D}_r)$, with ω the associated $(1, 1)$-form. If $\operatorname{Ric}\omega \geq \omega$ on $\mathbf{D}_r$, then $\omega \leq \omega_r$ on all of $\mathbf{D}_r$ (or equivalently, $ds^2 \leq ds_r^2$).*

Combining Lemmas 1.1 and 1.2 we arrive at the following strategy to verify a lower bound on the Kobayashi metric on a manifold M:

Proposition 1.4 (Negative curvature for a single metric). *Let F be a differential metric on T_M. To show that $F \leq F_M$, it suffices to show that $\operatorname{Ric}\omega_f \geq \omega_f$ for* *every* $f\colon \mathbf{D}_r \to M$ *holomorphic, where* ω_f *is the associated form to the pull-back metric* f^*F.

Unfortunately, $\operatorname{Ric}\omega_f \geq \omega_f$ rarely seems to occur. Ahlfors [1] gave a variant of his lemma which is more useful for our purposes.

Definition 1.4. Let $\{ds_1^2, \dots, ds_k^2\}$ be a family of Hermitian pseudo-metrics with associated $(1, 1)$-forms $\omega^1, \dots, \omega^k$. The family is *negatively curved* if for each z in $\mathbf{D}_r$ there exists an ω^i such that ω^i is maximal at z and $\operatorname{Ric}\omega^i \geq \omega^i$ at z.

Lemma 1.5 (Ahlfors' Lemma for negatively curved families). *Let* $\{ds_1^2, \dots, ds_k^2\}$ *be a negatively curved family of metrics on* $\mathbf{D}_r$, *with associated forms* $\omega^1, \dots, \omega^k$. *Then* $\omega^i \leq \omega_r$ *for all* i.

Thus we generalize the method above to include, in effect, the pull-backs of higher order jet metrics:

Proposition 1.6 (Negative curvature for families). *Let* F *be a differential metric on* T_M. *To show that* $F \leq F_M$, *it suffices to show there exists a negatively curved family* $\{\omega_f, \omega_{f''}, \dots, \omega_{f^{(n)}}\}$ *for every* $f\colon \mathbf{D}_r \to M$ *holomorphic, where* ω_f *is the associated form to the pull-back* f^*F *and* $\omega_{f''}, \dots, \omega_{f^{(n)}}$ *are* $(1, 1)$*-forms depending on higher order derivatives.*

2. Motivation for the model

In this section we describe a model for the Kobayashi metric on $\mathbf{P}_1 - \{a_1, a_2, a_3\}$, where the a_i are distinct, and begin the discussion of how to modify this model for the Kobayashi metric on $\mathbf{P}_2$ minus four hyperplanes.

Carlson and Griffiths [7] construct volume forms with negative curvature on manifolds minus divisors; restricted to $\mathbf{P}_1 - \{a_1, a_2, a_3\}$, these give a metric whose associated form ω has negative curvature. Their construction of a $(1, 1)$-form Ψ_0 is very explicit:

$$\Psi_0 = c\Pi_{1\leq i\leq 3}\{\varphi_0(a_i)[1/\varepsilon - \log\varphi_0(a_i)]^2\}^{-1}\Omega, \tag{2.0.1}$$

where c is a constant depending only on the a_i and ε (a small positive number), $\varphi_0(a_i)$ is the function on $\mathbf{P}_1$ which measures the chordal distance between z and a_i, and Ω is the $(1, 1)$-form associated to the Fubini-Study metric,

$$\Omega = \tfrac{i}{2\pi}\, dz\, d\bar{z}/(1 + |z|^2)^2, \tag{2.0.2}$$

in local coordinates. They designed this form along the lines of the Poincaré metric $\frac{i}{\pi}\, dz\, d\bar{z}/\{|z| \log|z|^2\}^2$ on the punctured disc, so that Ψ_0 blows up properly at the a_i. It is not hard to show that Ψ_0 has negative curvature; in fact

$$\operatorname{Ric}\Psi_0 \geq \Psi_0 + (1 - 12\varepsilon)\Omega. \tag{2.0.3}$$

Thus the metric corresponding to Ψ_0 furnishes a lower bound for the Kobayashi metric.

We want to modify this metric to give a model on

$$M = \mathbf{P}_2 - H_1 \cup \cdots \cup H_4,$$

where the H_i are hyperplanes in general position. We begin by restricting to a hyperplane L in $\mathbf{P}_2$, where L is distinct from the H_i, so that $L \cap M = L - \{a_1, \ldots, a_4\}$. Note that the a_i are not necessarily distinct; for some L there will be only three distinct points, and for three hyperplanes L—the ones that go through $H_i \cap H_j$ and $H_k \cap H_l$—there are only two points.

By the distance decreasing property (1.0.3), any lower bound for the Kobayashi metric on M would have to give a lower bound on the Kobayashi metric on $L - \{a_1, \ldots, a_4\}$. As long as L is not one of the three exceptional hyperplanes, we can pick three distinct points, a_1, a_2, and a_3, and look for a lower bound of the form given by Ψ_0 in (2.0.1). With a little work (see Lemma 5.2 below), we can see that the constant in (2.0.1) has the form

(2.0.4) $$c = 2\kappa_0^2 \varepsilon^{-4},$$

where κ_0 is a constant depending on the a_i. Of course, the a_i are not constant in our case; they depend on the hyperplane L. In fact, κ_0 satisfies $\varphi_0(a_i) \geq \kappa_0$ for all but at most one of $i = 1, 2, 3$ (cf. (5.1.3) below). Thus

(2.0.5) $$\kappa_0 \text{ goes to zero as } L \text{ goes through } H_i \cap H_j,$$

for $\{i, j\} \subset \{1, 2, 3\}$.

3. Distance functions and calculation of Ricci forms

We begin this section with the geometry of holomorphic curves in $\mathbf{P}_n$.

