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ALGEBRAIC DISTANCE GRAPHS AND RIGIDITY

M. HOMMA AND H. MAEHARA

ABSTRACT. An algebraic distance graph is defined to be a graph with vertices in E^n in which two vertices are adjacent if and only if the distance between them is an algebraic number. It is proved that an algebraic distance graph with finite vertex set is complete if and only if the graph is "rigid". Applying this result, we prove that (1) if all the sides of a convex polygon Γ which is inscribed in a circle are algebraic numbers, then the circumradius and all diagonals of Γ are also algebraic numbers, (2) the chromatic number of the algebraic distance graph on a circle of radius r is ∞ or 2 accordingly as r is algebraic or not. We also prove that for any $n > 0$, there exists a graph G which cannot be represented as an algebraic distance graph in E^n.

1. INTRODUCTION

Let X be a nonempty point set in a Euclidean space, and D a set of real numbers. Let $X(D)$ denote the graph with vertex set X in which two distinct points x, y are adjacent if and only if

$$|x - y| \in D.$$

The graph $X(D)$ is called the *D-distance graph* on X. Specifying D in various ways, there arise many interesting graphs. If $D = \{1\}$, we have unit distance graphs, e.g., [4, 9]. The case $D = [0, 1]$ is a generalization of unit interval graphs, e.g. [7, 8, 10]. Letting D be the set of integers, Anning and Erdős [1] proved that if $X(D)$ is a complete graph of infinite order, then all points of X lie on a line. Coloring problems of the real line for various D are discussed in [6].

In this paper we consider the case $D = A$, the set of all algebraic numbers. We call $X(A)$ the *algebraic distance graph* on X. We prove that for a finite set X, the algebraic distance graph $X(A)$ is complete if and only if $X(A)$ is a "rigid" graph. Applying this result, we have that if the sides of a convex polygon Γ, which is inscribed in a circle, are all algebraic numbers, then all diagonals of Γ and the radius of the circumcircle of Γ are also algebraic numbers.

Concerning the chromatic number of the algebraic distance graph on a circle C_r of radius r, we have

$$\chi(C_r(A)) = \begin{cases} \infty & \text{if } r \text{ is algebraic,} \\ 2 & \text{otherwise.} \end{cases}$$

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It is not difficult to see that every finite graph G is represented as an algebraic distance graph in Euclidean space. We also prove that for any integer $n > 0$, there is a finite graph G which cannot be represented as an algebraic distance graph in Euclidean n-space E^n.

2. Preliminary results

In this section the set D is not specified. Let $\overline{D} = R - D$, the complement of D in the reals. Throughout this paper, X stands for a nonempty point set in Euclidean space. The following lemma is clear.

Lemma 1. *A graph G is isomorphic to $X(D)$ if and only if $\overline{G}$ is isomorphic to $X(\overline{D})$, where $\overline{G}$ is the complement of the graph G.* □

Theorem 1. *Suppose that both D and $\overline{D}$ contain at least one positive number. Then every finite simple graph G is isomorphic to $X(D)$ for some X.*

Proof. Let $v_1, \dots, v_n$ be the vertices of G, and $A(G) = (a_{ij})$ be the adjacency matrix of G, i.e., $a_{ij} = 1$ or 0 accordingly as v_i, v_j are adjacent or not. If $t > 0$ is a sufficiently large number (which will be specified later), then the matrix $A(G) + tI$ (I: the identity matrix) is symmetric and positive definite. (Indeed, $t > n$ is sufficient because the minimum eigenvalue of $A(G)$ is greater than $-n$, see, e.g., [14].) Hence it can be represented as the product of an $(n \times n)$-matrix M and its transpose M':

$$A(G) + tI = M \cdot M'.$$

Let $\overline{v}_i$ denote the point in Euclidean n-space that corresponds to the ith row of M. Then the inner product $\langle \overline{v}_i, \overline{v}_j \rangle = t$ if $i = j$, and $= a_{ij}$ if $i \neq j$. Therefore,

$$|\overline{v}_i - \overline{v}_j|^2 = 2t - 2a_{ij}.$$

Now, since both D and $\overline{D}$ contain at least one positive number, there exists a frontier point $c > 0$ of D. Choose two positive numbers $a \in D$, $b \in \overline{D}$ sufficiently close to c so that $(1/a + 1/b)|a - b| < 1/n$. If $a < b$, we put

$$t = b^2/(b^2 - a^2), \qquad \tilde{v}_i = ((b^2 - a^2)/2)^{1/2}\overline{v}_i.$$

