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A TOPOLOGICAL CHARACTERIZATION OF R-TREES

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ABSTRACT. **R**-trees arise naturally in the study of groups of isometries of hyperbolic space. An **R**-tree is a uniquely arcwise connected metric space in which each arc is isometric to a subarc of the reals **R**. Actions on **R**-trees can be viewed as ideal points in the compactification of groups of isometries. As such they have applications to the study of hyperbolic manifolds. Our concern in this paper, however, is with the topological characterization of **R**-trees. Our main theorem is the following: *Let (X, p) be a metric space. Then X is uniquely arcwise connected and locally arcwise connected if, and only if, X admits a compatible metric d such that (X, d) is an* **R**-*tree.* Essentially, we show how to put a convex metric on a uniquely arcwise connected, locally arcwise connected, metrizable space.

1. INTRODUCTION

1.1. **History.** **R**-trees arise naturally in the study of groups of isometries of hyperbolic space. Actions on **R**-trees can be viewed as ideal points in the compactification of groups of isometries. As such, they have applications to the study of hyperbolic manifolds [T, CM, M, MS1, MS2, Be]. However, our concern in this paper is not with classification of group actions on **R**-trees but with the topological characterization of the **R**-trees themselves.

The first author became acquainted with **R**-trees in the course of a sequence of CBMS lectures [M] on them given by John W. Morgan in the summer of 1986 at UCLA. At the time, Morgan derived several topological properties of **R**-trees (Theorem 2.3, below) and conjectured that those properties (including unique arcwise connectedness and local arcwise connectedness; see §2.1 below for definitions) constituted a characterization. In conversation with Morgan, the first author suggested that the problem essentially was: *Could a convex metric be*

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put on a uniquely arcwise connected, locally arcwise connected metrizable space? and indicated Bing's convex metrization theorem for locally connected continua as a model [Bi].

Unique arcwise connectedness is a strong hypothesis, but care is required since the spaces involved are not assumed to be even locally compact. The reader may judge to what extent, in the end, the solution presented here to the problem posed by Morgan resembles Bing's work.

1.2. **R-segment** [M]. Let (X, d) be a metric space and $[x, y]$ an arc in X from x to y (degenerate, if $x = y$). We say that $[x, y]$ is an **R**-segment iff $d(x, \_)|[x, y]$ is an isometry onto the subarc $[0, d(x, y)] \subset \mathbf{R}^+$.

1.3. **R-tree** [M]. An **R**-tree is a metric space (X, d) such that the following axioms are satisfied:

A1: For all $x, y \in X$, x and y are the endpoints of an **R**-segment.
A2: If I and J are **R**-segments in X with a common endpoint, then $I \cap J$ is an **R**-segment.
A3: If I and J are **R**-segments in X with $I \cap J = \{x\}$, then $I \cup J$ is an **R**-segment.

1.4. **Preview.** Our main theorem is the **R**-tree Characterization Theorem (5.1): *Let (X, p) be a metric space. Then X is uniquely arcwise connected and locally arcwise connected iff X admits a compatible metric d such that (X, d) is an* **R**-*tree.* See §2 for further definitions.

The proof of the **R**-tree Characterization Theorem is in four parts. In §2 we provide some basic definitions, state the *if* direction of the theorem (Theorem 2.3, due to Morgan), and show that the proof in the *only if* direction reduces to finding a compatible *convex* metric d for X (Theorem 2.5).

The second part is presented in §3, where we show that a metric space (X, ρ) has a particularly nice decreasing sequence of covers (Theorem 3.3). No connectedness hypotheses are required in this section.

In §4 we present the third part, and heart, of the proof. We show that the sequence of covers produced in §3 allows us, on a uniquely arcwise connected space, to define a *radial distance function* $f: X \to [0, 1]$ (based at a fixed point $p \in X$). For $x \in X$, each cover induces a partition on the unique arc from p to x. We assign weights to the partition elements, sum them to get a radial distance relative to the nth cover, and take the limit over n to obtain f. This is essentially a counting process, but one has to be careful (meaning very local, rather than global) in assigning weights because of the absence of even local compactness.

The radial distance function, together with a standard *meet* function $\wedge$: $X \times X \to X$ (which can be defined on any uniquely arcwise connected space relative to a fixed base point), allows us to define a convex metric d on X finer than the given metric ρ (Theorem 4.9).

In §5, we use the local arcwise connectedness hypothesis for the first time to show that the finer convex metric d constructed in §4 is compatible with the given metric ρ on X. This fourth step concludes the proof of the **R**-tree Characterization Theorem. We finish with several questions.

The first author's conversations with Mladen Bestvina at the UCLA meeting were helpful. We have been informed that E. D. Tymchatyn and L. K. Mohler have independent proofs of the **R**-tree Characterization Theorem for the case of a separable metric space. Our conversations with them inspired a second look at an earlier version of this paper, which led us to a more general result with a simpler proof.

2. Some properties of **R**-trees

2.1. **Arcwise connectivity.** A space X is *arcwise connected* (abbreviated AC) iff for each pair of points x, $y \in X$, there is an arc $A \subset X$ one endpoint of which is x and the other endpoint of which is y. We say that A is an arc *from x to y*. A space X is *uniquely arcwise connected* (abbreviated UAC) iff X is AC and for each pair of points x, $y \in X$, there is a unique arc A from x to y. We often denote A by $[x, y]$. A space X is *locally arcwise connected* (abbreviated LAC) iff for each point $p \in X$ and each open set $U \subset X$ such that $p \in U$, there is an AC open set $V \subset U$ such that $p \in V$.

2.2. **Dimension.** In this paper we use large inductive dimension. For a metric space X, the large inductive dimension of X and the covering dimension of X are the same, though the small inductive dimension of X may be different. For separable metric spaces, all three are the same. Thus, by $\dim(X) = n$, or by "X is n-dimensional," we mean that the large inductive dimension of X is n. See [E2] for appropriate definitions.

2.3. **Theorem.** *Let (X, d) be an* **R**-*tree. Then X is UAC, LAC, contractible, and one-dimensional.*

Proof. The properties of **R**-trees stated in the above theorem are familiar to most researchers in the areas mentioned in the introduction. A proof of Theorem 2.3 appears in [M]. However, the proof in the preprint version of [M] shows only that an **R**-tree has small inductive dimension one. We refer the reader to [M] for the proof that an **R**-tree is UAC, LAC, and contractible.

The proof that X is one-dimensional depends on the following proposition. The proposition and its proof were communicated to us by F. Ancel. The idea of the proof is due to L. Rubin.

2.3.1. **Proposition.** *Let X be an* **R**-*tree, and let Y be a connected ANR. Then every map from a closed subset of X to Y extends to a map from X to Y.*

Proof. Let A be a closed subset of X, and let $f: A \to Y$ be a map. Since Y is an ANR, f extends to a map $g: U \to Y$, where U is a neighborhood of A in X. Fix $y_0 \in Y$. Let V be a component of U. Since V is an **R**-tree, V is contractible. Hence, there is a homotopy $G^V: V \times [0, 1] \to Y$ such that

$G_0^V = g|V$ and $G_1^V(V) = y_0$. Define the function $G\colon U \times [0, 1] \to Y$ by setting $G|V \times [0, 1] = G^V$ for each component V of U. Since an **R**-tree is locally connected, each component of U is an open subset of X. This ensures the continuity of G. Since $G_0 = g$ and $G_1(U) = y_0$, then $G|A \times [0, 1]$ is a homotopy from f to a constant map. Now, since Y is an ANR, the Borsuk homotopy extension principle [Bo, p. 94] implies that f extends to a map from X to Y.

Returning to the proof of Theorem 2.3, it is an immediate corollary to the above proposition that every map from a closed subset of X to S^1 extends to a map from X to S^1. Since the large inductive dimension of a metric space X is $\leq n$ iff every map from a closed subset of X to S^n extends to a map from X to S^n [E2, Theorem 1.9.3, p. 90], it follows that X is one-dimensional. □

2.4. **Convex metric.** Let (X, d) be a metric space. We say that d is a *convex* metric iff for all $x, z \in X$, there exists an arc $A \subset X$ from x to z such that for all $y \in A$, $d(x, z) = d(x, y) + d(y, z)$.

