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ASYMPTOTIC STABILITY OF PLANAR RAREFACTION WAVES FOR VISCOUS CONSERVATION LAWS IN SEVERAL DIMENSIONS

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ABSTRACT. This paper concerns the large time behavior toward planar rarefaction waves of the solutions for scalar viscous conservation laws in several dimensions. It is shown that a planar rarefaction wave is nonlinearly stable in the sense that it is an asymptotic attractor for the viscous conservation law. This is proved by using a stability result of rarefaction wave for scalar viscous conservation laws in one dimension and an elementary L^2-energy method.

0. INTRODUCTION

We will establish the asymptotic stability of planar rarefaction waves for scalar viscous conservation laws in two or more space dimensions. We consider n-dimensional scalar viscous conservation laws of the form

$$u_t + \sum_{i=1}^{n} (f_i(u))_{x_i} = \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j}, \qquad x \in \mathbf{R}^n, \ t > 0, \tag{1}$$

where $u \in \mathbf{R}^1$, $A = (a_{ij})$, called the viscosity matrix, is a constant positive definite matrix, and we assume that all the flux functions are smooth (say in C^n) and equation (1) is genuinely nonlinear in the x_1-direction [8], i.e., for a fixed constant $a > 0$,

$$f_1''(u) \geq a. \tag{2}$$

The initial data for equation (1) is

$$u(x, 0) = u_0(x) \tag{3}$$

satisfying

$$\lim_{x_1 \to \pm\infty} \|u(x_1, \cdot) - u_\pm\|_{L^\infty(\mathbf{R}^{n-1})} = 0, \tag{4}$$

where $u_\pm$, $u_- < u_+$, are two constants. A planar rarefaction wave (in x_1-direction) $u^r(x_1, t)$ is a solution of the following initial value problem for the

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corresponding inviscid equation

$$u_t + (f_1(u))_{x_1} = 0, \qquad x_1 \in \mathbf{R}^1, \; t > 0, \tag{5}$$

$$u(x_1, 0) = u_0^r(x_1), \tag{6}$$

where $u_0^r(x_1)$ satisfies

$$\lim_{x_1 \to \pm\infty} u_0^r(x_1) = u_\pm, \quad \text{and} \quad \frac{d}{dx_1} u_0^r(x_1) \geq 0, \quad \text{a.e.} \tag{7}$$

Since any rarefaction waves of (5) with same end states are time asymptotically equivalent (i.e., they converge to each other in L^∞-norm as t tends to infinity [6]), for definiteness, we will study a smooth rarefaction wave $u^r(x_1, t)$ of (5) with initial data $u_0^r(x_1)$ which satisfies

$$\frac{d}{dx_1} u_0^r(x_1) > 0, \quad \text{and} \quad \left| \frac{d^2}{dx_1^2} u_0^r(x_1) \right| \leq k_0 \frac{d}{dx_1} u_0^r(x_1), \qquad \forall x_1 \in \mathbf{R}^1 \tag{8}$$

for some positive constant k_0. Then our main result in this paper is the following stability theorem.

Theorem 1. *Suppose that $u^r(x_1, t)$ is a smooth planar rarefaction wave with initial data $u_0^r(x_1) \in C^n(\mathbf{R}^1)$ satisfying* (8). *Then there exists a constant δ such that if*

$$\|u_0(\cdot) - u_0^r(\cdot)\|_{H^{[n]/2+1}(\mathbf{R}^n)} \leq \delta, \tag{9}$$

then problem (1), (3) *has a global smooth solution $u(x, t)$ satisfying*

$$\lim_{t \to \infty} \|u(\cdot, t) - u^r(\cdot, t)\|_{L^\infty(\mathbf{R}^n)} = 0. \tag{10}$$

Remark 1. For the case that $a_{ij} = a_{ij}(u)$, it can be shown by checking our following proof that Theorem 1 still holds under the assumption that the strength of the wave, $u_+ - u_-$, is small.

Remark 2. The choice of x_1-direction is no loss of generality, since we can reduce a general situation to this case by a suitable change of coordinates.

Remark 3. In the following, we will only give a proof of Theorem 1 for the case $n = 2$, since the proof for $n > 2$ is identical.

Remark 4. In Theorem 1, we have no restriction on the strength of the planar rarefaction wave, this is in constrast to the complementary case of the stability of viscous scalar shock fronts in several dimensions [3] which is proved for weak waves only.

The proof of Theorem 1 is based on a stability result of rarefaction waves for scalar viscous conservation laws in one dimension and an elementary L^2-energy method. The one dimension stability of expansion waves for scalar viscous conservation laws was first established by Il′in and Oleinik [4] based on a maximum principle. Another approach using the semigroup argument was given in [1]. Our proof of multidimension stability in Theorem 1 has more in common with the proof of stability of weak rarefaction waves for systems of viscous hyperbolic conservation laws in one dimension which has recently been studied by many authors (see [12, 13, 7, 9, 10]).

The rest of the paper is organized as follows. In §1, by making use of a stability result for rarefaction wave in one space dimension, we can construct a planar solution $U(x_1, t)$ for equation (1) which approximates the smooth rarefaction wave $u^r(x_1, t)$; see Lemma 2. Then, we write the solution $u(x, t)$ of (1), (3) as a perturbation of $U(x_1, t)$ and reduce the proof of Theorem 1 to the energy estimates on the difference between $u(x, t)$ and $U(x_1, t)$. The basic stability estimate and higher order estimates are given in §2 and §3. Finally, we study the existence and large time behavior of the solution of (1), (3) by applying the a priori estimate derived in §3.

