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THE INITIAL-NEUMANN PROBLEM FOR THE HEAT EQUATION IN LIPSCHITZ CYLINDERS

RUSSELL M. BROWN

ABSTRACT. We prove existence and uniqueness for solutions of the initial-Neumann problem for the heat equation in Lipschitz cylinders when the lateral data is in L^p, $1 < p < 2+\varepsilon$, with respect to surface measure. For convenience, we assume that the initial data is zero. Estimates are given for the parabolic maximal function of the spatial gradient. An endpoint result is established when the data lies in the atomic Hardy space H^1. Similar results are obtained for the initial-Dirichlet problem when the data lies in a space of potentials having one spatial derivative and half of a time derivative in L^p, $1 < p < 2+\varepsilon$, with a corresponding Hardy space result when $p = 1$. Using these results, we show that our solutions may be represented as single-layer heat potentials. By duality, it follows that solutions of the initial-Dirichlet problem with data in L^q, $2-\varepsilon' < q < \infty$ and BMO may be represented as double-layer heat potentials.

INTRODUCTION

We study the initial-Neumann and initial-Dirichlet problems for the heat equation in cylindrical domains $(0, T)\times\Omega \subset \mathbf{R}^{n+1}$, $n \geq 2$, where $\Omega \subset \mathbf{R}^n$ is a bounded Lipschitz domain. For the Neumann problem, we consider lateral data from $L^p((0, T) \times \partial\Omega)$ for $1 < p < 2 + \varepsilon$ where ε depends on the Lipschitz constant of the domain and the initial data is taken to be zero. Our main result establishes existence and uniqueness with estimates in L^p for the parabolic maximal function of the spatial gradient.

For the Dirichlet problem, we will take data from a potential space $\mathscr{L}_1^p$ which consists of functions having one spatial and half of a time derivative in L^p. Again, we establish existence and uniqueness with estimates for the parabolic maximal function of the spatial gradient. For each of these problems, we will also consider an endpoint result when $p = 1$. In the Neumann problem, L^p is replaced by an atomic Hardy space, H^1, and in the Dirichlet problem, $\mathscr{L}_1^1$ is defined to be a potential space with densities in H^1. This extends results in [B1] where the case $p = 2$ is studied. The techniques used are similar to those

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of Dahlberg and Kenig in their study of boundary value problems for Laplace's equation in Lipschitz domains [DK].

With existence and uniqueness for the boundary value problems in hand, we turn to the question of representing our solutions as heat potentials. Here, the main result is that solutions of the Neumann problem with data in L^p, $1 < p < 2 + \varepsilon$, may be represented as single-layer heat potentials with densities in L^p and there is a Hardy space result for $p = 1$. Similar results are obtained for solutions of the Dirichlet problem when the data is in $\mathscr{L}_1^p$. We also show that solutions of the Dirichlet problem with data in L^p, $2 - \varepsilon/(1+\varepsilon) < p < \infty$ may be represented as double-layer heat potentials and an endpoint result is obtained for BMO. This is easy since the operator involved is essentially the adjoint of the operator we invert to establish a representation theorem for solutions of the Neumann problem. Existence and uniqueness of solutions to the Dirichlet problem with L^p-data has been established by Fabes and Salsa [FS].

It is easy to see that, except for the value of ε (which we do not attempt to compute), the range $1 < p < 2 + \varepsilon$ for the Neumann problem is sharp. The reader may find examples in [B1] which show that for any $\varepsilon > 0$, there is a Lipschitz domain D for which our estimates fail when $p = 2 + \varepsilon$.

We point out that the study of the layer potentials on Lipschitz domains is essentially different from the case of smooth or even C^1-domains. We recall that in the method of layer potentials, the solution of the boundary value problems is reduced to solving integral equations of the form $(\pm\frac{1}{2}I + K)(g) = f$. The operators $\pm\frac{1}{2}I + K$ arise as the boundary values of heat potentials. Because of the smoothness of the boundary, it is easy to see that $K : L^p((0, T) \times \partial\Omega) \to L^p((0, T) \times \partial\Omega)$ with small norm, at least when T is small and D is bounded. Hence the inverses of $\pm\frac{1}{2}I + K$ can be easily constructed as Neumann series. This argument on $C^{1,\alpha}$-domains is classical. However, on C^1-domains, the boundedness of the operators is a consequence of the deep theorem of Calderón [C1]. Shortly after Calderón's work, Fabes and Rivière [FR] were able to show that the classical argument for inverting the potential operators extended to the C^1-case. In contrast, there exist Lipschitz domains where the norms of these operators do not go to zero in any L^p-space (see [B1]). Hence, solving the integral equations directly would involve precise computations for the norm of the operators involved. A task which seems difficult.

The outline of this paper is as follows. §1 gives definitions and introduces the scale of spaces $\mathscr{L}_1^p$ from which we will take data when we study regularity in the Dirichlet problem. §2 gives, without proof, a review of the L^2-theory from [B1] and establishes a few minor extensions that will be needed later. §§3 and 4 give our main results in the special case of a domain lying above the graph of a Lipschitz function. §3 studies the Neumann problem with data in H^1 and the Dirichlet problem with data in $\mathscr{L}_1^1$. In §4 we give the L^p-theory for these problems and invert the potential operators on graph domains. §5 extends these results to bounded domains. In §6, we show that all caloric functions with zero initial values and parabolic maximal function in L^1 arise as solutions of the initial-Neumann problem with data from an atomic Hardy space.

Acknowledgments. The reader will recognize the influence of B. Dahlberg and C. Kenig and their work on the Neumann problem for Laplace's equation [DK]. Their insight was indispensable for carrying out this research. I thank Professor E. Fabes for introducing me to this subject and D. Jerison and M. Frazier for several helpful conversations while carrying out this research.

1. Notations, definitions and the spaces $\mathscr{L}_1^p$

We let $X = (X', x_n) = (x_0, x', x_n) = (x_0, x)$ denote points in $\mathbf{R}^{n+1}$. As indicated, we use '$\,'\,$' when the x_n variable is omitted and lower case letters when the x_0 or time variable is omitted. We let $\phi\colon \mathbf{R}^{n-1} \to \mathbf{R}$ be a Lipschitz function satisfying $\|\nabla'\phi\|_\infty \leq m$. We will use $D = \{(x', x_n)\colon x_n > \phi(x')\}$ to denote the region lying above the graph of ϕ. The cylinder $\mathbf{R} \times D$ will be denoted by D_∞ and we let $S = \mathbf{R} \times \partial D$ denote the boundary of D_∞. We will use Ω to denote bounded Lipschitz domains in $\mathbf{R}^n$, $n \geq 2$. To give our definition of a bounded Lipschitz domain, let $Z = \{(x', x_n)\colon |x_i| < r_0, i = 1, \dots, n-1 \text{ and } |x_n| < mr_0\}$ and let $\phi : \mathbf{R}^{n-1} \to (-mr_0, mr_0)$ be a Lipschitz function with $\|\nabla'\phi\|_\infty \leq m$. We say that (Z, ϕ) is a coordinate cylinder for $\Omega \subset \mathbf{R}^n$ if

$$100Z \cap \Omega = \{(x', x_n) : x_n > \phi(x')\} \cap 100Z$$

and

$$100Z \cap \partial\Omega = \{(x', x_n) : x_n = \phi(x')\} \cap 100Z$$

where rZ denotes the dilation of Z by a factor of r about its center. We say that a bounded connected open set is a Lipschitz domain with constant m and scale r_0 if there exists a finite covering of $\partial\Omega$ by coordinate cylinders $\{Z_i\colon i = 1, \dots, N\}$. The coordinate systems used to define each Z_i are allowed to differ by a rigid motion in $\mathbf{R}^n$. We let Ω_T and Ω_+ denote the cylinders $(0, T) \times \Omega$ and $(0, \infty) \times \Omega$ respectively. The lateral boundaries will be given by $\Sigma_T \equiv (0, T) \times \partial\Omega$ and $\Sigma_+ \equiv (0, \infty) \times \partial\Omega$. In §5 we will also need to refer to the exterior domain $\Omega^e \equiv \mathbf{R}^n \setminus \bar{\Omega}$. Points on S, Σ_T or Σ_+ will be denoted by P and Q and we will write $P = (p_0, p)$ when we need to distinguish the time variable.

We recall that the heat operator in $\mathbf{R}^{n+1}$ is $\partial_{x_0} - \Delta$ where $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ is the usual Laplacian in the n spatial variables. We let $\nabla = (\partial_{x_1}, \dots, \partial_{x_n})$ denote the gradient in $\mathbf{R}^n$ and ∇' be the gradient in $\mathbf{R}^{n-1}$.

Throughout this paper, we let c and C denote constants which may vary from line to line. When working on graph domains, these constants will depend only on the dimension n, m, α and p. On bounded domains, these constants may also depend on T and the collection of coordinate cylinders $\{Z_i\}$. We will use $A \approx B$ to mean that $c \le A/B \le C$.

For $r > 0$, we let $d_r(x_0, x) \equiv (r^2 x_0, rx)$ denote the parabolic dilation of order r. Observe that the heat equation satisfies

$$(\partial_{x_0} - \Delta)(u \circ d_r) = r^2(\partial_{x_0} u - \Delta u) \circ d_r.$$

Motivated by this, we introduce the parabolic metric

$$\delta(X; Y) \equiv |x_0 - y_0|^{1/2} + |x - y|$$

which is homogeneous with respect to the family of dilations d_r. We let $\delta(X; E) \equiv \inf_{Y \in E} \delta(X; Y)$ denote the parabolic distance between X and the set E. We will use $\delta(E; F)$ for the distance between two sets. We observe that, with dQ denoting surface measure, (S, δ, dQ) and (Σ_T, δ, dQ) form spaces of homogeneous type in the sense of Coifman and Weiss [CW]. We emphasize that throughout this paper, the usual inhabitants of spaces of homogeneous type such as atomic Hardy spaces and BMO will be defined using the metric δ. Thus we will be using the spaces often referred to as parabolic BMO or parabolic Hardy spaces.

For our domains D_∞, we introduce parabolic approach regions to the lateral boundary,

$$\Gamma(P, \alpha) \equiv \{Y \in D_\infty : \delta(Y; P) < (1 + \alpha)\delta(Y; S)\}, \qquad P \in S,$$

and for Ω_T we define similarly

$$\Gamma(P, \alpha) \equiv \{Y \in \Omega_T : \delta(Y; P) < (1 + \alpha)\delta(Y; \Sigma_T)\}, \qquad P \in \Sigma_T,$$

where $\alpha > 0$ will be called the aperture of $\Gamma(P, \alpha)$. Next, for a function u on D_∞ (or Ω_T) we define the parabolic maximal function by

$$N_\alpha(u)(P) \equiv \sup_{Y \in \Gamma(P, \alpha)} |u(Y)|$$

where $P \in S$ (or Σ_T). The definition of the parabolic maximal function for the exterior domain is obtained by simply replacing Ω_T by Ω_T^e in the definitions of the cones. As indicated in the introduction, the main goal of this paper is to bound the L^p-norm of the parabolic maximal function of the spatial gradient of the solution in terms of the Neumann or Dirichlet data of the solution.

We define parabolic cubes on $\mathbf{R} \times \mathbf{R}^{n-1}$ by $I_r(X') \equiv \{Y' : |y_i - x_i| < r,\ i = 1, \dots, n-1$ and $x_0 - r^2 < y_0 < x_0\}$. Using the coordinate map $\pi : S \to \mathbf{R} \times \mathbf{R}^{n-1}$ given by $\pi(x_0, x', \phi(x')) = (x_0, x')$, we let $I_r(Q) = \pi^{-1}(I_r(\pi(Q)))$. We also let $i_r(x')$ denote the $n-1$ dimensional cube $\{y' : |y_i - x_i| < r,\ i = 1, \dots, n-1\}$ and define cubes $i_r(q)$ on ∂D using the appropriate coordinate map. Similarly, we may define cubes on Σ_T using the coordinate maps in each of the coordinate cylinders when the sidelength r is small. We introduce local domains near the boundary. For $q \in \partial D$ and $r > 0$, set

$$\psi_r(q) = \{x \in D : |x_i - q_i| < r,\ \ i = 1, \dots, n-1,\ \ |x_n - q_n| < C_{n,m} r\}$$

where the constant $C_{n,m}$ is chosen so that $\psi_r(q)$ is a starshaped Lipschitz domain. This means that $\psi_r(q)$ is a Lipschitz domain which is starshaped convex with respect to some point x^* and $\partial\psi_r(q) \subset B(X^*, Cr) \setminus B(x^*, r/C)$ for some constant C. We let $\Psi_r(Q) = (q_0 - r^2, q_0) \times \psi_r(q)$ when $Q \in S$. We may make a similar construction in Ω_T if we use the coordinate system of one of the coordinate cylinders and r is small.

We introduce notation for mixed Sobolev spaces in $\mathbf{R}^{n+1}$. For an open set $\mathscr{O} \subset \mathbf{R}^m$ and $(a, b) \subset \mathbf{R}$, we define $W^{1/2,1}((a, b) \times \mathscr{O})$ to be the space of locally integrable functions f for which the seminorm

$$\|f\|^2_{W^{1/2,1}((a,b)\times\mathscr{O})} = \|f\|^2_{W^{1/2,0}((a,b)\times\mathscr{O})} + \|f\|^2_{W^{0,1}((a,b)\times\mathscr{O})}$$

is finite. The two expressions on the right are defined by

$$\|f\|^2_{W^{1/2,0}((a,b)\times\mathscr{O})} = \int_{\mathscr{O}} \int_a^b \int_a^b \frac{|f(t, y) - f(s, y)|^2}{|t - s|^2}\, ds\, dt\, dy',$$

$$\|f\|^2_{W^{0,1}((a,b)\times\mathscr{O})} = \int_a^b \int_{\mathscr{O}} |\nabla_x f(x_0, x)|^2\, dx\, dx_0.$$

We are omitting the L^2-norm in the above expressions because we will want to keep track of all the powers of r when we rescale.

We turn to a description of the spaces $\mathscr{L}^p_1(S)$ as in Fabes and Jodeit [FJ]. We begin by introducing the operator Λ which is a parabolic analogue of the Riesz potential of order one. Let

$$\lambda(X') = \begin{cases} 2(4\pi x_0)^{-n/2} \exp(-|x'|^2/4x_0), & x_0 > 0, \\ 0, & x_0 \le 0, \end{cases}$$

and then let

$$\Lambda(g)(X') = \lambda * g(X') = \int_{-\infty}^{x_0} \int_{\mathbf{R}^{n-1}} \lambda(X' - Y') g(Y') dY'.$$

For $f \in L^p$, $1 < p < n + 1$, this makes sense a.e. If $p \ge n + 1$, we replace $\lambda(X' - Y')$ by $\lambda(X' - Y') - \lambda(X'^* - Y')$ for some arbitrary point X'^*, thus $\Lambda(g)$ will only be defined modulo constants. For future reference, we record the following estimates for the kernel λ:

(1.1a) $$|\partial^\beta_{x_0} \partial^\alpha_{x'} \lambda(X')| \le C_{\alpha,\beta} \delta(X'; 0)^{-n-2\beta-|\alpha|},$$

(1.1b)
$$\begin{aligned} &|\partial^\beta_{x_0} \partial^\alpha_{x'} \lambda(X' - Y') - \partial^\beta_{x_0} \partial^\alpha_{x'} \lambda(X' - \bar{X}')| \\ &\quad \le C_{\alpha,\beta} r \delta(X'; \bar{X}')^{-n-2\beta-|\alpha|-1}, \qquad Y' \in I_r(X'),\ \delta(X'; I_r(\bar{X}')) \ge r. \end{aligned}$$

We define $\mathscr{L}^p_1(\mathbf{R} \times \mathbf{R}^{n-1})$, for $1 < p < \infty$, as $\Lambda(L^p) = \{f : f = \Lambda(g),\ g \in L^p(\mathbf{R} \times \mathbf{R}^{n-1})\}$. The norm of $f = \Lambda(g)$ is given by

$$\|f\|_{\mathscr{L}^p_1(\mathbf{R}\times\mathbf{R}^{n-1})} = \|g\|_{L^p(\mathbf{R}\times\mathbf{R}^{n-1})}.$$

To define these spaces on S, we set $\mathscr{L}_1^p(S) = \{f : f \circ \pi^{-1} \in \mathscr{L}_1^p(\mathbf{R} \times \mathbf{R}^{n-1})\}$. The norm in $\mathscr{L}_1^p(S)$ is given by $\|f\|_{\mathscr{L}_1^p(S)} = \|f \circ \pi^{-1}\|_{\mathscr{L}_1^p(\mathbf{R}\times\mathbf{R}^{n-1})}$.

We observe that

$$\Lambda : L^p(\mathbf{R} \times \mathbf{R}^{n-1}) \to L^{\frac{(n+1)p}{n+1-p}}(\mathbf{R} \times \mathbf{R}^{n-1}), \qquad 1 < p < n+1.$$

The proof given in [St, pp. 119–121] for the corresponding property of the Riesz potentials easily generalizes to the parabolic potential Λ. From this observation, we see that we have the embedding

$$\mathscr{L}_1^p(\mathbf{R} \times \mathbf{R}^{n-1}) \subset L^{\frac{(n+1)p}{n+1-p}}(\mathbf{R} \times \mathbf{R}^{n-1}), \qquad 1 < p < n+1.$$

There are also embeddings into Hölder spaces and BMO when $p \geq n+1$. However, since we will only prove theorems when $p < 2 + \varepsilon$ and $n \geq 2$, we omit further mention of these results and make the inessential assumption that $\varepsilon < n - 1$.

Letting $\hat{f}(\xi_0, \xi') = \int e^{-i(x_0\xi_0 + x'\xi')} f(x_0, x')\, dX'$, we observe that

$$\hat{\lambda}(\xi_0, \xi') = \frac{1}{\sqrt{|\xi'|^2 + i\xi_0}}.$$

This may be checked by first evaluating the Fourier transform in the spatial variables and then in the time variable. Thus it is easy to see that

$$\Lambda^{-1}(g) = \partial_{x_0}\Lambda(g) - \sum_{i=1}^{n-1} \partial_{x_i}\Lambda(\partial_{x_i} g)$$

defines a left inverse to Λ: $L^p(\mathbf{R} \times \mathbf{R}^{n-1}) \to \mathscr{L}_1^p(\mathbf{R} \times \mathbf{R}^{n-1})$ when $p < n + 1$. (If $p \geq n+1$, it takes a bit of fiddling to make sense of $\Lambda(g)$ for $g \in \mathscr{L}_1^p(\mathbf{R} \times \mathbf{R}^{n-1})$.) Noting that the operators $f \to \partial_{x_i}\Lambda$, $i = 1, \dots, n-1$, and $f \to \partial_{x_0}\Lambda^2(f)$ are parabolic singular integral operators, we have the following theorem from [FJ].

