

THE HEAT EQUATION FOR RIEMANNIAN FOLIATIONS

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In memory of Bruce L. Reinhart

ABSTRACT. Let \mathcal{F} be a Riemannian foliation on a closed oriented manifold M , with the transversal Laplacian Δ_B acting on the basic forms $\Omega_B^r(\mathcal{F})$ of degree $r \geq 0$. We construct the fundamental solution $e_B^r(x, y, t)$ for the basic heat operator $\partial/\partial t + \Delta_B$, and prove existence and uniqueness for the solution of the heat equation on $\Omega_B^r(\mathcal{F})$. As an application we give a new proof for the deRham-Hodge decomposition theorem for Δ_B in $\Omega_B^r(\mathcal{F})$, generalizing the approach to the classical deRham-Hodge theorem pioneered by Milgram and Rosenbloom.

1. INTRODUCTION

Let \mathcal{F} be a foliation on a closed oriented manifold M given by an exact sequence of vector bundles

$$0 \rightarrow L \rightarrow TM \xrightarrow{\pi} Q \rightarrow 0,$$

where L is the tangent bundle and Q the normal bundle of \mathcal{F} . The basic subcomplex $\Omega_B \equiv \Omega_B(\mathcal{F})$ of the deRham complex Ω_M is given by the forms ω satisfying $i(X)\omega = 0$ and $\theta(X)\omega = 0$ for all tangent vector fields $X \in \Gamma L$. The exterior differential d on Ω_M restricts by the Cartan formula $\theta(X) = di(X) + i(X)d$ to a differential $d_B: \Omega_B^r \rightarrow \Omega_B^{r+1}$. The basic cohomology $H_B \equiv H_B(\mathcal{F}) = H(\Omega_B, d_B)$ plays the role of the deRham cohomology of the leaf space M/\mathcal{F} of the foliation.

Let \mathcal{F} be a Riemannian foliation on M . A holonomy invariant transversal metric g_Q then gives rise to a basic Laplacian $\Delta_B = d_B\delta_B + \delta_B d_B: \Omega_B^r \rightarrow \Omega_B^r$ for all $r = 0, \dots, q$ up to the codimension q of the foliation. For the purposes of this paper we will use throughout a bundle-like metric g_M with basic mean curvature form κ , i.e., $\kappa \in \Omega_B^1$ (see §2). The formal adjoint $\delta_B: \Omega_B^r \rightarrow \Omega_B^{r-1}$ and therefore the Laplacian $\Delta_B: \Omega_B^r \rightarrow \Omega_B^r$ explicitly involve the mean curvature form. The harmonic basic forms \mathcal{H}_B^r of degree r are defined as the kernel of Δ_B on Ω_B^r .

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For a given form $\alpha_0 \in \Omega_B^r$, consider the initial value problem

$$(1.1) \quad \frac{\partial}{\partial t} \alpha(x, t) = -\Delta_B \alpha(x, t), \quad \lim_{t \downarrow 0} \alpha(x, t) = \alpha_0(x).$$

The main results of this paper can then be stated as follows.

Theorem. *Let \mathcal{F} be a transversally oriented Riemannian foliation on a closed oriented manifold (M, g_M) . Assume g_M to be a bundle-like metric with $\kappa \in \Omega_B^1$. Then the following hold.*

- (i) *There exists a unique solution α of (1.1), given in terms of the fundamental solution $e_B^r(x, y, t)$ of the basic heat operator $\partial/\partial t + \Delta_B$ by*

$$\alpha(x, t) = \int_M e_B^r(x, y, t) \wedge * \alpha_0(y).$$

- (ii) *Denote $\alpha(x, t) = [P_B(t)\alpha_0](x)$. Then there exists a uniform limit*

$$\lim_{t \uparrow \infty} P_B(t)\alpha_0 = H_B \alpha_0 \in \Omega_B^r,$$

and $H_B \alpha_0$ is Δ_B -harmonic.

- (iii) *The form*

$$G_B \alpha_0 = \int_0^\infty (P_B(t)\alpha_0 - H_B \alpha_0) dt$$

is well defined, and gives an operator $G_B: \Omega_B^r \rightarrow \Omega_B^r$ satisfying

$$\alpha_0 = \Delta_B G_B \alpha_0 + H_B \alpha_0.$$

The finite-dimensionality of $\mathcal{H}_B^r = \ker \Delta_B$ is a consequence of the proof-method. The identity in (iii) implies in usual fashion the orthogonal decomposition

$$\Omega_B^r \cong \text{im } d_B \oplus \text{im } \delta_B \oplus \mathcal{H}_B^r$$

and the isomorphism $H_B^r \cong \mathcal{H}_B^r$.