1. Preliminaries

We begin by considering the following Cauchy problem:

$$\begin{cases} U_t + (f_1(U))_{x_1} = a_{11} U_{x_1 x_1}, & x_1 \in \mathbf{R}^1, \ t > 0, \\ U(x_1, 0) = u_0^r(x_1). \end{cases} \tag{11}$$

Noting that $a_{11} > 0$ and (7), by the nonlinear stability of rarefaction waves for scalar conservation laws in one space dimension (see [4, 1]), we see that there exists a unique global (in time) smooth solution $U(x_1, t)$ to (11) which has the centered rarefaction wave of (5) determined by data (u_-, u_+) as a time asymptotic state in L^∞-norm. Since all rarefaction waves of (5) with the same end states are time asymptotically equivalent in L^∞-norm, we have

$$\lim_{t\to\infty} \|U(\cdot, t) - u^r(\cdot, t)\|_{L^\infty(\mathbf{R}^1)} = 0.$$

Furthermore, if we denote $U_{x_1}(x_1, t)$ by $w(x_1, t)$, then $w(x_1, t)$ satisfies

$$w_t + f_1'(U) w_{x_1} + f_1''(U) w^2 = a_{11} w_{x_1 x_1}. \tag{12}$$

Since $w(x_1, 0) = (u_0^r(x_1))_{x_1} > 0$ by (8), it follows from a maximum principle that $U(x_1, t)$ is strictly increasing in x_1 for each fixed $t > 0$, i.e.,

$$w(x_1, t) = \frac{\partial}{\partial x_1} U(x_1, t) > 0, \quad \forall x_1 \in \mathbf{R}^1, \ t > 0.$$

In other words, this says that the characteristic fields corresponding to the solution $U(x_1, t)$ of (11) is expansive, i.e.,

$$\frac{\partial}{\partial x_1}[f_1'(U(x_1, t))] > 0, \quad \forall x_1 \in \mathbf{R}^1, \ t > 0.$$

We list some properties of $U(x_1, t)$ in the following lemma which we will use later.

Lemma 2. *The Cauchy problem* (11) *has a unique smooth solution* $U(x_1, t)$ *satisfying*:

(13) (i) $\lim_{t\to\infty} \|U(\cdot, t) - u^r(\cdot, t)\|_{L^\infty(\mathbf{R}^1)} = 0$.

(14) (ii) $\frac{\partial}{\partial x_1} U(x_1, t) > 0, \quad \forall x_1 \in \mathbf{R}^1, \ t > 0$.

(iii) *There exists a positive constant* $K = K(k_0)$ *such that*

(15) $$\left|\frac{\partial^2}{\partial x_1^2} U(x_1, t)\right| \le K \frac{\partial}{\partial x_1} u(x_1, t), \quad \forall x_1 \in \mathbf{R}^1, \ t \ge 0.$$

Proof. We have already proved (i) and (ii). For (iii), we differentiate (12) with respect to x_1 and set $\phi = w_{x_1}$, then we get

$$\phi_t + f_1'(U)\phi_{x_1} + 3f_1''(U)w\phi + f_1'''(U)w^3 = a_{11}\phi_{x_1x_1};$$

thus

$$\phi_t - a_{11}\phi_{x_1x_1} + f_1'(U)\phi_{x_1} + 3f_1''(U)w\phi = -f_1'''(U)w^3.$$

If we define a linear parabolic differential operator L by

(16) $$L = \frac{\partial}{\partial t} - a_{11}\frac{\partial^2}{\partial x_1^2} + f_1'(U)\frac{\partial}{\partial x_1} + 3f_1''(U)w,$$

then

(17) $$L(\phi) = -f_1'''(U)w^3.$$

On other hand, we get from (12) that for any constant C

$$(Cw)_t + f_1'(U)(Cw)_{x_1} + f_1''(U)Cw^2 = a_{11}(Cw)_{x_1x_1},$$

so that

(18) $$L(Cw) = 2Cf_1''(U)w^2.$$

We now choose $|C| = K$ so large that

$$K \ge \max\left\{k_0, \frac{1}{2a}\max_{x_1\in\mathbf{R}^1} |f_1'''(U)w|\right\}.$$

We note that we can take $K < \infty$, since it can be proved that

$$\|w(\cdot, t)\|_{L^\infty(\mathbf{R}^1)} \le M_0, \quad \text{for all } t \ge 0,$$

where

$$M_0 = \|w(\cdot, 0)\|_{L^\infty(\mathbf{R}^1)} = \left\|\frac{d}{dx_1}u_0^r(\cdot)\right\|_{L^\infty(\mathbf{R}^1)},$$

which is a finite constant by our assumption. To see this, let $M(t)$ be the solution of the following problem

$$\frac{d}{d_t}M(t) + aM(t)^2 = 0, \qquad M(0) = M_0,$$

where a is the positive constant in (2). Then we have that $0 \le M(t) \le M_0$. Using (2), it is easy to check that $M(t)$ is an upper solution for nonlinear parabolic equation (12). Consequently we have

$$\|w(\cdot\,, t)\|_{L^\infty(\mathbf{R}^1)} \le M(t) \le M_0, \quad \text{for all } t \ge 0.$$

It follows from this and (17), (18) that

$$\begin{aligned} L(-Kw) &= -2Kf''(U)w^2 \le -2Kaw^2 \le -(f_1'''(U)w)w^2 \\ &= L(\phi) \le 2Kaw^2 \le 2Kf_1''(U)w^2 = L(Kw). \end{aligned}$$

Since we also have $|\phi(x_1, 0)| \le k_0 w(x_1, 0) \le Kw(x_1, 0)$ by (8), thus by a comparison theorem for a parabolic equation (cf. [11]), we have

$$|\phi(x_1, t)| \le Kw(x_1, t), \quad \forall x_1 \in \mathbf{R}^1, \ t \ge 0.$$

This completes the proof of (iii). We remark that (iii) also can be proved by the argument given in [14]. □

By (i) in Lemma 1, we see that, in order to prove Theorem 1, it would suffice to show that the smooth expansive planar wave $U(x_1, t)$ is an asymptotic attractor for the equation (1). Thus we will consider the solutions of (1), (3) in a neighborhood of $U(x_1, t)$. As we will see later, the advantage of using $U(x_1, t)$ instead of $u^r(x_1, t)$ is that $U(x_1, t)$ is an exact solution of (1) and this will enable us quite easily to estimate some terms which do not decay in x_2-direction.