Theorem 1.2. *Let* $f \in L^{\frac{p(n+1)}{n+1-p}}(\mathbf{R} \times \mathbf{R}^{n-1})$ *where* $1 < p < n+1$, *then* f *is in* $\mathscr{L}_1^p(\mathbf{R} \times \mathbf{R}^{n-1})$ *if and only if the weak derivatives* $\partial_{x_i} f$ *and* $\partial_{x_0}\Lambda(f)$ *are in* $L^p(\mathbf{R} \times \mathbf{R}^{n-1})$. *Furthermore,*

$$\|f\|_{\mathscr{L}_1^p(\mathbf{R}\times\mathbf{R}^{n-1})} \approx \|\partial_{x_0}\Lambda(f)\|_{L^p(\mathbf{R}\times\mathbf{R}^{n-1})} + \sum_{i=1}^{n} \|\partial_{x_i} f\|_{L^p(\mathbf{R}\times\mathbf{R}^{n-1})}.$$

Remark. When $p = 2$, we may replace $\|\partial_{x_0}\Lambda(f)\|_{L^2(\mathbf{R}\times\mathbf{R}^{n-1})}$ by the quantity

$$\|f\|^2_{W^{1/2,0}(\mathbf{R}\times\mathbf{R}^{n-1})} \approx \int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}} |\hat{f}(\xi_0, x')|^2 |\xi_0|\, d\xi_0 dx'$$

where $\hat{f}(\xi_0, x') = \int_{\mathbf{R}} e^{-ix_0\xi_0} f(x_0, x')\, dx_0$ is the partial Fourier transform in the x_0 variable. This second expression for the $W^{1/2,0}(\mathbf{R})$-seminorm may be found in [St, p. 139]. From this, it follows that we have $\mathscr{L}_1^2(\mathbf{R}\times\mathbf{R}^{n-1}) = W^{1/2,1}(\mathbf{R}\times\mathbf{R}^{n-1})$.

Next, we need to make the appropriate definition of $\mathscr{L}_1^1$. To do this, we first describe the atomic Hardy space on $H^1(\mathbf{R}\times\mathbf{R}^{n-1})$. Recall that a is an atom if a satisfies:

$$\operatorname{supp} a \subset I_r(X'), \text{ for some cube } I_r(X'), \tag{1.3a}$$

$$\|a\|_{L^2} \le r^{-(n+1)/2}, \tag{1.3b}$$

$$\int a\, dX' = 0. \tag{1.3c}$$

We define $H^1(\mathbf{R}\times\mathbf{R}^{n-1})$ to be the l^1-span of the atoms. This means that f is in $H^1(\mathbf{R}\times\mathbf{R}^{n-1})$ if and only if there exists a sequence of atoms $\{a_j\}$ and real numbers $\{\lambda_j\}$ satisfying $\sum|\lambda_j|<\infty$ such that $f=\sum\lambda_j a_j$. The norm of f in H^1 is given by

$$\|f\|_{H^1(\mathbf{R}\times\mathbf{R}^{n-1})} = \inf\sum|\lambda_j|$$

where the infinum is taken over all atomic decompositions of f. The definition given for $H^1(\mathbf{R}\times\mathbf{R}^{n-1})$ works equally well for $H^1(S)$ or $H^1(\Sigma_+)$.

We also recall the definition of a molecule for H^1. We say that A is molecule if for some $\bar{X}'\in\mathbf{R}\times\mathbf{R}^{n-1}$ and $\eta>0$, A satisfies

(1.4a)
$$\int_{\mathbf{R}\times\mathbf{R}^{n-1}} A(X')^2\, dX' \left(\int_{\mathbf{R}\times\mathbf{R}^{n-1}} A(X')^2 \delta(X';\bar{X}')^{(n+1)(1+\eta)}\, dX'\right)^{1/\eta} \le C(A)^{2+\frac{2}{\eta}},$$

$$\int_{\mathbf{R}\times\mathbf{R}^{n-1}} A(X')\, dX' = 0. \tag{1.4b}$$

It is well known that if A satisfies (1.4a), then $A\in L^1$ so that the second condition is sensible. Also, if A is a molecule, then $A\in H^1(\mathbf{R}\times\mathbf{R}^{n-1})$ and $\|A\|_{H^1}\le C_{n,\eta}C(A)$ (see [CW]).

We now define $\mathscr{L}_1^1(\mathbf{R}\times\mathbf{R}^{n-1}) = \{f: f=\Lambda(g),\ g\in H^1(\mathbf{R}\times\mathbf{R}^{n-1})\}$ and as before, we put $\mathscr{L}_1^1(S) = \{f: f\circ\pi^{-1}\in\mathscr{L}_1^1(\mathbf{R}\times\mathbf{R}^{n-1})\}$. The norms in these spaces are given by

$$\|\Lambda(g)\circ\pi^{-1}\|_{\mathscr{L}_1^1(S)} \equiv \|\Lambda(g)\|_{\mathscr{L}_1^1(\mathbf{R}\times\mathbf{R}^{n-1})} \equiv \|g\|_{H^1(\mathbf{R}\times\mathbf{R}^{n-1})}.$$

For applications, it will be necessary to have an atomic decomposition of this space. We first define the atoms. We say that function b defined on S is an atom which is parabolically smooth to order one (for short, a 1-atom) if b satisfies

$$\operatorname{supp} b\subset I_r(Q), \text{ for some cube on } S, \tag{1.5a}$$

$$\|b\|_{\mathscr{L}_1^2(S)} \le r^{-(n+1)/2}. \tag{1.5b}$$

In order to justify this definition, we will need the following proposition.

Proposition 1.6. *Let b be an* 1*-atom, then $b \in \mathscr{L}_1^1(S)$ and*

$$\|b\|_{\mathscr{L}_1^1(S)} \leq C.$$

Proof. It suffices to establish this proposition on $\mathbf{R} \times \mathbf{R}^{n-1}$. Thus suppose that b is a 1-atom supported in $I_r(\bar{X}') \subset \mathbf{R} \times \mathbf{R}^{n-1}$. Since $b \in \mathscr{L}_1^2$, we know that $b = \Lambda(g)$ with $\|g\|_{L^2} \leq r^{-(n+1)/2}$. To establish our proposition, it suffices to show that g is a molecule for H^1 with a bound for $C(g)$ depending only on the dimension. This follows easily from the representation of $g = \Lambda^{-1}(b)$ and Theorem 1.2. □

Our next theorem states that $\mathscr{L}_1^1$ is equal to the l^1-span of the 1-atoms.

Theorem 1.7. *A function f is $\mathscr{L}_1^1(S)$ if and only if there exists a decomposition of f into* 1*-atoms, $f = \sum \lambda_j b_j$ with $\sum |\lambda_j| < \infty$. Furthermore, the norm of f in $\mathscr{L}_1^1$ is comparable to $\inf \sum |\lambda_j|$ where the infinum is taken over all decompositions of f into* 1*-atoms.*

The proof of this result is simple. If $f = \Lambda(g)$, with $g \in H^1(\mathbf{R} \times \mathbf{R}^{n-1})$ we take an atomic decomposition of $g = \sum \lambda_j a_j$ satisfying $\sum |\lambda_j| \leq 2\|g\|_{H^1}$. Then we show that $B_j = \Lambda(a_j)$ is a 'molecule' for $\mathscr{L}_1^1$. The only impediment to carrying out this exercise is that we have not yet defined molecules for $\mathscr{L}_1^1$. Thus, after a glance at the behavior of $\Lambda(a)$, we say that B is a 1-molecule for $\mathscr{L}_1^1(S)$ if for some $Q \in S$, $\eta > 0$, $r > 0$, and $C(B) < \infty$, B satisfies

$$\|B\|_{W^{1/2,1}(I_{4r}(Q))} + r^{-1}\|B\|_{L^2(I_{4r}(Q))} \leq C(B) r^{-(n+1)/2}, \tag{1.8a}$$

$$\|B\|_{W^{1/2,1}(I_\rho(P))} + \rho^{-1}\|B\|_{L^2(I_\rho(P))} \leq C(B) \rho^{-(n+1)/2} \left(\frac{\rho}{r}\right)^{-\eta}, \quad \text{if } \delta(I_\rho(P)\,;\, I_r(Q)) \geq \rho. \tag{1.8b}$$

As expected, we have

Proposition 1.9. *Let B be a* 1*-molecule, then $B \in \mathscr{L}_1^1(S)$ and*

$$\|B\|_{\mathscr{L}_1^1(S)} \leq C_\eta C(B).$$

Proof. Again, it suffices to prove this proposition on $\mathbf{R} \times \mathbf{R}^{n-1}$. Let B be a molecule on $\mathbf{R} \times \mathbf{R}^{n-1}$ with center $\bar{X}'$ and radius r and assume that $C(B) = 1$. We begin by choosing a partition of unity, $1 = \sum_{k \geq 0,\, j=1,\ldots,N} \psi_{k,j}$ which satisfies: (1) Each $\psi_{k,j}$ is supported in a surface cube of sidelength $2^k r$, (2) $\operatorname{supp} \psi_{0,j} \subset I_{4r}(\bar{X}')$ for $j = 1, \ldots, N$, (3) when $k \geq 1$, $\delta(\operatorname{supp} \psi_{k,j}\,;\, I_r(\bar{X}')) \geq 2^k r$, and (4) for all k and j,

$$|\psi_{k,j}| \leq 1 \quad \text{and} \quad |\nabla' \psi_{k,j}| + 2^k r |\partial_{x_0} \psi_{k,j}| \leq C 2^{-k} r^{-1}.$$

We claim that there exists a constant $c > 0$ such that for all k and j, $b_{k,j} \equiv c2^{\eta k}\psi_{k,j}B$ is a 1-atom. This claim and Proposition 1.6 establish our result.

To prove the claim, we observe that each $b_{k,j}$ is supported in a cube of sidelength $2^k r$, thus we need to show that

$$\|\psi_{k,j}B\|_{\mathscr{L}_1^2(\mathbf{R}\times\mathbf{R}^{n-1})} \le C2^{-\eta k}(2^k r)^{-(n+1)/2}.$$

Using (1.8b), this estimate follows immediately from the following fact: Let ψ be supported in $I_\rho(X')$ and suppose that $|\psi| \le 1$ and $|\nabla'\psi| + \rho|\partial_{x_0}\psi| \le C\rho^{-1}$, then for $f \in W^{1/2,1}(I_\rho(X'))$, we have

$$\|\psi f\|_{\mathscr{L}_1^2(\mathbf{R}\times\mathbf{R}^{n-1})} \approx \|\psi f\|_{W^{1/2,1}(\mathbf{R}\times i_\rho(x'))} \le C(\|f\|_{W^{1/2,1}(I_\rho(X'))} + \rho^{-1}\|f\|_{L^2(I_\rho(X'))}). \tag{1.10}$$

The comparability is just the Remark after Theorem 1.2. To establish the inequality, only the estimate of the $W^{1/2,0}(\mathbf{R}\times i_\rho(x'))$ seminorm presents any difficulty. To establish the desired estimate,

$$\|\psi f\|_{W^{1/2,0}(\mathbf{R}\times i_\rho(x'))} \le C(\|f\|_{W^{1/2,0}(I_\rho(X'))} + \rho^{-1}\|f\|_{L^2(I_\rho(X'))}),$$

one notes that $f \to \psi f$ is bounded as a map on $L^2(I_\rho(X')) \to L^2(\mathbf{R}\times i_\rho(x'))$ and $L^2(I_\rho(X'))\cap W^{1,0}(I_\rho(X')) \to W^{1,0}(\mathbf{R}\times i_\rho(x'))$ and then interpolates. The stated dependence on ρ follows by rescaling. □

We are now ready to give

Proof of Theorem 1.7. Suppose that $f = \sum \lambda_j b_j$ where each b_j is a 1-atom and $\sum|\lambda_j| < \infty$. Proposition 1.6 guarantees that $f \in \mathscr{L}_1^1(S)$ and $\|f\|_{\mathscr{L}_1^1(S)} \le C\sum_{j=1}^{\infty}|\lambda_j|$.

To prove the converse, it suffices to work on $\mathbf{R}\times\mathbf{R}^{n-1}$. In fact, all we need to show is that if a is an atom for H^1, then $\Lambda(a)$ is a 1-molecule with $C(\Lambda(a)) \le C_n$. Thus suppose that a is an atom supported in $I_r(\bar{X}')$. Since $\|a\|_{L^2} \le r^{-(n+1)/2}$, the Remark after Theorem 1.2 implies that $\|\Lambda(a)\|_{W^{1/2,1}(I_{4r}(\bar{X}'))} \le Cr^{-(n+1)/2}$. The estimate $\|\Lambda(a)\|_{L^2(I_{4r}(\bar{X}'))} \le Cr^{(1-n)/2}$ follows from the embedding $\mathscr{L}_1^2 \subset L^{2n+2/(n-1)}$ and Hölder's inequality.

Before giving estimates for $\Lambda(a)$ off the support of a, we recall a simple interpolation inequality valid for functions f defined on an interval $(a, b) \subset \mathbf{R}$:

$$\|f\|_{W^{1/2}(a,b)} \le C\|f\|_{L^2(a,b)}^{1/2}\|\partial_t f\|_{L^2(a,b)}^{1/2}. \tag{1.11}$$

Next we claim that

$$|\Lambda(a)(X')| + \delta(X';\bar{X}')|\nabla'\Lambda(a)(X')| + \delta(X';\bar{X}')^2|\partial_{x_0}\Lambda(a)(X')| \le Cr\delta(X';\bar{X}')^{-n-1}, \qquad \delta(X';I_r(\bar{X}')) \ge r. \tag{1.12}$$

We prove the estimate for $\Lambda(a)(X')$, the other estimates follow from identical arguments. Since a has mean zero, for $\delta(X'; I_r(\bar{X}')) \geq r$ we have

$$|\Lambda(a)(X')| \leq \int_{I_r(\bar{X}')} |[\lambda(X'-Y')-\lambda(X'-\bar{X}')]a(Y')|\,dY' \leq Cr\|a\|_{L^1}\delta(X'; \bar{X}')^{-n-1}.$$

The second inequality follows from the estimate (1.1b) for λ.

Now let $I_\rho(X')$ satisfy $\delta(I_\rho(X'); I_r(\bar{X}')) \geq \rho$ where $\rho > r$. Using the interpolation inequality (1.11) and (1.12), we have

$$\begin{aligned}\|\Lambda(a)\|_{W^{1/2,0}(I_r(\bar{X}'))} &\leq \|\Lambda(a)\|^{1/2}_{L^2(I_r(\bar{X}'))}\|\partial_{x_0}\Lambda(a)\|^{1/2}_{L^2(I_r(\bar{X}'))}\\ &\leq C(r\rho^{-n-1})^{1/2}(r\rho^{-n-3})^{1/2} = C\rho^{-n-1}r/\rho,\end{aligned}$$

which is the bound in (1.8b) of the definition of a 1-molecule with $\eta = 1$. The estimates for $\Lambda(a)$ and $\nabla'\Lambda(a)$ are easier. $\square$

To define the spaces $\mathscr{L}^p_1(\Sigma_T)$, we introduce a partition of unity on $\partial\Omega$, $1 = \sum_{i=1}^N \psi_i(p)$ where each ψ_i is Lipschitz and $\operatorname{supp}\psi_i \subset 2Z_i$. For a function f defined on $(0, \infty) \times \partial\Omega$, we let $\tilde{f}$ denote the extension by zero to $\mathbf{R} \times \partial\Omega$,

$$\tilde{f}(Q) = \begin{cases} f(Q), & q_0 > 0, \\ 0, & q_0 \leq 0. \end{cases}$$

Now we may define

$$\mathscr{L}^p_1(\Sigma_+) = \{f : (\psi_i\tilde{f}) \circ \pi_i^{-1} \in \mathscr{L}^p_1(\mathbf{R} \times \mathbf{R}^{n-1}),\ i = 1, \dots N\}$$

and

$$\mathscr{L}^p_1(\Sigma_T) = \{f : f = \chi_{(0,T)} \cdot g,\ g \in \mathscr{L}^p_1(\Sigma_+)\}.$$

The norms are given by

$$\|f\|^p_{\mathscr{L}^p_1(\Sigma_+)} = \sum_{i=1}^N \|(\tilde{f}\psi) \circ \pi_i^{-1}\|^p_{\mathscr{L}^p_1(\mathbf{R}\times\mathbf{R}^{n-1})}$$

and

$$\|f\|_{\mathscr{L}^p_1(\Sigma_T)} = \inf\{\|g\|_{\mathscr{L}^p_1(\Sigma_+)} : g(q_0, q) = f(q_0, q) \text{ for } 0 < q_0 < T\}.$$

We conclude our discusssion of the spaces $\mathscr{L}^p_1$ by observing that it is easy to give an atomic decomposition of the space $\mathscr{L}^1_1(\Sigma_T)$. To do this, consider $(\psi_i f) \circ \pi_i^{-1} = g \in \mathscr{L}^1_1((0, T) \times \mathbf{R}^{n-1})$ and observe that g has an extension $\tilde{g} \in \mathscr{L}^1_1(\mathbf{R} \times \mathbf{R}^{n-1})$ which is zero for $x_0 < 0$. Applying Theorem 1.7, we have $\tilde{g} = \sum \lambda_j \tilde{b}_j$ with the 1-atoms $\tilde{b}_j$ supported in $x_0 \geq 0$. Choose η a Lipschitz function on $\mathbf{R} \times \mathbf{R}^{n-1}$ which is supported in $\pi_i((-T, 2T) \times (4Z_i \cap \partial\Omega))$ and which is identically one on $\pi_i((0, T) \times (2Z_i \cap \partial\Omega))$. It is easy to see that $b_j \equiv c\eta\tilde{b}_j$ is a 1-atom if we choose $c = c'(1+\max(T, r_0))^{-1}$. Hence, restricting to $\pi_i((0, 2T) \times (4Z_i \cap \partial\Omega))$, we have $g = \eta g = \sum c\lambda_i b_j$. Since the 1-atoms

b_j are supported in $\pi_i((0, 2T) \times (4Z_i \cap \partial\Omega))$, we may use the map π_i^{-1} to transfer this decomposition to Σ_T.

We now give the definitions of the spaces $\widetilde{BMO}(\Sigma_T)$ and $\tilde{H}^1(\Sigma_T)$ which will provide data for the endpoint cases of the initial-Dirichlet problem and the initial-Neumann problem, respectively. We first recall the definition of $BMO(\Sigma_T)$. A locally integrable function f on Σ_T is in $BMO(\Sigma_T)$ if

$$|I|^{-1} \int_I |f(Q) - f_I| \, dQ \le C \tag{1.13}$$

for all surface cubes $I \subset \Sigma_T$. In this definition, $f_I \equiv |I|^{-1} \int_I f \, dQ$ denotes the average of f over I. The BMO-norm of f, $\|f\|_{BMO(\Sigma_T)}$, is the smallest constant for which (1.13) holds.

The space $BMO(\Sigma_T)$ is too large for our purposes; to solve the initial-Dirichlet problem, we need to impose some restriction on the behavior near $x_0 = 0$. This restriction may be viewed as a compatibility condition with the homogeneous initial data. With this motivation, we let

$$\widetilde{BMO}(\Sigma_T) = \{f : f = g|_{\Sigma_T}, \text{ with } g \in BMO(\mathbf{R} \times \partial\Omega) \text{ and } g = 0 \text{ for } x_0 < 0\},$$

$$\tilde{H}^1(\Sigma_T) = \{f : f = g|_{\Sigma_T}, \text{ with } g \in H^1(\mathbf{R} \times \partial\Omega) \text{ and } g = 0 \text{ for } x_0 < 0\}.$$

The norms are given by

$$\|f\|_{\widetilde{BMO}(\Sigma_T)} = \inf_{\{g \,:\, f = g|_{\Sigma_T}\}} \|g\|_{BMO(\Sigma_T)}$$

and a similar definition for $\tilde{H}^1(\Sigma_T)$.

We compare these spaces with the standard spaces. From our definition, we see that functions in $\widetilde{BMO}(\Sigma_T)$ extend by zero to functions in $BMO((-\infty, T) \times\partial\Omega)$. In fact, one can see that

$$\begin{aligned} \|f\|_{\widetilde{BMO}(\Sigma_T)} &\approx \|\tilde{f}\|_{BMO((-\infty, T)\times\partial\Omega)} \\ &\approx \|f\|_{BMO(\Sigma_T)} + \sup_{\substack{I_r \subset \Sigma_T, \\ \delta(I_r; \{x_0 = 0\}) < r}} |f_{I_r}| \end{aligned}$$

where $\tilde{f}$ is the extension by zero of f. That is $\tilde{f} = f$ if $x_0 > 0$ and $\tilde{f} = 0$ if $x_0 \le 0$. This last expression is the norm used by [GNR] to define their space $B_0MOC(\Sigma_T)$. Hence, $\widetilde{BMO}(\Sigma_T) = B_0MOC(\Sigma_T)$.