Then it can easily be verified that $|\tilde{v}_i - \tilde{v}_j| = a$ if $a_{ij} = 1$, and $= b$ if $a_{ij} = 0$. Thus the D-distance graph on $\{\tilde{v}_1, \dots, \tilde{v}_n\}$ is isomorphic to G. If $a > b$, then we can prove that there exists a point set X such that $X(\overline{D})$ is isomorphic to $\overline{G}$. Then, applying Lemma 1, we have the theorem. □

Thus every finite graph is represented as $X(A)$ for some X. The dimension of the flat spanned by X is denoted by $\dim(X)$. For a graph G, let $\dim_A G$ denote the minimum value of $\dim(X)$ such that $X(A)$ is isomorphic to G. In §6, we will investigate $\dim_A G$.

3. Algebraic distance graphs and rigidity

A graph G with vertices in Euclidean space E^n is called a graph in E^n. Thus a graph G in E^n is a nonempty subset of E^n in which a number of pairs are distinguished as the edges. An *isotopy* of a graph G in E^n with vertex set X is a parametrized family of embeddings $f_t: X \to E^n$, $0 \le t < \varepsilon$, such that

(1) $f_0(x) = x$ for all $x \in X$,

(2) $f_t(x)$ is continuous on t,

(3) $|f_t(x) - f_t(y)| = |x - y|$ for all edges xy to G.

A graph G in E^n is said to be *flexible* if there is an isotopy f_t of G with

$$|f_t(x) - f_t(y)| \neq |x - y|$$

for some pair of points x, y of G and some t. If G is not flexible, then G is said to be *rigid* in E^n. For example, a complete graph in E^n is always rigid.

For more information on rigidity and flexibility of graphs, see, e.g., [2, 3, 13].

The next theorem will relate rigidity to algebraic distance graphs. A system of $(n+1)$ points p_i, $i = 0, \dots, n$, is said to be *affinely independent* if the vectors $\overrightarrow{p_0p_i}$, $i = 1, \dots, n$, are linearly independent. Thus n points in E^n are affinely independent if and only if they span a hyperplane. The following lemma is clear, and we omit the proof.

Lemma 2. *Let G_1, G_2 be two rigid graphs in E^n with vertex sets X_1, X_2, respectively. Suppose that $X_1 \cap X_2$ contains a set of n affinely independent points $p_1, \dots, p_n$. Then the following holds:*

(1) *If an isotopy $f_t: X_1 \to E^n$ of G_1 fixes p_i, $i = 1, \dots, n$, then f_t fixes all points of G_1.*

(2) *If a graph G with vertex set $X_1 \cup X_2$ contains G_1, G_2 as subgraphs, then G is rigid.* □

Theorem 2. *For a finite set $X \subset E^n$, the algebraic distance graph $X(A)$ is complete if and only if it is rigid.*

To prove this theorem, we need the following proposition concerning algebraic sets, the proof of which will be given in §7.

Proposition 1. *Let $\{f_1(x_1, \dots, x_N), \dots, f_m(x_1, \dots, x_N)\}$ be a collection of polynomials with coefficients in the set A of all real algebraic numbers. Let V denote the real algebraic set defined by $f_1 = \cdots = f_m = 0$. Suppose that $p = (s_1, \dots, s_N) \in V$ is an isolated point of V. Then $s_1, \dots, s_N$ are all algebraic numbers.*

Proof of Theorem 2. It is obvious that if $X(A)$ is complete, then it is rigid.