For $v \neq 0$ in $\mathbf{C}^{n+1}$, we will denote by $[v]$ the line spanned by v, so that $[v] \in \mathbf{P}_n$. The usual Hermitian inner product on $\mathbf{C}^{n+1}$ will be denoted by $\langle\ ,\ \rangle$. If $K \in \bigwedge^k \mathbf{C}^{n+1}$ and $L \in \bigwedge^l \mathbf{C}^{n+1}$, with $k \geq l$, then their *interior product*, denoted $\langle K, L\rangle$, is the unique $(k-l)$-vector satisfying

(3.0.1) $$\langle\langle K, L\rangle, M\rangle = \langle K, L \wedge M\rangle,$$

for all $M \in \bigwedge^{k-l} \mathbf{C}^{n+1}$. We will write $\|K, L\|$ for $\|\langle K, L\rangle\|$, so that if $v_0, \ldots, v_n$ is an orthonormal basis for $\mathbf{C}^{n+1}$ and A is a unit vector, then

(3.0.2) $$\|v_0 \wedge \cdots \wedge v_k, A\|^2 = |\langle v_0, A\rangle|^2 + \cdots + |\langle v_k, A\rangle|^2$$

is the norm of the projection of A on the subspace spanned by $v_0, \ldots, v_k$.

Let $f: \mathbf{D}_r \to \mathbf{P}_n$ be a holomorphic curve, and let f be represented by $F: \mathbf{D}_r \to \mathbf{C}^{n+1} - \{0\}$, that is, $f(z) = [F(z)]$. Let

(3.0.3) $$\Lambda_i = F \wedge \cdots \wedge F^{(i)};$$

then $f_{(i)} = [\Lambda_i]$ is the ith *associated curve*, $f_{(i)}: D_r \to \mathbf{P}(\bigwedge^{i+1} \mathbf{C}^{n+1})$. Note that $f_{(i)}$ does not depend on the choice of F. The curve f is *nondegenerate* if f lies in no hyperplane, or equivalently, if $\Lambda_n \neq 0$.

We put

$$\Omega_i = dd^c \log \|\Lambda_i\|^2, \tag{3.0.4}$$

the pull-back of the Fubini-Study metric on $\mathbf{P}(\bigwedge^{i+1} \mathbf{C}^{n+1})$ via $f_{(i)}$; this is independent of the choice of F. For convenience, we shall write $\Lambda_{-1} = 1$ and $\Omega_{-1} = 0$. Lemma 4.16 of [10] (cf. [2, 8, 16, 17]) determines $\operatorname{Ric}\Omega_i$:

Lemma 3.1. *If $f\colon \mathbf{D}_r \to \mathbf{P}_n$ is a holomorphic curve with $\Lambda_i \neq 0$, then*

$$\Omega_i = \tfrac{i}{2\pi}\{\|\Lambda_{i+1}\|^2 \|\Lambda_{i-1}\|^2 / \|\Lambda_i\|^4\}\, dz\, d\bar{z} \tag{3.1.1}$$

and thus

$$\operatorname{Ric}\Omega_i = \Omega_{i+1} - 2\Omega_i + \Omega_{i-1}, \tag{3.1.2}$$

for $0 \le i \le n$.

If A is a unit vector in $\mathbf{C}^{n+1}$, then $A^{\perp}$ is the hyperplane in $\mathbf{P}_n$ of all lines orthogonal to A. We define the jth distance function $\varphi_j = \varphi_j(A)$ by

$$\varphi_j = \|\Lambda_j, A\|^2 / \|\Lambda_j\|^2, \tag{3.1.3}$$

for all j such that Λ_j is not identically 0. This measures how close Λ_j is to lying in $A^{\perp}$. Note that if f is nondegenerate then φ_n is identically 1, and

$$0 \le \varphi_j \le \varphi_{j+1} \le 1. \tag{3.1.4}$$

In order to define metrics modeled after (2.0.1), we define $L_i = L_i(A, \varepsilon)$ by

$$L_i = 1/\varepsilon - \log \varphi_i, \tag{3.1.5}$$

where $1 \ge \varepsilon > 0$, and set

$$\Phi_j(A) = \varphi_{j+1}(A)\varphi_j(A)^{-1} L_j(A)^{-2}. \tag{3.1.6}$$

The main technical results we need from [10] are contained in the following lemma:

Lemma 3.2. *Let $f\colon \mathbf{D}_r \to \mathbf{P}_n - A^{\perp}$ be a holomorphic curve. Then*

$$dd^c \log \varphi_j = \{\varphi_{j+1}\varphi_{j-1}/\varphi_j^2\}\Omega_j - \Omega_j, \tag{3.2.1}$$

and

$$dd^c \log L_j^{-2} \ge 2\Phi_j \Omega_j - 4\varepsilon\Omega_j. \tag{3.2.2}$$

Proof. The first result is Lemma 5.17 of [10]. To prove (3.2.2), note that the calculation leading to [10, (5.21)] implies that

$$\begin{aligned} dd^c \log L_j^{-2} &\ge 2\{(\varphi_{j+1} - \varphi_j)\varphi_j^{-1} L_j^{-2} - L_j^{-1}\}\Omega_j \\ &\ge 2\varphi_{j+1}\varphi_j^{-1} L_j^{-2}\Omega_j - 2\{L_j^{-2} + L_j^{-1}\}\Omega_j. \end{aligned}$$

Now use that $L_j \ge 1/\varepsilon$. □

4. The model for the Kobayashi metric

According to the heuristic argument in (2.0.5), we must include in our model terms that go to zero when the tangent to the curve points to the intersection of two hyperplanes. So for A_1 and A_2 independent unit vectors, define $\varphi_1(A_1 \wedge A_2)$ by

$$\varphi_1(A_1 \wedge A_2) = \|\Lambda_1 , A_1 \wedge A_2\|^2 / \|\Lambda_1\|^2 . \tag{4.0.1}$$

Then

$$0 \le \varphi_1(A_1 \wedge A_2) \le \|A_1 \wedge A_2\|^2 \le 1 , \tag{4.0.2}$$

and $\varphi_1(A_1 \wedge A_2)$ measures the distance from the tangent line to the intersection of the hyperplanes $A_1^{\perp}$ and $A_2^{\perp}$.