On the other hand, the space $\tilde{H}^1(\Sigma_T)$ is easily seen to be the l^1-span of the restrictions to Σ_T of atoms supported on Σ_+. Thus this space is strictly larger that $H^1(\Sigma_T)$. For example, we have $L^2(\Sigma_T) \subset \tilde{H}^1(\Sigma_T)$ as long as $T < \infty$.

Our next task is to identify the dual of $\tilde{H}^1(\Sigma_T)$. To do this, we introduce the map $R(f)(p_0, p) = f(T - p_0, p)$ induced by reflection. Using R, we define a pairing

$$\langle f, g\rangle = \int_{\Sigma_T} f(Q) \, R(g)(Q) \, dQ, \qquad (f, g) \in \tilde{H}^1(\Sigma_T) \times \widetilde{BMO}(\Sigma_T). \tag{1.14}$$

We claim that the map $g \to \langle\cdot, g\rangle$ from $\widetilde{BMO}(\Sigma_T) \to \tilde{H}^1(\Sigma_T)^*$ is an isomorphism. To see this, we consider the case where f is an atom, a, with $\operatorname{supp} a \subset \Sigma_+$. Letting $\tilde{g}$ be the extension by zero of g, we have

$$|\langle a, g\rangle| = \left|\int_{\Sigma_+} a(Q)R(\tilde{g})(Q)\,dQ\right| \leq C\|\tilde{g}\|_{BMO((-\infty, T)\times\partial\Omega)} \leq C'\|g\|_{\widetilde{BMO}(\Sigma_T)}.$$

Hence, $\langle\cdot, \cdot\rangle$ which is initially defined say for $(f, g) \in (L^2 \cap \tilde{H}^1) \times \widetilde{BMO}$, may be extended to $\tilde{H}^1$ in the first variable and we have $\widetilde{BMO}(\Sigma_T) \subset \tilde{H}^1(\Sigma_T)^*$.

To establish the other inclusion, suppose $\lambda \in \tilde{H}^1(\Sigma_T)^*$. Then by restricting the domain of λ, λ induces a continuous linear functional on $H^1(\Sigma_T)$. Hence, there exists $g \in BMO(\Sigma_T)$, uniquely defined modulo constants, so that

$$\lambda(f) = \int_{\Sigma_T} f R(g)\,dQ, \qquad f \in H^1(\Sigma_T).$$

We wish to show that (some representative of) g lies in $\widetilde{BMO}(\Sigma_T)$. To do this, let $I_r(Q) \subset \Sigma_T$ be a cube which satisfies $\delta(I_r(Q); \{p_0 = T\}) < r$ and observe that $v = |I_r(Q)|^{-1}\chi_{I_r(Q)} \in \tilde{H}^1(\Sigma_T)$ and $\|v\|_{\tilde{H}^1(\Sigma_T)} \leq C$. Choosing a representative of g so that

$$\lambda(v) = |I_r(Q)|^{-1}\int_{I_r(Q)} R(g)\,dQ$$

holds for one cube $I_r(Q)$. It is easy to see that then $\lambda(f) = \langle f, g\rangle$ must hold for all $f \in \tilde{H}^1(\Sigma_T)$. Now as λ is continuous, it follows that the averages of $R(g)$ near $\{x_0 = T\}$, equivalently, the averages of g near $\{x_0 = 0\}$ are bounded. Thus $g \in \widetilde{BMO}(\Sigma_T)$ as desired.

Finally, in §6 we will need to know that $\tilde{H}^1(\Sigma_T)$ is a dual of some Banach space. We recall that in [CW, §4], VMO is defined to be the closure of C_0 in BMO. Similarly, we define $\widetilde{VMO}(\Sigma_T)$ to be the closure of $C_0(\Sigma_T)$ in $\widetilde{BMO}(\Sigma_T)$. It is easy to see that with this definition, $\widetilde{VMO}(\Sigma_T)$ consists of those functions in $VMO(\Sigma_T)$ which extend by zero to lie in $VMO((-\infty, T) \times \partial\Omega)$ and under the map $f \to \langle f, \cdot\rangle$, $\widetilde{VMO}(\Sigma_T)^* = \tilde{H}^1(\Sigma_T)$.

2. Statements of the boundary value problems and review of the L^2-theory

In this section, we summarize the results of [B1] for the Neumann problem with data in L^2. We remark that this paper only treats the more complicated situation where the domain is a bounded Lipschitz domain. The reader may consult the paper of Dahlberg and Kenig [DK] for the simplifications that arise when D is the region lying above the graph of a Lipschitz function.

We begin by studying the Neumann problem on graph domains. This is the problem of finding a smooth function in D_∞ which satisfies

$$\text{(NP)} \qquad \begin{cases} \partial_{x_0} u - \Delta u = 0, & \text{in } D_\infty \\ \partial_\nu u = f, & \text{on } S \end{cases}$$

where the datum f is specified. The restriction of ∇u to S is defined by means of parabolic limits:

$$\lim_{\Gamma(P,\alpha)\ni Y\to P} \nabla u(Y) \equiv \nabla u(P)$$

whenever this limit exists. The normal derivative of u at the boundary is then defined by $\partial_\nu u(P) = \langle \nabla u(Q), \nu(q)\rangle$ where $\nu(q)$ is the unit inner normal to ∂D at the point q and $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbf{R}^n$. We remark that we will always require solutions of this problem to satisfy $N_\alpha(\nabla u) \in L^p(S)$ for some p and α. This guarantees that the parabolic limits of ∇u exist a.e. with respect to surface measure on S (see [K, FS, B2]).

Theorem 2.1. *Let $f \in L^2(S)$, then there exists a solution of* (NP) *with datum f. This solution satisfies*

$$\|u\|^2_{W^{1/2,0}(S)} + \int_S N_\alpha(\nabla u)(P)^2\, dP \le C_{\alpha,m} \int_S f(P)^2\, dP.$$

Furthermore, this solution is unique in the class of functions with $N_\alpha(\nabla u) \in L^2(S)$.

Remark. On the unbounded domain D_∞, only the existence follows from the techniques in [B1], uniqueness will be established in Proposition 3.12.

We will also need a version of Theorem 2.1 for bounded domains. On the cylinders Ω_T, we study the initial-Neumann problem

$$\text{(INP)} \qquad \begin{cases} \partial_{x_0} u - \Delta u = 0, & \text{in } \Omega_T \\ \partial_\nu u = f, & \text{on } \Sigma_T \\ u(0, x) = 0, & x \in \Omega. \end{cases}$$

In [B1], the following is established:

Theorem 2.2. *Let $f \in L^2(\Sigma_T)$, then there exists a solution of* (INP) *with datum f which satisfies*

$$\|u\|^2_{W^{1/2,0}(\Sigma_T)} + \int_{\Sigma_T} N_\alpha(\nabla u)(P)^2 \le C \int_{\Sigma_T} f(P)^2\, dP$$

where $C = C(\alpha, T/r_0^2, m, \{Z_i\})$. Furthermore, this solution is unique in the class of functions with $N_\alpha(\nabla u) \in L^2(\Sigma_T)$.

We will need the following extension which gives estimates for caloric functions which do not have zero initial values. For this corollary, we introduce the truncated maximal function

$$N_{\alpha,r}(v)(P) \equiv \sup_{\Gamma_r(P,\alpha)} |v(Y)|$$

where $\Gamma_r(P, \alpha) = \Gamma(P, \alpha) \cap \{Y : \delta(Y; P) < r\}$. We also recall that the fundamental solution of the heat operator in $\mathbf{R}^{n+1}$ is given by

$$W(X) = \begin{cases} (4\pi x_0)^{-n/2} \exp(-|x|^2/4x_0), & x_0 > 0, \\ 0, & x_0 \le 0. \end{cases} \tag{2.3}$$

In order to make clear the dependence of the constants in the next Corollary, we restrict our attention to starshaped Lipschitz domains.

Corollary 2.4. *Let Ω be a starshaped Lipschitz domain of diameter r and suppose that $2r^2 < T$. Let u be caloric in Ω_T with $N_\alpha(\nabla u) \in L^2(\Sigma_T)$. Then $\partial_\nu u$ exists (in the sense of parabolic limits) and for any $\beta \in \mathbf{R}$, we have the estimate*

$$\|u\|^2_{W^{1/2,0}((2r^2, T)\times\partial\Omega)} + \int_{2r^2}^{T} \int_{\partial\Omega} N_{\alpha, r}(\nabla u)(P)^2 \, dP \le C \left(\int_{\Sigma_T} \partial_\nu u(P)^2 \, dP + r^{-3} \int_{\Omega_T} (u(X) - \beta)^2 \, dX \right)$$

where $C = C(T/r^2, \alpha, m)$.

Proof. We may assume that $\beta = 0$ and, by rescaling, we may assume that $r = 1$. We choose a function $\psi : (0, T) \to [0, 1]$ which is smooth and satisfies $\psi(x_0) = 0$ if $x_0 < 1/3$ and $\psi(x_0) = 1$ if $x_0 > 2/3$. For $X \in \Omega_T$, we define $f(X) = (\partial_{x_0} - \Delta)\psi(x_0)u(X) = \psi'(x_0)u(X)$ and let $u_1 = W * f$. Since $\nabla^2 u_1 \in L^2(\mathbf{R}^{n+1})$, it is easy to see that

$$\int_{\Sigma_T} |\nabla u_1(P)|^2 \, dP \le C \int_{\Omega_T} f(X)^2 \, dX \le C' \int_{\Omega_T} u(X)^2 \, dX. \tag{2.5}$$

Since $\psi'(x_0) = 0$ for $x_0 \ge 2/3$, it is trivial that

$$\|u_1\|^2_{W^{1/2,0}((2, T)\times\partial\Omega)} \le C \int_{\Omega_T} u(X)^2 \, dX \tag{2.6}$$

and

$$|\nabla u_1(X)|^2 \le C \int_{\Omega_T} u(X)^2 \, dX, \qquad x_0 \ge 1. \tag{2.7}$$

We define u_2 to be the solution of (INP) with data $\psi\partial_\nu u - \partial_\nu u_1$ on Σ_T. From Theorem 2.2, we have the estimate

$$\|u_2\|^2_{W^{1/2,0}(\Sigma_T)} + \int_{\Sigma_T} N_{\alpha, r}(\nabla u_2)(P)^2 \, dP \le C \int_{\Sigma_T} \partial_\nu u(P)^2 + \partial_\nu u_1(P)^2 \, dP.$$

Now as $\psi u = u_1 + u_2$, our corollary follows from (2.5), (2.6) and (2.7) after we note that when $q_0 > 2$, then $\Gamma_1(Q, \alpha) \subset \{X : x_0 > 1\}$ where $u = \psi u$. □

Remark. The uniqueness statement implicit in the proof of the above corollary requires a minor modification of the argument given in [B1, Corollary 2.11]. What is needed is to show that the energy estimate is valid for functions $u + v$

where u is caloric with $N_\alpha(\nabla u) \in L^2$ and $v \in W^{1,2}(\Omega_T)$. This is easily established by approximating v in the norm of $W^{1,2}(\Omega_T)$ by smooth functions.

We now state the Dirichlet problem for our cylinders D_∞ and the initial-Dirichlet problem for the cylinders Ω_T.

For a function f defined on S, we seek a solution of

$$\text{(DP)} \qquad \begin{cases} \partial_{x_0} u - \Delta u = 0, & \text{in } D_\infty \\ u = f, & \text{on } S \end{cases}$$

and for f defined on Σ_T, we study solutions of

$$\text{(IDP)} \qquad \begin{cases} \partial_{x_0} u - \Delta u = 0, & \text{in } \Omega_T \\ u = f, & \text{on } \Sigma_T \\ u(0, x) = 0, & x \in \Omega. \end{cases}$$

where the restriction of u to the lateral boundary is defined using parabolic limits.

When the data is in $\mathscr{L}_1^2$, we have the following result for (DP).

Theorem 2.8. *Let $f \in \mathscr{L}_1^2(S)$, then there exists a solution u of the Dirichlet problem with datum f which satisfies the estimate*

$$\int_S N_\alpha(\nabla u)(P)^2 \, dP \le C_{\alpha, m} \|f\|^2_{\mathscr{L}_1^2(S)}.$$

Furthermore, this solution is unique in the class of functions with $N_\alpha(\nabla u) \in L^2(S)$.

Again, the uniqueness will be established in Proposition 3.12, the existence is established by adapting the argument given in [B1] for bounded domains. We also have a version of Theorem 2.8 for bounded domains.

Theorem 2.9. *Let $f \in \mathscr{L}_1^2(\Sigma_T)$, then there exists a unique solution of* (IDP) *with datum f which satisfies*

$$\int_{\Sigma_T} N_\alpha(\nabla u)(P)^2 \, dP \le C \|f\|^2_{\mathscr{L}_1^2(\Sigma_T)}$$

where $C = C(\alpha, m, T/r_0^2)$.

For future applications, we record the following extension of Theorem 2.9.

Corollary 2.10. *Let Ω be a starshaped Lipschitz domain of diameter r. Let u be caloric in Ω_T and assume that $N_\alpha(\nabla u) \in L^2(\Sigma_T)$. Then for any $\beta \in \mathbf{R}$, we have*

$$\int_{2r^2}^T \int_{\partial\Omega} N_{\alpha, r}(\nabla u)(P)^2 \, dP \le C \left(\|u\|^2_{W^{1/2,1}((2r^2, T) \times \partial\Omega)} + r^{-3} \int_{\Omega_T} (u(X) - \beta)^2 \, dX \right)$$

where $C = C(\alpha, m, T/r^2, \{Z_i\})$.

The proof of this corollary is omitted since it is similar to that of Corollary 2.4.

3. Estimates for solutions with atomic data

In this section, we study solutions of the Neumann problem when the data is an atom. Using the L^2-theory outlined in §2 and well-known results from the theory of divergence form second order parabolic operators, we are able to deduce L^1-estimates for the parabolic maximal function of the spatial gradient. We also show that the restriction to S of solutions to (NP) with atomic data are 1-molecules. A parallel result is obtained for solutions of the Dirichlet problem when the datum is a 1-atom.

Since an atom lies in $L^2(S)$, the existence of a solution to (NP) for any atom follows from Theorem 2.1. Our first result gives more precise estimates for this solution.

Lemma 3.1. *Let a be an atom with support in $I_r(Q)$. Let u be the solution of* (NP) *with datum a. Then there exist constants $C = C(\alpha, m)$ and $\eta = \eta(m) > 0$ such that*

$$\int_S N_\alpha(\nabla u)(P)\, dP \leq C, \tag{3.2}$$

$$\int_S N_\alpha(\nabla u)(P)^2\, dP \times \left(\int_S N_\alpha(\nabla u)(P)^2 \delta(P; Q)^{(n+1)(1+\eta)}\, dP\right)^{1/\eta} \leq C, \tag{3.3}$$

(3.4) $\quad B \equiv u|_S$ *is a 1-molecule whose constant satisfies* $C(B) \leq C_m$

Our main tool in proving these estimates will be the theory of weak solutions of divergence form parabolic equations. We briefly recall what we will need. Let L denote a divergence form parabolic operator,

$$L = \partial_{x_0} - \operatorname{div} a\nabla.$$

We assume that the coefficient matrix $a = (a_{ij}(x))_{i,j=1,\ldots,n}$ has real bounded measurable entries and satisfies the ellipticity condition

$$\lambda^{-1}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2, \qquad \xi \in \mathbf{R}^n \text{ and a.e. } x \in \mathbf{R}^n.$$

where we are using the summation convention. We say that function u satisfies $Lu = 0$ in some open set $\mathscr{O} \subset \mathbf{R}^{n+1}$ if $\nabla u \in L^2_{\mathrm{loc}}(\mathscr{O})$ and

$$\int_{\mathscr{O}} (a_{ij}(x)\partial_{x_j} u(X)\partial_{x_i}\psi(X) - \partial_{x_0}\psi(X)u(X))\, dX = 0,$$

for ψ which are Lipschitz and compactly supported in $\mathscr{O}$.

To state several facts about solutions of these operators, we introduce the parabolic box $J_r(X) \equiv (x_0 - r^2, x_0) \times B(x, r)$ where $B(x, r)$ is the ball in $\mathbf{R}^n$ with center x and radius r. We first recall an L^∞-estimate. Let u satisfy $Lu = 0$ in $J_{2r}(X)$, then for $0 < p < \infty$

$$\sup_{Y \in J_r(X)} |u(Y)| \leq C_\lambda \left(\frac{1}{r^{n+2}} \int_{J_{2r}(X)} |u(Y)|^p\right)^{1/p}. \tag{3.5}$$

For $1 < p < \infty$, this was established in [M]. The extension to $p \leq 1$ uses an argument of Dahlberg and Kenig which may be found in [FSt, pp. 1004–1005]. Next we recall that if $Lu = 0$ in $J_R(X)$, then u is locally Hölder continuous there, [M]. Letting $\operatorname{osc}_J u = \sup_J u - \inf_J u$, we have the estimate

$$\operatorname*{osc}_{J_r(X)} u \leq C(r/R)^\eta \sup_{J_R(X)} u, \qquad \text{for } r \leq R \tag{3.6}$$

where C and $\eta > 0$ depend only on λ. We recall that Aronson [A1, Theorem 1] has constructed a fundamental solution $E(X; Y)$ for L and shown that for some $c = c_\lambda$, it satisfies

$$\begin{aligned} &c(x_0 - y_0)^{-n/2} \exp\left(\frac{-|x-y|^2}{c(x_0 - y_0)}\right) \leq E(X; Y) \\ &\leq c^{-1}(x_0 - y_0)^{-n/2} \exp\left(\frac{-c|x-y|^2}{(x_0 - y_0)}\right), \qquad x_0 > y_0. \end{aligned} \tag{3.7}$$

If we set $E(X; Y) = 0$ for $x_0 \leq y_0$, then $E(X; \cdot)$ is a solution of the adjoint equation $L^* = \partial_{x_0} + \operatorname{div} a^t \nabla$ in $\mathbf{R}^{n+1} \setminus \{X\}$. Next, we recall an elementary inequality, due to Cacciopoli in the elliptic case, whose proof may be found in [M]:

$$\int_{J_r(X)} |\nabla u(Y)|^2 \, dY \leq \frac{C}{r^2} \int_{J_{2r}(X)} (u(Y) - \beta)^2 \, dY \tag{3.8}$$

for $\beta \in \mathbf{R}$ and u a solution of $Lu = 0$ in $J_{2r}(X)$.