Now, we suppose that $u(x_1, x_2, t)$ is a solution of (1), (3). We decompose the solution as

$$u(x_1, x_2, t) = U(x_1, t) + V(x_1, x_2, t). \tag{19}$$

It follows from (11) and (9) that the Cauchy problem (1), (3) is equivalent to the following initial value problem:

$$V_t + (f_1'(U)V)_{x_1} + [Q(U, V)]_{x_1} + (f_2(U+V))_{x_2} = \sum_{i,j=1}^{2} a_{ij} V_{x_i x_j}, \tag{20}$$

$$V(x_1, x_2, 0) = V_0(x_1, x_2) \equiv u_0(x_1, x_2) - u_0^r(x_1) \in H^2(\mathbf{R}^2), \tag{21}$$

where $Q(U, V) = f_1(U+V) - f_1(U) - f_1'(U)V$ satisfying $|Q(U, V)| \le CV^2$ for some constant $C > 0$ if $|V|$ is small enough. Then, we need only to show that the Cauchy problem (20), (21) has a smooth global solution, which tends to zero as t approaches infinity uniformly with respect to x_1 and x_2. This is an initial value problem for a parabolic equation with initial data in $H^2(\mathbf{R}^2)$, so the local (in time) existence of solution is standard, and in order to get the global existence and large time behavior, we will need an a priori estimate on the solution of (20), (21). For this, we first define the solution space for (20) by

$$X(0, T) = \{\phi(x_1, x_2, t) \in C^0(0, T; H^2);\ \phi_{x_1}, \phi_{x_2} \in L^2(0, T; H^2)\} \tag{22}$$

with $T > 0$. We suppose that the solution $V(x_1, x_2, t)$ of (20), (21) belongs to $X(0, T)$ for some $T > 0$ and set

$$N(t) \equiv \sup_{\tau\in[0, t]} \|V(\cdot, \tau)\|_{H^2(\mathbf{R}^2)}. \tag{23}$$

In what follows, we always assume that $N(T) \leq \varepsilon_0$ for some positive constant ε_0. The desired a priori estimate on V will be derived in the following sections.

2. Basic estimate

In this section we derive the basic L^2-energy estimate on $V(x_1, x_2)$. For ease of notation, in what follows we will use $\|\cdot\|$ to denote the norm in $L^2(\mathbf{R}^2)$ and $\iiint$ the triple integral over $[0, t]\times\mathbf{R}^2$ unless otherwise stated. Multiplying (20) by V and integrating over $[0, t] \times \mathbf{R}^2$ gives

$$\begin{aligned}\frac{1}{2}\|V(\cdot, t)\|^2 &+ \iiint V[f_1'(U)V]_{x_1}\,dx_1\,dx_2\,d\tau \\ &+ \iiint V[Q(U, V)]_{x_1}\,dx_1\,dx_2\,d\tau \\ &+ \iiint V[f_2(U+V)]_{x_2}\,dx_1\,dx_2\,d\tau \\ &= \frac{1}{2}\|V(\cdot, 0)\|^2 + \iiint V\sum_{i,j=1}^{2} a_{ij}V_{x_ix_j}\,dx_1\,dx_2\,d\tau.\end{aligned} \tag{24}$$

Each term in (24) will be estimated separately as follows. First, by assumption that $A = (a_{ij})$ is positive definite, we see that there exists a constant $b > 0$ such that

$$\iiint V\sum_{i,j=1}^{2} a_{ij}V_{x_ix_j}\,dx_1\,dx_2\,d\tau \leq -b\int_0^t \|(V_{x_1}, V_{x_2})(\cdot, \tau)\|^2\,d\tau \tag{25}$$

as follows from integrating by parts. Next, taking into account (2) and (14), we may integrate by parts twice to get

$$\begin{aligned}\iiint V[f_1'(U)V]_{x_1}\,dx_1\,dx_2\,d\tau &\leq -\iiint f_1'(U)VV_{x_1}\,dx_1\,dx_2\,d\tau \\ &= \frac{1}{2}\iiint [f_1'(U)]_{x_1}V^2\,dx_1\,dx_2\,d\tau \\ &\geq \frac{a}{2}\iiint |U_{x_1}|V^2(x_1, x_2, t)\,dx_1\,dx_2\,d\tau.\end{aligned} \tag{26}$$

To estimate the third term on the left-hand side of (24), we write $Q(U, V) = g(U, V)V^2$, here $g(U, V)$ is a smooth bounded function by the Taylor formula. Thus, integration by parts leads to

$$\begin{aligned}
&\iiint V[Q(U,V)]_{x_1}\,dx_1\,dx_2\,d\tau = -\iiint g(U,V)V^2V_{x_1}\,dx_1\,dx_2\,d\tau \\
&\iiint g_u U_{x_1}(V^3/3)\,dx_1\,dx_2\,d\tau + \iiint g_v V_{x_1}(V^3/3)\,dx_1\,d_{x_2}\,d\tau \\
&\qquad \le C_1 \sup|V(x_1,x_2,t)| \iiint U_{x_1}V^2\,dx_1\,dx_2\,d\tau \\
&\qquad\quad + C_2\iiint |V_{x_1}V^3|\,dx_1\,dx_2\,d\tau
\end{aligned} \tag{27}$$

for some positive constants C_1 and C_2, where the sup is taken over $[0,t]\times\mathbf{R}^2$. We treat the last term on the right-hand side of (27) as follows: for fixed (x_1, x_2, t), we have by Cauchy inequality that

$$\begin{aligned}
V^2(x_1,x_2,t) &= \int_{-\infty}^{x_1}\frac{\partial}{\partial x_1}V^2(x_1,x_2,t)\,dx_1 \le 2\int_{-\infty}^{+\infty}|VV_{x_1}|\,dx_1 \\
&\le 2\|V(\cdot,x_2,\tau)\|_{L^2(\mathbf{R}^1)}\|V_{x_1}(\cdot,x_2,\cdot\tau)\|_{L^2(\mathbf{R}^1)},
\end{aligned}$$