Lemma 4.1. *Let $f\colon \mathbf{D}_r \to \mathbf{P}_n - A_1^{\perp} \cup A_2^{\perp}$ be a nonconstant holomorphic curve, with $A_1^{\perp}$ and $A_2^{\perp}$ distinct. Then*

$$\varphi_1(A_1)\varphi_0(A_2) + \varphi_1(A_2)\varphi_0(A_1) \ge \tfrac{1}{2}\varphi_1(A_1 \wedge A_2) . \tag{4.1.1}$$

Proof. Since f is nonconstant, Λ_1 is not identically 0; it suffices to prove the lemma at a point z_0 where Λ_1 is nonzero. Choose v_1 and v_2 orthonormal so that $v_1 \wedge v_2 = \Lambda_1(z_0)/\|\Lambda_1(z_0)\|$. Then we must show

$$\begin{aligned}\{\|v_1 , A_1\|^2 + \|v_2 , A_1\|^2\}\|v_1 , A_2\|^2 + \{\|v_1 , A_2\|^2 + \|v_2 , A_2\|^2\}\|v_1 , A_1\|^2 \\ \ge \tfrac{1}{2}|\langle v_1 , A_1\rangle\langle v_2 , A_2\rangle - \langle v_1 , A_2\rangle\langle v_2 , A_1\rangle|^2 .\end{aligned}$$

Now use that if a, b, c, and d are complex numbers, then the parallelogram law implies that $2|a|^2|d|^2 + 2|b|^2|c|^2 \ge |ad - bc|^2$. □

For short we will write

$$P(A_1 , A_2 , A_3) = \varphi_1(A_i \wedge A_j)\varphi_1(A_i \wedge A_k)\varphi_1(A_j \wedge A_k) ; \tag{4.1.2}$$

$P(A_1 , A_2 , A_3)$ is 0 if and only if the tangent to $f(z)$ points towards one of the $A_i^{\perp} \cap A_j^{\perp}$.

We need the following lemma to compute the Ricci form for our model.

Lemma 4.2. *Let $f\colon \mathbf{D}_r \to \mathbf{P}_n - A_1^{\perp} \cup A_2^{\perp} \cup A_3^{\perp}$ be a nonconstant holomorphic curve, with $A_1^{\perp}$, $A_2^{\perp}$, and $A_3^{\perp}$ distinct. Then*

$$\sum_{i=1}^{3} \frac{\varphi_1(A_i)}{\varphi_0(A_i)} \ge \frac{1}{8}\frac{P(A_1 , A_2 , A_3)}{\varphi_0(A_1)\varphi_0(A_2)\varphi_0(A_3)} . \tag{4.2.1}$$

Proof. Assume that we have renumbered so that $\varphi_0(A_3) \ge \varphi_0(A_i)$ at z_0 in $\mathbf{D}_r$ for $i = 1 , 2$. Then

$$2\varphi_0(A_3) \ge \varphi_0(A_i) + \varphi_0(A_3) \ge \varphi_1(A_i)\varphi_0(A_3) + \varphi_1(A_3)\varphi_0(A_i) ,$$

which implies that

$$\varphi_0(A_3) \ge \tfrac{1}{4}\varphi_1(A_i \wedge A_3) \ge \tfrac{1}{4}\varphi_1(A_1 \wedge A_3)\varphi_1(A_2 \wedge A_3) . \tag{4.2.2}$$

Thus

$$\sum_{i=1}^{3} \frac{\varphi_1(A_i)}{\varphi_0(A_i)} \geq \sum_{i=1}^{2} \frac{\varphi_1(A_i)}{\varphi_0(A_i)} \geq \frac{1}{2} \frac{\varphi_1(A_1 \wedge A_2)}{\varphi_0(A_1)\varphi_0(A_2)}$$
$$\geq \frac{1}{8} \frac{\varphi_1(A_1 \wedge A_2)\varphi_1(A_1 \wedge A_3)\varphi_1(A_2 \wedge A_3)}{\varphi_0(A_1)\varphi_0(A_2)\varphi_0(A_3)}. \quad \square$$

Now let $H_1, \dots, H_4$ be hyperplanes in general position in $\mathbf{P}_2$ and let $A_1, \dots, A_4$ be unit vectors such that $H_i = A_i^{\perp}$; general position of the H_i is equivalent to any three of the A_i being independent. Let $f\colon \mathbf{D}_r \to \mathbf{P}_2 - H_1 \cup \cdots \cup H_4$ be holomorphic. For any three of the A_i, define the basic model ω_{ijk} by

$$\omega_{ijk} = \varepsilon^{-s} P(A_i, A_j, A_k) \left\{ \prod_{l=i,j,k} \varphi_0(A_l) L_0(A_l)^2 \right\}^{a\varepsilon - 1} \Omega_0, \tag{4.2.3}$$

where a, $s \geq 0$ are constants which we will determine; and ε is thought of as small. This is the form Ψ_0 in (2.0.1) modified in two ways. First it has the term $P(A_i, A_j, A_k)$, which is zero when the tangent points to an intersection, as it must by (2.0.5). Second, the exponent of $\varphi_0(A_l)L_0(A_l)^2$ has been perturbed slightly, which turns out to be necessary in order to imbed ω_{ijk} in a negatively curved family.