2.5. **Theorem.** *Let (X, d) be a metric space. Then the following are equivalent*:

(1) *X is an* **R**-*tree.*
(2) *X is UAC, and d is convex.*
(3) *X is UAC, and each arc in X is an* **R**-*segment.*

Proof. The equivalence of (1) and (3) is well known. (It is used as the definition of an **R**-tree in [CM].) To see that (3) implies (2), observe that the metric d on an **R**-segment is convex by the isometry into $\mathbf{R}^+$. To show that (2) implies (3), let $[x, z]$ denote the unique arc from x to z in X. The function $d(x, \_)\colon [x, z] \to [0, d(x, z)]$ is an isometry, since for all $s < t \in [x, z]$, by convexity and by the metric $d_{\mathbf{R}}$ on $\mathbf{R}$, it follows that

$$d(s, t) = d(x, t) - d(x, s) = d_{\mathbf{R}}(d(x, s), d(x, t)).$$

Therefore, $[x, z]$ is an **R**-segment. □

3. Covers

Our goal in this section is to develop a particularly useful decreasing sequence of open covers for a metric space (X, ρ). The UAC and LAC hypotheses are not required in this section.

3.1. **Definitions.** Let $\mathscr{U}$ and $\mathscr{V}$ be covers of X.

3.1.1. **Star and mesh.** Let $A \subset X$. Define the *star* of A in $\mathscr{U}$ by

$$\mathrm{St}(A, \mathscr{U}) = \{U \in \mathscr{U} \mid A \cap U \neq \varnothing\}.$$

Define $\mathrm{St}^1(A, \mathscr{U}) = \mathrm{St}(A, \mathscr{U})$, and assume that $\mathrm{St}^{n-1}(A, \mathscr{U})$ is defined. Inductively, we define

$$\mathrm{St}^n(A, \mathscr{U}) = \mathrm{St}\left(\bigcup \mathrm{St}^{n-1}(A, \mathscr{U}), \mathscr{U}\right).$$

We define a cover $\mathrm{St}(\mathscr{U})$ of X by

$$\mathrm{St}(\mathscr{U}) = \{\bigcup \mathrm{St}(U, \mathscr{U}) \mid U \in \mathscr{U}\} .$$

In general, we define a cover

$$\mathrm{St}^n(\mathscr{U}) = \{\bigcup \mathrm{St}^n(U, \mathscr{U}) \mid U \in \mathscr{U}\} .$$

We define the *mesh* of $\mathscr{U}$ by

$$\mathrm{mesh}(\mathscr{U}) = \sup\{\mathrm{diam}(U) \mid U \in \mathscr{U}\} .$$

3.1.2. **Refinement.** We say $\mathscr{V}$ *refines* $\mathscr{U}$ iff for each $V \in \mathscr{V}$ there is a $U \in \mathscr{U}$ such that $V \subset U$. We say $\mathscr{V}$ *star-refines* $\mathscr{U}$ iff $\mathrm{St}(\mathscr{V})$ refines U.

3.1.3. **Boundary.** We define the *boundary* of a subset $A \subset X$ by

$$\mathrm{Bd}(A) = \mathrm{Cl}(A) \cap \mathrm{Cl}(X - A) ,$$

and we define

$$\mathrm{Bd}(\mathscr{U}) = \bigcup \{\mathrm{Bd}(U) \mid U \in \mathscr{U}\} .$$

3.1.4. **Spanning.** Let A and B be disjoint closed subsets of X. We say that a subset $\{U_1, U_2, \ldots, U_n\} \subset \mathscr{U}$ *spans* from A to B iff $U_1 \cap A \neq \varnothing$, $U_i \cap U_j \neq \varnothing$ for $|i - j| \leq 1$, and $U_n \cap B \neq \varnothing$. We say that $\mathscr{U}$ is *n-spanning* between A and B iff it takes at least n links of $\mathscr{U}$ to span from A to B.

3.1.5. **Neighborhood-finite.** We say that $\mathscr{U}$ is *neighborhood-finite* iff for each $x \in X$, there is an open set V containing x such that V meets only finitely many members of $\mathscr{U}$.

3.1.6 **Decreasing sequences of covers.** A sequence $\{\mathscr{U}_n\}_{n=0}^{\infty}$ of open covers of X is said to be a *decreasing sequence* iff $\mathrm{mesh}(\mathscr{U}_n) \to 0$ as $n \to \infty$, and for all $n > 0$, $\mathscr{U}_n$ refines $\mathscr{U}_{n-1}$.

3.2. *Remarks.* We indicate several immediate consequences of the above definitions.

3.2.1. If $\{\mathscr{U}_n\}_{n=0}^{\infty}$ is a decreasing sequence of neighborhood-finite open covers of X, then for all $x \in X$, for all $n \in \mathbf{N}$, there is a neighborhood V_x of x such that for all $i \leq n$, $\mathrm{St}(V_x, \mathscr{U}_i)$ is finite. Put $\mathscr{V}_n = \mathscr{U}_1 \cup \mathscr{U}_2 \cup \cdots \cup \mathscr{U}_n$. Then $\mathscr{V}_n$ is also neighborhood-finite.

3.2.2. For any neighborhood-finite open cover $\mathscr{U}$, $\mathrm{Bd}(\mathscr{U})$ is a closed set. In fact, no point of $\mathrm{Bd}(\mathscr{U})$ can be a limit point of boundary points from infinitely many different links of $\mathscr{U}$. For if x were a limit point of boundary points from infinitely many different links of $\mathscr{U}$, then every neighborhood of x would meet infinitely many links of $\mathscr{U}$, contradicting the neighborhood-finiteness of $\mathscr{U}$.

3.2.3. If $\mathscr{V}$ is a refinement of $\mathscr{U}$ and $\mathscr{U}$ is n-spanning between A and B, then so is $\mathscr{V}$. If $\mathscr{W}$ refines $\mathscr{V}$ and $\mathscr{V}$ star-refines $\mathscr{U}$, then $\mathscr{W}$ star-refines $\mathscr{U}$. If $\mathscr{W}$ star-refines $\mathscr{V}$ and $\mathrm{St}^n(\mathscr{V})$ refines $\mathscr{U}$, then $\mathrm{St}^{n+1}(\mathscr{W})$ refines $\mathscr{U}$.

3.3. **Theorem.** *Let X be a metric space, and let $p \in X$. Then there is a decreasing sequence $\{\mathscr{U}_n\}_{n=0}^{\infty}$ of neighborhood-finite, open covers of X such that $\mathscr{U}_0 = \{X\}$, and for $n > 0$*

(1) $\mathrm{St}^4(\mathscr{U}_n)$ *refines* $\mathscr{U}_{n-1}$;
(2) $\mathrm{mesh}(\mathrm{St}(\mathscr{U}_n)) < 2^{-n}$;
(3) *there is a unique* $U_n \in \mathscr{U}_n$ *such that* $p \in \mathrm{Cl}(U_n)$.

Proof. We prove the theorem by induction on n. Let $\mathscr{U}_0 = \{X\}$. Suppose that $\mathscr{U}_k$ has been constructed as required for all $k < n$. Since X is metric, it is paracompact [E1, Theorem 5.1.3, p. 373]. Each open cover of a paracompact space has a star-refinement [E1, Theorem 5.1.12, p. 377]. It follows from six applications of this that there is an open cover $\mathscr{V}_1$ of X such that $\mathrm{St}^6(\mathscr{V}_1)$ refines $\mathscr{U}_{n-1}$. Let $\mathscr{V}_2$ be any refinement of $\mathscr{V}_1$ of mesh less than 2^{-n-2}. By the Stone theorem [E1,Theorem 4.4.1, p. 349], there is a neighborhood-finite (called *locally finite* in [E1]) open refinement $\mathscr{V}_3$ of $\mathscr{V}_2$. Let

$$U_n = \bigcup \{ V \in \mathscr{V}_3 | p \in \mathrm{Cl}(V) \} ,$$

and let

$$\mathscr{U}_n = (\mathscr{V}_3 - \{ V \in \mathscr{V}_3 | p \in \mathrm{Cl}(V) \}) \cup \{U_n\} .$$

It is easy to check that $\mathrm{mesh}(\mathrm{St}(\mathscr{U}_n)) < 2^{-n}$, $\mathrm{St}^4(\mathscr{U}_n)$ refines $\mathscr{U}_{n-1}$, and $\mathscr{U}_n$ is neighborhood-finite. □

4. Convex metrics on UAC spaces

Assume in this section that (X, ρ) is a metric space, is UAC, and has a designated base point $p \in X$, and that $\{\mathscr{U}_n\}_{n=0}^{\infty}$ is a decreasing sequence of open covers of X guaranteed by Theorem 3.3. Our goal in this section is to use the sequence of covers to define a new metric d on X which we show in Theorem 4.9 is convex and finer than the given metric ρ. The LAC hypothesis on X is not needed in this section but is used in the proof of Theorem 5.1 to show that d is compatible with ρ.