We take this opportunity to give a version of the Poincaré inequality which holds for solutions of $Lu = 0$. For u a solution in $J_r(X)$, there exists $\beta \in \mathbf{R}$ such that for $1 \leq p < \infty$,

$$\int_{J_r(X)} |u(Y) - \beta|^p \, dY \leq r^p \int_{J_r(X)} |\nabla u(Y)|^p \, dY. \tag{3.9}$$

Recall that ∇ denotes the gradient in the spatial variables. To establish this estimate, assume that $r = 1$ and let $\eta \colon \mathbf{R}^n \to [0, 1]$ be a smooth function which is one on $B_{1/2}(x)$ and is supported in $B_1(X)$. Define $\beta(y_0) = (\int \eta \, dy)^{-1} \int \eta(y) u(y_0, y) \, dy$. Using the usual Poincaré inequality in $\mathbf{R}^n$, it follows that

$$\int_{B_1(x)} |u(y_0, y) - \beta(y_0)|^p \, dy \leq C \int_{B_1(x)} |\nabla u(y_0, y)|^p \, dy. \tag{3.10}$$

We observe that since u is a solution, we have

$$\begin{aligned} &\int_s^t \int_{B_1(x)} a(y_0, y) \nabla \eta(y) \nabla u(y_0, y) \, dy_0 \, dy \\ &= -\int_{B_1(x)} \eta(y)(u(t, y) - u(s, y)) \, dy = \beta(s) - \beta(t). \end{aligned}$$

Hence

$$\sup_{x_0-1<s<x_0} |\beta(x_0) - \beta(s)| \le \int_{J_1(X)} |\nabla u(Y)|\, dY. \tag{3.11}$$

Integrating (3.10) with respect to y_0 and using (3.11), we obtain (3.9) with $\beta = \beta(t)$ for any t satisfying $x_0 - r^2 < t < x_0$. We remark that (3.9) also holds at the boundary of a Lipschitz domain: that is, the inequality is also valid for the domains $\Psi_r(Q)$ instead of $J_r(X)$.

As our first application of these tools, we establish the uniqueness result which was promised in §2.

Proposition 3.12. *Let u be caloric in D_∞ and suppose that $N_\alpha(\nabla u) \in L^p((-\infty, T) \times \partial D)$ for some p satisfying $2 \le p < n+1$. If either $u = 0$ or $\partial_\nu u = 0$ on $S \cap \{x_0 < T\}$ then u is constant in $(-\infty, T) \times D$.*

Proof. For definiteness, we assume that $\partial_\nu u = 0$ on S. To prove the proposition under the assumption that $u = 0$ on S, one needs to replace the even reflection used below by an odd reflection. In this case, we also obtain the conclusion that $u = 0$ on D_∞.

To begin the proof, we use interior estimates to see that

$$\begin{aligned} &\delta(X; S)|\partial_{x_0} u(X)| + |\nabla u(X)| \\ &\quad \le C\delta(X; S)^{-(n+1)/p} \left(\int_{-\infty}^{x_0} \int_{\partial D} N_\alpha(\nabla u)(Q)^p\, dQ \right)^{1/p}. \end{aligned}$$

Hence, after possibly adding a constant to u, we obtain that

$$|u(X)| \le C(x_0)\delta(X; S)^{1-(n+1)/p}$$

where $C(x_0) \to 0$ as $x_0 \to -\infty$. In particular, u is bounded for $\delta(X; S) \ge 1$. Now we reflect u in S to obtain

$$\tilde{u}(x_0, x', x_n) = \begin{cases} u(x_0, x', x_n), & x_n \ge \phi(x'), \\ u(x_0, x', 2\phi(x') - x_n), & x_n < \phi(x'). \end{cases}$$

Since $\partial_\nu u = 0$ on S and $N_\alpha(\nabla u) \in L^2$, it is easy to see that $\tilde{u}$ is a weak solution of

$$\partial_{x_0} \tilde{u} - \operatorname{div} a \nabla \tilde{u} = 0, \quad \text{in } (-\infty, T) \times \mathbf{R}^{n+1}$$

where the matrix $a(x)$ is given by $a(x) = \mathbf{1}_n$ if $x_n \ge \phi(x')$ and

$$a(x) = \begin{pmatrix} \mathbf{1}_{n-1} & (2\nabla'\phi(x'))^t \\ 2\nabla'\phi(x') & 1 + 4|\nabla'\phi(x')|^2 \end{pmatrix}.$$

Now since $\tilde{u}$ is bounded away from S, we may use a variant of the Poincaré inequality, (3.9), and then the L^∞-estimate (3.5) to see that $\tilde{u}$ is bounded by $C(x_0)$ in $\{Y : y_0 < x_0\}$ when $x_0 < T$. Hence, writing

$$\tilde{u}(X) = \int_{\mathbf{R}^n} E(X; y_0, y)\tilde{u}(y_0, y)\, dy, \qquad x_0 > y_0,$$

it follows by letting $y_0 \to -\infty$ that $\tilde{u}(X) \equiv 0$. Here, E denotes the fundamental solution of the operator L. $\square$

Next, we give a fundamental lemma which replaces the asymptotic expansion of Serrin and Weinberger used in Dahlberg and Kenig's study of L^p-boundary value problems for Laplace's equation.

Lemma 3.13. *Let* $f\colon \mathbf{R}^n \to \mathbf{R}$ *satisfy*

$$|f(x)| \le (1+|x|)^{-n-2\eta} \quad \textit{and} \quad \int_{\mathbf{R}^n} f(x)\,dx = 0$$

where η *is the Hölder exponent in* (3.6). *Then the solution of*

$$\text{(IVP)} \qquad \begin{cases} Lu = 0, & \textit{in } (0,\infty)\times\mathbf{R}^n, \\ u(0,x) = f(x), & x \in \mathbf{R}^n, \end{cases}$$

satisfies $|u(X)| \le C(1+\delta(X;0))^{-n-\eta}$.

Proof. We refer the reader to [A2, §4] for a discussion of existence and uniqueness of solutions of (IVP). By the maximum principle, $|u(X)| \le 1$, hence it suffices to show that

$$|u(X)| \le C\delta(X;0)^{-n-\eta}, \qquad x_0 > 0. \tag{3.14}$$

To establish this estimate, we write

$$\begin{aligned} u(X) &= \int_{\mathbf{R}^n} E(X;0,y)f(y)\,dy \\ &= \int_{\{y\,:\,|y|<\delta(X;0)/10\}} [E(X;0,y) - E(X;0)]f(y)\,dy \\ &\quad + E(X;0)\int_{\{y\,:\,|y|<\delta(X;0)/10\}} f(y)\,dy \\ &\quad + \int_{\{y\,:\,|y|>\delta(X;0)/10\}} E(X;0,y)f(y)\,dy \\ &\equiv u_1(X) + u_2(X) + u_3(X). \end{aligned}$$

To estimate $u_1(X)$, we use the Hölder continuity of E in the adjoint variables and the upper bound for E in (3.7) obtaining

$$\begin{aligned} |u_1(X)| &\le C \sup_{\{Y\,:\,\delta(Y;0)<1/2\delta(X;0)\}} E(X;Y)\int_{\mathbf{R}^n}\left(\frac{|y|}{\delta(X;0)}\right)^{\eta}(1+|y|)^{-n-2\eta}\,dy \\ &\le C'\delta(X;0)^{-n-\eta}. \end{aligned}$$

To estimate u_2, we observe that since $\int f = 0$,

$$\left|\int_{\{y\,:\,|y|<\delta(X;0)/10\}} f(y)\,dy\right| \le \int_{\{y\,:\,|y|>\delta(X;0)/10\}} (1+|y|)^{-n-2\eta}\,dy \le C\delta(X;0)^{-\eta}.$$

Hence using the upper bound for E, it follows that

$$|u_2(X)| \le C\delta(X;0)^{-n-\eta}.$$

Finally, since $\int E(X;0,y)\,dy = 1$ for $x_0 > 0$, the estimate

$$|u_3(X)| \leq C\delta(X;0)^{-n-\eta}$$

is immediate. These estimates establish (3.14) and hence the lemma. □

Proof of Lemma 3.1. By rescaling and a translation, we may assume that a is supported in $I_1(0)$. Observe that u is constant for $x_0 < -1$ (see Proposition 3.12); we assume that this constant is zero. We reflect u to obtain $\tilde{u}$ as Proposition 3.12. Since $\partial_\nu u$ vanishes on $S \setminus \overline{I_1(0)}$ and $N_\alpha(\nabla u) \in L^2(S)$, it follows that $\tilde{u}$ is a weak solution of $L\tilde{u} = 0$ in $\mathbf{R}^{n+1} \setminus \overline{I_1(0)}$. We claim that $\tilde{u}$ satisfies

$$|\tilde{u}(X)| \leq C\delta(X;0)^{-n-\eta}, \qquad \text{if } \delta(X; I_1(0)) \geq 10. \tag{3.15}$$

To establish this estimate, we begin by using the argument from the proof of Proposition 3.12 to establish that $\tilde{u}$ is bounded away from $I_1(0)$. To be precise, we have

$$\sup_{J_{\frac{1}{2}}(X)} |\tilde{u}(Y)| \leq C\|N_\alpha(\nabla u)\|_{L^2((-\infty, x_0)\times\partial D)} \tag{3.16}$$

as long as $J_1(X) \cap I_1(0) = \varnothing$. Now we choose η a smooth function which vanishes for X such that $\delta(X; I_1(0)) < 1$ and satisfies $\eta(X) = 1$ when $\delta(X; I_1(0)) \geq 2$. From the above, it is clear that $\eta\tilde{u}$ lies in the class denoted by $\mathscr{E}^2((a,b)\times\mathbf{R}^n)$ in [A2] where (a,b) is a finite interval contained in $\mathbf{R}$. This means that

$$\int_a^b \int_{\mathbf{R}^n} (\eta\tilde{u})(X)^2 e^{-c|x|^2}\,dx\,dx_0 < \infty$$

for some constant c. Hence, from [A2, Theorem 9], we have

$$\begin{aligned}\eta\tilde{u}(X) = \int_{\mathbf{R}^{n+1}} & E(X;Y)(\tilde{u}(Y)\partial_{x_0}\eta(Y) - \langle a(y)\nabla\eta(Y), \nabla\tilde{u}(Y)\rangle) \\ & + \tilde{u}(Y)\langle a(y)\nabla\eta(Y), \nabla_y E(X;Y)\rangle\,dY + \int_{\mathbf{R}^n} E(X;t,y)\tilde{u}(t,y)\,dy\end{aligned} \tag{3.17}$$

where we are assuming that t is large in absolute value and negative. As we let $t \to -\infty$, we see from (3.16) that the last term in (3.17) vanishes. We use the Cauchy-Schwarz inequality and Cacciopoli's inequality, (3.8), in the adjoint variables to estimate the term involving ∇E. Now, the estimate for E, (3.7), implies that

$$|\tilde{u}(Y)| \leq Ce^{-c|x|^2} \qquad \text{for } \delta(X; I_1(0)) > 4 \text{ and } x_0 < 20. \tag{3.18}$$

Finally, using Green's identity we obtain that

$$\int_D u(10,x)\,dx = -\int_{-1}^{10}\int_{\partial D} \partial_\nu u(Q)\,dq\,dq_0 = -\int_S a(Q)\,dQ = 0. \tag{3.19}$$

From (3.18) and (3.19), we see that we may apply Lemma 3.13 to $v(x_0, x) \equiv u(10 + x_0, x)$. The estimate (3.15) follows from Lemma 3.13 and (3.18).

We are ready to begin the proof of the estimates (3.2) and (3.3). We first observe that from the L^2-theory we have

$$\int_{\{P\,:\,\delta(P;I_1(0))<16\}} N_\alpha(\nabla u)(P)^2\,dP \le C. \tag{3.20}$$

Let $S_k = \{P\colon 2^k < \delta(P; I_1(0)) \le 2^{k+1}\}$ for $k \ge 4$. It is easy to see that (3.2) and (3.3) will follow from

$$\int_{S_k} N_\alpha(\nabla u)(P)^2\,dP \le C_{\alpha,m} 2^{-k(n+1+2\eta)}, \qquad k \ge 4, \tag{3.21}$$

and (3.20).

To establish (3.21), we choose a covering of S_k by cubes $I_{r_k}(Q_{k,j})$, $k \ge 4$ and $j = 1, \dots, N$, which satisfy (i) $r_k = c_1 2^k$ and (ii) $\delta(I_{8r_k}(Q^+_{k,j}); I_1(0)) \ge c_2 2^k$ where $Q^+_{k,j} = Q_{k,j} + r_k^2 e_0$. We let $\Psi_{k,j}(\rho) = \Psi_{\rho r_k}(Q^+_{k,j})$ and observe that if $P \in I_{r_k}(Q_{k,j})$, then $\Gamma_{r_k}(P, \alpha) \subset \Psi_{k,j}(2)$. Applying Corollary 2.4 on $\Psi_{k,j}(\rho)$, integrating over $3 \le \rho \le 4$ and recalling that $\partial_\nu u = 0$ on $\partial\Psi_{k,j}(\rho) \cap S$, we obtain that

$$\begin{aligned}
&\int_{I_{r_k}(Q_{k,j})} N_{\alpha, r_k}(\nabla u)(P)^2\,dP \\
&\quad \le C\left(2^{-k}\int_{\Psi_{k,j}(4)} |\nabla u(X)|^2\,dX + 2^{-3k}\int_{\Psi_{k,j}(4)} u(X)^2\,dX\right) \\
&\quad \le C2^{-3k}\int_{\Psi_{k,j}(8)} u(X)^2\,dX \le C2^{-k(n+1+2\eta)}
\end{aligned} \tag{3.22}$$

We have applied Cacciopoli's inequality, (3.8), to $\tilde{u}$ for the second inequality. The third inequality follows from (3.15) and the properties of our cubes $I_{r_k}(Q_{k,j})$. Summing on $j = 1, \dots, N$, we obtain (3.21) for the truncated maximal function N_{α, r_k} on S_k. We now show that we have the same estimate for

$$\bar{N}_{\alpha, r_k}(\nabla u)(P) \equiv \sup_{Y \in \Gamma(P,\alpha)\setminus\Gamma_{r_k}(P,\alpha)} |\nabla u(Y)|, \qquad P \in S_k.$$

If $P \in S_k$ and $Y \in \Gamma(P, \alpha)\setminus\Gamma_{r_k}(P, \alpha)$, then for some small constant $c > 0$, we have $\delta(J_{c2^k}(Y); S) \ge c2^k$. Hence using interior estimates for caloric functions and (3.15) we obtain that

$$|\nabla u(Y)| \le C2^{-k} \sup_{X \in J_{c2^k}(Y)} |u(X)| \le C2^{-k(n+1+\eta)}.$$

Thus $\bar{N}_{\alpha, r_k}(\nabla u)(P) \le C2^{-k(n+1+\eta)}$ for $P \in S_k$. Integrating this last estimate over S_k completes the proof of (3.21).

We now indicate the proof of (3.4). The estimate (1.8a) for $\|u\|_{W^{1/2,1}(I_4(0))}$ follows from Theorem 2.1. The estimate for $\|u\|_{L^2(I_4(0))}$ follows from the estimate for $\|u\|_{W^{1/2,1}(I_4(0))}$ and a Poincaré inequality since $u = 0$ for $x_0 < -1$.

The estimates for $\nabla'(u \circ \pi^{-1})$ and u away from $I_1(0)$ are contained in (3.21) and (3.15), respectively. To estimate $\|u\|_{W^{1/2,0}(I_\rho(Q))}$ for $I_\rho(Q)$ away from $I_1(0)$, we repeat the argument which established (3.21) and use the estimate for the $W^{1/2,0}$-seminorm in Corollary 2.4. □

We now turn to solutions of the Dirichlet problem when the datum is a 1-atom. Since 1-atoms are elements of $\mathscr{L}_1^2(S)$, the existence of such solutions follows from Theorem 2.8.

Lemma 3.23. *Let b be a 1-atom supported on the cube $I_r(Q)$, then the solution of* (DP) *with datum b satisfies*

$$\int_S N_\alpha(\nabla u)(P)\,dP \le C, \tag{3.24}$$

(3.25)
$$\int_S N_\alpha(\nabla u)(P)^2\,dP \times \left(\int_S N_\alpha(\nabla u)(P)^2 \delta(P;Q)^{(n+1)(1+\eta)}\,dP\right)^{1/\eta} \le C,$$

(3.26)
$A \equiv \partial_\nu u|_S$ *is a molecule for $H_1(S)$ whose constant satisfies $C(A) \le C_m$.*

We will need the following estimate for weak solutions of divergence form parabolic equations: Let $Lu = 0$ in $J_{2r}(X)$, then

$$\|u\|^2_{W^{1/2,0}(J_r(X))} \le C_\lambda \int_{J_{2r}(X)} |\nabla u(Y)|^2 + r^{-2}(u(Y) - \beta)^2\,dY. \tag{3.27}$$

where $\beta \in \mathbf{R}$ is arbitrary. This estimate may be found in [LSU, Chapter III, Theorem 4.1].

Proof. By a translation and rescaling, we may assume that u is supported in $I_1(0) \subset S$. We perform an odd reflection of u to obtain

$$\tilde{u}(x_0, x', x_n) = \begin{cases} u(x_0, x', x_n), & x_n \ge \phi(x'), \\ -u(x_0, x', 2\phi(x') - x_n), & x_n < \phi(x'). \end{cases}$$

As in the proof of Lemma 3.1, $\tilde{u}$ satisfies $L\tilde{u} = 0$ in $\mathbf{R}^{n+1} \setminus I_1(0)$. We begin by claiming that $\tilde{u}$ satisfies

$$|\tilde{u}(X)| \le C\delta(X; I_1(0))^{-n-\eta}, \qquad \delta(X; I_1(0)) \ge 4.$$

This follows from Lemma 3.13 just as before. In particular, we observe that $\tilde{u}(10, \cdot)$ has mean zero since $\tilde{u}$ was constructed by an odd reflection.

The estimates (3.24), (3.25) and the estimate (1.4a) in the definition of a molecule follow as before. Observe that we will need to use (3.27) to bound the term $\|\tilde{u}\|^2_{W^{1/2,0}(\Psi_{k,j}(4))}$ which will arise on the right-hand side of our estimates. We leave the remainder of this portion of the argument to the reader.

Finally, we need to show that $\int_S \partial_\nu u(P)\,dP = 0$ to conclude the proof that $\partial_\nu u(P)$ is a molecule. This is the subject of the next proposition. □

Proposition 3.28. *Let u be caloric in D_∞ and suppose that $N_\alpha(\nabla u) \in L^1(S)$, then*

$$\int_S \partial_\nu u(Q)\, dQ = 0.$$

We remark that this result is a consequence of Theorem 6.1, but it seems worthwhile to give a direct proof.