consequently

$$\begin{aligned}
&\iiint |V_{x_1}V^3|\,dx_1\,dx_2\,d\tau = \iiint |V_{x_1}V|V^2\,dx_1\,dx_2\,d\tau \\
&\le 2\iint [\|V(\cdot,x_2,\tau)\|_{L^2(\mathbf{R}^1)}\|V_{x_1}(\cdot,x_2,\tau)\|_{L^2(\mathbf{R}^1)}\int |VV_{x_1}|dx_1]\,dx_2\,d\tau \\
&\le 2\iint \|V(\cdot,x_2,\tau)\|^2_{L^2(\mathbf{R}^1)}\|V_{x_1}(\cdot,x_2,\tau)\|^2_{L^2(\mathbf{R}^1)}\,dx_2\,d\tau \\
&\le 2\sup_{0\le\tau\le t,\,x_2\in\mathbf{R}^1}\|V(\cdot,x_2,\tau)\|^2_{L^2(\mathbf{R}^1)}\int_0^t\|V_{x_1}(\cdot,\tau)\|^2\,d\tau,
\end{aligned} \tag{28}$$

and so

$$\begin{aligned}
&\iiint V[Q(U,V)]_{x_1}\,dx_1\,dx_2\,d\tau \\
&\qquad \le C_1\sup|V(x_1,x_2,t)|\iiint U_{x_1}V^2\,dx_1\,dx_2\,d\tau \\
&\qquad\quad + 2C_2\sup_{0\le\tau\le t,\,x_2\in\mathbf{R}^1}\|V(\cdot,x_2\tau)\|^2_{L^2(\mathbf{R}^1)}\int_0^t\|V_{x_1}(\cdot,\tau)\|^2\,d\tau.
\end{aligned} \tag{29}$$

Finally, we estimate the last term on the left-hand side of (24). Since U does not depend on x_2, we may get after integrating by parts several times that

$$\begin{aligned}
&\iiint V[f_2(U+V)]_{x_2}\,dx_1\,dx_2\,d\tau = \iiint f_2'(U+V)VV_{x_2}\,dx_1\,dx_2\,d\tau \\
&\quad = -\frac{1}{2}\iiint f_2''(U+V)V^2V_{x_2}\,dx_1\,dx_2\,d\tau \\
&\quad = \frac{1}{6}\iiint f_2'''(U+V)V^3V_{x_2}\,dx_1\,dx_2\,d\tau,
\end{aligned}$$

consequently

$$\left|\iiint V[f_2(U+V)]_{x_2}\,dx_1\,dx_2\,d\tau\right| \le C_3 \iiint |VV_{x_2}|V^2\,dx_1\,dx_2\,d\tau$$

for some positive constant C_3. Therefore, in a similar way as for the estimate (28), we have

$$\begin{aligned}&\left|\iiint V[f_2(U+V)]_{x_2}\,dx_1\,dx_2\,d\tau\right| \\ &\quad\le 2 \sup_{0\le\tau\le t,\, x_1\in\mathbf{R}^1} \|V(x_1\,,\,\cdot\,,\,\tau)\|^2_{L^2(\mathbf{R}^1)} \int_0^t \|V_{x_2}(\cdot\,,\,\tau)\|^2\,d\tau.\end{aligned}\tag{30}$$

Collecting all the estimates (24), (25), (26), (29) and (30) together, we arrive at the following lemma.

Lemma 3. *There exist positive constants* ε_i $(i=1\,,\,2)$ *and* C_4 *depending only on the flux functions and viscosity matrix, such that if*

$$\sup_{0\le\tau\le t,\,(x_1,x_2)\in\mathbf{R}^2} |V(x_1\,,\,x_2\,,\,t)| \le \varepsilon_1\,,\tag{31}$$

$$\sup_{0\le\tau\le t}\left[\sup_{x_1\in\mathbf{R}^1} \|V(x_1\,,\,\cdot\,,\,\tau)\|^2_{L^2(\mathbf{R}^1)} + \sup_{x_2\in\mathbf{R}^1} \|V(\cdot\,,\,x_2\,,\,\tau)\|^2_{L^2(\mathbf{R}^1)}\right] \le \varepsilon_2\,,\tag{32}$$

then, for all $t\in[0\,,\,T]$, *we have*

$$\begin{aligned}&\|V(\cdot\,,\,t)\|^2 + \iiint |U_{x_1}|V^2(x_1\,,\,x_2\,,\,t)\,dx_1\,dx_2\,d\tau \\ &\quad + \int_0^t \|(V_{x_1}\,,\,V_{x_2})(\cdot\,,\,\tau)\|^2\,d\tau \le C_4\|V(\cdot\,,\,0)\|^2.\end{aligned}\tag{33}$$

Next, we notice that the conditions (31) and (32) will be satisfied if $N(T)$ is suitably small. In fact, it follows from Sobolev inequality that there exists an absolute constant C_5, such that

$$\sup_{0\le\tau\le t,\,(x_1,x_2)\in\mathbf{R}^2} |V(x_1\,,\,x_2\,,\,t)| \le C_5 N(T).\tag{34}$$

To estimate the left-hand side of (32), we set

$$\theta(x_2\,,\,t) = \|V(\cdot\,,\,x_2\,,\,\tau)\|^2_{L^2(\mathbf{R}^1)}\,,$$

then

$$\begin{aligned}\theta(x_2\,,\,t) &\le \int\left|\frac{\partial}{\partial x_2}\theta(x_2\,,\,t)\right|\,dx_2 = \int\left|\frac{\partial}{\partial x_2}\int V^2(x_1\,,\,x_2\,,\,t)\,dx_1\right|\,dx_2 \\ &\le 2\iint\left|VV_{x_2}\right|(x_1\,,\,x_2\,,\,t)\,dx_1\,dx_2 \le \|V(\cdot\,,\,t)\|^2 + \|V_{x_2}(\cdot\,,\,t)\|^2 \\ &\le N^2(T).\end{aligned}$$

Thus

$$\sup_{0\le\tau\le t,\,x_2\in\mathbf{R}^1} \|V(\cdot\,,\,x_2\,,\,\tau)\|^2_{L^2(\mathbf{R}^1)} \le N^2(T).\tag{35}$$