4.1. **Meet function.** Let X be a UAC metric space with base point $p \in X$. We define a *meet* function $\wedge : X \times X \to X$ by

$$x \wedge y = c \quad \text{iff } [p, x] \cap [p, y] = [p, c] .$$

Note that the definition of $\wedge$ is relative to the base point p. Also note that $[p, c]$ may be degenerate. That $x \wedge y$ is well defined follows from the fact that X is UAC.

4.2. **Partitioning arcs relative to a sequence of covers.**

4.2.1. **Definition.** For each $x \in X - \{p\}$, for each $n \in \mathbf{N} = \{0, 1, 2, \ldots\}$, define the partition $\mathscr{P}_x^n = \{p = p_{0,x}^n < p_{1,x}^n < \cdots < p_{m(n,x),x}^n \leq x\}$ by induction

as

$$p^n_{0,x} = p\,, \quad \text{and}$$
$$p^n_{i+1,x} = \min\left\{ y \in [p\,,x] \,|\, p^n_{i,x} < y \text{ and } y \notin \bigcup \operatorname{Star}(p^n_{i,x}\,, \mathscr{U}_n) \right\},$$

where $<$ denotes the natural order on the arc $[p\,,x]$.

When it is clear that the point x is fixed, we will omit the subscript x in the above notation. Similarly, n will be omitted when it is clear from the context which n is intended.

4.2.2. **Proposition.** *$\mathscr{P}^n_x$ is finite for all $x \in X - \{p\}$ and for all $n \in \mathbf{N}$.*

Proof. Suppose not. Then for some x there are an integer n and an infinite sequence $p^n_0 < p^n_1 < \cdots$ in $\mathscr{P}^n_x \subset [p\,,x]$. Let $y = \lim p^n_i$. There is a $U \in \mathscr{U}_n$ such that $y \in U$. Since $\mathscr{U}_n$ is neighborhood-finite, let O be a neighborhood of y such that $O \subset U$ and O intersects only finitely many members $U_1\,, \ldots\,, U_k \in \mathscr{U}_n$. Choose j_0 such that $[p^n_{j_0}\,, y] \subset 0$. Then $p^n_{j_0+1} > y$, which is a contradiction. □

4.2.3. **3-spanning for partitions.** Let $\mathscr{A} = \{c = a_0 < a_1 < \cdots < a_m \le d\}$ and $\mathscr{B} = \{c = b_0 < b_1 < \cdots < b_n \le d\}$ be two ordered sequences of points on the arc $[c\,,d]$. We say $\mathscr{B}$ is *3-spanning* in $\mathscr{A}$ iff for all $j \in \{0\,, \ldots\,, m-1\}$ there is an $i \in \{1\,, \ldots\,, n-2\}$ such that

$$a_j < b_i < b_{i+1} < b_{i+2} < a_{j+1}\,.$$

4.3. **Theorem.** *For all $x \in X - \{p\}$, for all $n \in \mathbf{N}$, $\mathscr{P}^n_x$ satisfies*

(1) *$\mathscr{P}^n_x$ is 2^{-n}-dense in $[p\,,x]$ (except $n = 0$),*
(2) *$\mathscr{P}^{n+1}_x$ is 3-spanning in $\mathscr{P}^n_x$, and*
(3) *for all $y \in X - \{p\}$, $\mathscr{P}^n_x$ and $\mathscr{P}^n_y$ coincide up to $x \wedge y$.*

Proof. It follows from Definition 4.2.1 that up to $x \wedge y$, $\mathscr{P}^n_x$ and $\mathscr{P}^n_y$ are the same on $[p\,,x]$ and $[p\,,y]$, respectively. For $n > 0$, the cover $\mathscr{U}_n$ used to define $\mathscr{P}^n_x$ has $\operatorname{mesh}(\operatorname{Star}(\mathscr{U}_n)) < 2^{-n}$; since

$$[p^n_{i,x}\,, p^n_{i+1,x}] \subset \operatorname{Cl}\left(\bigcup \operatorname{Star}(p^n_{i,x}\,, \mathscr{U}_n)\right),$$

it follows that $\operatorname{diam}([p^n_{i,x} p^n_{i+1,x}]) < 2^{-n}$. Thus, $\mathscr{P}^n_x$ is 2^{-n}-dense in $[p\,,x]$.

Fix $x \in X - \{p\}$. To establish (2), suppose by way of contradiction that there are $p^n_j < p^n_{j+1}$ such that $|\mathscr{P}^{n+1} \cap (p^n_j\,, p^n_{j+1})| < 3$. Without loss of generality, suppose $|\mathscr{P}^{n+1} \cap (p^n_j\,, p^n_{j+1})| = 2$.

There is a $p^{n+1}_k \in \mathscr{P}^{n+1}$ maximal with respect to $p^{n+1}_k \le p^n_j$. On the arc $[p^{n+1}_k\,, p^n_{j+1}]$ we then have

$$p^{n+1}_k \le p^n_j < p^{n+1}_{k+1} < p^{n+1}_{k+2} < p^n_{j+1}\,.$$

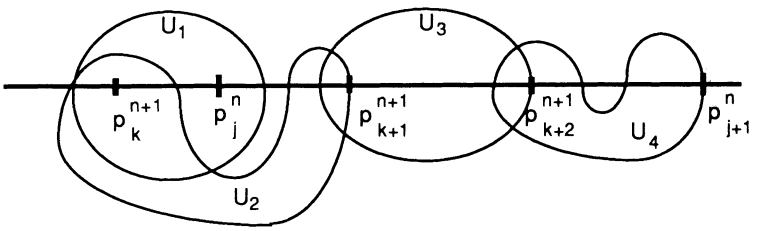

FIGURE 4.1

Therefore, there are links $U_1, U_2, U_3, U_4 \in \mathscr{U}_{n+1}$ such that $p_j^n \in U_1$, $p_k^{n+1} \in U_1 \cap U_2$, $p_{k+1}^{n+1} \in \mathrm{Bd}(U_2) \cap U_3$, $p_{k+2}^{n+1} \in \mathrm{Bd}(U_3) \cap U_4$, and $p_{j+1}^n \in \mathrm{Cl}(U_4)$. See Figure 4.1.

Since $\mathscr{U}_n$ is neighborhood-finite, $\mathrm{St}(p_j^n, \mathscr{U}_n)$ is finite. By Theorem 3.3, there exists a $V \in \mathscr{U}_n$ such that $p_j^n \in U_1 \subset \bigcup \mathrm{St}^4(U_1, \mathscr{U}_{n+1}) \subset V$. By definition $p_{j+1}^n \notin V$. However, $U_1, U_2, U_3, U_4 \in \mathrm{St}^3(U_1, U_{n+1})$ and $p_{j+1}^n \in \mathrm{Cl}(U_4) \subset \mathrm{St}^4(U_1, \mathscr{U}_{n+1}) \subset V$, which is a contradiction. □

4.4. Weight functions.

4.4.1. **Induced chain covers on arcs.** For each $n \in \mathbf{N}$ and $x \in X - \{p\}$, put

$$\begin{aligned}
&U_{i,x}^n = [p_{i,x}^n, p_{i+1,x}^n], && \text{for } i < m(n, x),\\
&U_{m(n,x),x}^n = \varnothing, && \text{if } p_{m(n,x),x}^n = x, \text{ and}\\
&U_{m(n,x),x}^n = [p_{m(n,x),x}^n, x], && \text{otherwise};\\
&l(n, x) = m(n, x) - 1, && \text{if } p_{m(n,x),x}^n = x,\\
&l(n, x) = m(n, x), && \text{otherwise};
\end{aligned}$$

$$\mathscr{C}_x^n = \{U_{i,x}^n \mid i = 0, 1, \dots, l(n, x)\}.$$

Then $U_{l(n,x),x}^n$ always denotes the last link of $\mathscr{C}_x^n$. Let $U, V \in \mathscr{C}_x^n$. By $U < V$ we mean that U precedes V in the chain order (the order on $\mathscr{C}_x^n$ associated with the index i above.) As in Definition 4.2.1, we will omit x and/or n from the above notation when it is clear from context which x and/or n is intended.