Note that although each of the ω_{ijk} may be 0 at any point z_0, they cannot all be 0 without forcing f to be tangent to Δ:

Lemma 4.3. *If $f'(z_0) \neq 0$ and all the ω_{ijk} are 0 at z_0, then f is tangent at z_0 to Δ_{ij} for some i, j, where*

$$\Delta_{ij} = (H_i \cap H_j) \oplus (H_k \cap H_l) \tag{4.3.1}$$

is a diagonal of the quadrilateral determined by $H_1, \dots, H_4$.

Proof. Since $f'(z_0) \neq 0$ if and only if $\Lambda_1 \neq 0$ at z_0, we can choose v_1 and v_2 orthonormal so that $f(z_0) = [v_1]$ and $v_1 \wedge v_2 = \Lambda_1(z_0)/\|\Lambda_1(z_0)\|$. That all the ω_{ijk} are 0 implies that all the $P(A_i, A_j, A_k)$ are 0. By (4.1.2), we can renumber so that $\varphi_1(A_1 \wedge A_2) = 0$ at z_0; thus the line $[v_1, v_2]$ spanned by v_1 and v_2 in $\mathbf{P}_2$ contains $H_1 \cap H_2$.

Some other $\varphi_1(A_i \wedge A_j)$ must also be 0 at z_0; we show that it is $\varphi_1(A_3 \wedge A_4)$. If, for example, $\varphi_1(A_1 \wedge A_3)$ were 0, then $[v_1, v_2]$ would contain $H_1 \cap H_3$. Thus $[v_1, v_2]$ would equal H_1, which would imply that $f(z_0) \in H_1$, a contradiction. This same argument shows that since $\varphi_1(A_3 \wedge A_4) = 0$, then $[v_1, v_2] = (H_1 \cap H_2) \oplus (H_3 \cap H_4) = \Delta_{12}$. $\square$

Let ds^2_{ijk} be the Hermitian pseudo-metric on $\mathbf{D}_r$ whose associated $(1, 1)$-form is ω_{ijk}. Since ds^2_{ijk} is defined canonically in terms of f and its first derivative there is a unique differential metric F_{ijk} on $\mathbf{P}_n - H_1 \cup \cdots \cup H_4$, which we now write out explicitly, such that

$$f^* F_{ijk} = ds^2_{ijk}. \tag{4.3.2}$$

Indeed, in local coordinates let $x = [(1, x_1, x_2)]$ and let the tangent vector ξ at x be given by $\xi = (0, \xi_1, \xi_2)$. Define $\tilde{\varphi}_0(A) = \tilde{\varphi}_0(A, x)$ on $\mathbf{P}_2$ by

$$\tilde{\varphi}_0(A) = \|x, A\|^2/\|x\|^2, \tag{4.3.3}$$

and define $\tilde{\varphi}_1(A_1 \wedge A_2) = \tilde{\varphi}_1(A_1 \wedge A_2, x, \xi)$ on the tangent space by

$$\tilde{\varphi}_1(A_1 \wedge A_2) = \|x \wedge \xi, A_1 \wedge A_2\|^2/\|x \wedge \xi\|^2. \tag{4.3.4}$$

Note that $\tilde{\varphi}_1(A_1 \wedge A_2, x, \lambda\xi) = \tilde{\varphi}_1(A_1 \wedge A_2, x, \xi)$ for any nonzero scalar λ. Then

$$\begin{aligned} F_{ijk} = {} & \varepsilon^{-s}\tilde{\varphi}_1(A_i \wedge A_j)\tilde{\varphi}_1(A_i \wedge A_k)\tilde{\varphi}_1(A_j \wedge A_k) \\ & \cdot \left\{ \prod_{l=i,j,k} \tilde{\varphi}_0(A_l)[1/\varepsilon - \log \tilde{\varphi}_0(A_l)]^2 \right\}^{a\varepsilon - 1} ds^2_{\mathbf{P}}, \end{aligned} \tag{4.3.5}$$

where $ds^2_{\mathbf{P}}$ is the Fubini-Study metric on $\mathbf{P}_2$.

The main result of this paper is the following theorem.

Theorem 4.4. *Let $M = \mathbf{P}_2 - H_1 \cup \cdots \cup H_4$ and let F_M be the infinitesimal Kobayashi metric on M. Then for $s < 4$, $a \geq 48$, and ε small enough, the Kobayashi metric satisfies the following estimate*:

$$\max\{F_{123}, F_{124}, F_{134}, F_{234}\} \leq F_M. \tag{4.4.1}$$

In particular, $F_M(x, \xi) > 0$ unless $x \in \Delta_{ij}$ and ξ is tangent to Δ_{ij} for $\{i, j\} \subset \{1, 2, 3\}$.

Corollary 4.5. *$\mathbf{P}_2 - H_1 \cup \cdots \cup H_4$ is hyperbolic modulo $\Delta = \Delta_{12} \cup \Delta_{13} \cup \Delta_{14}$.*

As the first step in the proof of Theorem 4.4, we compute $\operatorname{Ric}\omega_{ijk}$. We note that if g is a holomorphic function, then $dd^c \log |g|^2 = 0$. Thus

$$dd^c \log \varphi_0(A) = -\Omega_0, \tag{4.5.1}$$

and

$$dd^c \log \varphi_1(A_1 \wedge A_2) = -\Omega_1. \tag{4.5.2}$$

Lemma 4.6. *For fixed $a \geq 0$ and $0 \leq s < 4$, if ε is small enough then*

$$\operatorname{Ric}\omega_{ijk} \geq \omega_{ijk} - 2\Omega_1 + \tfrac{1}{2}\Omega_0. \tag{4.6.1}$$

Proof. We prove the result for ω_{123}. So let $\alpha = a\varepsilon$ in the definition of ω_{ijk} (4.2.3), and denote $P(A_1, A_2, A_3)$ by P_{123}. Then

(4.6.2)

$$\operatorname{Ric}\omega_{ijk} \geq -3\Omega_1 + (1-\alpha)\left\{3(1-4\varepsilon)\Omega_0 + \sum_{1\leq i\leq 3} 2\Phi_0(A_i)\Omega_0\right\} + \Omega_1 - 2\Omega_0.$$