Similarly

$$\sup_{0\le\tau\le t,\, x_1\in\mathbf{R}^1} \|V(x_1\,,\,\cdot\,,\,\tau)\|^2_{L^2(\mathbf{R}^1)} \le N^2(T). \tag{36}$$

As a consequence of (34)–(36) and Lemma 3, we arrive at the following basic a priori estimate

Proposition 4 (a priori estimate). *There exist positive constants* δ_0 $(\le \varepsilon_0)$ *and* C_4 *independent of* T, *such that if* $N(T) \le \delta_0$, *then the basic energy estimate* (33) *holds.*

3. Higher estimates

In this section we will establish energy estimates on higher order derivatives of U, which will yield a time uniform estimate on U with which we can obtain the global existence of solution by a standard continuity argument. These estimates will be derived by making use of the basic estimate (33). First, we estimate the first derivative of U and we have

Lemma 5. *There exists a constant* $C_\sigma > 0$ *such that if* $N(T) \le \delta_0$, *then*

$$\begin{aligned} &\|V(\cdot\,,\,t)\|^2_{H^1(\mathbf{R}^2)} + \iiint |U_{x_1}|V^2(x_1\,,\,x_2\,,\,t)\,dx_1\,dx_2\,d\tau \\ &\quad + \int_0^t \|(V_{x_1}\,,\,V_{x_2})(\cdot\,,\,\tau)\|^2_{H^1(\mathbf{R}^2)}\,d\tau \le C_\sigma \|V(\cdot\,,\,0)\|^2_{H^1(\mathbf{R}^2)}\,. \end{aligned} \tag{37}$$

Proof. Since the estimate on V_{x_1} is similar to but somewhat easier than that on V_{x_2}, we will only derive the estimate on V_{x_2}. For this, we differentiate (20) with respect to x_2, multiply the resulting equation by V_{x_2} and integrate over $[0\,,\,t] \times \mathbf{R}^2$. We find

$$\begin{aligned} &\frac{1}{2}\|V_{x_2}(\cdot\,,\,t)\|^2 + b\int_0^t \|(V_{x_1x_2}\,,\,V_{x_2x_2})(\cdot\,,\,\tau)\|^2\,d\tau \\ &\quad \le \frac{1}{2}\|V_{x_2}(\cdot\,,\,0)\|^2 + \left|\iiint V_{x_2}[f_1'(U)V]_{x_1x_2}\,dx_1\,dx_2\,d\tau\right| \\ &\qquad + \left|\iiint V_{x_2}[Q(U\,,\,V)]_{x_1x_2}\,dx_1\,dx_2\,d\tau\right| \\ &\qquad + \left|\iiint V_{x_2}[f_2(U+V)]_{x_2x_2}\,dx_1\,dx_2\,d\tau\right|. \end{aligned} \tag{38}$$

We now estimate the last three terms on the right-hand side of (38). (In what follows, we will denote by C any positive constant which does not depend on

T.) First, using integration by parts and Cauchy inequality, we can get

$$
\begin{aligned}
(39)\quad & \left|\iiint V_{x_2}[f_1'(U)V]_{x_1x_2}\,dx_1\,dx_2\,d\tau\right| = \left|\iiint V_{x_2x_2}[f_1'(U)V]_{x_1}\,dx_1\,dx_2\,d\tau\right| \\
& = \left|\iiint V_{x_2x_2}[f_1''(U)U_{x_1}V + f_1'(U)V_{x_1}]\,dx_1\,dx_2\,d\tau\right| \\
& \le \frac{b}{6}\int_0^t \|V_{x_2x_2}(\cdot,\tau)\|^2\,d\tau + C\iiint |U_{x_1}|V^2(x_1,x_2,t)\,dx_1\,dx_2\,d\tau \\
& \quad + C\int_0^t \|V_{x_1}(\cdot,\tau)\|^2\,d\tau.
\end{aligned}
$$

Continuing,

$$
\begin{aligned}
(40)\quad & \left|\iiint V_{x_2}[Q(U,V)]_{x_1x_2}\,dx_1\,dx_2\,d\tau\right| = \left|\iiint V_{x_2x_2}[(U,V)]_{x_1}\,dx_1\,dx_2\,d\tau\right| \\
& \le C\iiint |V_{x_2x_2}| \subset |U_{x_1}|V^2 + V^2|V_{x_1}| + |VV_{x_1}|)\,dx_1\,dx_2\,d\tau \\
& \le \frac{b}{6}\int_0^t \|V_{x_2x_2}(\cdot,\tau)\|^2\,d\tau + C\iiint |U_{x_1}|V^2(x_1,x_2,t)\,dx_1\,dx_2\,d\tau \\
& \quad + C\int_0^t \|V_{x_1}(\cdot,\tau)\|^2\,d\tau,
\end{aligned}
$$

where we have used estimate (34) and the assumption $N(T) \le \varepsilon_0$. In a similar way, one can obtain

$$
\begin{aligned}
& \left|\iiint V_{x_2}\,[f_2(U+V)]_{x_2x_2}\,dx_1\,dx_2\,d\tau\right| \\
(41)\quad & \qquad = \left|\iiint V_{x_2x_2} f_2'(U+V)V_{x_2}\,dx_1\,dx_2\,d\tau\right| \\
& \qquad \le \frac{b}{6}\int_0^t \|V_{x_2x_2}(\cdot,\tau)\|^2\,d\tau + C\int_0^t \|V_{x_1}(\cdot,\tau)\|^2\,d\tau.
\end{aligned}
$$

It follows from (38)–(41) that

$$
\begin{aligned}
& \|V_{x_2}(\cdot,t)\|^2 + b\int_0^t \|(V_{x_1x_2}, V_{x_2x_2})(\cdot,\tau)\|^2\,d\tau \\
(42)\quad & \le \|V_{x_2}(\cdot,0)\|^2 + C\left[\iiint |U_{x_1}|V^2(x_1,x_2,t)\,dx_1\,dx_2\,d\tau \right. \\
& \qquad \left. + \int_0^t \|(V_{x_1}, V_{x_2})(\cdot,\tau)\|^2\,d\tau\right].
\end{aligned}
$$

Similar estimate holds for V_{x_1}. Now, the lemma follows from (42) and Proposition 4. □

Next, we estimate the second derivatives of V.