Let $\mathscr{U} = \{U_1, U_2, \dots, U_n\}$ be a chain. Let $U \in \mathscr{U}$. By U^+ (respectively, U^-) we denote the immediate successor (respectively, immediate predecessor) of U in $\mathscr{U}$, should such a successor (respectively, predecessor) exist; otherwise, we let U^+ (respectively, U^-) be the empty set.

4.4.2. *Remark.* It follows from Theorem 4.3(3) that if $U_{i,x}^n \in \mathscr{C}_x^n$, $U_{i,y}^n \in \mathscr{C}_y^n$, and $p_{i+1,x}^n < x \wedge y$, then $p_{i+1,x}^n = p_{i+1,y}^n$. So $U_{i,x}^n = U_{i,y}^n$. That is, $\mathscr{C}_x^n$ and $\mathscr{C}_y^n$ have the same links prior to $x \wedge y$.

4.4.3. **Containment operator.** Now $\mathscr{C}_x^n$ need not refine $\mathscr{C}_x^{n-1}$. We define a containment operator $\partial_x^n : \mathscr{C}_x^n \to \mathscr{C}_x^{n-1}$ as follows: for each link $U \in \mathscr{C}_x^n$, if there is a unique link $V \in \mathscr{C}_x^{n-1}$ such that $U \subset V$, then define $\partial_x^n(U) = V$. Otherwise, there is a pair of consecutive links V, $V^+ \in \mathscr{C}_x^{n-1}$ such that $U \subset V \cup V^+$. Define $\partial_x^n(U) = V$. Usually we will just write ∂U for $\partial_x^n(U)$, since context will make clear which n and x are intended.

4.4.4. **Counting links.** For all $n > 0$ and $x \in X - \{p\}$, for each $U \in \mathscr{C}_x^n$ define

$$b_U = \left|\{W \in \mathscr{C}_x^n \mid \partial W = \partial U \text{ and } W \leq U\}\right| ,$$

and for each $V \in \mathscr{C}_x^{n-1}$ define

$$t_V = \left|\{W \in \mathscr{C}_x^n \mid \partial W = V\}\right| .$$

4.4.5. **Definition of weight functions.** For each $x \in X - \{p\}$ and $n \in \mathbf{N}$ define a *weight function* $w_{n,x} : \mathscr{C}_x^n \to [0, 1]$ inductively by $w_{0,x}(U) = 1$ for $U \in \mathscr{C}_x^0$, and for $n > 0$ and $U \in \mathscr{C}_x^n$,

$$w_{n,x}(U) = \begin{cases} 2^{-b_U}\left(w_{n-1,x}(\partial U)\right) & \text{if } b_U > 1 \text{ or } (\partial U)^- = \varnothing, \\ 2^{-1}\left(w_{n-1,x}(\partial U)\right) + 2^{-t_{(\partial U)^-}}\left(w_{n-1,x}\left((\partial U)^-\right)\right) & \\ \qquad\qquad\qquad\qquad \text{if } b_U = 1 \text{ and } (\partial U)^- \neq \varnothing. \end{cases}$$

As in Definitions 4.2.1 and 4.4.1, we will omit x and/or n in the above notation when convenient. Note that the weight of an individual link $U \in \mathscr{C}_x^n$ depends only on the link W of $\mathscr{C}_x^{n-1}$ with which ∂ associates U, its order in the portion of the chain $\mathscr{C}_x^n$ associated with W, and whether or not U is the first link of $\mathscr{C}_x^n$ to be associated with W.

4.5. *Remark*: *Carryover.* We observe an important consequence of our definition of the weight functions. Let $V \in \mathscr{C}_x^n$ and suppose V^- exists. Then

$$\sum_{\partial U = V} w_{n+1}(U) = \left(1 - 2^{-t_V}\right)\left(w_n(V)\right) + 2^{-t_{V^-}}\left(w_n\left(V^-\right)\right) .$$

Note that in the above it is reasonable to think of $2^{-t_{V^-}}(w_n(V^-))$ as the "remaining weight from level n" of V^-, which we will call *carryover*. Also note that by 3-spanning (Theorem 4.3(2)), $t_{V^-} > 2$.

Consequently, the last link $U_{l(n)}$ of $\mathscr{C}_x^n$ will *never* have its full weight at the n level preserved at the $n+1$ level, since it has no successor. (It is not V^- for any $V \in \mathscr{C}_x^n$, so there is no link for its carryover to be carried over to.) And the next to the last link $U_{l(n)-1}$ of $\mathscr{C}_x^n$ *may* not have its full weight at the n level preserved at the $n+1$ level (since it is possible that $\{U \in \mathscr{C}_x^{n+1} | \partial U = U_{l(n)}\} = \varnothing$). But every link of $\mathscr{C}_x^n$ before $U_{l(n)-1}$ *will* have its full weight at the n level preserved at the $n+1$ level, though a portion of that weight will be carried over to its immediate successor.

4.6. **Radial distance function.**

4.6.1. **Definition.** For each $n \in \mathbf{N}$, define $f_n : X \to [0, 1]$ by $f_n(p) = 0$, and for $x \neq p$,

$$f_n(x) = \sum_{U \in \mathscr{C}_x^n} w_{n,x}(U) .$$

Define the *radial distance function* $f : X \to [0, 1]$ by

$$f(x) = \lim_{n \to \infty} f_n(x) .$$

It is clear that $f(p) = 0$. Our lemmas below are generally stated for all $x \in X - \{p\}$. We prove in Corollary 4.6.4 below that f is well defined. The radial distance function $f(x)$ is intended to represent the distance from the base point p to the point $x \in X$. In Corollary 4.7.4 we show that f is strictly increasing on any arc $[p, z]$. It is also continuous on $[p, z]$ in the ρ metric, though we do not prove this fact directly.

4.6.2. **Lemma.** *For all $x \in X - \{p\}$ and $n > 0$, for all $U \in \mathscr{C}_x^n$,*

$$0 < w_{n,x}(U) \leq (\tfrac{3}{4})^{n-1} \cdot \tfrac{1}{2} ,$$

with strict inequality for $n > 1$.

Proof. Fix $x \in X - \{p\}$. It is clear that $w_{n,x}(U) > 0$ for all $n > 0$ and $U \in \mathscr{C}_x^n$. We proceed by induction on n. For $n = 1$, $b_U \geq 1$ and $(\partial U)^- = \varnothing$ for all $U \in \mathscr{C}^1$. Thus, $w_1(U) \leq \frac{1}{2}$. Assume $n > 1$ and the lemma is true for all $k < n$. Let $U \in \mathscr{C}^n$. By hypothesis, the maximum weight of any element in C^{n-1} is $(\frac{3}{4})^{n-2} \cdot \frac{1}{2}$. Hence, if $b_U > 1$ or $(\partial U)^- = \varnothing$, then

$$w_n(U) = 2^{-b_U}\left(w_{n-1}(\partial U)\right) \leq 2^{-b_U} \cdot (\tfrac{3}{4})^{n-2} \cdot \tfrac{1}{2} \leq \tfrac{1}{2} \cdot (\tfrac{3}{4})^{n-2} \cdot \tfrac{1}{2} < (\tfrac{3}{4})^{n-1} \cdot \tfrac{1}{2} .$$

Otherwise, since $t_{(\partial U)^-} > 2$ by 3-spanning.

$$\begin{aligned}
w_n(U) &= 2^{-1}\left(w_{n-1}(\partial U)\right) + 2^{-t_{(\partial U)^-}}\left(w_{n-1}\left((\partial U)^-\right)\right) \\
w_n(U) &< 2^{-1}\left(w_{n-1}(\partial U)\right) + 2^{-2}\left(w_{n-1}\left((\partial U)^-\right)\right) \\
w_n(U) &< (\tfrac{1}{2} + \tfrac{1}{4})(\tfrac{3}{4})^{n-2} \cdot \tfrac{1}{2} \\
w_n(U) &< (\tfrac{3}{4})^{n-1} \cdot \tfrac{1}{2}. \quad \square
\end{aligned}$$