Now

$$(4.6.3)\quad \sum_{1\le i\le 3} 2\Phi_0(A_i) \ge 2\varepsilon^{-4}\left\{\prod_{1\le i\le 3} L_0(A_i)^{-2}\right\}\sum \varphi_1(A_i)\varphi_0(A_i)^{-1}$$

$$\ge \frac{1}{4}\varepsilon^{-4}P_{123}\left\{\prod_{1\le i\le 3}\varphi_0(A_i)L_0(A_i)^2\right\}^{-1} \quad \text{by (4.2.1)}.$$

Since $x\{1/\varepsilon - \log x\}^k \le \varepsilon^{-k}$ for $0 < x \le 1$ and $0 < \varepsilon \le 1/k$, then

$$(4.6.4)\quad \varphi_j(A)L_j(A)^k \le \varepsilon^{-k},$$

so that

$$(4.6.5)\quad \left\{\prod_{1\le i\le 3}\varphi_0(A_i)L_0(A_i)^2\right\}^{-1} \ge \varepsilon^{6\alpha}\left\{\prod_{1\le i\le 3}\varphi_0(A_i)L_0(A_i)^2\right\}^{\alpha-1}.$$

Trivially, $1-\alpha \ge \frac{1}{2}$ for small enough, so

$$\begin{aligned}\operatorname{Ric}\omega_{123} &\ge (1-\alpha)\left\{3(1-4\varepsilon)\Omega_0 + \tfrac{1}{4}\varepsilon^{6\alpha-4}P_{123}\left\{\prod\varphi_0(A_i)L_0(A_i)^2\right\}^{\alpha-1}\Omega_0\right\}\\ &\quad - 2\Omega_1 - 2\Omega_0 \quad \text{by (4.6.2), (4.6.3), and (4.6.5)}\\ &\ge \tfrac{1}{8}\varepsilon^{6\alpha-4+s}\omega_{123} - 2\Omega_1 + (1-3\alpha-12\varepsilon)\Omega_0 \quad \text{by (4.2.3)}\\ &\ge \omega_{ijk} - 2\Omega_1 + \tfrac{1}{2}\Omega_0,\end{aligned}$$

when ε is small enough. □

Remark 4.7. We need $\alpha < 1/3$ in order that the coefficient of Ω_0 be positive; thus even if we did not need α to be small, ω_{ijk} would still blow up at least as fast as $\varphi_0^{-2/3}$ when φ_0 approaches 0.

5. Construction of a negatively curved family

To prove our main result, Theorem 4.4, we construct auxiliary metrics so that ω_{ijk} is a member of a negatively curved family. The basic ingredient is a $(1,1)$-form on $\mathbf{P}_2 - H_1 \cup \cdots \cup H_4$ motivated by value distribution theory [9]:

$$(5.0.1)\quad \Psi_1 = C\varepsilon^{-3}\left\{\prod_{1\le i\le 4}\varphi_0(A_i)L_0(A_i)^2L_1(A_i)^2\right\}^{2\varepsilon-1/3}\Omega_0^{2/3+2\varepsilon}\Omega_1^{1/2-2\varepsilon},$$

where C is a constant depending only on the A_i.

In many ways the form Ψ_1 is the most natural generalization of the form Ψ_0 (2.0.1) which models the Kobayashi metric on $\mathbf{P}_1 - \{a_1, a_2, a_3\}$; Ψ_1 actually has negative curvature (Lemma 5.2 below). Indeed, Borel's Theorem (Theorem 0.1) follows immediately from Ahlfor's Lemma and the existence of Ψ_1. Unfortunately, Ψ_1 alone is not enough for our purposes. It is not the pull-back of a metric on $\mathbf{P}_2 - H_1 \cup \cdots \cup H_4$, but rather is the pull-back of a 2-jet differential metric, because of the Ω_1 term.

We would like to use $\{\omega_{ijk}, \Psi_1\}$ as our family of pseudo-metrics. The idea is that when $\operatorname{Ric}\omega_{ijk}$ is less than ω_{ijk}, then by (4.6.1), Ω_1 must be large compared to Ω_0 and ω_{ijk}. Thus ω_{ijk} should be smaller than Ψ_1 at such points. The problem with this approach—as it stands—is that ω_{ijk} blows up at least as fast as $\varphi_0^{-2/3}$, but Ψ_1 blows up at best as $\varphi_0(A_i)^{-1/3}$. Thus ω_{ijk} may be greater than Ψ_1 even at a point where Ω_1 is large.

To get around this difficulty, we consider the $(1, 1)$-form ρ_{ij} defined by

$$\rho_{ij} = \varepsilon^{-2}\varphi_1(A_i \wedge A_j)\left\{\prod_{l=i,j}\varphi_0(A_l)L_0(A_l)^2\right\}^{-1}\Omega_0. \tag{5.0.2}$$

Then

$$\operatorname{Ric}\rho_{ij} \geq \rho_{ij} - 8\varepsilon\Omega_0, \tag{5.0.3}$$

just as in Lemma 4.6. Note that this is the best we can do; no metric involving only two hyperplanes can actually have negative curvature, since $\mathbf{P}_2$ minus two hyperplanes is biholomorphic to $\mathbf{C} \times (\mathbf{C} - \{0\})$.