Lemma 6. *If* $N(T) \le \delta_0$, *then*

$$\sum_{|\alpha|=2} \|\partial_x^\alpha V(\cdot, t)\|^2 + \int_0^t \sum_{|\alpha|=3} \|\partial_x^\alpha V(\cdot, \tau)\|^2 \, d\tau \le \sum_{|\alpha|=2} \|\partial_x^\alpha V(\cdot, 0)\|^2 + C \left[\iiint |U_{x_1}| V^2(x_1, x_2, t)\, dx_1\, dx_2\, d\tau + \int_0^t \|(V_{x_1}, V_{x_2})(\cdot, \tau)\|^2_{H^1(\mathbf{R}^2)} \, d\tau \right]. \tag{43}$$

Proof. We first estimate $V_{x_1x_1}$. Differentiating (20) twice with respect to x_1 and multiplying the resulting equation by $V_{x_1x_1}$, we find

$$V_{x_1x_1} \left\{ V_{x_1x_1t} + (f_1'(U)V]_{x_1x_1x_1} + [Q(U, V)_{x_1x_1x_1} + [f_2(U+V)]_{x_2x_1x_1} \right\} = \sum_{i,j=1}^{2} a_{ij} V_{x_1x_1} V_{x_ix_jx_1x_1}.$$

Integrating the above equation over $[0, t] \times \mathbf{R}^2$, we find after some manipulations and integration by parts that

$$\begin{aligned} &\frac{1}{2}\|V_{x_1x_1}(\cdot, t)\|^2 + b\int_0^t \|(V_{x_1x_1x_1}, V_{x_1x_1x_2})(\cdot, \tau)\|^2 \, d\tau \\ &\quad \le \frac{1}{2}\|V_{x_1x_1}(\cdot, 0)\|^2 + \iiint V_{x_1x_1x_1}[f_2(U+V)]_{x_2x_1}\, dx_1\, dx_2\, d\tau \\ &\quad + \iiint V_{x_1x_1x_1}[Q(U, V)]_{x_1x_1}\, dx_1\, dx_2\, d\tau \\ &\quad - \iiint V_{x_1x_1}[f_1'(U)V]_{x_1x_1x_1}\, dx_1\, dx_2\, d\tau. \end{aligned} \tag{44}$$

We denote the last three terms on the right-hand side of (44) by I_1, I_2 and I_3 respectively. I_1 can be estimated quite easily as before, indeed, it follows from Cauchy inequality that

$$\begin{aligned} I_1 &\equiv \iiint V_{x_1x_1x_1}[f_2(U+V)]_{x_2x_1}\, dx_1\, dx_2\, d\tau \\ &= \iiint V_{x_1x_1x_1}[f_2'(U+V)V_{x_2x_1} + f_2''(U+V)U_{x_1}V_{x_2} \\ &\qquad + f_2''(U+V)V_{x_1}V_{x_2}]\, dx_1\, dx_2\, d\tau \\ &\le \varkappa \int_0^t \|V_{x_1x_1x_1}(\cdot, \tau)\|^2 \tau + C(\varkappa)\int_0^t (\|V_{x_2}(\cdot, \tau)\|^2 + \|V_{x_1x_2}(\cdot, \tau)\|^2)\, d\tau \\ &\quad + C(\varkappa)\iiint |V_{x_1}|^2 |V_{x_2}|^2 (x_1, x_2, \tau)\, dx_1\, dx_2\, d\tau, \end{aligned}$$

where $\varkappa$ is a small positive constant to be chosen later. By Sobolev inequality, we have

$$
\begin{aligned}
&\iiint |V_{x_1}|^2|V_{x_2}|(x_1, x_2, t)\,dx_1\,dx_2\,d\tau \\
&\quad \le \iint \Big[\|V_{x_1}(\cdot, x_2, \tau)\|_{L^2(\mathbf{R}^1)} \\
&\qquad \times \|V_{x_1x_1}(\cdot, x_2, \tau)\|_{L^2(\mathbf{R}^1)}\|V_{x_2}(\cdot, x_2, \tau)\|^2_{L^2(\mathbf{R}^1)}\Big]\,dx_2\,d\tau \\
&\quad \le CA_1\left\{\int_0^t \|V_{x_1x_1}(\cdot, \tau)\|^2\,d\tau + \iint \|V_{x_2}(\cdot, x_2, \tau)\|^4_{L^2(\mathbf{R}^1)}\,dx_2\,d\tau\right\} \\
&\quad \le CA_1\int_0^t \|V_{x_1x_1}(\cdot, \tau)\|^2\,d\tau + CA_1B_1^2\int_0^t \|V_{x_2}(\cdot, \tau)\|^2\,d\tau,
\end{aligned}
\tag{45}
$$

where

$$
A_1 = \sup_{0\le\tau\le T,\, x_2\in\mathbf{R}^1} \|V_{x_1}(\cdot, x_2, \tau)\|^2_{L^2(\mathbf{R}^1)},
$$

$$
B_1 = \sup_{0\le\tau\le T,\, x_2\in\mathbf{R}^1} \|V_{x_2}(\cdot, x_2, \tau)\|^2_{L^2(\mathbf{R}^1)}.
$$

Consequently

$$
\begin{aligned}
I_1 \le{}& \varkappa\int_0^t \|V_{x_1x_1x_1}(\cdot, \tau)\|^2\,d\tau + C(\varkappa)(1 + A_1B_1^2)\int_0^t \|V_{x_2}(\cdot, \tau)\|^2\,d\tau \\
&+ C(\varkappa)\int_0^t \|V_{x_1x_2}(\cdot, \tau)\|^2\,d\tau + C((\varkappa)A_1\int_0^t \|V_{x_1x_1}(\cdot, \tau)\|^2\,d\tau.
\end{aligned}
\tag{46}
$$