4.6.3. **Lemma.** *For all $x \in X - \{p\}$, for all $n \in \mathbf{N}$,*

$$f_n(x) - (\tfrac{3}{4})^{n-1} < f_{n+1}(x) < f_n(x) .$$

Proof. Fix $x \in X - \{p\}$ and $n \in \mathbf{N}$. By definition

$$f_{n+1}(x) = \sum_{U \in \mathscr{C}^{n+1}} w_{n+1}(U) .$$

Regrouping the sum, we can write

$$f_{n+1}(x) = \sum_{V \in \mathscr{C}^n} \sum_{U \in \mathscr{C}^{n+1} \& \partial U = V} w_{n+1}(U) .$$

Applying Remark 4.5, we then have

$$f_n(x) = \sum_{V \in \mathscr{C}^n} w_n(V) > f_{n+1}(x)$$

and, assuming without loss of generality that $|\mathscr{C}_x^n| \geq 2$,

$$f_{n+1}(x) > \sum_{V \in \mathscr{C}^n \& V < U_{l(n)-1}} w_n(V) = f_n(x) - [w_n(U_{l(n)}) + w_n(U_{l(n)-1})] .$$

Thus,

$$f_n(x) - [w_n(U_{l(n)}) + w_n(U_{l(n)-1})] < f_{n+1}(x) < f_n(x) .$$

Hence, by Lemma 4.6.2,

$$f_n(x) - (\tfrac{3}{4})^{n-1} < f_{n+1}(x) < f_n(x) . \quad \square$$

4.6.4. **Corollary.** *$f(x)$ is well defined, nonnegative, and if $x \neq p$*

$$f_n(x) - 4 \cdot (\tfrac{3}{4})^{n-1} \leq f(x) < f_n(x) .$$

Proof. Let $x \in X - \{p\}$. It follows from Lemma 4.6.2 that $f_n(x) > 0$, and it follows from Lemma 4.6.3 that $f_{n+1}(x) < f_n(x)$. Hence, $f(x) = \lim f_n(x)$ exists and $f(x) \geq 0$.

In fact, by Lemma 4.6.3 and a telescoping sum, for $n < m$

$$f_n(x) - f_m(x) < \sum_{k=n}^{m-1} (\tfrac{3}{4})^{k-1} .$$

Extending the sum to the limit, we have

$$f_n(x) - \sum_{k=n}^{\infty} (\tfrac{3}{4})^{k-1} \leq f(x) < f_n(x) ,$$

$$f_n(x) - 4 \cdot (\tfrac{3}{4})^{n-1} \leq f(x) < f_n(x) . \quad \square$$

4.6.5. **Lemma.** *For all $x \in X - \{p\}$, for all $n \in \mathbf{N}$, $w_{n,x}(U_{1,x}^n) = 2^{-n}$.*

Proof. Since $U_{1,x}^n$ has no predecessor in $\mathscr{C}_x^n$, this follows by induction on n from the definition of the weight functions. $\square$

4.7. **Bounds on f_n and f.**

4.7.1. **Lemma.** *For all $x \in X$ and $n \in \mathbf{N}$ such that $|\mathscr{C}_x^n| \geq 2$,*

$$f_n(x) - [w_{n,x}(U_{l(n,x),x}^n) + w_{n,x}(U_{l(n,x)-1,x}^n)] \leq f(x) < f_n(x) .$$

Proof. Fix $x \in X - \{p\}$ such that $|\mathscr{C}_x^n| \geq 2$. It suffices to show that for all $j \geq 0$,

$$f_{n+j}(x) > f_n(x) - [w_n(U_{l(n)}) + w_n(U_{l(n)-1})] .$$

If $j = 0$, the result is trivial.

If $j = 1$, it follows from the proof of Lemma 4.6.3 that

$$f_{n+1}(x) > f_n(x) - [w_n(U_{l(n)}) + w_n(U_{l(n)-1})] .$$

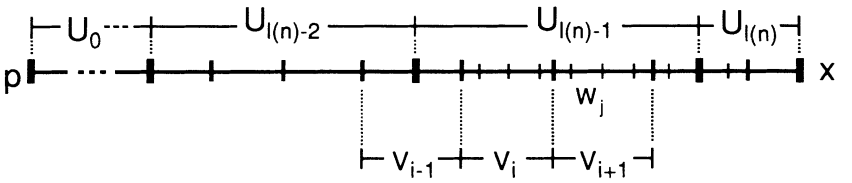

FIGURE 4.2

If $j = 2$, we must use 3-spanning to make sure that full weight is preserved at the $n+2$ level for all but possibly the last two links of $\mathscr{C}^n$. Refer to Figure 4.2 in what follows.

The first link V_i of $\mathscr{C}^{n+1}$ for which $\partial V_i = U_{l(n)-1}$ incorporates the carryover of $U_{l(n)-2}$. By 3-spanning, there is another link V_{i+1} of $\mathscr{C}^{n+1}$ for which $\partial V_i = U_{l(n)-1}$. Then the first link W_j of $\mathscr{C}^{n+2}$ for which $\partial W_j = V_{i+1}$ incorporates the carryover of V_i. Thus, at the least, the full weight at level n of $U_0, \ldots, U_{l(n)-2}$ plus half the weight at level n of $U_{l(n)-1}$ is preserved at level $n+2$. Hence, we have

$$f_{n+2}(x) > f_n(x) - [w_n(U_{l(n)}) + w_n(U_{l(n)-1})] .$$

Inductively, using 3-spanning of the $n+j$ level in the $n+j-1$ level, we obtain

$$f_{n+j}(x) > f_n(x) - [w_n(U_{l(n)}) + w_n(U_{l(n)-1})] . \quad \square$$

4.7.2. **Lemma.** *For all $x, y \in X$, for all $U \in \mathscr{C}_x^n$, for all $V \in \mathscr{C}_y^n$ such that $U^- = V^-$, $w_{n,x}(U) = w_{n,y}(V)$.*

Proof. The lemma is trivial for $n = 0$. Suppose $n > 0$ and the lemma is true for all $k < n$. Then

$$U^- = [p_{j-1,x}^n p_{j,x}^n] = [p_{j-1,y}^n, p_{j,y}^n] = V^- .$$

Hence, $U = [p_{j,x}^n, p_{j+1,x}^n]$ and $V = [p_{j,y}^n, p_{j+1,y}^n]$ have at least the point $p_j^n = p_{j,x}^n = p_{j,y}^n$ in common. See Figure 4.3.

Let $P_{s,x}^{n-1}$ be maximal with respect to $p_{s,x}^{n-1} \le p_j^n$. Note that $p_{s,x}^{n-1} = p_{s,y}^{n-1} = p_s^{n-1}$. First suppose $s \neq 0$. Let $W = [p_{s-1}^{n-1}, p_s^{n-1}] \in \mathscr{C}_x^{n-1} \cap \mathscr{C}_y^{n-1}$. Let W_x^+ (respectively, W_y^+) denote the successor of W with respect to $\mathscr{C}_x^n$ (respectively, $\mathscr{C}_y^n$). By 3-spanning and our choice of p_s^{n-1}, it follows that $U \subset W_x^+$ and $V \subset W_y^+$. Hence, $\partial U = W_x^+$ and $\partial V = W_y^+$. Thus, $(\partial U)^- = W = (\partial V)^-$. On the other hand, if $s = 0$, (so $p_s^{n-1} = p$), then $(\partial U)^- = \varnothing = (\partial V)^-$.