There are two published proofs [EH, KT2] of the deRham-Hodge decomposition theorem in Ω_B^r . The proof in [EH] does not involve the mean curvature assumption stated above. But the fundamental estimates necessary in certain applications, as, e.g., in [KRT], seem inaccessible through that method. On the other hand, the proofs in [KT2] and in this paper, using an explicit elliptic extension of Δ_B to the full deRham complex, do provide the necessary control to prove the desired estimates.

For the case of a point foliation, the heat equation approach described above is Milgram and Rosenbloom's approach to the classical deRham-Hodge decomposition theorem [MR]. A stochastic interpretation of this can be seen in Yosida [Y]. The new technical aspect in the present context is that the basic forms do not constitute all sections of a vector bundle, and the usual elliptic theory does not apply directly. The technical device to handle this situation is to extend $\Delta_B: \Omega_B(\mathcal{F}) \rightarrow \Omega_B(\mathcal{F})$ to a genuine elliptic operator $\hat{\Delta}: \Omega_M \rightarrow \Omega_M$ on the

full deRham complex, and to exploit the standard elliptic theory for $\tilde{\Delta}$. An explicit construction of such an extension was given in [KT2]. This is where the assumption of the basic mean curvature is involved (see §2). The authors do not know if the construction of the fundamental solution of the heat equation holds in a more general context. But what is known is that the mean curvature hypothesis holds for the following interesting classes of foliations: Riemannian submersions for appropriate metrics, E. Cartan's isoparametric families of surfaces, and foliations by the orbits of isometric Lie group actions. A typical example of the latter kind is the flow defined by a nonsingular Killing field. The mean curvature form is invariant under the flow, hence constant along the orbits.

In the construction of the fundamental solution $e_B'(x, y, t)$ of the basic heat operator $\partial/\partial t + \Delta_B$, the necessary convergence properties arise from the existence of the strongly elliptic extension $\tilde{\Delta}$ (see §4). However, since $\tilde{\Delta}$ need not be selfadjoint, this does not directly follow from corresponding properties of the fundamental solution of the heat operator $\partial/\partial t + \tilde{\Delta}$. For the case of basic functions (forms of degree 0) treated in [EK] these difficulties do not arise, since the ordinary Laplacian already preserves basic functions. In this case no assumption on the mean curvature is needed.

Comparing the heat flows $e^{-t\Delta_B}$ and $e^{-t\tilde{\Delta}}$, an important consequence of these constructions is the Invariance Theorem 4.11, stating that $e^{-t\tilde{\Delta}}$ acting on the full deRham complex Ω_M leaves the subcomplex Ω_B invariant, and coincides on Ω_B with $e^{-t\Delta_B}$. This constitutes a parabolic version of Hadamard's descent method for hyperbolic equations [H].

We expect that this heat equation approach will prove to be useful for the discussion of the transversal index problem for foliations. Another application is given in [RT], where an almost Lie foliation structure is deformed to a Lie foliation structure by the heat flow method.

We wish to contrast this discussion of the heat equation in transversal geometry with the heat flow along the leaves of \mathcal{F} . The latter is defined by the leafwise Laplacian Δ_0 . A deRham-Hodge decomposition for this leafwise elliptic operator has been established in [AT] for an appropriately defined Sobolev space of forms on M .

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2. CALCULUS IN Ω_B

The ingredients for the calculus in Ω_B are as follows [KT1, KT2]. Let $\nu \in \Omega_B^q$ be the transversal (closed) volume form associated to the transversal holonomy invariant Riemannian metric g_Q . The characteristic form $\chi_{\mathcal{F}}$ of \mathcal{F} is the p -form ($p = \text{dimension of leaves}$) given by $\chi_{\mathcal{F}} = *\nu$, expressed in terms of the star operator associated to g_M . The Riemannian volume form

$\mu \in \Omega_M^n$ is then given by $\mu = \nu \wedge \chi_{\mathcal{F}}$. These forms define the compatible orientations used throughout this paper.

The Weingarten map $W(Z): L \rightarrow L$ for $Z \in \Gamma L^\perp$ is given by $W(Z)X = -\pi^\perp(\nabla_X^M Z)$ for $X \in \Gamma L$, where ∇^M denotes the Riemannian connection of g_M , and π^\perp the tangential projection to L with kernel L^\perp . The mean curvature 1-form κ is then defined by $\kappa(Z) = \text{Tr } W(Z)$ for $Z \in \Gamma L^\perp$ and $\kappa(X) = 0$ for $X \in \Gamma L$ (the factor $1/p$ has been suppressed throughout). $\chi_{\mathcal{F}}$ and κ are related by Rummler’s formula

$$(2.1) \quad d\chi_{\mathcal{F}} + \kappa \wedge \chi_{\mathcal{F}} \equiv 0 \pmod{\mathcal{F}\text{-trivial forms}},$$

i.e, modulo forms evaluating to zero on p -vectors in $\bigwedge^p L$.