Proof. We assume that the the origin lies in S. Choose η, a smooth function, which is supported in $\{X\colon \delta(X;0) \le 1\}$ and satisfies $\eta(X) = 1$ for $X \in \{X\colon \delta(X;0) \le \frac{1}{2}\}$. Let $\eta_R(X) = \eta(x_0/R^2, x/R)$. We use the divergence theorem and obtain, for $\beta \in \mathbf{R}$, that

$$\int_S \eta_R(P)\partial_\nu u(P)\, dP = -\int_{D_\infty} \nabla\eta_R(Y)\nabla u(Y) - (u(Y)-\beta)\partial_{x_0}\eta_R(Y)\, dY. \tag{3.29}$$

A simple geometric argument shows that

$$\int_{\{X\,:\,R/2<\delta(X;0)<R\}} |v(X)|\, dX \le C_\alpha R \int_{\{P\,:\,R/4<\delta(P;0)<2R\}} N_\alpha(v)(P)\, dP. \tag{3.30}$$

Using (3.30) and $|\nabla\eta_R| \le C/R$, we may estimate the first term on the right of (3.29) by

$$\int_{D_\infty} |\nabla\eta_R(Y)\nabla u(Y)|\, dY \le C \int_{\{P\,:\,R/4<\delta(P;0)<2R\}} N_\alpha(\nabla u)(P)\, dP. \tag{3.31}$$

To handle the second term, we claim the following extension of (3.9): Let u be caloric in D_∞, then there exists $\beta_R \in \mathbf{R}$ and $C > 0$ (independent of R) such that

$$\begin{aligned}\int_{\{Y\in D_\infty\,:\,R/2<\delta(Y;0)<R\}} &|u(Y)-\beta_R|\, dY \\ &\le CR \int_{\{Y\in D_\infty\,:\,R/C<\delta(Y;0)<CR\}} |\nabla u(Y)|\, dY.\end{aligned} \tag{3.32}$$

This follows from (3.9) and a covering argument as in [Ko, pp. 167–168]. We use a larger annulus on the right-hand side of (3.32), since the set $\{Y \in D_\infty : R/2 < \delta(Y;0) < R\}$ may not be connected. Thus from (3.30) and (3.32) we may estimate the second term on the right of (3.29) by

$$\int_{D_\infty} |\partial_{x_0}\eta_R(Y)(u(Y)-\beta_R)|\, dY \le C \int_{\{P\,:\,R/2C<\delta(P;0)<2CR\}} N_\alpha(\nabla u)(P)\, dP. \tag{3.33}$$

Using (3.31) and (3.33) in (3.29), recalling that $N_\alpha(\nabla u) \in L^1(S)$ and then letting $R \to \infty$, we obtain that $\int \partial_\nu u(P)\, dP = 0$ as desired. □

4. Solutions of the boundary value problems in graph domains

In this section, we give our results for solutions of (NP) with data in $L^p(S)$, $1 < p < 2 + \varepsilon_m$, and $H^1(S)$. We also study solutions of (DP) with data in

$\mathscr{L}_1^p(S)$, $1 \le p < 2 + \varepsilon_m$. Existence for $p = 1$, follows immediately from the estimates of §3 for solutions with atomic data. The range $1 < p < 2$ then follows by interpolation. We also show that these solutions of (DP) and (NP) may be represented as classical single-layer heat potentials. The method we present is that of Verchota [V] which he used to study layer potentials for Laplace's equation on Lipschitz domains. We will give two applications of the method of layer potentials. The first is to complete the proof of Theorems 4.12 and 4.14 by establishing the existence of solutions when $2 < p < 2+\varepsilon_m$. This is done using a perturbation argument due to Calderón. The second is to establish that solutions of (DP) may represented as double-layer potentials when the data is in $L^p(S)$, $2-\varepsilon < p < \infty$ and in $BMO(S)$. This follows by duality from the representation of solutions of (NP) as single-layer potentials. The existence for solutions with BMO-data on Lipschitz domains is new. This extends results on C^1-domains due to Grimaldi-Piro, Neri and Ragnedda [GNR].

Lemma 4.1. *Let u be caloric in D_∞ and suppose that $N_\alpha(\nabla u) \in L^p(S)$, $1 \le p < n+1$. Then after replacing u by $u+\beta$, for some $\beta \in \mathbf{R}$, we have*

$$\left(\int_S |u(P+te_n)|^q \, dP\right)^{\frac{1}{q}} \le Ct^{(n+1)(\frac{1}{n+1}-\frac{1}{p}+\frac{1}{q})}\|N_\alpha(\nabla u)\|_{L^p(S)}$$

for $0 \le \frac{1}{q} < \frac{1}{p} - \frac{1}{n+1}$, with the usual interpretation when $q = \infty$. The constant C depends on M, α, p and q.

Proof. From interior estimates, we have

$$|(\partial_{x_n} u)(P+te_n)| \le Ct^{-\frac{n+1}{p}}\left(\int_{-\infty}^{p_0}\int_{\partial D} N_\alpha(\nabla u)(Q)^p \, dQ\right)^{\frac{1}{p}}$$

and when α is large, we trivially have that $|\partial_{x_n} u(P+te_n)| \le N_\alpha(\nabla u)(P)$. Hence,

$$\begin{aligned} |u(Q+te_n)| &\le \int_t^\infty |(\partial_{x_n} u)(Q+se_n)| \, ds \\ &\le CN_\alpha(\nabla u)(Q)^{(1-\theta)}\int_t^\infty s^{-\theta(\frac{n+1}{p})} \, ds \|N_\alpha(\nabla u)\|_{L^p(S)}^\theta, \qquad 0 \le \theta \le 1. \end{aligned}$$

If we also assume that $\theta\frac{n+1}{p} > 1$, we have

$$|u(Q+te_n)| \le CN_\alpha(\nabla u)(Q)^{1-\theta}t^{1-\theta\frac{n+1}{p}}\|N_\alpha\nabla u)\|_{L^p(S)}^\theta.$$

Raising each side of this inequality to the power $p/(1-\theta)$, and integrating with respect to Q, we obtain

$$\left(\int_S |u(Q+te_n)|^{\frac{p}{1-\theta}} \, dQ\right)^{\frac{1-\theta}{p}} t^{1-\theta\frac{n+1}{p}} = Ct^{1-\theta\frac{n+1}{p}}\|N_\alpha(\nabla u)\|_{L^p(S)}.$$

Putting $q = p/(1-\theta)$ and translating the restriction that $1 \ge \theta > p/(n+1)$ into a restriction on q gives the lemma when α is large. The restriction that α is large may be removed, since it is well known that parabolic maximal functions defined using different cone openings have comparable L^p norms. □

Theorem 4.2. *Let* $N_\alpha(\nabla u) \in L^p$, *for* $1 \le p < n+1$ *and suppose that* $\partial_\nu u = 0$ *a.e. on* S. *Then* u *is a constant.*

Proof. We fix $X \in D_\infty$ and let $X^* = (X', 2\phi(x') - x_n)$. We let $G(Y) = E(X; Y) + E(X^*; Y)$ where $E(X; Y)$ is the fundamental solution in $\mathbf{R}^{n+1}$ of the parabolic operator introduced in §3. We claim that if u is normalized as in Lemma 4.1, then we have

$$u(X) = \int_S G(Q)\partial_\nu u(Q)\, dQ.$$

This and our assumption that the normal derivative of u vanishes on S imply that $u = 0$ since $X \in D_\infty$ is arbitrary.

To establish the claim, we let $u_t(X) = u(X + te_n)$. Next we let R satisfy $R > 100\delta(X; 0)$. We fix a smooth function $\chi \in C_0^\infty(\{Y : \delta(Y; 0) < 2\})$ which satisfies $\chi(X) = 1$ if $\delta(X; 0) \le 1$ and let $\chi_R(Y) = \chi(y_0/R^2, y/R)$. For t small, we have

$$\begin{aligned} u_t(X) = &\int_S G(Q)\chi_R(Q)\partial_\nu u_t(Q)\, dQ \\ &+ \int_S G(Q) u_t(Q)\partial_\nu \chi_R(Q)\, dQ \\ &+ \int_{D_\infty} G(Y) 2\langle \nabla u(Y), \nabla \chi_R(Y)\rangle\, dY \\ &+ \int_{D_\infty} G(Y) u(Y)(\Delta \chi_R(Y) - \partial_{y_0}\chi_R(Y))\, dY \\ \equiv &\ \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned}$$

Recall that $E(X; Y) \le \delta(X; Y)^{-n}$, hence $G(\cdot) \in L^p(S)$ for $\frac{n+1}{n} < p \le \infty$. This, our assumption that $N_\alpha(\nabla u) \in L^p(S)$, $1 \le p < n+1$, and the dominated convergence theorem imply that

$$\lim_{R\to\infty} \mathrm{I} = \int_S G(Q)\partial_\nu u_t(Q)\, dQ.$$

Thus, we wish to show that the expressions II, III and IV go to zero as $R \to \infty$.

To do this, we let $F(R) = \{Y : |\nabla \chi_R(Y)| + |\partial_{y_0}\chi_R(Y)| \ne 0\}$ and observe that when R is large, $G(Y) \le C_X R^{-n}$ for $Y \in F(R)$. Using this and the estimate $|\nabla \chi_R| \le C/R$, we may estimate II by

$$|\mathrm{II}| \le CR^{-n-1}\int_{S\cap F(R)} |u_t(Q)|\, dQ \le C\left(R^{-n-1}\int_S |u_t(Q)|^q\, dQ\right)^{1/q}$$

where q is as in Lemma 4.1. Thus II vanishes as $R \to \infty$ since $u_t \in L^q(S)$. Using the same estimates for G and $\nabla \chi_R$, we have

$$|\mathrm{III}| \le \int_0^R \left(R^{-n-1}\int_S N_\alpha(\nabla u)(Q)^p\, dQ\right)^{\frac{1}{p}} ds \le R^{1-\frac{n+1}{p}} \|N_\alpha(\nabla u)\|_{L^p(S)}$$

which goes to zero as $R \to \infty$. To estimate the last term we will need that $|\Delta\chi_R| + |\partial_{y_0}\chi_R| \le CR^{-2}$. Using this we have

$$\begin{aligned}
|\mathrm{IV}| &\le CR^{-n-2}\int_{F(R)} |u_t(Y)|\,dY \le CR^{-1}\int_0^R \left(R^{-n-1}\int_S |u_t(Q+se_n)|^q\,dQ\right)^{\frac{1}{q}} ds \\
&\le CR^{-(1+\frac{n+1}{q})}\int_t^{R+t} s^{(n+1)(\frac{1}{n+1}-\frac{1}{p}+\frac{1}{q})}\,ds \\
&\le C_{p,q,n}R^{-(1+\frac{n+1}{q})}R^{2+\frac{n+1}{q}-\frac{n+1}{p}} = CR^{1-\frac{n+1}{p}}
\end{aligned}$$

where the last inequality is valid if $2+\frac{n+1}{q}-\frac{n+1}{p} > 0$, which will hold if q is close to $\frac{p(n+1)}{n+1-p}$. Thus we have

$$u_t(X) = \int_S G(Q)\partial_\nu u_t(Q)\,dQ$$

letting $t \to 0$ and using the Lebesgue dominated convergence theorem establishes the claim. □

Theorem 4.3. *Let u be caloric, satisfy $N_\alpha(\nabla u) \in L^p(S)$, $1 \le p < n+1$, and suppose that $u = 0$ a.e. on S in the sense of nontangential limits, then $u \equiv 0$.*

Proof. We begin by showing that our hypotheses imply that u is already normalized as in Lemma 4.1. To do this, we wish to show that $u(X) \to 0$ as $\delta(X;S) \to \infty$. As a first step, we claim that

$$\|u_t\|_{L^p(S)} \le Ct\|N_\alpha(\nabla u)\|_{L^p(S)} \tag{4.4}$$

where, as before, $u_t(P) = u(P+te_n)$. To establish (4.4), we observe that for a.e. $P \in S$,

$$|u(P+te_n)| = \lim_{\varepsilon\to 0}\left|\int_\varepsilon^t (\partial_{x_n}u)(P+se_n)\,ds\right| \le tN_\alpha(\nabla u)(P)$$

since $u = 0$ a.e. on S. We raise this inequality to the pth power and integrate over S which gives (4.4).

To show that u tends to zero as $\delta(X;S) \to \infty$, we fix $X \in D_\infty$ and let $\rho = c\delta(X;S)$ where the constant c is chosen so that $\delta(J_\rho(X);S) \ge \rho$. Using interior estimates, we have

$$\begin{aligned}
|u(X)| &\le C\rho^{-n-2}\int_{J_\rho(X)} |u(Y)|\,dY \le C\sup_{c\rho<s<\rho/c}\left(\frac{1}{\rho^{n+1}}\int_S |u_s(Q)|^p\,dQ\right)^{\frac{1}{p}} \\
&\le C\rho^{1-\frac{n+1}{p}}\|N_\alpha(\nabla u)\|_{L^p(S)}
\end{aligned}$$

where the last inequality is (4.4). Since $p < n+1$, this last expression goes to zero as $\delta(X;S) \to \infty$.

We will show that $\partial_\nu u = 0$, then our uniqueness result for (NP) and our observation that u is already normalized will imply that $u = 0$ on D_∞. To proceed, we fix b, a compactly supported Lipschitz function on S, and note that b is (a multiple of) a 1-atom. Hence, from Lemma 3.1, there exists a solution of the Dirichlet problem for the adjoint heat equation

$$\text{(DP*)} \qquad \begin{cases} \partial_{x_0} v + \Delta v = 0, & \text{in } D_\infty \\ v = b, & \text{on } S. \end{cases}$$

From the proof of Lemma 3.1, we see that v satisfies

$$|v(X)| \le \delta(X;0)^{-n-\alpha}, \qquad \text{for } \delta(X;0) \text{ large} \tag{4.5}$$

and

$$N_\alpha(\nabla v) \in L^1(S) \cap L^2(S). \tag{4.6}$$

Next we observe that v satisfies the maximum principle,

$$|v(X)| \le \|b\|_{L\infty(S)} < \infty, \qquad X \in D_\infty. \tag{4.7}$$

This may be proved by adapting the simple argument given in [GT, Theorem 8.1] for weak solutions of elliptic equations. From the estimate (4.7) and interior estimates for caloric functions, we obtain that

$$|\nabla v(X)| \le C\delta(X;S)^{-1}\|b\|_{L^\infty(S)}.$$

With these estimates for v, we are ready to proceed. Let χ_R be a cutoff function as in the proof of Theorem 4.2 and apply Green's identity to v_s and $\chi_R u_t$. This gives

$$\begin{aligned} \int_S u_t(Q)\chi_R(Q)\partial_\nu v_s(Q)\,dQ = & \int_S v_s(Q)\chi_R(Q)\partial_\nu u_t(Q)\,dQ \\ & + \int_S v_s(Q)u_t(Q)\partial_\nu \chi_R(Q)\,dQ \\ & + \int_{D_\infty} 2v_s(Y)\langle \nabla u_t(Y), \nabla\chi_R(Y)\rangle\,dY \\ & + \int_{D_\infty} v_s(Y)u_t(Y)(\Delta\chi_R(Y) - \partial_{y_0}\chi_R(Y))\,dY \\ \equiv & \ \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned}$$

We now let $R \to \infty$. To evaluate the limit of the left-hand side, we observe that (4.6) and (4.8) imply that $\partial_\nu v_s \in L^q(S)$ when $s > 0$ and $1 \le q \le \infty$ and the estimate (4.4) gives that $u_t \in L^p(S)$. Hence the dominated convergence theorem gives that the integral on the left converges to

$$\int_S u_t(Q)\partial_\nu v_s(Q)\,dQ$$

as $R \to \infty$. Since v_s satisfies the same estimates as G (see (4.5) and (4.6)), we may evaluate the limit as $R \to \infty$ of the expressions I to IV in the same manner as the like-numbered terms in the proof of Theorem 4.2. After letting $R \to \infty$, we obtain

$$\int_S u_t(Q)\partial_\nu v_s(Q))\,dQ = \int_S v_s(Q)\partial_\nu u_t(Q)\,dQ. \tag{4.9}$$

We now let $t \to 0$. Since $\partial_\nu v_s \in L^q(S)$, $1 \le q \le \infty$, and (4.4) implies that $\|u_t\|_{L^p(S)} \to 0$ for some p, the lefthand side of (4.9) converges to zero. The right-hand side of (4.9) converges to the obvious limit using the dominated convergence theorem and the estimate

$$|v_s(Q)||\partial_\nu u_t(Q)| \le C_b(1+\delta(Q;0))^{-n-\alpha} N_\alpha(\nabla u)(Q). \tag{4.10}$$

To establish (4.10), we use (4.5) and (4.7) to bound v_s. Our restriction that $p < n+1$ guarantees that the right-hand side of (4.10) is in L^1. Finally, we use (4.10) again, and let $s \to 0$. This gives

$$\int_S b(Q)\partial_\nu u(Q)\, dQ = 0.$$

Since b is an arbitrary compactly supported Lipschitz function, it follows that $\partial_\nu u = 0$ as desired. □

Theorem 4.11. *Let $f \in H^1(S)$, then there exists a solution to* (NP) *with datum f which satisfies*

$$\|u\|_{\mathscr{L}_1^1(S)} + \int_S N_\alpha(\nabla u)(P)\, dP \le C_{\alpha,m}\|f\|_{H^1(S)}.$$

Furthermore, this solution is unique subject to the condition that $N_\alpha(\nabla u)$ is in $L^1(S)$ for some α.

Proof. Existence with estimates follows immediately from Lemma 3.1 and the uniqueness was treated in Theorem 4.2. □

We can now give our result for the Neumann problem with L^p-data.

Theorem 4.12. *There exists $\varepsilon > 0$ depending only on m such that for $f \in L^p(S)$, $1 < p < 2+\varepsilon$, there exists a unique solution of the Neumann problem* (NP) *with datum f. This solution satisfies*

$$\|u\|_{\mathscr{L}_1^p(S)} + \|N_\alpha(\nabla u)(P)\|_{L^p(S)} \le C_{\alpha,m}\|f\|_{\mathscr{L}_1^p(S)}.$$

We give the proof for $1 < p < 2$ here. When $p > 2$, the result is a corollary of the invertibility of the potential operator on L^2.

Proof for $1 < p < 2$. The existence in the range $1 < p < 2$ with L^p-estimates for $N_\alpha(\nabla u)$ follows by interpolating between the results for $H^1(S)$ and $L^2(S)$ given in Theorems 4.11 and 2.1 respectively. See [CW, Theorem D] for the relevant interpolation theorem. The estimate for u in $\mathscr{L}_1^p(S)$ may be established by interpolation applied to the map $f \to \Lambda^{-1}(u|_S \circ \pi^{-1})$ which takes $L^p(S)$ to $L^p(\mathbf{R} \times \mathbf{R}^{n-1})$. □

We now carry out the same program for solutions of the Dirichlet problem with data in $\mathscr{L}_1^p(S)$, $1 \le p < 2+\varepsilon$.

Theorem 4.13. *Let $f \in \mathscr{L}_1^1(S)$, then there exists a solution of (DP) with datum f. This solution satisfies*

$$\|\partial_\nu u\|_{H^1(S)} + \|N_\alpha(\nabla u)\|_{L^1(S)} \le C_{\alpha,m}\|f\|_{\mathscr{L}_1^1(S)}.$$

Furthermore, this solution is unique in the class of functions with $N_\alpha(\nabla u) \in L^1(S)$.

Proof. Existence follows immediately from Lemma 3.23 and the 1-atomic decomposition of $\mathscr{L}_1^1(S)$ in Theorem 1.7. Uniqueness is established in Theorem 4.3. □

Theorem 4.14. *Let $f \in \mathscr{L}_1^p(S)$ with $1 < p < 2+\varepsilon$. Then there exists a solution of the Dirichlet problem with datum f. This solution satisfies*

$$\int_S N_\alpha(\nabla u)(P)^p \, dP \le C_{\alpha,m,p}\|f\|^p_{\mathscr{L}_1^p(S)}.$$

Furthermore, this solution is unique in the class of functions with $N_\alpha(\nabla u) \in L^p(S)$.

Proof for $1 < p < 2$. As for the Neumann problem, existence for $1 < p < 2$ follows by interpolating between the $\mathscr{L}_1^2$ and the $\mathscr{L}_1^1$ results in Theorems 2.8 and 4.3 respectively. The uniqueness is established in Theorem 4.3. □

We now turn to the study of the layer potentials. Verchota's argument exploits the jump of the normal derivative of the single-layer potential across the boundary. Thus we alter our notation so that we can refer to the domains above and below the graph of ϕ simultaneously without causing excessive confusion. We let $D^+ = D = \{x : x_n > \phi(x')\}$ and introduce $D^- \equiv \{x : x_n < \phi(x')\}$. We continue to let S be the common boundary of D^+_∞ and D^-_∞. Also, $\nu(Q)$ will always denote the normal to S which points into D^+_∞. We do not distinguish the parabolic maximal functions for D^+_∞ and D^-_∞, but we will be careful to specify the domain of v in $N_\alpha(v)$ so that the reader can determine which parabolic maximal function is intended.