Suppose that $X(A)$ is rigid. Let $k = \dim(X)$. Then without loss of generality, we may assume that X is in Euclidean k-space. Let $X = \{p_1, p_2, \dots, p_n\}$, and $p_i = (s_{i,1}, \dots, s_{i,k})$, $i = 1, 2, \dots, n$. We may further suppose that

$p_1, p_2, \ldots, p_{k+1}$ span the k-space and

$$\begin{aligned} p_1 &= (0, \ldots, 0), \\ p_2 &= (s_{2,1}, 0, \ldots, 0), \\ p_3 &= (s_{3,1}, s_{3,2}, 0, \ldots, 0), \\ &\cdots \\ p_k &= (s_{k,1}, \ldots, s_{k,k-1}, 0). \end{aligned}$$

If $|p_i - p_j|$ is an algebraic number, then denote this number by e_{ij}, that is, e_{ij} is the "edge length". Now consider the following system of polynomial equations:

$$(*) \qquad \begin{cases} (x_{i,1} - x_{j,1})^2 + \cdots + (x_{i,k} - x_{j,k})^2 - e_{ij}^2 = 0 \\ \qquad\qquad \text{for all "edges" of } X(A), \\ x_{i,i} = \cdots = x_{i,k} = 0 \quad \text{for } i = 1, \ldots, k. \end{cases}$$

These equations define a real algebraic set V in kn-space E^{kn}, and the point $p = (p_1, \ldots, p_n)$ is a point of V. For a moment, suppose that p is an isolated point of V. Then by Proposition 1, all coordinates of p are algebraic numbers, and hence the distance between any two points of X is an algebraic number; therefore, $X(A)$ is a complete graph.

Thus our remaining task is to show that p is an isolated point of V. Suppose, on the contrary, that $V - \{p\}$ contains points arbitrarily close to p. Then "the curve selection lemma" (see Milnor [11, p. 25]) asserts that there exists a real analytic curve $x: [0, \varepsilon) \to V$ with $x(0) = p$ and $x(t) \neq p$ for $t > 0$. Denote by $f_t(p_i)$ the projection of $x(t) \in E^k \times \cdots \times E^k = E^{kn}$ onto the ith factor E^k, that is,

$$x(t) = (f_t(p_1), \ldots, f_t(p_n)) \in E^k \times \cdots \times E^k = E^{kn}.$$

Then, since $x(t)$ $(0 \leq t < \varepsilon)$ lies on the algebraic set V defined by $(*)$, $f_t: X \to E^k$ is an isotopy of the graph $X(A)$. And since $X(A)$ is rigid, we have

$$(1) \qquad |f_t(p_i) - f_t(p_j)| = |p_i - p_j| \quad \text{for all } i \neq j.$$

Furthermore, since the equations $x_{i,i} = \cdots = x_{i,k} = 0$, $i = 1, \ldots, k$, hold on the algebraic set V, the last $k - i + 1$ coordinates of $f_t(p_i)$ are zero for $i = 1, \ldots, k$, that is,

$$(2) \qquad f_t(p_i) = (\underbrace{*, \ldots, *}_{i-1}, \underbrace{0, \ldots, 0}_{k-i+1})$$

First, we show that f_t fixes p_i, $i = 1, \ldots, k$. This is done by induction on i. For $i = 1$, $f_t(p_1) = (0, \ldots, 0)$ by (2), whence $f_t(p_1) = p_1$. Suppose $f_t(p_i) = p_i$ for $i < j \leq k$. By (2) we may regard $\{f_t(p_1), \ldots, f_t(p_j)\}$ as a subset of $(j-1)$-space E^{j-1}, and by (1) we may regard f_t as an isotopy of the complete graph on $\{f_t(p_1), \ldots, f_t(p_j)\}$. Then since f_t fixes $p_1, \ldots, p_{j-1}$ which are affinely independent, it follows from Lemma 2(1) that $f_t(p_j) = p_j$. Thus, f_t fixes $p_1, \ldots, p_k$. Now since these k points are affinely independent,

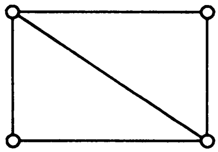

FIGURE 1

applying Lemma 2(1) again, we have $f_t(q) = q$ for all q of X. Therefore, $x(t) = p$, a contradiction. □

Example 1. Let G be the graph obtained from the complete graph K_4 by removing one edge (see Figure 1). Since any graph in the plane isomorphic to G is rigid, as is easily seen, G cannot be represented by an algebraic graph in the plane. However, it can be represented in 3-space. (How ?) Hence $\dim_A G = 3$.