To give an alternative formulation of (2.1), consider the filtration [KT1, (2.1)]

$$F^r \Omega^m = \{\omega \in \Omega^m \mid i(X_1) \cdots i(X_{m-r+1})\omega = 0\}$$

for $X_1, \dots, X_{m-r+1} \in \Gamma L$. This decreasing filtration breaks off above the codimension q of \mathcal{F} . The \mathcal{F} -trivial $(p+r)$ -forms are precisely $F^{r+1} \Omega^{p+r}$. Thus (2.1) is equivalent to

$$(2.2) \quad d\chi_{\mathcal{F}} + \kappa \wedge \chi_{\mathcal{F}} = \varphi_0 \in F^2 \Omega^{p+1}.$$

We assume throughout that $\kappa \in \Omega_B^1$. This implies $d\kappa = 0$ [KT2, Appendix].

The star operator $*$: $\Omega_B^r \rightarrow \Omega_B^{q-r}$ associated to g_Q is related to the star operator $*$: $\Omega_M^r \rightarrow \Omega_M^{n-r}$ associated to g_M by the formulas

$$(2.3) \quad \bar{*}\alpha = (-1)^{p(q-r)} *(\alpha \wedge \chi_{\mathcal{F}}),$$

$$(2.4) \quad *\alpha = \bar{*}\alpha \wedge \chi_{\mathcal{F}}$$

for $\alpha \in \Omega_B^r$. The global scalar product $\langle \cdot, \cdot \rangle$ on Ω_M^r restricts on Ω_B^r to the expression

$$\langle \alpha, \alpha' \rangle_B = \int_M \alpha \wedge \bar{*}\alpha' \wedge \chi_{\mathcal{F}}.$$

With respect to this scalar product, the formal adjoint $\delta_B: \Omega_B^r \rightarrow \Omega_B^{r-1}$ of $d_B = d|_{\Omega_B}$ is given on $\alpha \in \Omega_B^r$ by [KT1, (4.14)]

$$(2.5) \quad \delta_B \alpha = (-1)^{q(r+1)+1} \bar{*}(d_B - \kappa \wedge) \bar{*}\alpha.$$

This is where the hypothesis of a basic mean curvature κ is involved. The basic Laplacian is then $\Delta_B = \delta_B d_B + d_B \delta_B$.

3. Δ_B AND COMPLETE ORTHOGONAL SYSTEM FOR $L^2(\Omega_B)$

The idea in [KT2] is to study Δ_B by constructing a strongly elliptic extension $\tilde{\Delta}: \Omega_M^r \rightarrow \Omega_M^r$. Let $\delta: \Omega_M^r \rightarrow \Omega_M^{r-1}$ be the formal adjoint of d , which is given by

$$\delta \omega = (-1)^{n(r+1)+1} * d * \omega.$$

The relation between $\delta\alpha$ and $\delta_B\alpha$ for $\alpha \in \Omega_B^r$ is then given by [KT2, Proposition 3.3]

$$(3.1) \quad \delta\alpha = \delta_B\alpha + *\gamma(\alpha),$$

where

$$\gamma(\alpha) = (-1)^{(p+1)(r+1)+qr} \bar{*}\alpha \wedge \varphi_0.$$

Let $\tilde{\gamma}: \Omega_M^r \rightarrow \Omega_M^{n-r+1}$ be the bounded linear operator given by [KT2, (3.11)]

$$(3.2) \quad \tilde{\gamma}(\omega) = (-1)^{(q+1)(p+r)+1} *(\omega \wedge \chi_{\mathcal{F}}) \wedge \varphi_0,$$

and $\tilde{\eta}: \Omega_M^r \rightarrow \Omega_M^r$ the differential operator of order one or less given by [KT2, (3.7)]

$$(3.3) \quad \tilde{\eta}(\omega) = *\tilde{\gamma}(d\omega) + d*\tilde{\gamma}(\omega).$$

Using the ordinary Laplacian $\Delta = \Delta_M: \Omega_M^r \rightarrow \Omega_M^r$, now let

$$(3.4) \quad \tilde{\Delta} = \Delta - \tilde{\eta}: \Omega_M^r \rightarrow \Omega_M^r.$$

Then $\tilde{\Delta}$ satisfies by [KT2, (3.6)]

$$(3.5) \quad \tilde{\Delta}|_{\Omega_B} = \Delta_B.$$

The main feature of the extension $\tilde{\Delta}$ of Δ_B is that it differs from the ordinary Laplacian Δ by lower order terms (the correction term $\tilde{\eta}(\omega)$ is of order one or less). In particular the symbol is unchanged and $\tilde{\Delta}$ is strongly elliptic. We use the existence of $\tilde{\Delta}$ in the proof of the following result.

