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REMARKS AND CORRECTIONS FOR "GROUPS ACTING ON AFFINE ALGEBRAS"

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We repair several foolish mistakes concerning tensor products in the paper cited above [2]. I am indebted to (and flattered by) the careful attention paid to these results by Nikolaus Vonessen, who uncovered the errors and suggested corrections. Secondly, we point out the similarity between a theorem about dynamics of actions on a torus due to Mañé and a proposition of G. Bergman. Finally, we connect the Faithful Units Conjecture to more traditional versions of the Nullstellensatz. My thanks to Robert Snider for pointing me in the right direction, here.

We begin with a more transparent proof of Lemma 14 of [2]. Suppose that H is a commutative F-algebra and $k \supseteq L \supseteq F$ is a tower of fields. If I is an ideal of $k \otimes_F H$ such that $I = k(I \cap (L \otimes_F H))$ then

$$(k \otimes_F H)/I = k \otimes_L (L \otimes H/I \cap (L \otimes H)).$$

Lemma (see Lemma 14 of [2]). *Assume R is an affine k-algebra and B is an ideal of R. Then there is a subfield L of k which is finitely generated over its prime field and an affine L-algebra S in R such that $R = k \otimes_L S$ and $R/B = k \otimes_L (S/B \cap S)$.*

Proof. Think of R as the homomorphic image of a polynomial algebra,

$$R = k[X_1, \dots, X_n]/I.$$

According to the Hilbert Basis Theorem, both I and the inverse image C of B in $k[X_1, \dots, X_n]$ are finitely generated. Let L be the field generated over the prime field by the coefficients in k of both finite lists of polynomials. Then $I = k(I \cap L[X_1, \dots, X_n])$ and $C = k(C \cap L[X_1, \dots, X_n])$. Now apply the observation above twice, with the prime field playing the role of F. The algebra S will be the image of $L[X_1, \dots, X_n]$ in R. □

In Lemma 16 of [2] one must assume that S is G-stable. Part of the argument in Lemma 17 is just plain wrong.

Lemma (see paragraph two, Lemma 17 of [2]). *Assume that R is an affine k-algebra and G is a countable group of k-automorphisms of R. If B is an ideal of R then there is a subfield $K \subseteq k$ and a countable K-algebra T such that*

(i) $R = k \otimes_K T$,

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(ii) $R/B = k \otimes_K (T/T \cap B)$, *and*

(iii) T *is stabilized by* G.

Proof. First choose L and S using the previous lemma. S is generated as an L-algebra by some finite set $x_1, \dots, x_m$. For each $g \in G$ and each i write ${}^g x_i$ as a k-linear combination of monomials in $x_1, \dots, x_m$. Adjoin this countable collection of coefficients to L to obtain a field extension K whose cardinality is countable. Certainly $K \otimes_L S$ (regarded inside $k \otimes_L S = R$) is G-stable. The result follows from the associativity of the tensor product when we set $T = K \otimes_L S$. □

We turn, now, to the extraordinary connection between theorems of Mañé and Bergman.

Theorem [3]. *Let* T *be the* n*-dimensional complex torus. Suppose* f *is a continuous automorphism of* T *with no eigenvalue of modulus one such that no proper toral subgroup is invariant under any power of* f. *Then every nonconstant rectifiable curve contains a point with a dense orbit.*

Theorem [1]. *Let* A *be the free abelian group of rank* n *and assume* G *is a subgroup of* $GL(A)$. *If* G *and all of its subgroups of finite index act rationally irreducibly on* A *then all of the invariant nonzero ideals of the group algebra* $k[A]$ *have finite codimension.*

First, the reader should notice that the hypothesis on f amounts to the rational irreducibility of the action of $\langle f \rangle$ and all of it subgroups of finite index on the character group of T, which can be identified with A. One Zariski analogue of the conclusion from Mañé's Theorem is that given a nonmaximal prime ideal P of $k[A]$, there exists a maximal ideal $\mathscr{M}$ containing P such that $\bigcap_G^g \mathscr{M} = 0$. We discuss whether this follows from the hypothesis of Bergman's Theorem.