4. A FEW APPLICATIONS

Let T be a triangle with sides a, b, c all algebraic numbers. Then by Heron's formula, the area of T is also algebraic. Let r be the radius of the circumcircle of T. Then by the sine law, $c/(\sin\gamma) = 2r$, where γ is the angle opposite the side c. Hence the area of T is represented as $(1/2)ab(\sin\gamma) = abc/(4r)$. Therefore the circumradius r of T is also an algebraic number. The next theorem genrealizes this result.

Theorem 3. *Let Δ be an n-dimensional simplex in E^n whose sides are all algebraic numbers. Then the radius of the circumsphere of Δ (i.e., the sphere passing through all vertices of Δ) is also an algebraic number.*

Proof. Let r be the radius of the circumsphere of Δ. We regard Δ as a simplex lying on a hyperplane H in E^{n+1}. Let L be the line passing through the circumcenter of Δ, and perpendicular to the hyperplane H. Let p, q be the two points on L, each at distance

$$(m - r^2)^{1/2}$$

from the circumcenter, where m is a sufficiently large integer. Then the vertices of Δ and p, q, induce together an algebraic distance graph G, which is rigid by Lemma 2(2). Hence by Theorem 2, G is a complete graph, and hence $|p - q| = 2(m - r^2)^{1/2}$ is algebraic. Therefore, r is algebraic. □

By a *polygon* in the plane, we mean a closed polygonal curve possibly having self-intersections. Let O denote the origin. The *area* of a polygon $\Gamma = p_1p_2\cdots p_n$ is defined by

$$\text{area}(\Gamma) = \sum_{i=1}^{n} \varepsilon_i(\text{area of } \Delta O p_i p_{i+1}) \qquad (p_{n+1} := p_1),$$

where $\varepsilon_i = +1$ if $O \to p_i \to p_{i+1} \to O$ is counter-clockwise, and $\varepsilon_i = -1$ otherwise. Note that for a convex polygon, this definition agrees with the usual definition of area in absolute value. The *winding number* wind(Γ) of a polygon $\Gamma = p_1 \cdots p_n$ around the origin O is defined by

$$\text{wind}(\Gamma) = \frac{1}{2\pi} \sum_{i=1}^{n} \angle p_i O p_{i+1},$$

where $\angle p_i O p_{i+1}$ is the signed angle. If Γ passes through the origin, then we define $\text{wind}(\Gamma) = 0$. For example, a convex polygon Γ has winding number ± 1 if Γ contains the origin O inside, and $\text{wind}(\Gamma) = 0$ otherwise.

For a polygon $\Gamma = p_1 p_2 \cdots p_n$ in the xy-plane in E^3, the "suspension graph" of Γ with poles $q_1 = (0, 0, z)$, $q_2 = (0, 0, -z)$ is the graph with vertices $q_1, q_2, p_1, \dots, p_n$ and edges $p_i p_{i+1}$, $q_j p_i$, $i = 1, \dots, n$, $j = 1, 2$. The following theorem is a special case of the results of Connelly [5, Theorems 1, 2].

Theorem C (Connelly). *Let Γ be a polygon in the xy-plane in E^3 and G be the suspension graph of Γ with poles $q_1 = (0, 0, z)$ and $q_2 = (0, 0, -z)$. If G is flexible, then*

$$\text{area}(\Gamma) = 0 \quad \textit{and} \quad \text{wind}(\Gamma) = 0. \quad \square$$

If the length of every side of a polygon is an algebraic number, then it will be called an *algebraic polygon.*

Theorem 4. *Let Γ be an algebraic polygon inscribed in a circle with center at the origin. If* $(\text{area}(\Gamma), \text{wind}(\Gamma)) \neq (0, 0)$, *then the radius r of the circle and the all diagonals of Γ are also algebraic numbers.*

Proof. Let $p_1, \dots, p_n$ be the vertices of the polygon Γ. We may suppose that the circle lies in the xy-plane in E^3 with center at the origin. Take an integer $m > r$, and put

$$q_1 = (0, 0, (m - r^2)^{1/2}), \qquad q_2 = (0, 0, -(m - r^2)^{1/2}).$$

Let $X = \{p_1, \dots, p_n, q_1, q_2\}$. Then, since $|p_i - q_j| = m$, the graph $X(A)$ contains, as a spanning subgraph, the suspension graph of Γ with poles q_1, q_2. However, since $(\text{area}(\Gamma), \text{wind}(\Gamma)) \neq (0, 0)$, the suspension graph of Γ is rigid by Theorem C. Hence $X(A)$ is also rigid, and hence $X(A)$ is complete by Theorem 2. Therefore, all diagonals of Γ and $|q_1 - q_2| = 2(m - r^2)^{1/2}$ are algebraic numbers. $\square$

We denote by C_r the circumference of a circle of radius r.