3.6. Proposition. *There exists a complete orthonormal system (COS) for $L^2(\Omega_B^r)$, consisting of smooth eigenforms of Δ_B in Ω_B^r .*

Proof. First we note the obvious identities

$$\langle \Delta_B\alpha, \beta \rangle = \langle \alpha, \Delta_B\beta \rangle, \quad \langle \Delta_B\alpha, \alpha \rangle = \|d_B\alpha\|_{L^2}^2 + \|\delta_B\alpha\|_{L^2}^2$$

for $\alpha, \beta \in \Omega_B^r$, which imply that Δ_B defines a symmetric, positive operator

$$\Delta_B: D(\Delta_B) = \Omega_B \subset L^2(\Omega_B) \rightarrow L^2(\Omega_B).$$

We consider the symmetric operator $\Delta_1 = \Delta_B + I$,

$$\Delta_1: D(\Delta_1) = \Omega_B \subset L^2(\Omega_B) \rightarrow L^2(\Omega_B),$$

which is semibounded from below with lower bound 1, i.e.,

$$\langle \Delta_1\alpha, \alpha \rangle \geq \|\alpha\|_{L^2}^2 \quad \text{for } \alpha \in \Omega_B.$$

Let Δ_1^F denote the Friedrichs extension of Δ_1 [CO, p.11]

$$\Delta_1^F: D(\Delta_1^F) \subset L^2(\Omega_B) \rightarrow L^2(\Omega_B),$$

which by definition is the adjoint Δ_1^* of Δ_1 restricted to the domain $D(\Delta_1^F) = D(\Delta_1^*) \cap H_1$. Here H_1 denotes the Hilbert space completion of Ω_B with respect to the scalar product

$$\langle \alpha, \beta \rangle_1 = \langle \Delta_1\alpha, \beta \rangle \quad \text{for } \alpha, \beta \in \Omega_B.$$

Note that since

$$\|\alpha\|_1^2 = \langle \Delta_1 \alpha, \alpha \rangle = \|d_B \alpha\|_{L^2}^2 + \|\delta_B \alpha\|_{L^2}^2 + \|\alpha\|_{L^2}^2,$$

the new norm $\|\cdot\|_1$ is equivalent to the Sobolev H_1 -norm on Ω_B ; hence $H_1 \cong H_1(\Omega_B)$.

It follows from its construction that the Friedrichs extension Δ_1^F is still symmetric and surjective, and hence selfadjoint. Furthermore, since Δ_1^F has the same lower bound 1 as Δ_1 , Δ_1^F is also injective. Hence we have the selfadjoint inverse

$$G_1^F = (\Delta_1^F)^{-1} : R(\Delta_1^F) = L^2(\Omega_B) \rightarrow D(\Delta_1^F) \subset L^2(\Omega_B),$$

which is a bounded map into $D(\Delta_1^F)$. Since by Rellich's lemma the inclusion $H_1 \cong H_1(\Omega_B) \hookrightarrow L^2(\Omega_B)$ is compact, the composition

$$G_1^F : L^2(\Omega_B) \rightarrow L^2(\Omega_B)$$

is a compact selfadjoint operator. Thus G_1^F has eigenvalues $|\mu_1| \geq |\mu_2| \geq \dots \downarrow 0$ with corresponding eigenforms $\{\psi_i\}$ constituting a COS for $L^2(\Omega_B)$ [GL, p. 43].

We prove now that the ψ_i are smooth eigenforms of Δ_B . First note that $G_1^F \psi_i = \mu_i \psi_i$ implies that $\psi_i \in D(\Delta_1^F) \subset H_1(\Omega_B)$ and $\Delta_1^F \psi_i = \mu_i^{-1} \psi_i$. Now consider the elliptic operator $\tilde{\Delta}_1 = \tilde{\Delta} + I$, and let $\tilde{\Delta}_1^*$ denote its formal adjoint. Then we observe by the definition of distribution derivatives and (3.5) that for any $\alpha \in \Omega_B$

$$\langle \tilde{\Delta}_1 \psi_i, \alpha \rangle = \langle \psi_i, \tilde{\Delta}_1^* \alpha \rangle = \langle \psi_i, \Delta_1 \alpha \rangle = \langle \Delta_1^F \psi_i, \alpha \rangle = \langle \mu_i^{-1} \psi_i, \alpha \rangle;$$