Since $(\partial U)^- = (\partial V)^-$, it follows that $(\partial U)^{--} = (\partial V)^{--}$. Therefore, by the induction hypothesis,

$$w_{n-1,x}(\partial U) = w_{n-1,y}(\partial V) \quad \text{and} \quad w_{n-1,x}((\partial U)^-) = w_{n-1,y}((\partial V)^-) .$$

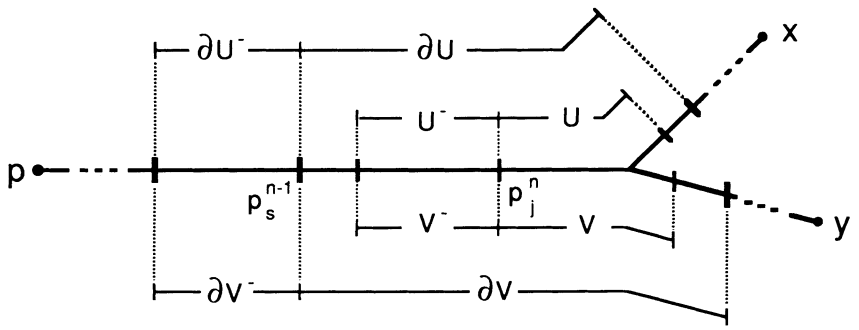

FIGURE 4.3

As $\mathscr{C}_x^n$ and $\mathscr{C}_y^n$ agree prior to $x \wedge y$, we have $b_U = b_V$ and $t_{(\partial U)^-} = t_{(\partial V)^-}$. It then follows directly from the definition of weights that $w_{n,x}(U) = w_{n,y}(V)$. □

4.7.3. **Lemma.** *Suppose $x, y \in X$ with $p \leq x < y$. Suppose there is a link $V \in \mathscr{C}_y^n$ not meeting any link of $\mathscr{C}_x^n$ and that V^{++} exists. Then*

$$f(y) - f(x) \geq w_{n,y}(V) > 0.$$

Proof. Without loss of generality, suppose that $p < x$ and that

$$\mathscr{C}_x^n = \{U_1, \ldots, U_{k-1}, U_k\} \quad \text{and}$$
$$\mathscr{C}_y^n = \{U_1, \ldots, U_{k-1}, U_k', U_{k+1}, \ldots, U_j, \ldots, U_r\},$$

where $r - k > 2$, and U_j (where $r - 2 \geq j \geq k + 1$) is the link V in the statement of the lemma. Note that $U_k \subset U_k'$, but they need not be equal. By Lemma 4.7.1, we have

$$f(y) \geq f_n(y) - (w_{n,y}(U_r) + w_{n,y}(U_{r-1})).$$

Then, as $r - 2 \geq j \geq k + 1$, we have

$$f(y) \geq \sum_{U < U_k'} w_{n,y}(U) + w_{n,y}(U_k') + w_{n,y}(U_j).$$

By Lemmas 4.7.2 and 4.7.1, we then have

$$f(y) \geq \sum_{U < U_k} w_{n,x}(U) + w_{n,x}(U_k) + w_{n,y}(U_j),$$
$$f(y) \geq f_n(x) + w_{n,y}(U_j),$$
$$f(y) \geq f(x) + w_{n,y}(U_j). \quad \square$$

4.7.4. **Corollary.** *For all $z \in X - \{p\}$, f is strictly increasing on $[p, z]$.*

Proof. Suppose $x < y$ with $x, y \in [p, z]$. Then for some sufficiently large n, the conditions of Lemma 4.7.3 are satisfied. Thus, $f(y) > f(x)$. □

4.7.5. **Lemma.** *For all* $x \in X - \{p\}$, *for all* $n \in \mathbf{N}$, *for all* $U \in \mathscr{C}_x^n$ *such that* U^- *exists,* $w_{n,x}(U) \geq 2^{-n} \cdot w_{n,x}(U^-)$.

Proof. The lemma is trivial for $n = 0$ and follows directly from the definition of weight functions for $n = 1$. Let $n > 1$, and assume the lemma is true for all $k < n$. There are three cases. Only the third case uses the induction hypothesis.

Case 1. Suppose $b_U = 1$. Then $(\partial U)^- = \partial U^-$ and $b_{U^-} > 2$ by 3-spanning. Also, $b_{U^-} = t_{(\partial U)^-}$. Consequently, we have

$$\begin{aligned} w_n(U) &= \tfrac{1}{2} w_{n-1}(\partial U) + 2^{-t_{(\partial U)^-}} \cdot w_{n-1}\left((\partial U)^-\right) \\ &> 2^{-b_{U^-}} \cdot w_{n-1}\left(\partial U^-\right) = w_n\left(U^-\right). \end{aligned}$$

Case 2. Suppose $b_U \neq 1 \neq b_{U^-}$. Then $\partial U = \partial U^-$ and $b_U = b_{U^-} + 1$. Hence,

$$w_n(U) = \tfrac{1}{2} w_n\left(U^-\right).$$

Case 3. Suppose $b_{U^-} = 1$. Then $b_U = 2$ and $\partial U = \partial U^-$ by 3-spanning. If $(\partial U^-)^- = \varnothing$, then

$$w_n(U) = 2^{-b_U} \cdot w_{n-1}(\partial U) = \tfrac{1}{2} w_n\left(U^-\right).$$

Otherwise, we have

$$\begin{aligned} w_n(U^-) &= \tfrac{1}{2} w_{n-1}(\partial U^-) + 2^{-t_{(\partial U^-)^-}} \cdot w_{n-1}((\partial U^-)^-), \\ w_n(U^-) &\leq \tfrac{1}{2} w_{n-1}(\partial U^-) + \tfrac{1}{4} w_{n-1}((\partial U^-)^-). \end{aligned}$$

Since $w_{n-1}((\partial U^-)^-) \leq 2^{n-1} \cdot w_{n-1}(\partial U^-)$ by the induction hypothesis,

$$\begin{aligned} w_n\left(U^-\right) &\leq \tfrac{1}{2} w_{n-1}\left(\partial U^-\right) + \tfrac{1}{4} \cdot 2^{n-1} \cdot w_{n-1}\left(\partial U^-\right), \\ w_n\left(U^-\right) &\leq (\tfrac{1}{2} + 2^{n-3}) w_{n-1}\left(\partial U^-\right). \end{aligned}$$

Since $w_n(U) = \frac{1}{4} w_{n-1}(\partial U)$, $\partial U = \partial U^-$, and $n > 1$, we then have

$$\begin{aligned} w_n\left(U^-\right) &\leq (\tfrac{1}{2} + 2^{n-3}) \cdot 4 \cdot w_n(U), \\ w_n\left(U^-\right) &\leq (2 + 2^{n-1}) w_n(U) \leq 2^n \cdot w_n(U). \quad \square \end{aligned}$$

4.7.6. *Remark.* At this point, we could define an *overlap* function $\theta\colon X \times X \to \mathbf{R}^+$ by $x\theta y = f(x \wedge y)$. It could then be shown that θ satisfies the *rooted tree properties* defined by Alperin and Bass [AB, B], from which it follows by a theorem of theirs that X admits an $\mathbf{R}$-tree metric. (They use $\wedge$ to symbolize the overlap function.) It would remain to be shown that this $\mathbf{R}$-tree metric is compatible with the given metric ρ on X. Instead, we proceed below to define an $\mathbf{R}$-tree metric d on X directly in terms of f and $\wedge$, and we show that d is compatible with ρ in Theorems 4.9 and 5.1.

4.8. **Distance function.**

4.8.1. **Definition.** Define a *distance function* $d: X \times X \to [0, 2]$ by

$$d(x, y) = f(x) + f(y) - 2f(x \wedge y),$$

and define $d_n: X \times X \to [0, 2]$ by

$$d_n(x, y) = f_n(x) + f_n(y) - 2f_n(x \wedge y).$$

We prove in Theorem 4.9 below that d is a convex metric on X, finer than the given metric ρ. We do not need d_n to define d, but d_n is useful in the proof of Theorem 5.1 because, by the following lemma and corollary, d_n approximates d.

4.8.2. **Lemma.** $d_n(x, y) - 8 \cdot (\frac{3}{4})^{n-1} \leq d(x, y) \leq d_n(x, y) + 8 \cdot (\frac{3}{4})^{n-1}$.

Proof. Follows easily from the definitions of d and d_n and Corollary 4.6.4. □

4.8.3. **Corollary.** $d(x, y) = \lim d_n(x, y)$.

4.9. **Theorem.** *Let (X, ρ) be a UAC metric space. Then there is a convex metric d on X. Moreover, the topology on X induced by the metric d is finer than the topology on X induced by the metric ρ.*

Proof. Let $p \in X$, let $\{\mathscr{U}_n\}_{n=0}^{\infty}$ be the decreasing sequence of covers of X guaranteed by Theorem 3.6, and let d be the distance function constructed relative to p and $\{\mathscr{U}_n\}_{n=0}^{\infty}$ in §§4.1–4.8. We first verify that d is a metric.