Corollary 1. *Let Γ be a convex polygon inscribed in a circle C_r. If the sides of Γ are all algebraic numbers, then the radius r and the all diagonals of Γ are also algebraic numbers.* $\square$

Corollary 2. *Let Γ be an algebraic polygon inscribed in a circle of center O and transcendental radius. Then* $\text{area}(\Gamma) = \text{wind}(\Gamma) = 0$. $\square$

5. The algebraic distance graph on a circle

In this section we study the (infinite) algebraic distance graph $C_r(A)$ on C_r, the circumference of a circle of radius r.

Theorem 5. *Suppose $r > 0$ is algebraic. Then every connected component of $C_r(A)$ is complete.*

Proof. Let p_0 be the center of C_r. If $p_1p_2p_3$ is a path in the graph $C_r(A)$, then p_i, $i = 0, 1, 2, 3$, induce a rigid algebraic distance graph, whence $|p_1 - p_3| \in A$, and p_1 and p_3 are adjacent in $C_r(A)$. From this it follows easily that if two vertices of $C_r(A)$ are connected by a path, then the two vertices are adjacent. Therefore, every connected component of $C_r(A)$ is complete. □

Next, we are going to show that if $r > 0$ is transcendental, then $C_r(A)$ contains no odd cycle. We begin with a lemma.

Lemma 3. *Let a_i, $i = 1, \dots, n$, be n distinct algebraic numbers, and let w be a transcendental number such that $w > a_i$, $i = 1, \dots, n$. Then n real numbers $(w - a_i)^{1/2}$, $i = 1, \dots, n$, are linearly independent over the field A of real algebraic numbers.*

Proof. Suppose that there is a nontrivial linear combination of the $(w - a_i)^{1/2}$'s such that

$$b_1(w - a_1)^{1/2} + \cdots + b_n(w - a_n)^{1/2} = 0, \qquad b_i \in A.$$

Let $f(x)$ be the function obtained from the left-hand side of the above formula by replacing w by x. Then $f(x)$ is a nontrivial algebraic function over A (that is, $f(x)$ is algebraic over the field of rational functions $A(x)$). Hence there exists an irreducible polynomial over $A(x)$

$$F(x, y) = g_0(x)y^k + g_1(x)y^{k-1} + \cdots + g_k(x), \qquad g_i(x) \in A(x),$$

such that $F(x, f(x)) = 0$. Note that $g_k(x) \neq 0$. Without loss of generality, we may assume that the $g_i(x)$'s are all polynomials over A. Now, since $f(w) = 0$, we have

$$0 = F(w, f(w)) = g_k(w),$$

which is a contradiction since w is transcendental. □

Theorem 6. *Let $r > 0$ be a transcendental number. Then the algebraic distance graph $C_r(A)$ contains no odd cycle.*

Proof. Without loss of generality, we may assume that the center of the circle C_r is at the origin O. We show that every cycle of $C_r(A)$ is of even order. Consider a cycle of $C_r(A)$, and let $\Gamma = p_1 \cdots p_n$ be the corresponding algebraic polygon inscribed in C_r. We must show that n is even. Let $c_i = |p_i - p_{i+1}|$, $i = 1, \dots, n-1$, $c_n = |p_n - p_1|$. Then by Corollary 2, the area of Γ is zero, and since the area of the triangle Op_ip_{i+1} is $(c_i/2)(r^2 - (c_i/2)^2)^{1/2}$, we have

$$(*) \qquad 0 = \text{area}(\Gamma) = \sum_{i=1}^{i=n} \varepsilon_i (c_i/2)(r^2 - (c_i/2)^2)^{1/2},$$