that is, $\langle (\tilde{\Delta}_1 - \mu_i^{-1} I) \psi_i, \alpha \rangle = 0$ for any $\alpha \in \Omega_B$. Note that $(\tilde{\Delta}_1 - \mu_i^{-1} I) \psi_i \in H_{-1}(\Omega_B)$, the completion of Ω_B with respect to the Sobolev H_{-1} -norm $\|\cdot\|_{-1}$ on Ω_B defined by

$$\|\alpha\|_{-1} = \sup\{|\langle \alpha, \beta \rangle| : \|\beta\|_1 \leq 1, \beta \in \Omega_B\}.$$

Since the pairing $\langle \cdot, \cdot \rangle : H_{-1}(\Omega_B) \times H_1(\Omega_B) \rightarrow \mathbf{R}$ is continuous and nondegenerate, it then follows that ψ_i satisfies the elliptic equation

$$(\tilde{\Delta}_1 - \mu_i^{-1} I) \psi_i = 0.$$

This implies the regularity of ψ_i [GL, p. 23]; that is, $\psi_i \in \Omega_M$ and hence $\psi_i \in H_1(\Omega_B) \cap \Omega_M$. Since by [KT2, Corollary 4.14] we have $H_1(\Omega_B) \cap \Omega_M = \Omega_B$, this implies $\psi_i \in \Omega_B$. Now $\tilde{\Delta}_1 \psi_i = \mu_i^{-1} \psi_i$ implies by (3.5) that $\Delta_B \psi_i = \lambda_i \psi_i$ with $\lambda_i = \mu_i^{-1} - 1$. \square

Note that since Δ_B is a positive operator, we have $\lambda_i \geq 0$. From $|\mu_1| \geq |\mu_2| \geq \dots \downarrow 0$ it follows that $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \uparrow \infty$ (except of course for the case of a finite-dimensional Ω_B' , an exception we will not point out repeatedly). Furthermore, the multiplicities of the eigenvalues λ_i cannot increase too rapidly by the following fact.

3.7. Lemma. *Let $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \uparrow \infty$ be the eigenvalues of Δ_B in Ω_B^r . Then there exists a constant $C > 0$ and an exponent $\delta > 0$ such that $\lambda_j \geq C \cdot j^\delta$, provided $j \geq j_0$ is large.*

This is proved by the same argument as in [GL, p. 44].

4. HEAT EQUATION

For a given initial r -form $\alpha_0 \in \Omega_B^r$, we consider the initial value problem for the heat equation on Ω_B^r :

$$(4.1) \quad \frac{\partial}{\partial t} \alpha(x, t) = -\Delta_B \alpha(x, t), \quad \lim_{t \downarrow 0} \alpha(x, t) = \alpha_0(x).$$

We are going to prove that (4.1) has a unique solution $\alpha(x, t) \in \Omega_B^r$, $t > 0$, which is given in terms of the fundamental solution $e_B^r(x, y, t)$ of the basic heat operator $L_B = \partial/\partial t + \Delta_B$, to be constructed below, by

$$(4.2) \quad \alpha(x, t) = \int_M e_B^r(x, y, t) \wedge * \alpha_0(y).$$

For $(x, y, t) \in M \times M \times (0, \infty)$ we define

$$(4.3) \quad e_B^r(x, y, t) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \psi_i(x) \otimes \psi_i(y) \in \bigwedge^r T_x^* M \otimes \bigwedge^r T_y^* M,$$

where $\{\psi_i\}_{i=1}^{\infty}$ is a COS for $L^2(\Omega_B^r)$ consisting of smooth eigenforms ψ_i of Δ_B corresponding to the eigenvalues λ_i (Proposition 3.6).

4.4. Proposition. *$e_B^r(x, y, t)$ is a well-defined smooth double form of bidegree (r, r) on $M \times M$, depending smoothly on the parameter $t > 0$.*

Proof. Let k and s be positive integers such that $2s > k + n/2$. Then, by the a priori coercive estimate for $\tilde{\Delta}$ combined with the Sobolev embedding lemma, we get the fundamental estimate

$$\|\psi_i\|_{C^k} \leq C(\|\tilde{\Delta}^s \psi_i\|_{L^2} + \|\psi_i\|_{L^2}) \leq C(1 + \lambda_i^s).$$