Let $x, y \in X$. By Definition 4.8.1,

$$d(x, y) = f(x) - f(x \wedge y) + f(y) - f(x \wedge y).$$

Since $x \wedge y \leq x$ and $x \wedge y \leq y$, it follows from Corollary 4.7.4 that

$$f(x) - f(x \wedge y) \geq 0 \quad \text{and} \quad f(y) - f(x \wedge y) \geq 0.$$

Consequently, $d(x, y) \geq 0$. Moreover, if $x = y$, then $x = x \wedge y = y$. Hence, $d(x, y) = 0$. Conversely, if $d(x, y) = 0$, then

$$f(x) - f(x \wedge y) = 0 \quad \text{and} \quad f(y) - f(x \wedge y) = 0,$$

since their sum is 0 and both are nonnegative. Hence,

$$f(x) = f(x \wedge y) \quad \text{and} \quad f(y) = f(x \wedge y).$$

Since f is strictly increasing on $[p, x]$ and on $[p, y]$ by Corollary 4.7.4, $x = x \wedge y$ and $y = x \wedge y$. Consequently, $x = y$.

Since $x \wedge y = y \wedge x$, it follows that $d(x, y) = d(y, x)$, for all $x, y \in X$.

To establish the triangle inequality, let $x, y, z \in X$. We claim that $d(x, z) \leq d(x, y) + d(y, z)$. Let $v = x \wedge z$ and let b be the last point of the arc

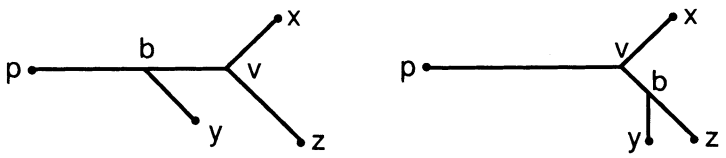

FIGURE 4.4

$[p, y]$ that lies in $[p, x] \cup [p, z]$. There are three cases. See Figure 4.4. We freely apply Lemma 4.7.3 and Corollary 4.7.4 where required.

Case 1. Suppose $b \leq v$. By definition,

$$\begin{aligned} d(x, z) &= f(x) + f(z) - 2f(v), \\ d(x, y) &= f(x) + f(y) - 2f(b), \quad \text{and} \\ d(y, z) &= f(y) + f(z) - 2f(b). \end{aligned}$$

Adding the last two equations we have

$$d(x, y) + d(y, z) = f(x) + f(z) + 2f(y) - 4f(b).$$

But since $f(y) \geq f(b)$, we have

$$d(x, y) + d(y, z) \geq f(x) + f(z) - 2f(b).$$

Since $f(v) \geq f(b)$, we have

$$d(x, y) + d(y, z) \geq f(x) + f(z) - 2f(v) = d(x, z).$$

Case 2. Suppose $v < b \leq z$. Then

$$\begin{aligned} d(x, y) &= f(x) + f(y) - 2f(v) \quad \text{and} \\ d(y, z) &= f(y) + f(z) - 2f(b). \end{aligned}$$

Adding these two equations we have

$$d(x, y) + d(y, z) = f(x) + f(z) + 2f(y) - 2f(b) - 2f(v).$$

Since $f(y) \geq f(b)$, we have

$$d(x, y) + d(y, z) \geq f(x) + f(z) - 2f(v) = d(x, z).$$

Case 3. Suppose $v < b \leq x$. This case is symmetric to Case 2.

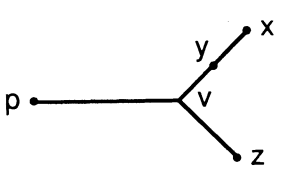

FIGURE 4.5

We now verify that d is convex. Let A be an arc in X and let $x, y, z \in A$ such that $y \in [x, z]$. Without loss of generality, assume that $x \wedge z = v \in [y, z]$. See Figure 4.5. Then $x \wedge y = y$ and $y \wedge z = v$. Thus, we have

$$d(x, y) = f(x) + f(y) - 2f(y) \quad \text{and}$$
$$d(y, z) = f(y) + f(z) - 2f(v).$$

Adding these two equations, we have

$$d(x, y) + d(y, z) = f(x) + f(z) - 2f(v) = d(x, z).$$

Therefore, d is convex.

In order to prove that the topology on X induced by the metric d is finer than the topology on X induced by the metric ρ it suffices to prove that if $\varnothing \neq A \subset X$ and $x \notin \mathrm{Cl}(A)$ (where closure is taken in the topology induced by the original metric ρ), then $d(x, A) > 0$.

Let x, A be as above. Let $\eta = \rho(x, A)$. Choose n such that $\mathrm{mesh}(\mathscr{U}_n) < \eta/10$. Then $\mathscr{U}_n$ is 9-spanning from x to A.

Suppose that $x = p$. Let $a \in A$. Since $\mathscr{U}_n$ is 9-spanning from p to a, there are more than three links in $\mathscr{C}_a^n$, including $[p, p_{1,a}^n] = U_{1,a}^n$, $U_{2,a}^n$, and $U_{3,a}^n$. It follows from Lemmas 4.6.5 and 4.7.3 that

$$f(a) \geq w_{n,a}(U_{1,a}^n) = 2^{-n} > 0.$$

Since for each $a \in A$, $d(x, a) = d(p, a) = f(a)$, we have $d(x, A) > 0$.

Suppose that $x \neq p$. Let $\varepsilon = 2^{-n} \cdot \min\{w_{n,x}(U) | U \in \mathscr{C}_x^n\}$. Let $a \in A$ and let $b = x \wedge a$. It suffices to show that $d(x, a) \geq \varepsilon$. There are two cases. See Figure 4.6.

Case 1. Suppose that at least 4 links of $\mathscr{C}_x^n$ are required to cover $[b, x]$ irreducibly. Let U be the first link of $\mathscr{C}_x^n$ such that $U \cap [p, a] = \varnothing$. By Lemma 4.7.3, $f(x) - f(b) \geq w_{n,x}(U) > \varepsilon$. Therefore,

$$d(x, a) = f(x) - f(b) + f(a) - f(b) > \varepsilon + 0 = \varepsilon.$$

Case 2. Suppose that at most 3 links of $\mathscr{C}_x^n$ are required to cover $[b, x]$ irreducibly. Since $\mathscr{U}_n$ is 9-spanning from x to A, it takes at least 4 links of $\mathscr{C}_a^n$

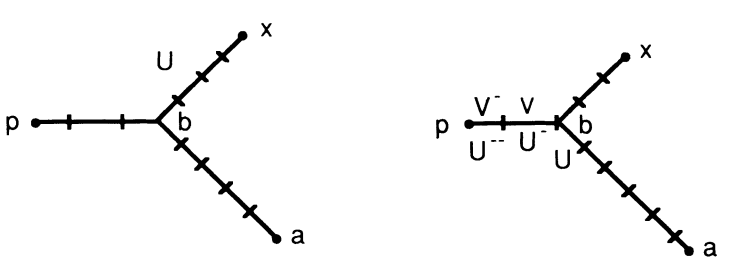

FIGURE 4.6

to span from a to b. Let U be the first link of $\mathscr{C}_a^n$ such that $U \cap [p, x] \subset \{b\}$. If U^- does not exist, then $p \in U$. Hence, by Lemmas 4.7.2 and 4.6.5

$$w_{n,a}(U) = w_{n,x}(U_{1,x}^n) = 2^{-n}.$$

Consequently, by Lemma 4.7.3,

$$f(a) - f(b) \geq w_{n,a}(U) = 2^{-n} > \varepsilon.$$

If U^- exists, then U^- meets $[p, x]$. Hence, there is a link $V \in \mathscr{C}_x^n$ such that $V^- = U^{--}$. By Lemmas 4.7.3, 4.7.5, and 4.7.2, we then have

$$f(a) - f(b) \geq w_{n,a}(U) \geq 2^{-n} \cdot w_{n,a}(U^-) = 2^{-n} \cdot w_{n,x}(V) \geq \varepsilon.$$

Therefore,

$$d(x, a) = f(x) - f(b) + f(a) - f(b) \geq 0 + \varepsilon = \varepsilon. \quad \square$$

5. Characterizing R-trees

5.1. **R-tree Characterization Theorem.** *Let (X, ρ) be a metric space. Then the following are equivalent:*

(1) *X is UAC and LAC.*

(2) *There is a convex metric d on X, equivalent to ρ, such that (X, d) is an* **R**-*tree.*

Proof. That (2) implies (1) is Theorem 2.3. To establish that (1) implies (2), let $p \in X$, let $\{\mathscr{U}_n\}_{n=0}^{\infty}$ be a decreasing sequence of covers of X guaranteed by Theorem 3.3, and let d be the distance function constructed relative to p and $\{\mathscr{U}_n\}_{n=0}^{\infty}$ in §4. By Theorem 4.9, d is a convex metric on X. By Theorem 2.5, (X, d) is an **R**-tree. We claim that d is equivalent to ρ. By Theorem 4.9, the topology on X induced by the metric d is finer than the topology on X induced by the metric ρ. Hence, it suffices to show that if $A \subset X$ and $x \in \mathrm{Cl}(A)$ (where closure is taken in the original topology induced by ρ), then $d(x, A) = 0$. It is only in this step that we require the LAC hypothesis.