This is no hope when k is algebraic over a finite field; in this case every maximal ideal has a finite orbit. However, there is a theorem here whenever k satisfies the Faithful Units Conjecture [2]: for every affine k-domain R and every finitely generated group A of units of R, there exists a maximal ideal $\mathscr{M}$ of R such that the image of A in $R/\mathscr{M}$ is faithful.

Assume k is such a field and that P is a nonmaximal prime ideal of $k[A]$. Let $E = \{a \in A | a - 1 \in P\}$. Then A/E imbeds multiplicatively in $k[A]/P$. By the Faithful Units Conjecture there is a maximal ideal $\mathscr{M}$ containing P such that A/E maps faithfully into $\mathscr{M}$. If $\mathscr{M}$ has a finite orbit then its stabilizer H has finite index in G and acts as a group of k-automorphisms on the finite field extension $k[A]/\mathscr{M}$ over k. By elementary Galois theory some subgroup of finite index in H fixes each element of A/E. According to the hypothesis of Bergman's Theorem and the fact that our proposed theorem has no meaning unless $\operatorname{rank} A > 1$, we must have A/E finite. In this case P would have finite codimension in $k[A]$, and so be maximal. Thus $\mathscr{M}$ has an infinite orbit. This time the conclusion of Bergman's Theorem tells us that $\bigcap_G^g \mathscr{M} = 0$.

In [2] we mentioned that the Faithful Units Conjecture holds for uncountable fields k. One can do much better by invoking Vamos' variation of Hilbert's Nullstellensatz.

Theorem [4]. *If* $\operatorname{tr.deg.}(k|F) \geq m$ *and* $M = F[X_1, \dots, X_m]\backslash\{0\}$ *then every maximal ideal of* $k[X_1, \dots, X_m]_M$ *has finite* k*-codimension.*

One can always enlarge the group A in the Faithful Units Conjecture by inverting an additional finite number of members of R.

Corollary. *Assume that* A *spans* R *as a vector space over* k *and* F *is the prime subfield of* k. *If* $\operatorname{tr.deg.}(k|F) \geq \operatorname{rank} A$ *then the conjecture holds.*

Proof. Let S be the ring generated by F and A. We shall actually prove that there exists a maximal ideal $\mathscr{M}$ of R such that $\mathscr{M} \cap S = 0$. This will show that all of S survives in $R/\mathscr{M}$.

S^* denotes the set of nonzero elements of S. It clearly suffices to prove that R_{S^*} has a maximal ideal with finite k-codimension. (Its intersection with R is prime, has finite codimension, and misses S^*.)

Choose $a_1, \dots, a_m \in A$ algebraically independent over F so that the field of fractions of S, namely S_{S^*}, is algebraic over $F(a_1, \dots, a_m)$. Set $M = F[a_1, \dots, a_m]\backslash\{0\}$. Since $F(a_1, \dots, a_m) \subseteq k[a_1, \dots, a_m]_M$ we see that both A and S_{S^*} are integral over $k[a_1, \dots, a_m]_M$. The assumption that A spans R over k now implies that R_{S^*} is integral over $k[a_1, \dots, a_m]_M$. By standard results about integral extensions we conclude that it suffices to show that $k[a_1, \dots, a_m]_M$ has a maximal ideal of finite k-codimension.

It is not difficult to see that $k[a_1, \dots, a_m]_M$ is a k-algebra homomorphic image of $k \otimes_F F(a_1, \dots, a_m)$, i.e., that $k[a_1, \dots, a_m]_M$ is the image of $k[X_1, \dots, X_m]_M$. Also, $\operatorname{tr.deg.}(k|F) \geq \operatorname{rank} A \geq m$. Apply Vamos' Theorem. $\square$

References

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