The argument above, comparing ω_{ijk} to Ψ_1, fails exactly when one of the $\varphi_0(A_i)^{-1}$ is large, that is, when $\operatorname{Ric}\rho_{ij}$ is much greater than ρ_{ij}. Yet ω_{ijk} is made up virtually of ρ_{ij}, ρ_{ik}, and ρ_{jk}. This suggests, since $\operatorname{Ric}\rho_{ij}$ is off by just $\varepsilon\Omega_0$, that we modify ρ_{ij} by a metric whose Ric exceeds $\varepsilon\Omega_0$, namely that we modify by Ψ_1. Thus we define $\tilde{\rho}_{ij}$ by

$$\tilde{\rho}_{ij} = \rho_{ij}^{1-a\varepsilon}\Psi_1^{a\varepsilon}. \tag{5.0.4}$$

We show in this section that $\{\omega_{ijk}, \tilde{\rho}_{ij}, \tilde{\rho}_{ik}, \tilde{\rho}_{jk}\}$ is a negatively curved family, thereby proving Theorem 4.4 by means of Lemma 1.5. To do this, we first calculate $\operatorname{Ric}\Psi_1$, then calculate $\operatorname{Ric}\tilde{\rho}_{ij}$, and finally show that $\operatorname{Ric}\omega_{ijk} \geq \omega_{ijk}$ when $\omega_{ijk} \geq \tilde{\rho}_{ij}, \tilde{\rho}_{ik}, \tilde{\rho}_{jk}$.

The next lemma is the crucial step in computing $\operatorname{Ric}\Psi_1$; it is a variation of a lemma by Ahlfors [2].

Lemma 5.1 (Sum into product). *Let $A_1, \ldots, A_N$ be unit vectors in general position in $\mathbf{C}^{n+1}$, $N \geq n+1$. Then there exist constants $0 < \kappa_0, \ldots, \kappa_n \leq 1$ (depending only on the A_i) such that for any holomorphic curve $f\colon \mathbf{D}_r \to \mathbf{P}_n - A_1^{\perp} \cup \cdots \cup A_N^{\perp}$ and any λ, $0 \leq \lambda \leq 1/(n-j) - 1/N$,*

$$\sum_{i=1}^{N}\Phi_j(A_i) \geq \{\kappa_j\varepsilon^{-2}\}^{N/(n-j)-\lambda N-1}\prod_{i=1}^{N}\{\Phi_j(A_i)\}^{1/(n-j)-\lambda} \tag{5.1.1}$$

for each j, $0 \leq j < n$, where Φ_j was defined in (3.1.6).

Proof. Let $v_0, \ldots, v_j$ be orthonormal in $\mathbf{C}^{n+1}$. Since the A_i are in general position, at most $n-j$ of the A_i can be orthogonal to all the v_k. By compactness of the space of orthonormal bases, there exists a constant κ_j, $1 \geq \kappa_j > 0$, such that for each $v_0, \dots, v_j$,

$$\|v_0 \wedge \cdots \wedge v_j, A_i\|^2 \geq \kappa_j \tag{5.1.2}$$

for all but at most $n-j$ of the A_i. Now fix $z_0 \in \mathbf{D}_r$, then

$$\varphi_j(A_i) \geq \kappa_j \quad \text{at } z_0 \tag{5.1.3}$$

for all but at most $n-j$ of the A_i. By renumbering, we may assume that $\varphi_j(A_i) \geq \kappa_j$ at z_0 for $i > n-j$. By (3.1.6) and $L_j \geq 1/\varepsilon$, we have

$$\Phi_j(A_i) \leq \kappa_j^{-1}\varepsilon^2 \tag{5.1.4}$$

at z_0 for $i > n-j$.

We put $\mu = 1/(n-j) - \lambda$ and $\nu = \lambda(n-j)/\{N-(n-j)\}$, and note that $\mu \geq \nu$. We now use the inequality connecting arithmetic and geometric means:

$$\lambda_1 x_1 + \cdots + \lambda_k x_k \geq x_1^{\lambda_1} \cdots x_k^{\lambda_k} \tag{5.1.5}$$

for all $\lambda_i, x_i \geq 0$ with $\lambda_1 + \cdots + \lambda_k = 1$. Thus we obtain

$$\begin{aligned}\sum_{i=1}^{N} \Phi_j(A_i) &\geq \prod_{i=1}^{n-j} \Phi_j(A_i)^{\mu} \prod_{i=n-j+1}^{N} \Phi_j(A_i)^{\nu} \geq \prod_{i=1}^{N} \Phi_j(A_i)^{\mu} \prod_{i=n-j+1}^{N} \Phi_j(A_i)^{\nu-\mu} \\ &\geq \prod_{i=1}^{N} \Phi_j(A_i)^{\mu} \prod_{i=n-j+1}^{N} \{\kappa_j^{-1}\varepsilon^2\}^{\nu-\mu} \\ &\geq \{\kappa_j \varepsilon^{-2}\}^{N/(n-j)-\lambda N-1} \prod_{i=1}^{N} \Phi_j(A_i)^{\mu}. \quad \square\end{aligned}$$

Using Lemma 5.1, we can compute $\operatorname{Ric}\Psi_1$.

Lemma 5.2. *Let* $A_1, \dots, A_4$ *be in general position in* $\mathbf{C}^3$. *Let* Ψ_1 *be the* $(1, 1)$*-form given by* (5.0.1), *where* $C = \frac{1}{3}\kappa_0^{2/3}\kappa_1$. *Then for* ε *small enough,*

$$\operatorname{Ric}\Psi_1 \geq \Psi_1 + \tfrac{1}{6}\Omega_0 + \tfrac{2\varepsilon}{3}\Omega_1. \tag{5.2.1}$$

Proof. Using Lemmas 3.1 and 3.2, we see that

$$\begin{aligned}\operatorname{Ric}\Psi_1 &\geq (\tfrac{1}{3} - 2\varepsilon)\left\{4\Omega_0 + 2\sum \Phi_0(A_i)\Omega_0 - 16\varepsilon\Omega_0 + 2\sum \Phi_1(A_i)\Omega_1 - 16\varepsilon\Omega_1\right\} \\ &\quad + (\tfrac{2}{3} + 2\varepsilon)(\Omega_1 - 2\Omega_0) + (\tfrac{1}{3} - 2\varepsilon)(\Omega_0 - 2\Omega_1) \\ &\geq (\tfrac{1}{3} - 20\varepsilon)\Omega_0 + (2\varepsilon/3)\Omega_1 + \tfrac{1}{3}\left\{\sum \Phi_0(A_i)\Omega_0 + \sum \Phi_1(A_i)\Omega_1\right\}.\end{aligned}$$