where $\varepsilon = +1$ if the orientation of the triangle Op_ip_{i+1} is counter-clockwise, $\varepsilon_i = -1$ otherwise. Let $a_1, \dots, a_m$ be the distinct numbers in $\{c_1/2, \dots, c_n/2\}$ and let $w = r^2$. Then, since $(w - a_i^2)^{1/2}$, $i = 1, \dots, m$, are linearly independent over A by Lemma 3, it follows from $(*)$ that for each j $(1 \le j \le m)$,

$$0 = \sum_{c_i/2=a_j} \varepsilon_i(c_i/2) = a_j \sum_{c_i/2=a_j} \varepsilon_i.$$

Hence the number of subscripts i such that $c_i/2 = a_j$ is even for every j. Therefore n is even. □

Corollary 3. *If $r > 0$ is transcendental, then $C_r(A)$ is a bipartite graph.* □

Corollary 4. *The chromatic number of $C_r(A)$ is*

$$\chi(C_r(A)) = \begin{cases} \infty & \text{if } r \text{ is algebraic}, \\ 2 & \text{otherwise.} \end{cases}$$ □

The algebraic distance graph on C_r with r transcendental has even cycle of arbitrary order $2n$.

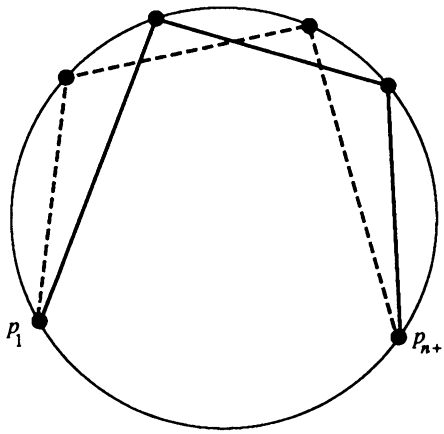

FIGURE 2

Example 2. For any integer $n > 1$, the graph $C_r(A)$, $r > 0$, contains a cycle of order $2n$. This can be seen as follows: Take a path $P = p_1p_2 \cdots p_{n+1}$ of order $n + 1$ in $C_r(A)$ (see Figure 2). This is clearly possible. Then reflect this path with respect to the perpendicular bisector of the line segment p_1p_{n+1} (in Figure 2, the reflected path is indicated by dotted line). These two paths together make a cycle of order $2n$. Of course we must choose a path P so that P and its reflection share only two vertices in common.

Theorem 7. *For any $r > 0$, the graph $C_r(A)$ is disconnected.*

Proof. First suppose that r is an algebraic number. Let $p, q \in C_r$ be two points such that $|p - q|$ is a transcendental number. Then by Theorem 5,

p and q cannot belong to the same component of C_r. Hence $C_r(A)$ is disconnected. Now suppose that r is transcendental and $C_r(A)$ is connected. Let $P = p_1p_2\cdots p_n$ be a path of minimum order that connects a pair of diametrically opposite points. Since the diameter $2r$ is transcendental, n is greater than 2. Let $Q = q_1q_2\cdots q_n$ be the path obtained by rotating P around the center O of C_r through the angle π. Then $q_1 = p_n$ and $q_n = p_1$. Since the path P is a path of minimum order that connects p_1, p_n, the paths P, Q share only two points in common. Note that

$$\angle p_1Op_2 + \cdots + \angle p_{n-1}Op_n = \angle q_1Oq_2 + \cdots + \angle q_{n-1}Oq_n = (2m+1)\pi$$

for some integer m. Let Γ be the algebraic polygon

$$p_1p_2\cdots p_{n-1}q_1q_2\cdots q_{n-1}.$$

Then, $\mathrm{wind}(\Gamma) = (2m+1) \neq 0$, which contradicts Corollary 2. □

6. The minimum dimensional representations

First note that if H is an "induced" subgraph of G, then $\dim_A H \leq \dim_A G$. This is not the case for a subgraph. For example, $\dim_A K_4 = 1$, but $\dim_A(K_4 -$ one edge$) = 3$ as already seen.

Theorem 8. *For a complete bipartite graph* $K(m, n)$ *with order* $m+n \geq 3$, *we have* $\dim_A K(m, n) = 2$.