Let $A \subset X$ and $x \in \mathrm{Cl}(A)$. Let $\varepsilon > 0$ be given. By Lemma 4.6.2 we may choose an $N \in \mathbf{N}$ so that for all $n > N$, for all $z \in X$, for all $U \in \mathscr{C}_z^n$, $w_{n,z}(U) < \varepsilon/2$. By Lemma 4.8.2, it suffices to show that for arbitrarily large n, there is an $a \in A$ such that $d_n(x, a) < \varepsilon$. Let $M \in \mathbf{N}$ be given, and choose $n > \max\{M, N\}$. There are two cases.

Case 1. Suppose that $x \notin \mathrm{Bd}(\mathscr{U}_n)$. By Remark 3.2.2, $\mathrm{Bd}(\mathscr{U}_n)$ is closed. Since X is LAC, we may choose an arcwise connected neighborhood V of x such that

$$\mathrm{Cl}(V) \cap \mathrm{Bd}(\mathscr{U}_n) = \varnothing.$$

Since $x \in \mathrm{Cl}(A)$, $V \cap A \neq \varnothing$. Let $a \in V \cap A$. Since V is AC, $x \wedge a \in V$. Moreover, since we have chosen V to miss $\mathrm{Bd}(\mathscr{U}_n)$, $V \cap (\mathscr{P}_x^n \cup \mathscr{P}_a^n) = \varnothing$. Hence, by Lemma 4.7.2, $f_n(x) = f_n(x \wedge a) = f_n(a)$. Therefore, $d_n(x, a) = 0$.

Case 2. Suppose that $x \in \mathrm{Bd}(\mathscr{U}_n)$. Since $\mathscr{U}_n$ is neighborhood-finite, the sets $\mathrm{St}(x, \mathscr{U}_n)$ and $\mathscr{B} = \{U \in \mathscr{U}_n | x \in \mathrm{Bd}(U)\}$ are both finite. Since x has a neighborhood which meets only the links of $\mathrm{St}(x, \mathscr{U}_n) \cup \mathscr{B}$, we may choose an AC neighborhood V of x such that

(a) $\mathrm{Cl}(V) \subset \bigcap \mathrm{St}(x, \mathscr{U}_n)$, and
(b) for all $U \in \mathscr{U}_n - \mathrm{St}(x, \mathscr{U}_n)$, $U \cap V \neq \varnothing$ implies $U \in \mathscr{B}$.

Let $a \in V \cap A$. Since V is AC, $[x, a] = [x \wedge a, a] \cup [x \wedge a, x] \subset V$, so $x \wedge a \in V$. See Figure 5.1.

We claim that

(c) $|\mathscr{P}_a^n \cap [x \wedge a, a]| \leq 1$, and
(d) $|\mathscr{P}_x^n \cap [x \wedge a, x]| \leq 1$.

To establish (c), suppose by way of contradiction that

$$p_{j,a}^n, p_{k,a}^n \in \mathscr{P}_a^n \cap [x \wedge a, a]$$

for some $j < k$. By our choice of V, there is a $U \in \mathrm{St}(x, \mathscr{U}_n)$ such that $p_{j,a}^n \in V \subset U$. But

$$[p_{j,a}^n, p_{k,a}^n] \subset [x \wedge a, a] \subset V \subset U \subset \bigcup \mathrm{St}(p_{j,a}^n, \mathscr{U}_n),$$

which, since $j + 1 \leq k$, is a contradiction of the definition of $p_{j+1,a}^n$. The proof of (d) is similar.

Since $\mathscr{C}_x^n$, $\mathscr{C}_{x \wedge a}^n$, and $\mathscr{C}_a^n$ agree up to $x \wedge a$, it follows from (c) and (d) that $|\mathscr{C}_x^n| - |\mathscr{C}_{x \wedge a}^n| \leq 1$ and $|\mathscr{C}_a^n| - |\mathscr{C}_{x \wedge a}^n| \leq 1$. Then by Lemma 4.7.2 (applied in each of the several cases that result, depending on what $|\mathscr{P}_a^n \cap [x \wedge a, a]|$ and $|\mathscr{P}_x^n \cap [x \wedge a, x]|$ are), we have that $f_n(x)$ and $f_n(x \wedge a)$ differ by at most the weight of one link, as do $f_n(a)$ and $f_n(x \wedge a)$. Consequently, by our choice of n, we have

$$f_n(x) - f_n(x \wedge a) < \varepsilon/2 \quad \text{and} \quad f_n(a) - f_n(x \wedge a) < \varepsilon/2.$$

It follows that

$$d_n(x, a) = f_n(x) - f_n(x \wedge a) + f_n(a) - f_n(x \wedge a) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \square$$

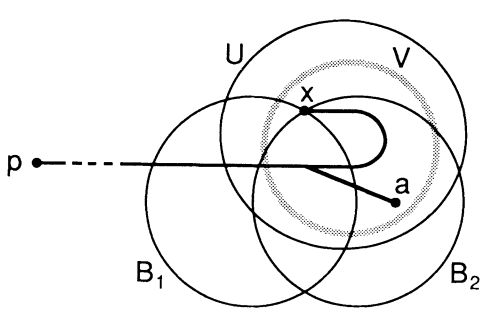

FIGURE 5.1

5.2. **Dimension of UAC, LAC spaces.** Since an **R**-tree has dimension one by Theorem 2.3, we have the following corollary to our main theorem:

5.2.1. **Corollary.** *If X is a UAC, LAC metric space, then X is one-dimensional.*

5.3. **Endpoints.** Suppose X is a UAC, LAC metric space. Let E be the set of endpoints of X. Then E is totally disconnected. This does not mean E is zero-dimensional. Nishiura and Tymchatyn [NT] have an example of a UAC, LAC, separable metric space whose set of endpoints is one-dimensional. It is based on a construction of Lelek [L] (going back to Sierpinski). Corollary 5.2.1 shows that the Nishiura-Tymchatyn example cannot be generalized to obtain an n-dimensional set of endpoints $(n > 1)$.

5.4. **Convex metrics on LAC spaces.** Our proof makes essential use of the UAC assumption on X. One wonders if weaker conditions might suffice to put a convex metric on a LAC space. In asking Question 5.4.2 below we have in mind the following definition of a convex metric:

5.4.1. **Definition.** Let (X, d) be a metric space. We say that d is a *convex* metric iff for all $x, y \in X$, there exists an arc $[x, y] \subset X$ from x to y such that the function $d(x, \_)|[x, y]: [x, y] \to [0, d(x, y)] \subset \mathbf{R}^+$ is an isometry.

5.4.2. *Question.* Let (X, ρ) be a connected LAC metric space. What additional conditions, if any, are required for there to exist a convex metric d on X compatible with ρ?

For example, it is clear that the plane E^2 minus a finite set of points has a compatible convex metric. It seems likely that E^2 minus a countable dense set of points has a compatible convex metric.

Something that might help in answering Question 5.4.2 is the following question due to E. D. Tymchatyn:

5.4.3. *Question.* Let (X, ρ) be a locally connected metric space. Does X have a decreasing sequence of neighborhood-finite open covers whose elements are connected?

For completeness, we repeat one of the questions concerning group actions on **R**-trees from [CM]:

5.5. *Question.* Which finitely generated groups act freely on **R**-trees?

Shalen [MS1, CM] conjectures that such a group must be the free product of abelian groups and surface groups.

5.6. **Compactifying and embedding R-trees.** In a subsequent paper, we expect to show the following:

5.6.1. **Theorem.** *If X is a separable* **R***-tree, then X embeds in the universal smooth dendroid.*

5.6.2. **Theorem.** *There is a universal separable* **R***-tree.*

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