Next, we estimate I_2. Straightforward calculation gives

$$
\begin{aligned}
I_2 \equiv{}& \iiint V_{x_1x_1x_1}[Q(U, V)]_{x_1x_1}\,dx_1\,dx_2\,d\tau \\
\le{}& C\iiint |V_{x_1x_1x_1}|\Big[|U_{x_1}|^2V^2 + |U_{x_1}V_{x_1}|V^2 + |U_{x_1}V_{x_1}V| \\
&\qquad + |V_{x_1x_1}|V^2 + |V_{x_1x_1}V| + V_{x_1}^2V^2 + |V|V_{x_1}^2 + V_{x_1}^2\Big]\,dx_1\,dx_2\,d\tau \\
&+ \iiint g(U, V)V_{x_1x_1x_1}U_{x_1x_1}V^2\,dx_1\,dx_2\,d\tau,
\end{aligned}
$$

where $g(U, V)$ is a smooth function. By Cauchy inequality, (34) and the assumption $N(T) \le \varepsilon_0$, one can verify that

(47)

$$
\begin{aligned}
I_2 \le{}& \varkappa\int_0^t \|V_{x_1x_1x_1}(\cdot, \tau)\|^2\,d\tau \\
&+ C(\varkappa)\left\{\iiint |U_{x_1}|V^2(x_1, x_2, t)\,dx_1\,dx_2\,d\tau + \int_0^t \|V_{x_1}(\cdot, \tau)\|^2_{H^1(\mathbf{R}^2)}\,d\tau \right.\\
&\qquad\qquad \left. + \iiint |V_{x_1}|^4(x_1, x_2, \tau)\,dx_1\,dx_2\,d\tau\right\} \\
&+ \iiint g(U, V)V_{x_1x_1x_1}U_{x_1x_1}V^2\,dx_1\,dx_2\,d\tau,
\end{aligned}
$$

where $\varkappa$ is as before. We now follow the similar argument as in (45) to show

$$\begin{aligned}(48)\qquad &\iiint |V_{x_1}|^4(x_1, x_2, \tau)\,dx_1\,dx_2\,d\tau\\ &\le CA_1\int_0^t \|V_{x_1x_1}(\cdot, \tau)\|^2\,d\tau + CA_1^3\int_0^t \|V_{x_2}(\cdot, \tau)\|^2\,d\tau.\end{aligned}$$

By using (iii) of Lemma 2 and (34), we can estimate the last integral on the right-hand side of (47) as follows:

$$\begin{aligned}&\iiint g(U, V)V_{x_1x_1x_1}U_{x_1x_1}V^2\,dx_1\,dx_2\,d\tau\\ &\le \iiint |g(U, V)||V_{x_1x_1x_1}|(kU_{x_1})V^2\,dx_1\,dx_2\,d\tau\\ (49)\qquad &\le \varkappa\int_0^t \|V_{x_1x_1x_1}(\cdot, \tau)\|^2\,d\tau\\ &\quad + C(\varkappa)\iiint |U_{x_1}|V^2(x_1, x_2, t)\,dx_1\,dx_2\,d\tau,\end{aligned}$$

so that

$$\begin{aligned}(50)\quad I_2 \le 2\varkappa&\int_0^t \|V_{x_1x_1x_1}(\cdot, \tau)\|^2\,d\tau\\ &+ C(\varkappa)\left\{\iiint |U_{x_1}|V^2(x_1, x_2, t)\,dx_1\,dx_2\,d\tau\right.\\ &\qquad + (1 + A_1^3)\int_0^t \|V_{x_2}(\cdot, \tau)\|^2_{H^1(\mathbf{R}^2)}\,d\tau\\ &\qquad\qquad \left.+ (1 + A_1)\int_0^t \|V_{x_1x_1}(\cdot, \tau)\|^2\,d\tau\right\}.\end{aligned}$$

Finally we estimate I_3.

$$\begin{aligned}(51)\quad I_3 &\equiv -\iiint V_{x_1x_1}[f_1'(U)V]_{x_1x_1x_1}\,dx_1\,dx_2\,d\tau\\ &= \iiint V_{x_1x_1x_1}[f_1'(U)V]_{x_1x_1}\,dx_1\,dx_2\,d\tau\\ &\le C\iiint |V_{x_1x_1x_1}|\left[U_{x_1}^2|V| + |U_{x_1}V_{x_1}| + |V_{x_1x_1}| + |U_{x_1x_1}V|\right]dx_1\,dx_2\,d\tau\\ &\le 2\varkappa\int_0^t \|V_{x_1x_1x_1}(\cdot, \tau)\|^2\,d\tau\\ &\quad + C(\varkappa)\left\{\iiint |U_{x_1}|V^2(x_1, x_2, t)\,dx_1\,dx_2\,d\tau\right.\\ &\qquad\qquad \left.+ \int_0^t \|V_{x_1}(\cdot, \tau)\|^2_{H^1(\mathbf{R}^2)}\,d\tau + \int_0^t \|V_{x_1x_1}(\cdot, \tau)\|^2\,d\tau\right\}.\end{aligned}$$

where we have used Cauchy inequality and (iii) of Lemma 2 as in the estimate of I_2.

We notice that in a similar way as in the proof of (35) and (36) one can check that

$$
\begin{aligned}
A_1^2 &\le \sup_{0\le\tau\le T}\left[\|V_{x_1}(\cdot,\tau)\|^2+\|V_{x_1x_2}(\cdot,\tau)\|^2\right]\le N(T)^2,\\
B_1^2 &= \sup_{0\le\tau\le T}\left[\|V_{x_2}(\cdot,\tau)\|^2+\|V_{x_2x_2}(\cdot,\tau)\|^2\right]\le N(T)^2.
\end{aligned}
\tag{52}
$$

Thus, if we now choose $\varkappa = b/8$, then it follows from (44), (46) and (50)–(52) that for $N(T)\le\delta_0$

$$
\begin{aligned}
&\|V_{x_1x_1}(\cdot,t)\|^2+b\int_0^t\|(V_{x_1x_1x_1},V_{x_1x_1x_2})(\cdot,\tau)\|^2\,d\tau\\
&\quad\le V_{x_1x_1}(\cdot,0)\|^2+C\left[\iiint|U_{x_1}|V^2(x_1,x_2,t)\,dx_1\,dx_2\,d\tau\right.\\
&\qquad\left.+\int_0^t\|(V_{x_1},V_{x_2})(\cdot,\tau)\|^2_{H^1(\mathbf{R}^2)}\,d\tau\right].
\end{aligned}
$$