To define our potential operators, we recall that W, the fundamental solution of the heat operator, was defined in (2.3). For appropriate densities $f \in L^p(S)$ or $H^1(S)$, we let

$$\mathscr{S}(f)(X) = \int_{-\infty}^{x_0}\int_{\partial D} W(X-Q)f(Q)\,dQ.$$

If p is large, we replace $W(X-Q)$ by $W(X-Q) - W(X^*-Q)$ for some point $X^* \notin S$, thus $\mathscr{S}(f)$ is defined up to a constant. Observe that the single-layer potential is defined everywhere in $\mathbf{R}^{n+1} \setminus S$ and a.e. with respect to n-dimensional surface measure on S. We also will study the double-layer potential on S. This is defined by

$$\mathscr{D}(f) = \int_{-\infty}^{x_0}\int_{\partial D} \partial_{\nu(Q)} W(X-Q)f(Q)\,dQ, \qquad X \in \mathbf{R}^{n+1} \setminus S.$$

We let $\mathscr{S}^+(f)$, $\mathscr{S}^-(f)$ and $\mathscr{S}^b(f)$ denote the restrictions of $\mathscr{S}(f)$ to D^+_∞, D^-_∞ and S respectively. Similarly, we let $\mathscr{D}^+(f)$ and $\mathscr{D}^-(f)$ denote the restrictions of $\mathscr{D}(f)$ to D^+_∞ and D^-_∞.

To discuss the boundary values of the double-layer potential and of the gradient of the single-layer potential, we introduce the boundary potential operators

$$K_i(f)(P) = \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{p_0 - \varepsilon} \int_{\partial D} \partial_{p_i} W(P-Q) f(Q)\, dQ, \qquad i = 1, \dots, n,$$
$$K(f)(P) = \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{p_0 - \varepsilon} \int_{\partial D} \partial_{\nu(Q)} W(P-Q) f(Q)\, dQ,$$
$$K'(f)(P) = \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{p_0 - \varepsilon} \int_{\partial D} \partial_{\nu(P)} W(P-Q) f(Q)\, dQ.$$

The boundedness properties of these operators are summarized in

Theorem 4.15. *The maps*

$$K_i,\ K,\ K' : L^p(S) \to L^p(S), \qquad 1 < p < \infty,$$
$$K' : H^1(S) \to H^1(S), \quad K : BMO(S) \to BMO(S)$$

are bounded.

The proof of this theorem is essentially the same as in the case of C^1-domains discussed in [FR] . Of course, one must substitute the result of [CMM] on the Cauchy integral on Lipschitz curves for Calderón's result [C1] cited by Fabes and Rivière.

With these results for the boundedness of the boundary potential operators, we may prove the following theorems on the boundary behavior of the heat potentials. The proofs are the same as in the C^1-case and are omitted.

Theorem 4.16. *The single-layer potential $\mathscr{S}$ satisfies*

(1) $\partial_{x_0} \mathscr{S}(f) - \Delta \mathscr{S}(f) = 0$ *in* $\mathbf{R}^{n+1} \setminus S$.
(2) $\lim_{\Gamma^\pm(P,\alpha) \ni X \to P} \partial_{x_i} \mathscr{S}^\pm(f)(X) = \mp \frac{1}{2} \nu_i(P) f(P) + K_i(f)(P)$, *for* $i = 1, \dots, n$ *and a.e.* $P \in S$.
(3) *The maps* $\mathscr{S}^b : L^p(S) \to \mathscr{L}^p_1(S)$ *and* $\mathscr{S}^b : H^1(S) \to \mathscr{L}^1_1(S)$ *are bounded.*
(4) $\|N_\alpha(\nabla \mathscr{S}^\pm(f))\|_{L^p(S)} \le C_{\alpha,m} \|f\|_{L^p(S)}$, *for* $1 < p < \infty$ *and* $\|N_\alpha(\nabla \mathscr{S}^\pm(f))\|_{L^1(S)} \le C_{\alpha,m} \|f\|_{H^1(S)}$.

We have the corresponding result for the double-layer potential.

Theorem 4.17. *The double-layer potential satisfies*

(1) $\partial_{x_0} \mathscr{D}(f) - \Delta \mathscr{D}(f) = 0$ *in* $\mathbf{R}^{n+1} \setminus S$,
(2) $\lim_{\Gamma^\pm(P,\alpha) \ni X \to P} \mathscr{D}^\pm(f)(X) = \pm \frac{1}{2} f(P) + K(f)(P)$, *for a.e.* $P \in S$,

(3) $\|N_\alpha(\mathscr{D}^\pm(f))\|_{L^p(S)} \le C_{\alpha,m}\|f\|_{L^p(S)}$, *for* $1 < p < \infty$. *If* $f \in BMO(S)$, *then* $\|N_{\alpha,r}(\mathscr{D}^\pm(f) - c(f))\|_{L^2(I_r(Q))} \le C\|f\|_{BMO(S)}$ *where*

$$c(f) = |I_r(Q)|^{-1}\int_{I_r(Q)} f\,dQ.$$

After this rather lengthy preamble, we can finally do something.

Theorem 4.18. *There exists* $\varepsilon > 0$ *such that the maps*

$$\begin{aligned} \pm\tfrac{1}{2}I + K' &: L^p(S) \to L^p(S), && 1 < p < 2+\varepsilon, \\ \pm\tfrac{1}{2}I + K' &: H^1(S) \to H^1(S), \\ \mathscr{S}^b &: L^p(S) \to \mathscr{L}_1^p(S), && 1 < p < 2+\varepsilon, \\ \mathscr{S}^b &: H^1(S) \to \mathscr{L}_1^1(S), \end{aligned}$$

are invertible. In each case, the norms of the inverses depend only on p *and* m.

Remark. The case $p = 2$ of this theorem was established in [B1] and was used to prove the results quoted in §2.

Proof. We first consider the case $1 \le p < 2$. For definiteness, we give the argument which shows that $\mathscr{S}^b\colon H^1(S) \to \mathscr{L}_1^1(S)$ is invertible. The other cases are similar. We begin with the jump relation

$$f = \partial_\nu\mathscr{S}^-(f) - \partial_\nu\mathscr{S}^+(f), \qquad f \in H^1(S).$$

This is a consequence of part (2) of Theorem 4.16. Hence, we have

$$\|f\|_{H^1(S)} \le \|\partial_\nu\mathscr{S}^+(f)\|_{H^1(S)} + \|\partial_\nu\mathscr{S}^-(f)\|_{H^1(S)} \le C\|\mathscr{S}^b(f)\|_{\mathscr{L}_1^1(S)} \tag{4.19}$$

where we have used Theorem 4.11 in D_∞^+ and in D_∞^- to bound the normal derivatives of $\mathscr{S}(f)$ by the $\mathscr{L}_1^1(S)$ norm of $\mathscr{S}^b(f)$.

To proceed we introduce the one-parameter family of domains, $D_t \equiv \{(x', x_n)\colon x_n > \phi(x')\}$, for $0 \le t \le 1$ and let $\mathscr{S}_t$ be the single layer potential for $\mathbf{R}\times D_t$. We have the estimate (4.19) uniformly in t for $0 \le t \le 1$ and it is easy to see that when $t = 0$, $\mathscr{S}_0^b = \frac{1}{2}\Lambda$. One can show that $t \to \mathscr{S}_t$ is a continuous map from $[0, 1]$ into $\mathscr{L}(H^1(S), \mathscr{L}_1^1(S))$. Hence, the method of continuity implies that $\mathscr{S}^b = \mathscr{S}_1^b\colon H^1(S) \to \mathscr{L}_1^1(S)$ is invertible as desired.

To carry out this argument for the operator $(-\frac{1}{2}I + K')(f) = \partial_\nu\mathscr{S}^+(f)$, we begin with (4.19) and then use Theorem 4.13 in D_∞^+ to conclude that

$$\|f\|_{H^1(S)} \le C\|(-\tfrac{1}{2}I + K')(f)\|_{H^1(S)}.$$

Then proceed as before.

We now consider the invertibility of

$$\mathscr{S}^b : L^p(S) \to \mathscr{L}_1^p(S), \quad \text{when } 2 < p < 2+\varepsilon.$$

This is equivalent to establishing the invertibility of the map $T\colon L^p(\mathbf{R}\times\mathbf{R}^{n-1}) \to L^p(\mathbf{R}\times\mathbf{R}^{n-1})$ which is defined by $T(f) = \Lambda^{-1}[(\mathscr{S}^b(f\circ\pi))\circ\pi^{-1}]$. It is clear that T is uniformly bounded on L^p, for say $4/3 < p < 4$. Since $\mathscr{S}^b\colon L^2(S) \to \mathscr{L}_1^2(S)$ is invertible, it follows that $T\colon L^2(\mathbf{R}\times\mathbf{R}^{n-1}) \to L^2(\mathbf{R}\times\mathbf{R}^{n-1})$ is invertible. Hence the argument of [C2, pp. 39–40] applied to $T\circ T^*$ and $T^*\circ T$ implies that these operators on invertible on L^p for $|1/2 - 1/p| < \varepsilon_m$. But then T must also be invertible, as desired. A similar argument establishes the invertibility of the operators $\pm\frac{1}{2}I + K'$. □

We can now complete the proofs of Theorems 4.12 and 4.14.

Proofs of Theorems 4.12 *and* 4.14 *when* $2 < p < 2+\varepsilon$. If $f \in \mathscr{L}_1^p(S)$ with $2 < p < 2+\varepsilon_m$ then using Theorem 4.18, we may construct the solution of (DP) with datum f as

$$u(X) = \mathscr{S}^+((\mathscr{S}^b)^{-1}(f))(X).$$

The desired estimates for u follow from Theorem 4.16. The uniqueness has already been established. In a similar manner, we may construct solutions of (NP) using the invertibility of $-\frac{1}{2}I + K'$. □

We now give our representation for solutions of the Dirichlet problem with data in $L^p(S)$ and in $BMO(S)$. Existence and uniqueness for this problem with data in $L^p(S)$ has been treated by Fabes and Salsa [FS].

Corollary 4.20. *Let* $2-\varepsilon < p < \infty$. *The maps*

$$\pm\tfrac{1}{2}I + K\colon L^p(S) \to L^p(S), \qquad \pm\tfrac{1}{2}I + K\colon BMO(S) \to BMO(S)$$

are invertible. Hence we may represent solutions of (DP) *with data in these spaces as double-layer potentials and we have the estimates of Theorem* 4.17 *for these solutions.*

Proof. Let R be the reflection defined by $R(f)(p_0, p) = f(-p_0, p)$. A simple calculation shows that

$$\pm\tfrac{1}{2}I + K = R\circ(\pm\tfrac{1}{2}I + K')^*\circ R.$$

Hence, the invertibility of $\pm\frac{1}{2}I+K$ follows immediately from Theorem 4.18. □

5. Boundary value problems in bounded domains

In this section, we give the arguments needed to deduce results on bounded domains from the results on graph domains studied in §3 and §4. Throughout this section and the next, we allow constants to depend on the collection of coordinate cylinders used to cover $\partial\Omega$ and T, as well as α, m and p.

We begin by stating and proving a uniqueness theorem for solutions of (IDP) or (INP) with $N_\alpha(\nabla u)$ in $L^p(S)$. To do this, we quote a lemma of Calderón which we will use to approximate our Lipschitz domain by subdomains.

Lemma 5.1 [C2, pp. 34–35]. *Let Ω be a Lipschitz domain and let $\alpha\colon \mathbf{R}^n \to \{x\colon |x| \le 1\}$ be a Lipschitz vector field satisfying $\langle \alpha, \nu\rangle \ge \delta > 0$ a.e. on $\partial\Omega$. There exists $t_0 > 0$ such that for $0 < t < t_0$,*

(1) $\Omega_t \equiv \Omega \setminus \{x\colon x = q + s\alpha(q),\ 0 < s < t\}$ *is a Lipschitz domain.*
(2) $\partial\Omega_t = \{x\colon x = q + t\alpha(q)\}$.
(3) $\Phi_t(q) \equiv q + t\alpha(q)$ *defines a Lipschitz map* $\Phi_t : \partial\Omega \to \partial\Omega_t$ *with Lipschitz inverse.*
(4) *for a.e.* $q \in \partial\Omega$, $|\nu_t(\Phi_t(q)) - \nu(q)| \le Ct$ *where* ν_t *denotes the unit inner normal to* $\partial\Omega_t$.
(5) *If* α *is large, then* $(q_0, \Phi_t(q)) \in \Gamma((q_0, q); \alpha)$.

Theorem 5.2. *Let u be caloric in Ω_T and zero initially. If $N_\alpha(\nabla u) \in L^p(\Sigma_T)$, $1 \le p \le \infty$, and $\partial_\nu u = 0$ on Σ_T, then $u = 0$ in Ω_T. In the exterior domain, we have uniqueness if we make the additional assumption that $|u(X)| \le e^{\gamma|x|^2}$ for $|x|$ large and $0 \le x_0 < T$.*

Remark. If $p \ge 2$, then this result is trivial since u clearly satisfies the energy estimate. It is only when $p < 2$ that the following argument is needed.

Proof. We begin by considering uniqueness in the interior domain Ω_T. Let $\Omega_s \subset \Omega$ be the approximating domains constructed in Lemma 5.1. We have

$$\int_{\Omega_{s,T}} |\nabla u(Y)|^2\,dY + \sup_{0<t<T} \frac{1}{2}\int_{\Omega_s} u(t, y)^2\,dy \le \int_{\Sigma_{s,T}} |u(P)\partial_{\nu_s} u(P)|\,dP. \tag{5.3}$$

We wish to show that the right-hand side of this inequality goes to zero as $k \to \infty$. We claim that our hypothesis that $\partial_\nu u = 0$ implies that $u \in L^\infty(\Omega_T)$. Thus since $N_\alpha(\nabla u) \in L^p(\Sigma_T)$ and $\partial_\nu u = 0$ on Σ_T it follows that $\int_{\Sigma_{s,T}} |u(P)\partial_{\nu_s} u(P)|\,dP \to 0$ as $s \to 0$. Whence it follows from (5.3) that $u = 0$.

To establish the claim that u is bounded, we fix a coordinate cylinder, Z_i, and let L be the parabolic operator associated with reflection in the graph $\{x_n = \phi_i(x')\}$ as in §3. We let $\chi(y) \in C_c^\infty(\mathbf{R}^n)$ be a smooth function which satisfies $\chi(y) = 1$ for $y \in 2Z_i$ and $\chi(y) = 0$ for $y \notin 4Z_i$. Let $X \in (0, T) \times (\Omega \cap Z_i)$ and let $X^* = (x_0, 2\phi_i(x') - x_n)$ be the reflection of X. Then as in the proof of Theorem 4.2 (uniqueness in NP) we have

$$\begin{aligned} u(X) = {} & \int_0^T \int_{(4Z_i\setminus 2Z_i)\cap\Omega} G(Y)[2\langle \nabla u(Y), \nabla\chi(Y)\rangle + u(Y)\Delta\chi(Y)]\,dY \\ & + \int_0^T \int_{\partial\Omega\cap(4Z_i\setminus 2Z_i)} u(Q)\partial_\nu \chi(Q)\,dQ \end{aligned}$$

where $G(Y) = E(X; Y) + E(X^*; Y)$ and $E(\cdot\,; \cdot)$ is the Green's function in $\mathbf{R}^{n+1}$ for the operator L. Since $G(Y) \le C\,\mathrm{dist}(Z_i; (2Z_i)^c)^{-n}$ when $y \notin 2Z_i$ and $X \in Z_i$, it follows that $u(X)$ is uniformly bounded for $X \in Z_i$. Repeating this argument in each coordinate cylinder, it follows that u is bounded near Σ_T and hence in all of Ω_T.

To establish uniqueness in the exterior domain, we need to show that u decays rapidly at infinity and then we will have the analogue of (5.3). The rest of the argument is the same.

We show that when T is small, then u does decay rapidly. To see this, we note that the Gauss-Weierstrass kernel satisfies

$$|W(X)| \le C_{x_0} \exp(-|x|^2/8x_0), \quad \text{for } |x| > 1.$$

Thus if $1/8x_0 > \gamma$, our growth assumption on u is sufficient to establish the representation

$$(\eta u)(X) = \int_0^{x_0} \int_{\Omega^e} W(X-Y)[2\langle \nabla u(Y), \nabla \eta(y)\rangle + u(Y)\Delta\eta(y)]\, dY.$$

where η is a smooth function on $\mathbf{R}^n$ which vanishes in a neighborhood of $\partial\Omega$ and is identically one near infinity. Since the integrand is compactly supported it follows that $|u(X)| \le C\exp(-c|x|^2)$ when $|x|$ is large and $0 < x_0 < 1/(16\gamma)$, say. Hence, u decays sufficiently fast at infinity to guarantee that the boundary terms at infinity vanish when we establish the energy estimate, (5.3). This observation suffices to complete the proof when T is small. We may deduce uniqueness for arbitrary T by repeatedly applying the result for small T. □

Theorem 5.4. *Let $N_\alpha(\nabla u) \in L^p(\Sigma_T)$, $1 \le p \le \infty$. Suppose that u is a solution of* (IDP) *with zero datum. Then $u \equiv 0$. In the exterior domain, we have the same conclusion if we make the additional assumption that $|u(X)| \le e^{\gamma|x|^2}$ for some $\gamma < \infty$.*

Proof. We will establish that $\partial_\nu u = 0$; then uniqueness in (INP) will yield that $u = 0$ in Ω_T. We remark that the proof of uniqueness in the exterior domain may be carried out using the arguments given below for Ω_T and the observation at the end of Theorem 5.2.

To show that $\partial_\nu u$ is zero, we choose a Lipschitz function b whose support is compactly contained in Σ_T. We let v be the solution of

$$\text{(IDP*)} \qquad \begin{cases} \partial_{x_0} v + \Delta v = 0, & \text{in } \Omega_T, \\ v(T, x) = 0, & x \in \Omega, \\ v(Q) = b(Q), & Q \in \Sigma_T, \end{cases}$$

which is given by Theorem 2.9. We observe that $N_\alpha(\nabla v) \in L^2(\Sigma_T)$. Hence v coincides with the weak solution of this problem which is known to be Hölder continuous for some exponent $\beta > 0$ (see [A2, Theorem D, p. 617] and the references cited there). Letting $\rho = c\delta(X;\Sigma)$ where c is chosen so that $\delta(J_\rho(X);\Sigma_T) \ge c\delta(X;\Sigma_T)$, we have

$$(5.5) \qquad |\nabla v(X)| \le C\delta(X;\Sigma)^{-1} \sup_{Y\in J_\rho(X)} |v(X) - v(Y)| \le C_b \delta(X;\Sigma)^{\beta-1}.$$

The last constant depends on the Hölder seminorm of v and hence on b.

Next, we observe that for s small,

$$\|u\|_{L^1(\Sigma_{s,T})} \leq C\|N_\alpha(\nabla u)\|_{L^1(\Sigma_T)} \leq |s|C'_T\|N_\alpha(\nabla u)\|_{L^p(\Sigma_T)} \tag{5.6}$$

To see this, we use the fundamental theorem of calculus to see that for a.e. $Q \in \Sigma_T$, we have

$$|u(Q+s\alpha(q))| = \lim_{\varepsilon\to 0^+}\left|\int_\varepsilon^s \langle\alpha(q), \nabla u(Q+s\alpha(q))\rangle\, ds\right| \leq |s|N_\alpha(\nabla u)(Q).$$

Hence, integrating over Σ_T, and setting $Q_s = \Phi_s(Q)$, we obtain

$$\int_{\Sigma_{s,T}} |u(Q_s)|\, dQ_s \leq C\int_{\Sigma_T} |u(\Phi_s(Q))|\, dQ \leq C|s|\int_{\Sigma_T} N_\alpha(\nabla u)(Q)\, dQ$$

which gives (5.6).