Thus, for any $m, n > 0$, there exists a flexible graph in the plane which is isomorphic to $K(m, n)$.

Proof. It is clear that $\dim_A K(m, n) > 1$ for $m+n \geq 3$. We represent $K(m, n)$ as an algebraic distance graph in the plane E^2. Let $p_i = ((i+\pi)^{1/2}, 0)$, $i = 1, \ldots, m$; $q_j = (0, (j+3-\pi)^{1/2})$, $j = 1, \ldots, n$. Then $|p_i - q_j|$ is algebraic, but $|p_i - p_j|$, $|q_i - q_j|$ are not algebraic unless $i = j$. Hence the algebraic distance graph on $\{p_1, \ldots, p_m, q_1, \ldots, q_n\}$ is isomorphic to $K(m, n)$. □

Now there arises a question: as a function on finite graphs G, is $\dim_A G$ unbounded?

Lemma 4. *Let* W, X, Y, Z *be four sets in Euclidean space such that*

(1) $Z = X \cup Y$, $W = X \cap Y$,

(2) $W(A)$, $X(A)$, $Y(A)$ *are complete, but* $Z(A)$ *is not complete, and*

(3) $\dim(W) \geq i-2$.

Then $\dim(Z) \geq i$, *and* $\max\{\dim(X), \dim(Y)\} \geq i-1$.

Proof. By (3), there exist $i-1$ affinely independent points in W. If Z is contained in $(i-1)$-space E^{i-1}, then by Lemma 2(2), $Z(A)$ must be rigid and hence complete, a contradiction. Therefore, $\dim(Z) \geq i$. Since $\dim(W) \geq i-2$, there must be a point z in Z such that $\dim(W \cup \{z\}) \geq i-1$. Hence $\max\{\dim(X), \dim(Y)\} \geq i-1$. □

Theorem 9. *Denote by* G_n *the complete* n*-partite graph* $K(2, \ldots, 2)$. *Then* $\dim_A G_n \geq n$.

Thus, $\dim_A G$ *is unbounded as a function on finite graphs* G.

Proof. Let U_n be a point set such that $U_n(A)$ is isomorphic to G_n. Then, corresponding to the nested sequence of induced subgraphs

$$G_2 \subset G_3 \subset \cdots \subset G_{n-1} \subset G_n,$$

there is a nested sequence of subsets

$$U_2 \subset U_3 \subset \cdots \subset U_{n-1} \subset U_n$$

such that $U_i(A)$ is isomorphic to G_i. Then $|U_{i+1} - U_i| = 2$. We claim that for each $i = 2, \ldots, n$, U_i contains subsets W_i, X_i, Y_i, Z_i satisfying the conditions of Lemma 4. This is proved by induction on i. For $i = 2$, take a path $p_1 p_2 p_3$ in $U_2(A)$, and let

$$W_2 = \{p_2\}, \quad X_2 = \{p_1, p_2\}, \quad Y_2 = \{p_2, p_3\}, \quad Z_2 = \{p_1, p_2, p_3\}.$$

Then these four sets satisfy the conditions. Suppose that U_i contains subsets W_i, X_i, Y_i, Z_i satisfying (1)–(3). We may suppose $\dim(X_i) \geq i - 1$. Let $\{x, y\} = U_{i+1} - U_i$. Then the algebraic distance graphs induced by $X_i \cup \{x\}$ and by $X_i \cup \{y\}$ are both complete. Hence letting

$$W_{i+1} = X_i, \quad X_{i+1} = X_i \cup \{x\}, \quad Y_{i+1} = X_i \cup \{y\}, \quad Z_{i+1} = X_i \cup \{x, y\},$$

we have the $(i+1)$-case.