Note also that for $t > 0$ and $\lambda > 0$ we have the estimate

$$e^{-t\lambda} \lambda^s \leq t^{-s} C(s) e^{-t\lambda/2}.$$

By applying these two inequalities to the RHS in (4.3), we get the following estimate:

$$(4.5) \quad \|e_B^r(x, y, t)\|_{C^k} \leq t^{-s(k)} C(k) \sum_{i=1}^{\infty} e^{-t\lambda_i/2}.$$

But by Lemma 3.7, for large $j > j_0$, we have also the estimate $\lambda_j > C \cdot j^\delta$ for some $C > 0$ and $\delta > 0$. This implies that the series on the RHS of (4.5) can be bounded by (some constant multiple of) the series $\sum_{j=1}^{\infty} e^{-t \cdot j^\delta/2}$, which converges absolutely for each $t > 0$. Since k is arbitrary, this shows that $e_B^r(x, y, t)$ is a well-defined smooth (r, r) -form on $M \times M \times (0, \infty)$. \square

Note that the integrand in (4.2) is a double form of bidegree (r, n) , which integrates for fixed x and t to an element in $\wedge^r T_x^* M$. It is now verified without difficulty that $\alpha(x, t)$ in (4.2) gives rise to a solution of the initial value problem (4.1).

For $\alpha \in \Omega_B^r$ and $t > 0$ we now define

$$(4.6) \quad [P_B(t)\alpha](x) = \int_M e_B^r(x, y, t) \wedge * \alpha(y) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \langle \alpha, \psi_i \rangle_B \psi_i(x).$$

Note that $P_B(t): \Omega_B^r \rightarrow \Omega_B^r$ for all $t > 0$. Moreover $P_B(t)$ is a symmetric operator for all $t > 0$:

$$(4.7) \quad \langle P_B(t)\alpha, \beta \rangle_B = \langle \alpha, P_B(t)\beta \rangle_B \quad \text{for } \alpha, \beta \in \Omega_B^r.$$

In fact, let $f(s, t) = \langle P_B(t)\alpha, P_B(s)\beta \rangle_B$. Then

$$\begin{aligned} \frac{\partial}{\partial t} f(t, s) &= \left\langle \frac{\partial}{\partial t} P_B(t)\alpha, P_B(s)\beta \right\rangle_B = \langle -\Delta_B P_B(t)\alpha, P_B(s)\beta \rangle_B \\ &= \langle P_B(t)\alpha, -\Delta_B P_B(s)\beta \rangle_B = \left\langle P_B(t)\alpha, \frac{\partial}{\partial s} P_B(s)\beta \right\rangle_B = \frac{\partial}{\partial s} f(t, s). \end{aligned}$$

It follows that $f(t, s)$ is a function of $t + s$. For $s = 0$ this yields the desired result in the form $f(t, 0) = f(0, t)$.

For a given $\omega_0 \in \Omega_M^r$, we can also consider the initial value problem for the heat equation on Ω_M^r :

$$(4.8) \quad \frac{\partial}{\partial t} \omega(x, t) = -\tilde{\Delta} \omega(x, t), \quad \lim_{t \downarrow 0} \omega(x, t) = \omega_0(x).$$

It is well known, see, e.g., [GR], that the fundamental solution $e_M^r(x, y, t)$ for the heat operator $\partial/\partial t + \tilde{\Delta}$ exists, and that the unique solution of (4.8) is given by

$$(4.9) \quad \omega(x, t) = \int_M e_M^r(x, y, t) \wedge * \omega_0(y).$$

For $\omega \in \Omega_M^r$ and $t > 0$ let

$$(4.10) \quad [P(t)\omega](x) = \int_M e_M^r(x, y, t) \wedge * \omega(y).$$

This defines $P(t): \Omega_M^r \rightarrow \Omega_M^r$. The following result may be considered a parabolic analog of Hadamard’s descent method for hyperbolic equations [H].

4.11. Invariance Theorem. (i) If $\alpha \in \Omega_B^r$, then $P(t)\alpha \in \Omega_B^r$ for all $t > 0$.

(ii) $P(t)|_{\Omega_B} = P_B(t)$ for all $t > 0$.