Now we change sums into products, using $\lambda = 1/2-(1-6\varepsilon)/(2+6\varepsilon)$ in the first sum ($\varepsilon \leq 1/15$ corresponds to the necessary condition $\lambda \leq 1/4$) and $\lambda = 0$ in the second:

$$\begin{aligned}
\tfrac{1}{3}&\left\{\sum \Phi_0(A_i)\Omega_0 + \sum \Phi_1(A_i)\Omega_1\right\} \\
&\geq \tfrac{1}{3}\left[\left\{(\kappa_0\varepsilon^{-2})^{1-4\lambda}\prod \Phi_0(A_i)^{1/2-\lambda}\right\}\Omega_0 + \left\{(\kappa_1\varepsilon^{-2})^3\prod \Phi_1(A_i)\right\}\Omega_1\right] \\
&\geq \tfrac{1}{3}\left[\left\{(\kappa_0\varepsilon^{-2})^{1-4\lambda}\prod \Phi_0(A_i)^{1/2-\lambda}\right\}\Omega_0\right]^{2/3+2\varepsilon} \\
&\quad\cdot\left[\left\{(\kappa_1\varepsilon^{-2})^3\prod \Phi_1(A_i)\right\}\Omega_1\right]^{1/3-2\varepsilon} \quad \text{by (5.1.5)} \\
&\geq \tfrac{1}{3}(\kappa_0\varepsilon^{-2})^{2/3-10\varepsilon}(\kappa_1\varepsilon^{-2})^{1-6\varepsilon}\prod \Phi_0(A_i)^{1/3-2\varepsilon}\Phi_1(A_i)^{1/3-2\varepsilon}\Omega_0^{2/3+2\varepsilon}\Omega_1^{1/3-2\varepsilon} \\
&\geq C\varepsilon^{-3}\left\{\prod \Phi_0(A_i)\Phi_1(A_i)\right\}^{1/3-2\varepsilon}\Omega_0^{2/3+2\varepsilon}\Omega_1^{1/3-2\varepsilon} \\
&\geq C\varepsilon^{-3}\left\{\prod \varphi_0(A_i)L_0(A_i)^2L_1(A_i)^2\right\}^{2\varepsilon-1/3}\Omega_0^{2/3+2\varepsilon}\Omega_1^{1/3-2\varepsilon} \\
&= \Psi_1.
\end{aligned}$$

Thus

$$\operatorname{Ric}\Psi_1 \geq \Psi_1 + (\tfrac{1}{3} - 20\varepsilon)\Omega_0 + \tfrac{2\varepsilon}{3}\Omega_1, \tag{5.2.2}$$

which implies (5.2.1) for ε small enough. □

Lemma 5.3. *Define* $\tilde{\rho}_{ij}$ *by* (5.0.4). *If* $a \geq 48$, *then for any* ε *small enough,*

$$\operatorname{Ric}\tilde{\rho}_{ij} \geq \tilde{\rho}_{ij}. \tag{5.3.1}$$

Proof. Let $\alpha = a\varepsilon$. Then using (5.1.5), we see that

$$\begin{aligned}
\operatorname{Ric}\tilde{\rho}_{ij} &\geq (1-\alpha)\operatorname{Ric}\rho_{ij} + \alpha\operatorname{Ric}\Psi_1 \\
&\geq (1-\alpha)(\rho_{ij} - 8\varepsilon\Omega_0) + \alpha\left\{\Psi_1 + \tfrac{1}{6}\Omega_0 + \tfrac{2\varepsilon}{3}\Omega_1\right\} \\
&\geq \tilde{\rho}_{ij} + \left\{\tfrac{\alpha}{6} - (1-\alpha)8\varepsilon\right\}\Omega_0 + \alpha\tfrac{2\varepsilon}{3}\Omega_1 \\
&\geq \tilde{\rho}_{ij} \quad \text{if } a \geq 48. \quad \square
\end{aligned}$$

We conclude the proof of Theorem 4.4 by showing that ω_{ijk} is a member of a negatively curved family.

Theorem 5.4. *If* $a \geq 48$ *and* $0 \leq s < 4$, *then for any* ε *small enough, the family of pseudo-metrics determined by* $\{\omega_{ijk}, \tilde{\rho}_{ij}, \tilde{\rho}_{ik}, \text{and } \tilde{\rho}_{jk}\}$ *has negative curvature.* (*Note*: a *is a parameter in both* ω_{ijk} *and* $\tilde{\rho}_{ij}$.)

Proof. If f is degenerate ($\Lambda_2 \equiv 0$), then $\Omega_1 \equiv 0$, so $\operatorname{Ric}\omega_{ijk} \geq \omega_{ijk}$. So assume f is nondegenerate, and we may assume that i, j, and k are 1, 2, and 3. Since $\tilde{\rho}_{12}$, $\tilde{\rho}_{13}$, and $\tilde{\rho}_{23}$ have negative curvature when a is large enough, we need only show that whenever $\omega_{123} \geq \tilde{\rho}_{12}, \tilde{\rho}_{13}, \tilde{\rho}_{23}$ at z_0, then $\operatorname{Ric}\omega_{123} \geq \omega_{123}$ at z_0. To do this, it suffices to show that $\Omega_0 \geq 4\Omega_1$, by Lemma 4.6.