Applying Green's identity in $\Omega_{s,T}$, we have

$$\int_{\Sigma_{s,T}} u(Q_s)\partial_{\nu_s} v(Q_s)\, dQ_s = \int_{\Sigma_{s,T}} v(Q_s)\partial_{\nu_s} u(Q_s)\, dQ_s. \tag{5.7}$$

Now (5.5) and (5.6) imply that the left-hand side of (5.7) goes to zero as $s \to 0$. The continuity of v, our assumption that $N_\alpha(\nabla u) \in L^p(\Sigma_T)$, Lemma 5.1 and the dominated convergence theorem imply that the right-hand side of (5.7) converges to $\int_{\Sigma_T} b\partial_\nu u$ as $s \to 0$. Hence, we have

$$\int_{\Sigma_T} \partial_\nu u(Q)b(Q)\, dQ = 0.$$

Since b is an arbitrary compactly supported Lipschitz function, we have that $\partial_\nu u = 0$ as desired. □

Lemma 5.8. *Let u be a solution of the heat equation in Ω_T with $N_\alpha(\nabla u) \in L^2(\Omega_T)$. Suppose that either $\partial_\nu u = 0$ or $u = 0$ on $I_{16r}(Q)$, then*

$$\int_{I_r(Q)} N_{\alpha,r}(\nabla u)(P)^2\, dP \leq C_\alpha r^{-n-4}\left(\int_{\Psi_{16r}(Q)} |\nabla u(Y)|\, dY\right)^2.$$

Proof. We give the proof under the assumption that $\partial_\nu u = 0$ on $I_{16r}(Q)$. The case where $u = 0$ is handled by a similar argument. We apply Corollary 2.4 on $\Psi_{sr}(Q)$ for $3 < s < 4$. Integrating in s gives

$$\int_{I_r(Q)} N_{\alpha,r}(\nabla u)(P)^2\, dP \leq Cr^{-1}\left(\int_{\Psi_{4r}(Q)} |\nabla u(X)|^2 + r^{-2}(u(X)-\beta)^2\, dX\right).$$

Extending u by an even reflection as in Proposition 3.12, we may use the the Caccioppoli inequality (3.8), the L^∞-estimate (3.5), and the Poincaré inequality (3.9) to obtain the desired estimate. □

We will also need the following lemma to control lower order terms. Though not particularly difficult, we give a proof to make clear the dependence of the constants on T.

Lemma 5.9. *let u be caloric in Ω_T and zero initially, then*

$$\int_{\Omega_{\varepsilon,T}} |\partial_{y_0}^{\beta}\partial_y^{\gamma}u(Y)|\,dY \le C\varepsilon^{-2\beta-\gamma}\min(1\,,\,T/\varepsilon^2)\int_{\Sigma_T} N_\alpha(u)(P)\,dP.$$

The constant C depends on α, β, and γ as well as the Lipschitz character of $\partial\Omega$, but not on T.

Proof. Our starting point is the estimate

$$\int_{\Omega} |v(t\,,\,y)|\,dy \le C_\alpha \int_{\partial\Omega} N_\alpha(v)(t\,,\,p)\,dp \tag{5.10}$$

which is valid for any continuous function. Next, we observe that interior estimates imply that

$$N_{\alpha/2}(\delta(\cdot\,;\,\Sigma_T)^{2\beta+\gamma}\partial_{y_0}^{\beta}\partial_y^{\gamma}u)(P) \le C_{\alpha\,,\,\beta\gamma}N_\alpha(u)(P).$$

Using this observation in (5.10) and integrating over $(0\,,\,T)$ yields

$$\int_{\Omega_{\varepsilon,T}} |\partial_{y_0}^{\beta}\partial_y^{\gamma}u(Y)|\,dY \le C\varepsilon^{-2\beta-\gamma}\int_{\Sigma_T} N_\alpha(u)(P)\,dP$$

which establishes our lemma for $T/\varepsilon^2 > 1$. When T is small, we observe that since u is zero initially,

$$\int_0^T |\partial_{y_0}^{\beta}\partial_y^{\gamma}u(y_0\,,\,y)|\,dy_0 \le T\int_0^T |\partial_{y_0}^{\beta+1}\partial_y^{\gamma}u(y_0\,,\,y)|\,dy_0.$$

Integrating this over Ω_ε and applying (5.10) with β replaced by $\beta+1$ yields the lemma for $T/\varepsilon^2 < 1$. □

We now begin our study of existence of solutions for the boundary value problems in bounded Lipschitz domains. This is not particularly difficult given our results for graph domains. One remark is in order. We will initially prove our results on Ω_T, with T small. This will be enough to establish the invertibility of the boundary potential operators with T small. Using the fact that the kernels of the potentials are supported in $x_0 > 0$ and a well-known time stepping argument, we may deduce the invertibility of the potential operators when T is large from the case when T is small. This, in turn, will remove the restriction that T is small from Propositions 5.11 and 5.20, Theorems 5.22 and 5.23 and Corollaries 5.15, 5.16 and 5.21.

Proposition 5.11. *Let a be an atom supported in Σ_+. There exists a positive number T_0, independent of a, and a solution to* (INP) *in Ω_T with lateral datum a. For $T < T_0$, we have the estimate*

$$\|N_\alpha(\nabla u)\|_{L^1(\Sigma_T)} \le C \tag{5.12}$$

and the function $u|_{\Sigma_T}$ lies $\mathscr{L}_1^1(\Sigma_T)$ and satisfies $\|u\|_{\mathscr{L}_1^1(\Sigma_T)} \le C$.

Proof. Since atoms lie in $L^2(\Sigma_T)$, the existence of a solution u follows from Theorem 2.2. The main step toward establishing the L^1-estimate for $N_\alpha(\nabla u)$

is the following claim: Let $T < \operatorname{diam}(\Omega)^2$, then there exists a constant C, independent of T, and $\varepsilon > 0$ such that

$$\int_{\Sigma_T} N_\alpha(\nabla u)(P)\,dP \le C\left(1 + \int_{\Omega_{\varepsilon,T}} |\nabla u(Y)|\,dY\right) \tag{5.13}$$

where $\Omega_{\varepsilon,T} = (0, T) \times \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. Assuming this claim for the moment, we show that we can choose T_0 small so that (5.12) is true.

Using Lemma 5.9 in (5.13) gives

$$\int_{\Sigma_T} N_\alpha(\nabla u)(P)\,dP \le C + C'T \int_{\Sigma_T} N_\alpha(\nabla u)(P)\,dP$$

where the dependence on ε is omitted since we have already fixed ε. Choosing T_0 small, we may absorb the last term on the right of this inequality into the left and obtain (5.12) as desired.

We turn to the proof of (5.13). We assume that $\operatorname{supp} a \subset (0, \infty) \times Z$ where Z is a coordinate cylinder for $\partial\Omega$. We may construct a graph domain D which satisfies $64Z \cap \Omega \subset D$, $\partial\Omega \cap \partial D \supset 64Z \cap \partial\Omega$ and the Lipschitz constant for D is bounded by that of Ω. We let u_1 be the solution of (NP) in D_∞ with datum a and set $u_2 = u - u_1$. We write $\Sigma_T = \Sigma_{1,T} \cup \Sigma_{2,T}$ where $\Sigma_{1,T} = \Sigma_T \cap [(0, T) \times 16Z]$ and $\Sigma_{2,T} = \Sigma_T \setminus \Sigma_{1,T}$. We have

$$\int_{\Sigma_T} N_\alpha(\nabla u)(P)\,dP \le \int_{\Sigma_{1,T}} N_\alpha(\nabla u_1) + N_\alpha(\nabla u_2)\,dP + \int_{\Sigma_{2,T}} N_\alpha(\nabla u)(P)\,dP.$$

On $\Sigma_{1,T}$, u_2 has zero Neumann data, so we may use Hölder's inequality and Lemma 5.8 to estimate $N_{\alpha,r}(\nabla u_2)$ there. On $\Sigma_{2,T}$, u has zero Neumann data, so again Lemma 5.8 is applicable. In each case, the supremum of $|\nabla u|$ over $\Gamma(P, \alpha) \setminus \Gamma_r(P, \alpha)$ is bounded by the integral over Ω_T. Finally, Lemma 3.1 states that $\|N_\alpha(\nabla u_1)\|_{L^1(\Sigma_{1,T})} \le C$. Combining these estimates gives

(5.14)

$$\begin{aligned}\int_{\Sigma_T} N_\alpha(\nabla u)(P)\,dP &\le C\left(1 + \int_{\Omega_T} |\nabla u(Y)|\,dY + \int_{(0,T)\times(64Z\cap\Omega)} |\nabla u_2(Y)|\,dY\right)\\ &\le C'\left(1 + \int_{\Omega_T} |\nabla u(Y)|\,dY + \int_{(0,T)\times(64Z\cap\Omega)} |\nabla u_1(Y)|\,dY\right)\end{aligned}$$

where we have used $\nabla u_2 = \nabla u - \nabla u_1$ to obtain the second inequality. Since $16Z \cap \Omega \subset D$, one easily obtains that

$$\int_{(0,T)\times(\Omega\cap 64Z)} |\nabla u_1(Y)|\,dY \le C \int_S N_\alpha(\nabla u_1)(P)\,dP \le C'.$$

Finally, for ε small, we have

$$\int_{\Omega_T\setminus\Omega_{\varepsilon,T}} |\nabla u(Y)|\,dY \le C\varepsilon \int_{\Sigma_T} N_\alpha(\nabla u)(P)\,dP.$$

Combining (5.14) and these last two observations and then choosing ε small establishes (5.13).

We turn to the proof that $u|_{\Sigma_T}$ lies in $\mathscr{L}_1^1(\Sigma_T)$. Since u_1 comes from a 1-molecule on S, we may choose a cutoff function η and decompose ηu_1 into 1-atoms as in the proof of Proposition 1.9. Since $\mathscr{L}_1^2(\Sigma_T) \subset \mathscr{L}_1^1(\Sigma_T)$, to complete the proof of the theorem, it suffices to show that

$$\|u\|_{L^2(\Sigma_{2,T})} + \|u\|_{W^{1/2,1}(\Sigma_{2,T})} \leq C$$

and

$$\|u_2\|_{L^2(\Sigma_{1,T})} + \|u_2\|_{W^{1/2,1}(\Sigma_{1,T})} \leq C.$$

We may estimate $\|u\|_{W^{1/2,1}(\Sigma_{2,T})}$ and $\|u_2\|_{W^{1/2,1}(\Sigma_{1,T})}$ by using Corollary 2.4 on subdomains of Ω_T, the reverse Hölder estimate for ∇u in Lemma 5.8 and our L^1-estimate for $N_\alpha(\nabla u)$. Since T is finite, the estimates for $\|u\|_{L^2(\Sigma_{2,T})}$ and $\|u_2\|_{L^2(\Sigma_{1,T})}$ are trivial consequences of the $W^{1/2,1}$-estimates and the fact that $u = 0$ for $x_0 < 0$. □

Corollary 5.15. *Let $T < T_0$ and let $f \in \tilde{H}^1(\Sigma_T)$, then there exists a solution of* (IDP) *in Ω_T with datum f. This solution satisfies*

$$\|N_\alpha(\nabla u)\|_{L^1(\Sigma_T)} + \|u\|_{\mathscr{L}_1^1(\Sigma_T)} \leq C\|f\|_{\tilde{H}^1(\Sigma_T)}.$$

Next, we consider the Neumann problem in the exterior domain, $\Omega^e \equiv \mathbf{R}^n \setminus \bar{\Omega}$.

Corollary 5.16. *Let $T < T_0$ and suppose that $f \in \tilde{H}^1(\Sigma_T)$. Then there exists a unique solution of* (INP) *in the domain Ω_T^e. This solution satisfies*

$$\|N_\alpha(\nabla u)\|_{L^1(\Sigma_T)} + \|u\|_{\mathscr{L}_1^1(\Sigma_T)} \leq C\|f\|_{\tilde{H}^1(\Sigma_T)}$$

and $|u(X)| \leq C\|f\|_{\tilde{H}^1(\Sigma_T)} \exp(-c|x|^2)$ as $|x| \to \infty$.

Proof. The uniqueness has been treated in Theorem 5.2. We will deduce the existence from our previous result, Corollary 5.15. To show existence, we assume initially that $f \in L^2(\Sigma_T) \cap \tilde{H}^1(\Sigma_T)$ and hence from the L^2-theory, there exists a solution with datum f. We apply Corollary 5.15 to u in the bounded domain $(0, T) \times \Omega^R$ where $\Omega^R \equiv \{x \in \Omega^e : |x| < R\}$ for some large R. This gives

$$\|N_{\alpha, R/2}(\nabla u)\|_{L^1(\Sigma_T)} + \|u\|_{\mathscr{L}_1^1(\Sigma_T)} \leq C(\|f\|_{\tilde{H}^1(\Sigma_T)} + \|\partial_\nu u\|_{\tilde{H}^1((0,T)\times\partial B_R)}) \tag{5.17}$$

with $B_R = \{|x| < R\}$. Next, we observe that from Lemma 5.9 and interior estimates it follows that

$$\sup_{\{X \in \Omega_T^e : \delta(X;\Sigma_T) = R/4\}} |\nabla u(X)| \leq CT\|N_{\alpha, R/2}(\nabla u)\|_{L^1(\Sigma_T)}.$$

Hence, using the maximum principle to compare u with a translate of the fundamental solution W, it follows that

(5.18) $$|\nabla u(X)| \leq C\|N_{\alpha,R/2}(\nabla u)\|_{L^1(\Sigma_T)} \exp(-c|x|^2), \qquad \delta(X;\Sigma_T) > R/4.$$

Thus it follows that

(5.19)
$$\|\partial_\nu u\|_{\tilde{H}^1((0,T)\times\partial B_R)} \leq CT^{1/2}\|\nabla u\|_{L^\infty((0,T)\times\partial B_R)} \leq CT^{3/2}\|N_{\alpha,R/2}(\nabla u)\|_{L^1(\Sigma_T)}$$

where we have used that $L^\infty(\Sigma_T) \subset \tilde{H}^1(\Sigma_T)$. Combining (5.17) and (5.19), we obtain the desired estimate for $N_{\alpha,R/2}$. We may use (5.18) to bound the supremum over $\Gamma_{R/2}(P,\alpha)\setminus\Gamma(P,\alpha)$ and this inequality also gives the desired behavior at infinity. To remove the restriction that $f \in L^2(\Sigma_T)\cap\tilde{H}^1(\Sigma_T)$, observe that such functions are dense in $\tilde{H}^1(\Sigma_T)$. □

We now turn our attention to the (IDP) with data in $\mathcal{L}_1^p(\Sigma_T)$.

Proposition 5.20. *Let u be the solution of* (IDP) *in Ω_T with lateral data b, a 1-atom supported in Σ_+. Then for $T < T_0$, u satisfies*

$$\|N_\alpha(\nabla u)\|_{L^1(\Sigma_T)} \leq C$$

and $\partial_\nu u$ lies in $\tilde{H}^1(\Sigma_T)$ and satisfies $\|\partial_\nu u\|_{\tilde{H}^1(\Sigma_T)} \leq C$.

Proof. The proof is essentially the same as that of Proposition 5.11 and is omitted. □

As before, our proposition quickly leads to existence.

Corollary 5.21. *Let $f \in \mathcal{L}_1^1(\Sigma_T)$ with $T < T_0$. There exists a solution of* (IDP) *in Ω_T with data f. This solution satisfies*

$$\|N_\alpha(\nabla u)\|_{L^1(\Sigma_T)} + \|\partial_\nu u\|_{\tilde{H}^1(\Sigma_T)} \leq C\|f\|_{\mathcal{L}_1^1(\Sigma_T)}.$$

A similar result holds for (IDP) *in the exterior domain Ω_T^e where the unique solution is bounded by $C\|f\|_{\tilde{H}^1(\Sigma_T)}\exp(-c|x|^2)$.*

By interpolating, we may establish existence for our boundary value problems when $1 < p < 2$.

Theorem 5.22. *Let $T < T_0$ and suppose $f \in L^p(\Sigma_T)$, $1 < p < 2$. There exists a solution of* (INP) *in Ω_T with data f. This solution satisfies*

$$\|N_\alpha(\nabla u)\|_{L^p(\Sigma_T)} + \|u\|_{\mathcal{L}_1^p(\Sigma_T)} \leq C\|f\|_{L^p(\Sigma_T)}.$$

The same result holds for the exterior (INP) *in Ω_T^e where the unique solution is bounded by $C\|f\|_{L^p(\Sigma_T)}\exp(-c|x|^2)$ as $|x| \to \infty$.*

Theorem 5.23. *Let $T < T_0$ and suppose $f \in \mathcal{L}_1^p(\Sigma_T)$, $1 < p < 2$. Then there exists a unique solution of* (IDP) *with datum f. This solution satisfies*

$$\|N_\alpha(\nabla u)\|_{L^p(\Sigma_T)} \leq C\|f\|_{\mathcal{L}_1^p(\Sigma_T)}.$$

The same result holds for the exterior (IDP) *and in addition, the unique solution decays as* $\exp(-c|x|^2)$ *as* $|x| \to \infty$.

We now begin our study of the potential operators. The boundedness properties of these operators is the same as on graph domains, thus we do not repeat Theorems 4.6–4.8 in the setting of bounded domains. We begin with a classical argument which shows that the potential operators are injective. We remark that we will use $\mathcal{S}^+$ for the restriction of the single-layer potential to Ω_T and $\mathcal{S}^-$ for the restriction to Ω_T^e.

Proposition 5.24. *Let* $g \in L^p(\Sigma_T)$ *or* $\tilde{H}^1(\Sigma_T)$ *and suppose that one of the following holds on* Σ_T

$$\mathcal{S}^b(g) = 0, \quad (+\tfrac{1}{2}I + K')(g) = 0, \quad (-\tfrac{1}{2}I + K')(g) = 0,$$

then $g = 0$.

Proof. For definiteness, we assume that $(-\frac{1}{2}I + K')(g) = \partial_\nu \mathcal{S}^+(g) = 0$ and show that $g = 0$. The proofs of the other assertions are essentially the same.

Since $\partial_\nu \mathcal{S}^+(g) = 0$ on Σ_T, uniqueness in the interior (INP) (Proposition 5.1) implies that $\mathcal{S}^+(g) \equiv 0$ and thus $\mathcal{S}^b(g) = 0$. But then uniqueness in the exterior (IDP) implies that $\mathcal{S}^-(g) = 0$ and in particular $\partial_\nu \mathcal{S}^-(g) = 0$. From the jump relation $g = \partial_\nu \mathcal{S}^-(g) - \partial_\nu \mathcal{S}^+(g)$, we conclude that $g = 0$. □

Theorem 5.25. *Let* $0 < T < \infty$, *there exists an* $\varepsilon > 0$ *such that the maps*

$$\begin{aligned}
\pm\tfrac{1}{2}I + K' &: L^p(\Sigma_T) \to L^p(\Sigma_T), \qquad 1 < p < 2 + \varepsilon,\\
\pm\tfrac{1}{2}I + K' &: H^1(\Sigma_T) \to H^1(\Sigma_T),\\
\mathcal{S}^b &: L^p(\Sigma_T) \to \mathcal{L}_1^p(\Sigma_T), \qquad 1 < p < 2 + \varepsilon,\\
\mathcal{S}^b &: H^1(\Sigma_T) \to \mathcal{L}_1^1(\Sigma_T),
\end{aligned}$$

are invertible. The norms of the inverses depend on T *as well as* m, p *and* $\{Z_i\}$.

Proof. We show that the operator $-\frac{1}{2}I + K' = \partial_\nu \mathcal{S}^+$ is invertible on $\tilde{H}^1(\Sigma_T)$, the other results for $1 \le p < 2$ are similar. Then we will show how to adapt Calderón's argument to handle $p > 2$.