Proof. For a given $\alpha \in \Omega_B$, we have a solution $P_B(t)\alpha$ of (4.1), and a solution $P(t)\alpha$ of (4.8). Since $P_B(t)\alpha \in \Omega_B$ and $\tilde{\Delta}|_{\Omega_B} = \Delta_B$, it follows that $P_B(t)\alpha$ is also a solution of (4.8). By the uniqueness of the solution to (4.8), it follows that $P(t)\alpha = P_B(t)\alpha$. This proves (i) and (ii). \square

4.12. Corollary. *The solution to the initial value problem (4.1) is unique. In particular, $P_B(t)$ has the semigroup property*

$$P_B(t_1) \circ P_B(t_2) = P_B(t_1 + t_2) \quad \text{for } t_1, t_2 > 0.$$

It is now easy to prove the following supplementary fact:

$$(4.13) \quad \alpha \in \Omega_B^r, \quad d_B \alpha = 0 \Rightarrow d_B(P_B(t)\alpha) = 0 \text{ for all } t \geq 0.$$

Proof. By assumption $d_B P_B(0)\alpha = d_B \alpha = 0$. We show that $d_B P_B(t)\alpha$ is a solution of the heat equation (4.1). Namely

$$\frac{d}{dt} d_B P_B(t)\alpha = d_B \left(\frac{d}{dt} P_B(t)\alpha \right) = d_B(-\Delta_B P_B(t)\alpha) = -\Delta_B d_B P_B(t)\alpha.$$

By the uniqueness of the solution of the initial value problem (4.1), it follows that $d_B P_B(t)\alpha = 0$ for all $t \geq 0$. \square

5. LONG TIME BEHAVIOR OF SOLUTIONS

We now examine $P_B(t)\alpha$ for $\alpha \in \Omega_B$ in its dependence on t .

5.1. Lemma. *Let $\alpha \in \Omega_B^r$. Then $\|P_B(t)\alpha\|$ is a nonincreasing function of t for $t > 0$.*

Proof. We have

$$\begin{aligned} \frac{d}{dt} \|P_B(t)\alpha\|^2 &= 2 \left\langle \frac{d}{dt} P_B(t)\alpha, P_B(t)\alpha \right\rangle = -2 \langle \Delta_B P_B(t)\alpha, P_B(t)\alpha \rangle \\ &= -2[\|d_B P_B(t)\alpha\|^2 + \|\delta_B P_B(t)\alpha\|^2] \leq 0. \quad \square \end{aligned}$$

Next we turn to the discussion of the behavior of $P_B(t)\alpha$ for $t \rightarrow \infty$.

5.2. Theorem. *Let $\alpha \in \Omega_B^r$.*

- (i) $P_B(t)\alpha$ converges uniformly for $t \uparrow \infty$.
- (ii) $H_B \alpha = \lim_{t \uparrow \infty} P_B(t)\alpha \in \Omega_B^r$ is Δ_B -harmonic.

Proof. (i) By Lemma 5.1, there exists $\lim_{t \uparrow \infty} \|P(t)\alpha\|^2 = a \geq 0$. But by Corollary 4.12 and (4.7)

$$\begin{aligned} &\|P_B(t + 2h)\alpha - P_B(t)\alpha\|^2 \\ &= \|P_B(t + 2h)\alpha\|^2 + \|P_B(t)\alpha\|^2 - 2\langle P_B(t + 2h)\alpha, P_B(t)\alpha \rangle \\ &= \|P_B(t + 2h)\alpha\|^2 + \|P_B(t)\alpha\|^2 - 2\|P_B(t + h)\alpha\|^2, \end{aligned}$$

which approaches 0 for $t \uparrow \infty$. Thus $\{P_B(t)\alpha\}_{t \geq 0}$ has the Cauchy property, hence converges in $L^2(\Omega_B^r)$ for $t \uparrow \infty$.

To prove uniform convergence, we fix $h > 0$. Then we have for all $t \in (0, \infty)$ and $x \in M$

$$\begin{aligned} |P_B(t + 2h)\alpha - P_B(t + h)\alpha|(x) &= |P_B(h)(P_B(t + h)\alpha - P_B(t)\alpha)|(x) \\ &= \left| \int_M e_B^r(x, y, h) \wedge *(P_B(t + h)\alpha - P_B(t)\alpha)(y) \right| \\ &\leq \sup_{x \in M} \|e_B^r(x, \cdot, h)\| \cdot \|P_B(t + h)\alpha - P_B(t)\alpha\|, \end{aligned}$$

which goes to 0 for $t \uparrow \infty$ by the preceding observation. It follows that $P_B(t)\alpha$ converges uniformly on M for $t \uparrow \infty$. We define $H_B\alpha$ as this uniform limit.