As usual we set $\alpha = a\varepsilon$, and renumber so that $\varphi_0(A_3) \geq \varphi_0(A_1)$, $\varphi_0(A_2)$ at z_0. Since, by assumption, $\omega_{123} \geq \tilde{\rho}_{12}$ at z_0, then

(5.4.1)

$$\begin{aligned}
&\varepsilon^{-s} P_{123} \left\{ \prod_{1 \leq i \leq 3} \varphi_0(A_i) L_0(A_i)^2 \right\}^{\alpha - 1} \Omega_0 \\
&\quad \geq \left\{ \varepsilon^{-2} \varphi_1(A_1 \wedge A_2) \left\{ \prod_{i=1,2} \varphi_0(A_i) L_0(A_i)^2 \right\}^{-1} \Omega_0 \right\}^{1-\alpha} \\
&\quad\quad \cdot \left\{ C\varepsilon^{-3} \left\{ \prod_{1 \leq i \leq 4} \varphi_0(A_i) L_0(A_i)^2 L_1(A_i)^2 \right\}^{2\varepsilon - 1/3} \Omega_0^{2/3 + 2\varepsilon} \Omega_1^{1/3 - 2\varepsilon} \right\}^{\alpha}.
\end{aligned}$$

Now let

$$\lambda = \tfrac{1}{3} - 2\varepsilon > 0. \tag{5.4.2}$$

Since $1 \geq \varphi_0(A_3) \geq \frac{1}{4} \varphi_1(A_1 \wedge A_3) \varphi_1(A_2 \wedge A_3)$ by (4.2.2) and $\varphi_1(A_i \wedge A_j) \leq 1$, then (5.4.1) implies that

$$\begin{aligned}
\Omega_0^{\alpha\lambda} &\geq \tfrac{1}{4} \varepsilon^{s - \alpha - 2} \left\{ \prod_{i=1,2} \varphi_0(A_i) L_0(A_i)^2 \right\}^{-\alpha\lambda} L_0(A_3)^{2(1 - \alpha(1+\lambda))} \\
&\quad \cdot \left\{ C \left\{ \varphi_0(A_4) L_0(A_4)^2 \prod_{1 \leq i \leq 4} L_1(A_i)^2 \right\}^{-\lambda} \right\}^{\alpha} \Omega_1^{\alpha\lambda}.
\end{aligned}$$

Now we use that $\varphi_1 \geq \varphi_0$, so $L_0 \geq L_1 \geq \frac{1}{\varepsilon}$, hence

$$\begin{aligned}
&L_0(A_3)^{2(1 - \alpha(1+\lambda))} \left\{ \prod_{1 \leq i \leq 4} L_1(A_i)^2 \right\}^{-\alpha\lambda} \\
&\quad \geq L_0(A_3)^{2(1 - \alpha(1 + 2\lambda))} \left\{ \prod_{i \neq 3} L_0(A_i)^2 \right\}^{-\alpha\lambda} \\
&\quad \geq \varepsilon^{-2(1 - \alpha(1 + 2\lambda))} \left\{ \prod_{i \neq 3} L_0(A_i)^2 \right\}^{-\alpha\lambda}.
\end{aligned}$$

Thus

$$\Omega_0^{\alpha\lambda} \geq \tfrac{1}{4} C^{-\alpha\lambda} \varepsilon^{s - 4 + \alpha(1 + 4\lambda))} \left\{ \prod_{i \neq 3} \varphi_0(A_i) L_0(A_i)^4 \right\}^{-\alpha\lambda} \Omega_1^{\alpha\lambda}.$$

Using the estimate (4.6.4) on $\varphi_0(A_i)L_0(A_i)^4$, we obtain

$$\begin{aligned}\Omega_0^{\alpha\lambda} &\geq \tfrac{1}{4}C^{-\alpha\lambda}\varepsilon^{s-4+\alpha(1+16\lambda)}\Omega_1^{\alpha\lambda}\\ &\geq \tfrac{1}{4}C^{-\alpha\lambda}\varepsilon^{(s-4)/2}\Omega_1^{\alpha\lambda},\end{aligned}$$

which implies that

$$\begin{aligned}\Omega_0 &\geq C^{-1}\left\{\tfrac{1}{4}\varepsilon^{(s-4)/2}\right\}^{1/(\alpha\lambda)}\Omega_1\\ &\geq 4\Omega_1 \quad \text{for } \varepsilon \text{ small enough.}\end{aligned}$$

Now using Lemma 4.6, we see that $\operatorname{Ric}\omega_{123} \geq \omega_{123}$ at any point where $\omega_{123} \geq \tilde{\rho}_{12}, \tilde{\rho}_{13}, \tilde{\rho}_{23}$. □

6. $\mathbf{P}_n$ MINUS HYPERPLANES

For $\mathbf{P}_n$, $n > 2$, the case of $n+2$ hyperplanes in general position is not well understood. The construction in (4.2.3) does generalize, but to the pull-back on an $(n-1)$-jet differential metric. That is, we can find a $(1, 1)$-form involving $\Omega_0, \ldots, \Omega_{n-1}$ which can be imbedded in a negatively curved family. The problem is to find the corresponding k-jet differential metrics for all k, $0 \leq k \leq n-2$.

One of the reasons this is so difficult is that for $\mathbf{P}_2$ minus hyperplanes the Kobayashi metric is actually zero on the exceptional set Δ, but for $\mathbf{P}_n$ minus hyperplanes the Kobayashi metric is only zero in some directions in Δ. For example, in $\mathbf{P}_3$, Δ is the union of the planes Δ_{ij} which go through a line $(H_i \cap H_j)$ and a point $(H_k \cap H_l \cap H_m)$. Thus $\Delta_{ij} - H_1 \cup \cdots \cup H_5$ is a $\mathbf{P}_2$ minus 4 lines, three of which go through a point, and hence is biholomorphic to $(\mathbf{C} - \{0\}) \times (\mathbf{C} - \{0, 1\})$. It is difficult to construct a model on $\mathbf{P}_3$ which is nonzero off Δ, restricts to 0 in the first factor on Δ_{ij}, and gives a model for the Poincaré metric on the second factor.

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