Therefore $\dim(U_n) \geq \dim(Z_n) \geq n$. □

Example 3.

$$\dim_A G_n = \dim_A K\underbrace{(2, \ldots, 2)}_{n} = n \quad \text{for } n \leq 4.$$

Proof. We show only the case $n = 4$. In the plane E^2, take four points p_i, $i = 1, 2, 3, 4$, on the circle of radius π centered at the origin such that $|p_1 - p_2|$, $|p_2 - p_3|$, $|p_3 - p_4|$, $|p_4 - p_1|$ are all algebraic, see Example 2. Then by Theorem 3, $|p_2 - p_4|$ and $|p_1 - p_3|$ are transcendental numbers. Take four points q_i, $i = 1, 2, 3, 4$, with similar property on the circle of, this time, radius $(m - \pi^2)^{1/2}$, centered at the origin, where m is a sufficiently large integer. Since the radius $(m - \pi^2)^{1/2}$ is transcendental, it also follows from Theorem 3 that $|q_2 - q_4|$ and $|q_1 - q_3|$ are transcendental. Now, in $E^4 = E^2 \times E^2$, let X be the set of eight points

$$\begin{array}{cccc} (p_1, 0), & (p_2, 0), & (p_3, 0), & (p_4, 0), \\ (0, q_1), & (0, q_2), & (0, q_3), & (0, q_4). \end{array}$$

Then it can be easily verified that $X(A)$ is isomorphic to G_4.

7. Proof of Proposition 1

As usual, the symbols Q, R, C stand for the set of rational numbers, the set of real numbers, the set of complex numbers, respectively. The algebraic closure of the field Q is denoted by $\overline{Q}$. Thus, $A = \overline{Q} \cap R$.

Let $\mathfrak{P}$ be the prime ideal of $A[x_1, \dots, x_N]$ defined as the kernel of the A-algebra homomorphism

$$A[x_1, \dots, x_N] \to R,$$

$$x_i \to s_i$$

where s_i is the ith coordinate of the isolated point $p = (s_1, \dots, s_N)$ of the algebraic set V. Let us denote by $F(\mathfrak{P})$ the quotient field $A[x_1, \dots, x_N]/\mathfrak{P}$. Since $A \subset F(\mathfrak{P}) \subset R$, the field A is algebraically closed in $F(\mathfrak{P})$, that is, $F(\mathfrak{P})$ is a regular extension of A. Hence the ideal $\mathfrak{P}$ is absolutely prime; that is, for every extension K of A, the ideal $\mathfrak{P} \cdot K[x_1, \dots, x_N]$ is prime (see [16, Chapter VII, Theorem 39]).

For each field $K = \overline{Q}$, R, C, let W_K denote the algebraic set in K^N defined by the ideal $\mathfrak{P} \cdot K[x_1, \dots, x_N]$. Then $p \in W_R \subset W_C$, and since W_C is defined over A and $\overline{Q}$ is the algebraic closure of A, we have

$$\emptyset \neq W_{\overline{Q}} \subset W_C. \tag{1}$$

Since $\mathfrak{P}$ is absolutely prime,

$$W_C \text{ is an irreducible algebraic set in } C^N. \tag{2}$$

Suppose, for a moment, that the algebraic dimension, $\dim W_C$, of W_C is equal to zero. Then, since W_C is irreducible, we have $W_C = \{p\}$, and hence by (1), $W_{\overline{Q}} = \{p\}$, i.e., $p = (s_1, \dots, s_N) \in \overline{Q}^N$.

Now we show that $\dim W_C = 0$. Suppose, on the contrary, $\dim W_C > 0$. In this case, any isolated point of W_R is a singular point of W_C. This fact can be proved by using the implicit function theorem; see [15, Chapter II, 2.3] for details. Then, since $p \in W_R \subset V$, p is an isolated point of W_R, and hence p is a singular point of W_C. Let

$$g_1, \dots, g_M \in A[x_1, \dots, x_N]$$

be a system of generators of $\mathfrak{P}$ and let r be the rank of the $M \times N$ matrix $(\partial g_i/\partial x_j)$ evaluated on W_C. Then, since p is a singular point of W_C, the values of all $r \times r$ minors of the matrix $(\partial g_i/\partial x_j)$ at p are zero. Now, by the definitions of $\mathfrak{P}$ and W_C, we have that

$$g \in A[x_1, \dots, x_N],\ g(p) = 0 \text{ implies that } g = 0 \text{ on } W_C$$

(i.e., p is an "A-generic point" in the sense of Mumford [12]). Therefore, all $r \times r$ minors vanish on W_C, a contradiction. □

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Department of Mathematics, Ryukyu University, Nishihara, Okinawa, Japan