(ii) To prove that $H_B\alpha$ is Δ_b -harmonic, we observe first that for fixed $t > 0$

$$\begin{aligned} |P_B(t+h)\alpha - P_B(t)H_B\alpha|(x) &= |P_B(t)(P_B(h)\alpha - H_B\alpha)|(x) \\ &\leq \sup_{x \in M} \|e_B^r(x, \cdot, t)\| \cdot \|P_B(h)\alpha - H_B\alpha\|, \end{aligned}$$

which implies for $h \uparrow \infty$.

$$0 = \lim_{h \uparrow \infty} (P_B(t+h)\alpha - P_B(t)H_B\alpha) = H_B\alpha - P_B(t)H_B\alpha.$$

Hence we have the invariance property

$$(5.3) \quad P_B(t)H_B\alpha = H_B\alpha.$$

It now follows from (5.3) that

$$\Delta_B H_B\alpha = \Delta_B P_B(t)H_B\alpha = -\frac{\partial}{\partial t} P_B(t)H_B\alpha = -\frac{\partial}{\partial t} H_B\alpha = 0$$

and $H_B\alpha$ is indeed Δ_B -harmonic. \square

Note that since $\tilde{\Delta}|\Omega_B = \Delta_B$, the Δ_B -harmonic forms \mathcal{H}_B form a subspace of the finite-dimensional space \mathcal{H} of $\tilde{\Delta}$ -harmonic forms. Thus \mathcal{H}_B is itself finite-dimensional.

For $\alpha \in \Omega_B^r$ define now

$$(5.4) \quad G_B\alpha(x) = \int_0^\infty (P_B(t)\alpha - H_B\alpha)(x) dt.$$

5.5. Lemma. $G_B\alpha$ is well defined, and $G_B\alpha \in \Omega_B^r$.

Proof. Let $k = \dim \mathcal{H}_B^r$. Then in (4.3) we have $\lambda_1 = \dots = \lambda_k = 0$ and $\psi_1, \dots, \psi_k \in \mathcal{H}_B^r$. It follows from (4.6) that

$$H_B\alpha = \lim_{t \uparrow \infty} P_B(t)\alpha = \sum_{i=1}^k \langle \psi_i, \alpha \rangle_B \psi_i \quad \text{in } \Omega_B^r,$$

since the contribution of the positive eigenvalues $\lambda_{k+1} \leq \lambda_{k+2} \leq \dots$ disappears for $t \uparrow \infty$. Hence we have

$$P_B(t)\alpha - H_B\alpha = \sum_{i=k+1}^\infty e^{-\lambda_i t} \langle \psi_i, \alpha \rangle_B \psi_i \quad \text{in } \Omega_B^r.$$

For fixed $h > 0$, and each $x \in M$, $t \in (0, \infty)$ we then have

$$\begin{aligned} |P_B(t+h)\alpha - H_B\alpha|(x) &= |P_B(h)(P_B(t)\alpha - H_B\alpha)|(x) \\ &= \left| \int_M e_B^r(x, y, h) \wedge *(P_B(t)\alpha - H_B\alpha)(y) \right| \\ &\leq \left(\sup_x \|e_B^r(x, \cdot, h)\| \right) \cdot \|(P_B(t)\alpha - H_B\alpha)\| \\ &= c \left\| \sum_{i=k+1}^\infty e^{-\lambda_i t} \langle \psi_i, \alpha \rangle_B \psi_i \right\| \\ &\leq ce^{-\lambda_{k+1}t} \left\| \sum_i \langle \psi_i, \alpha \rangle_B \psi_i \right\| \leq c\|\alpha\|e^{-\lambda_{k+1}t}. \end{aligned}$$

It follows that

$$\begin{aligned} \left| \int_0^{t+h} (P_B(t+h)\alpha - H_B\alpha)(x) dt \right| &\leq \int_0^{t+h} |P_B(t+h)\alpha - H_B\alpha|(x) dt \\ &\leq c\|\alpha\| \cdot \int_0^{t+h} e^{-\lambda_{k+1}t} dt. \end{aligned}$$

Since the last term converges as $t \uparrow \infty$, it follows that the RHS in (5.4) exists as a limit for $t \uparrow \infty$ of \int_0^t , uniformly in x . Since the integrand is a basic r -form for each t , it follows that $G_B\alpha \in \Omega_B^r$. \square

Finally we calculate, using the harmonicity of $H_B\alpha$,

$$\begin{aligned} \Delta_B G_B\alpha &= \Delta_B \int_0^\infty (P_B(t)\alpha - H_B\alpha) dt = \int_0^\infty \Delta_B (P_B(t)\alpha - H_B\alpha) dt \\ &= \int_0^\infty (\Delta_B P_B(t)\alpha) dt = \int_0^\infty \left(-\frac{d}{dt} P_B(t)\alpha \right) dt \\ &= \lim_{t \downarrow 0} P_B(t)\alpha - \lim_{t \uparrow \infty} P_B(t)\alpha = \alpha - H_B\alpha. \end{aligned}$$

The identity

$$(5.6) \quad \alpha = \Delta_B G_B\alpha + H_B\alpha$$

completes now the proof of the theorem stated in the Introduction. \square

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