Similar estimates hold for $V_{x_1x_2}$ and $V_{x_2x_2}$. Thus the proof of Lemma 6 is considered complete. □

Combining Lemma 6 with Proposition 5, we have proved the following time-uniform estimate

Proposition 7 (time-uniform estimate). *There exists constant $C_4>0$ independent of T, such that if $N(T)\le\delta_0$, then for $t\in[0,T]$, it holds that*

$$
\begin{aligned}
&\|V(\cdot,t)\|^2_{H^2(\mathbf{R}^2)}+\iiint|U_{x_1}|V^2(x_1,x_2,t)\,dx_1\,dx_2\,d\tau\\
&\quad+\int_0^t\|(V_{x_1}V_{x_2})(\cdot,\tau)\|^2_{H^2(\mathbf{R}^2)}\,d\tau\le C_4\|V(\cdot,0)\|^2_{H^2(\mathbf{R}^2)}.
\end{aligned}
\tag{53}
$$

4. Asymptotic behavior

The global existence of unique solutions for problem (20), (21) and its large time behavior is an immediate consequence of Proposition 6. Indeed, combining the standard theory of the existence and uniqueness of the local (in time) solution for parabolic equations with the time-uniform estimate (53), one can extend the local solution for (20), (21) globally by the usual continuity process and show that the estimate (53) holds forever (cf. [5, 3, 2]). Thus we have

Proposition 8. *There exist positive constants δ $(\le\delta_0)$ and C_4 such that if $\|V(\cdot,0)\|^2_{H^2(\mathbf{R}^2)}=\|u_0(\cdot)-u_0^r(\cdot)\|^2_{H^2(\mathbf{R}^2)}\le\delta$, then the initial value problem* (20), (21) *has a unique global solution $V\in X(0,+\infty)$ satisfying*

$$
\begin{aligned}
&\sup_{t\ge0}\|V(\cdot,t)\|^2_{H^2(\mathbf{R}^2)}+\int_0^{+\infty}\|U_{x_1}^{1/2}V(\cdot,t)\|^2\,dt\\
&\quad+\int_0^{+\infty}\|(V_{x_1}V_{x_2})(\cdot,t)\|^2_{H^2(\mathbf{R}^2)}\,d\tau\le C_4\|V(\cdot,0)\|^2_{H^2(\mathbf{R}^2)}.
\end{aligned}
\tag{54}
$$

To complete the proof of Theorem 1, by Proposition 8 and (i) of Lemma 2, it suffices to show that

$$\lim_{t\to+\infty}\|V(\cdot,t)\|_{L^\infty(\mathbf{R}^2)}=0. \tag{55}$$

In order to prove (55), first, we have from the inequality (54) and the equation (20) that

$$\int_0^{+\infty}\left\{\|(V_{x_1},V_{x_2})(\cdot,t)\|^2+\left|\frac{\partial}{\partial t}\|(V_{x_1},V_{x_2})(\cdot,t)\|^2\right|\right\}dt<+\infty,$$

from which it follows

$$\lim_{t\to+\infty}\|(V_{x_1},V_{x_2})(\cdot,t)\|^2=0. \tag{56}$$

On the other hand, we can get

$$\|V(\cdot,t)^2_{L^\infty(\mathbf{R}^2)}\le 2\|V(\cdot,t)\|\|V_{x_2}(\cdot,t)\|+2\|V_{x_1x_2}(\cdot,t)\|\|V_{x_1}(\cdot,t)\|. \tag{57}$$

In fact, for any fixed $(x_1,x_2,t)\in\mathbf{R}^2\times\mathbf{R}^+$, we have

$$\begin{aligned}V^2(x_1,x_2,t)&=2\int_{-\infty}^{x_1}\frac{\partial}{\partial x_1}V^2(x_1,x_2,t)\,dx_1\le 2\int_{-\infty}^{+\infty}|V|V_{x_1}|(x_1,x_2,t)\,dx_1\\&\le\|V(\cdot,x_2,t)\|^2_{L^2(\mathbf{R}^1)}+\|V_{x_1}(\cdot,x_2,t)\|^2_{L^2(\mathbf{R}^1)},\end{aligned}$$

thus

$$\|V(\cdot,t)\|^\infty_{L^2(\mathbf{R}^2)}\le\sup_{x_2\in\mathbf{R}^1}\|V(\cdot,x_2,t)\|^2_{L^2(\mathbf{R}^1)}+\sup_{x_2\in\mathbf{R}^1}\|V_{x_1}(\cdot,x_2,t)\|^2_{L^2(\mathbf{R}^1)}.$$

Furthermore we have by using Cauchy inequality that

$$\begin{aligned}\sup_{x_2\in\mathbf{R}^1}\|V(\cdot,x_2,t)\|^2_{L^2(\mathbf{R}^1)}&=\sup_{x_2\in\mathbf{R}^1}\int_{-\infty}^{x_2}\frac{\partial}{\partial x}\|V(\cdot,x_2,t)\|^2_{L^2(\mathbf{R}^1)}\,dx_2\\&\le 2\iint_{\mathbf{R}^2}|V||V_{x_2}|(x_1,x_2,t)\,dx_1,\ dx_2\\&\le 2\|V(\cdot,t)\|\|V_{x_2}(\cdot,t)\|.\end{aligned}$$

Similarly, one may get

$$\sup_{x_2\in\mathbf{R}^1}\|V_{x_1}(\cdot,x_2,t)\|^2_{L^2(\mathbf{R}^1)}\le 2\|V_{x_1x_2}(\cdot,t)\|\|V_{x_1}(\cdot,t)\|,$$

and (57) is proved. It follows from (56) and (57) that (55) holds, therefore we have the desired asymptotic behavior (10). This completes the proof of Theorem 1. □

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