Using the jump relation, $g = \partial_\nu \mathcal{S}^-(g) - \partial_\nu \mathcal{S}^+(g)$ and the estimates of Corollaries 5.21 and 5.15, we obtain

$$\|g\|_{\tilde{H}^1(\Sigma_T)} \le C\|(-\tfrac{1}{2}I + K')(g)\|_{\tilde{H}^1(\Sigma_T)}, \qquad T < T_0.$$

Since $-\frac{1}{2}I + K'$ is invertible on $L^2(\Sigma_T)$ [B1, Proposition 7.2], and $L^2(\Sigma_T)$ is dense in $\tilde{H}^1(\Sigma_T)$ it follows that the image of $-\frac{1}{2}I + K'$ is dense in $\tilde{H}^1(\Sigma_T)$. This establishes the result when T is small.

We now remove the restriction that T is small. Let $f = f_1 \in \tilde{H}^1(\Sigma_T)$ with T arbitrary. By our result for T_0 small, there exists $g_1 \in \tilde{H}^1(\Sigma_{T_0})$ such that

$(-\frac{1}{2}I+K')(g_1)=f_1$ on Σ_{T_0}. By the definition of our Hardy space, there exists an extension of g_1, $\bar{g}_1\in\tilde{H}^1(\Sigma_T)$ with $\|\bar{g}_1\|_{\tilde{H}^1(\Sigma_T)}\le 2\|g_1\|_{\tilde{H}^1(\Sigma_{T_0})}$. Let

$$f_2=f_1-(-\tfrac{1}{2}I+K')(\bar{g}_1)$$

it is clear that f_2 is supported on $[T_0, T]\times\partial\Omega$ and that $f_2\in\tilde{H}^1((T_0, T)\times\partial\Omega)$. Hence there exists g_2 supported in $(T_0, 2T_0)\times\partial\Omega$ such that $(-\frac{1}{2}I+K')(g_2)=f_2$ on Σ_{2T_0}. Whence $(-\frac{1}{2}I+K')(\bar{g}_1+g_2)=f_1$ on Σ_{2T_0}. Continuing this process, we see that $-\frac{1}{2}I+K'$ is invertible on $\tilde{H}^1(\Sigma_T)$ for any finite T.

We now turn to p in the range $2<p<2+\varepsilon$. The invertibility of $\pm\frac{1}{2}I+K'$ follows from Calderón's argument given in the proof of Theorem 4.9. To establish the invertibility of $\mathcal{S}^b\colon L^p(\Sigma_T)\to\mathcal{L}_1^p(\Sigma_T)$ we give a minor modification of Calderón's argument. Let $\{\psi_1, \dots, \psi_N\}$ be the partition of unity on $\partial\Omega$ used to define the norm of the spaces $\mathcal{L}_1^p(\Sigma_T)$. We define a map

$$V\colon L^p(\Sigma_T)\to L^p((0, T)\times\mathbf{R}^{n-1})^N$$

by $V(g)_i=\Lambda^{-1}((\psi_i\mathcal{S}^b(g))\circ\pi^{-1})$. It is clear that V is bounded for all p, $1<p<\infty$. Since is S^b is injective, so is V and from the invertibility of $\mathcal{S}^b\colon L^2(\Sigma_T)\to\mathcal{L}_1^2(\Sigma_T)$ and the definition of the $\mathcal{L}_1^2(\Sigma_T)$-norm it follows that V has closed range when $p=2$. Hence, $V^*\circ V$ is an isomorphism on $L^2(\Sigma_T)$. By Calderón's result, $V^*\circ V$ is also invertible on $L^p(\Sigma_T)$ for p near 2. This implies that V and hence $\mathcal{S}^b$ have closed range for $2<p<2+\varepsilon$. Now, the argument given in [B1, Proposition 6.2 and 7.2] shows that $\mathcal{S}^b\colon L^p(\Sigma_T)\to\mathcal{L}_1^p(\Sigma_T)$ has dense range. Hence we are done. □

Corollary 5.26. *The maps*

$$\pm\tfrac{1}{2}I+K: L^p(\Sigma_T)\to L^p(\Sigma_T), \qquad 2-\varepsilon<p<\infty$$
$$\pm\tfrac{1}{2}I+K: \widetilde{BMO}(\Sigma_T)\to\widetilde{BMO}(\Sigma_T)$$

are invertible.

Proof. These follow from duality and the fact that $R\circ(\pm\frac{1}{2}I+K')^*\circ R=\pm\frac{1}{2}I+K$ as in §4. □

We conclude §5 by listing the representations of solutions of initial-boundary value problems as classical heat potentials. The results in the next corollary follow immediately from the invertibility of the boundary potential operators which was established in Theorem 5.25 and Corollary 5.26.

Corollary 5.27. *We have the following representations for solutions of* (INP) *in* Ω_T:

$u(X)=\mathcal{S}^+((-\frac{1}{2}I+K')^{-1}(f))(X)$, $\quad f\in L^p(\Sigma_T)$, $1<p<2+\varepsilon$, *or* $\tilde{H}^1(\Sigma_T)$,

and in Ω_T^e

$u(X) = \mathscr{S}^{-}((+\frac{1}{2}I + K')^{-1}(f))(X)$, $f \in L^p(\Sigma_T)$, $1 < p < 2+\varepsilon$, *or* $\tilde{H}^1(\Sigma_T)$.
We have the following representations for solutions of (IDP) *in* Ω_T

$$u(X) = \mathscr{S}^{+}((\mathscr{S}^{b})^{-1}(f))(X), \quad f \in \mathscr{L}_1^p(\Sigma_T),$$

and

$u(X) = \mathscr{D}^{+}((+\frac{1}{2}I+K)^{-1}(f))(X)$, $f \in L^p(\Sigma_T)$, $2-\varepsilon < p < \infty$, *or* $\widetilde{BMO}(\Sigma_T)$,
with similar results for (IDP) *in* Ω_T^e.

6. Characterization of a Hardy space

Motivated by the Stein-Weiss theory of conjugate harmonic functions in several variables, we introduce

$$H_N^1(\Omega_T) = \{\nabla u : u \text{ is caloric in } \Omega_T, \text{ zero initially and } N_\alpha(\nabla u) \in L^1(\Sigma_T)\}.$$

Our goal is to identify this space with a space of functions on the lateral boundary.

Theorem 6.1. *We have* $\nabla u \in H_N^1(\Omega_T)$ *if and only if* $\partial_\nu u$ *exists and lies in* $\tilde{H}^1(\Sigma_T)$. *Furthermore,*

$$\|\partial_\nu u\|_{\tilde{H}^1(\Sigma_T)} \approx \|N_\alpha(\nabla u)\|_{L^1(\Sigma_T)}.$$

Before continuing, we recall the definitions of Carleson measures and of certain fractional derivatives which are adapted to the study of the heat equation. We say a positive measure μ on Ω_T is a Carleson measure if

$$\int_{\Psi_r(Q)} d\mu(X) \le C \int_{I_r(Q)} dQ \tag{6.2}$$

for each cube $I_r(Q) \subset \Sigma_T$. The Carleson norm of μ, $\|\mu\|_c$, is the smallest constant C for which (6.2) holds. For a smooth function f which decays rapidly at $-\infty$, we let

$$\partial_t^\alpha f(t) = \partial_t I_{1-\alpha}(f)(t)$$

where I_β is a fractional integral which may be defined for $0 < \beta \le 1$ by

$$I_\beta(f)(t) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^t f(s)(t-s)^{\beta-1}\, ds.$$

In this definition, Γ denotes the usual Gamma function. We recall two properties of these derivatives that will be useful later. First, we note that if $\alpha+\beta \le 1$, then $\partial_t^\alpha \partial_t^\beta = \partial_t^{\alpha+\beta}$. Next, we set

$$I_\beta^*(f)(t) = \frac{1}{\Gamma(\beta)} \int_t^\infty f(s)(s-t)^{\beta-1}\, ds$$

for f smooth and vanishing rapidly at ∞ and then put $\partial_t^{\alpha*} f(t) = \partial_t I_{1-\alpha}^*(f)(t)$. With these definitions, we may integrate fractional derivatives by parts

$$\int_{\mathbf{R}} f \partial_t^\alpha g\, dt = -\int_{\mathbf{R}} \partial_t^{\alpha*} f g\, dt.$$

The main step of the proof of Theorem 6.1 is an extension of a result of Varopoulos: A function in $VMO(\mathbf{R}^n)$ has an extension to $\mathbf{R}^n \times (0, \infty)$ whose gradient is a Carleson measure. To apply this result in our setting, we must include some time derivative in the gradient. The obvious candidate would be a half-order time derivative, since this is of the same parabolic order as one spatial derivative. However, I am unable to control the L^1-norm of $N_\alpha(\partial_{x_0}^{1/2} u)$ by the L^1-norm of $N_\alpha(\nabla u)$. Fortunately, a version of Varopolous's theorem holds for fractional derivatives of any order if we compensate by multiplying by a power of the distance to the boundary and, as we shall see in Lemma 6.13, it is fairly easy to control the parabolic maximal function of $\delta(X\,;\Sigma_T)^{2\beta-1}\partial_{x_0}^\beta u$ when $\beta > \frac{1}{2}$.

We first consider the extension theorem for the half space $\{x_n > 0\}$. When we flatten out the boundary of a Lipschitz domain, the change of measure means that we must consider BMO on $\mathbf{R}\times\mathbf{R}^{n-1}$ with respect to a weight which is bounded above and below. Thus, we let σ denote a measure on $\mathbf{R}\times\mathbf{R}^{n-1}$ which satisfies

$$r^{n+1}/C_\sigma \le \sigma(I_r) \le C_\sigma r^{n+1} \tag{6.3}$$

for all cubes in $\mathbf{R}\times\mathbf{R}^{n-1}$. This restrictive assumption on σ allows us to follow the outline of Varopolous's argument without change. Also, we only prove this result for functions which are continuous and compactly supported since such functions are dense in VMO.

Proposition 6.4. *Let $f \in C_c(\mathbf{R}\times\mathbf{R}^{n-1})$, then there exists g on $\mathbf{R}\times\mathbf{R}^{n-1}$ and v a smooth function on $\mathbf{R}^{n+1}_+$ such that*

(1) $f(X') = v(X', 0) + g(X')$, *and*
(2) $\|g\|_{L^\infty(\mathbf{R}\times\mathbf{R}^{n-1})} \le C\|f\|_{BMO(\mathbf{R}\times\mathbf{R}^{n-1})}$,
(3) *for $0 < \beta \le 1$, the measure $d\mu(X) = (|\nabla v(X)| + x_n^{2\beta-1}|\partial_{x_0}^\beta v(X)|)\,dX$ is a Carleson measure with*

$$\|\mu\|_c \le C\|f\|_{BMO(\mathbf{R}\times\mathbf{R}^{n-1}\,;\sigma)}.$$

Furthermore, if f is supported in $[0, r^2]\times\overline{i_r(x')}$, then we may choose v to be supported in $\{(Y', y_n) : y' \in [0, 2r^2]\times\overline{i_{2r}(x')},\ 0 < y_n < 2r\}$ and g to be supported in $[0, r^2]\times\overline{i_2r(x')}$.

To give the proof of this proposition, we will need some more notation. We first define the mesh of dyadic parabolic cubes. Let $I_{2^k} = (0, 4^k)\times(0, 2^k)^{n-1}$. For $k \in \mathbf{Z}$, we let

$$\mathscr{D}_k = \{I_{2^k} + \gamma : \gamma \in d_{2^k}(\mathbf{Z}\times\mathbf{Z}^{n-1})\}$$

where d_{2^k} is the parabolic dilation introduced in §1. We let $\mathscr{D} = \bigcup_{k\in\mathbf{Z}}\mathscr{D}_k$. We recall that if I and J are two dyadic cubes, then either $I\cap J = \varnothing$, $I \subset J$ or

$J \subset I$. If $I \in \mathscr{D}_k$, then there exists a unique predecessor $J \in \mathscr{D}_{k+1}$ such that $I \subset J$. We let $\tilde{I} = J$ and we denote the jth predecessor of I by $\tilde{I}^j \in \mathscr{D}_{k+j}$. For a cube $I_r(X')$, we let $l(I_r(X')) = r$ denote its sidelength. Finally, for a cube I, we let

$$T(I) = \{(X', x_n) : X' \in I,\ 0 < x_n < l(I)\}$$

denote the tent over I. The sets $T(I)$ play the same role as the domains $\Psi_r(Q)$: μ is a Carleson measure if and only if $\mu(T(I)) \le C\sigma(I)$ for all cubes I.

With this notation available we can begin:

Proof of Proposition 6.4. We recall a decomposition of compactly supported functions in BMO due to Garnett: Given $f \in C_c(\mathbf{R} \times \mathbf{R}^{n-1})$, we may find a decomposition of $f = \sum c_\alpha \chi_{J_\alpha} + g$ where $\|g\|_{L^\infty(\mathbf{R}\times\mathbf{R}^{n-1})} \le C\|f\|_{BMO(\mathbf{R}\times\mathbf{R}^{n-1})}$, the cubes $\{J_\alpha\} \subset \mathscr{D}$ and the constants $\{c_\alpha\}$ satisfy

$$\sum_{\{\alpha : J_\alpha \subset I\}} \sigma(J_\alpha) \le C\sigma(I) \quad \text{for all cubes } I, \tag{6.5}$$

$$|c_\alpha| \le C\|f\|_{BMO(\mathbf{R}\times\mathbf{R}^{n-1};\sigma)}. \tag{6.6}$$

Also if I and J are disjoint dyadic cubes which have a face in common and satisfy $\frac{1}{2} \le l(I)/l(J) \le 2$, then

$$\left| \sum_{\substack{J_\alpha \supset I \\ J_\alpha \cap J = \varnothing}} c_\alpha - \sum_{\substack{J_\alpha \supset J \\ J_\alpha \cap I = \varnothing}} c_\alpha \right| \le C\|f\|_{BMO(\mathbf{R}\times\mathbf{R}^{n-1};\sigma)}. \tag{6.7}$$

Finally, if f is supported in $[0, r^2] \times \overline{i_r(x')}$, then the cubes will satisfy $J_\alpha \subset (0, 2r^2) \times i_{2r}(x')$. The decomposition of f satisfying (6.5) and (6.6) is established in [GJ, Theorem 2.1]. The statement (6.7) is in [Va, p. 288]. It is easy to see from Garnett and Jones argument that the decomposition does not increase the support.

Now we may define the function v by

$$v = \sum \chi_{T(J_\alpha)}.$$

It is easy to see that v is an extension of $f - g$ which satisfies the support property of the proposition. While v is not smooth, it is easy to smooth v out. We concentrate on establishing the Carleson measure condition in (2). We will confine ourselves to showing that $x_n^{2\beta-1}|\partial_{x_0}^\beta v(X)|dX$ is a Carleson measure when $\beta < 1$. The result for the spatial gradient or for $\beta = 1$ was treated by Varopoulos.

It suffices to check the Carleson condition for dyadic cubes. Thus we fix a cube $I \in \mathscr{D}$ and wish to show that

$$\int_{T(I)} x_n^{2\beta-1}|\partial_{x_0}^\beta v(X)|\, dX \le \sigma(I). \tag{6.8}$$

We let $\pi: \mathbf{R} \times \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ be the projection onto the spatial variables: $\pi(x_0, x') = x'$. It is clear that to establish (6.8), we only need to consider the contribution to v from cubes in $\mathscr{J} \equiv \{J_\alpha : \pi(J_\alpha) \cap \pi(I) \neq \varnothing\}$. (We only need to consider cubes in $\mathscr{J}$ which do not lie above I, but it seems simpler not to make this distinction.) We will split the cubes in $\mathscr{J}$ into three cases. In order to do this, we construct a family of disjoint dyadic cubes $\mathscr{B} = \bigcup_{k\ge 1} \mathscr{B}_k$ which satisfy: (1) $\mathbf{R} \times \pi(I) \subset \bigcup_{J\in\mathscr{B}} \bar{J}$, (2) If $I \in \mathscr{B}_k$, then $l(J) = 2^k l(I)$. (3) If $J \in \mathscr{B}_k$ with $k \ge 2$, then $\delta(J; I) \ge 2^{-1/2} 2^k l(I)$. (4) $\#\mathscr{B}_k \le 6$ for all k. To construct this collection we recall that $\tilde{I}^k$ is the kth predecessor of I and let $\tilde{I}^{k\prime} = \tilde{I}^k \pm (2^k l(I))^2 e_0$ where we choose the sign which makes $\tilde{I}^{k\prime}$ closest to I. Now we define $\mathscr{B}_1 = \{\tilde{I} + j(2l(I))^2 :$ this cube is in $\tilde{I}^2 \cup \tilde{I}^{2\prime}\}$ and for $k \ge 2$, we put $\mathscr{B}_k = \{\tilde{I}^k + j(2^k l(I))^2 :$ for j such that this cube is in $(\tilde{I}^{k+1} \cup \tilde{I}^{k+1\prime}) \setminus (\tilde{I}^k \cup \tilde{I}^{k\prime})\}$. Now we may define

$$\begin{aligned}
\mathscr{J}_1 &= \{J_\alpha : l(J_\alpha) \le l(I)\},\\
\mathscr{J}_2 &= \{J_\alpha : l(J_\alpha) > l(I) \text{ and } K \subseteq J_\alpha \text{ for some } K \in \mathscr{B}\},\\
\mathscr{J}_3 &= \{J_\alpha : l(J_\alpha) > l(I) \text{ and } J_\alpha \subsetneq K \text{ for some } K \in \mathscr{B}\}.
\end{aligned}$$

It is clear that $\mathscr{J}$ is the disjoint union of the $\mathscr{J}_i$'s. We estimate the contribution to (6.8) from $\mathscr{J}_1$, $\mathscr{J}_2$ and $\mathscr{J}_3$ separately.

Before beginning the study of these cases, we observe that if $J = (x_0, x_0 + 4^k) \times j_{2^k}(x')$, then we have the following estimate for fractional derivatives of $\chi_{T(J)}$ when $y \in j_{2^k}(x')$:

(6.9)
$$|\partial^\beta_{x_0} \chi_{T(J)}(y_0, y)| \le C \begin{cases} |x_0 - y_0|^{-\beta} + |x_0 + 4^k - y_0|^{-\beta}, & |x_0 - y_0| < 2\cdot 4^k,\\ l(J)^2 |x_0 - y_0|^{-\beta-1}, & |x_0 - y_0| \ge 2 \cdot 4^k, \end{cases}$$

and of course this derivative vanishes if $y \notin j_{2^k}(x')$.

Contribution from cubes in $\mathscr{J}_1$. let $I_j = I + jl(I)^2 e_0$ for $j \in \mathbf{Z}$. Each $J_\alpha \in \mathscr{J}_1$ is contained in I_j for some j. When $j = -1,\ 0,\ 1$, we have

$$\begin{aligned}
&\sum_{J_\alpha \subset I_j} \int_{T(I)} |c_\alpha| x_n^{2\beta-1} |\partial^\beta_{x_0} \chi_{T(J_\alpha)}(X)|\, dX\\
&\le C \sum_{J_\alpha \subset I_j} l(J_\alpha)^{n-1+2\beta} \left[\int_0^{2l(J_\alpha)^2} s^{-\beta}\, ds + l(J_\alpha)^2 \int_{2l(J_\alpha)^2}^\infty s^{-\beta-1}\, ds \right] \qquad (6.10)\\
&\le C' \sum_{J_\alpha \subset I_j} l(J_\alpha)^{n+1} \le C'' \sigma(I)
\end{aligned}$$

where we have used (6.5) and (6.9). When $|j| \ge 2$, the first sum in (6.10) is bounded by

$$C\sum_{J_\alpha\subset I_j} l(J_\alpha)^{n-1+2\beta} l(J_\alpha)^2 \int_0^{l(I)^2} (s-|j|l(I)^2)^{-\beta-1}\,ds$$

$$\le C'\sum_{J_\alpha\subset I_j} l(J_\alpha)^{n+1}\left(\frac{l(J_\alpha)}{l(I)}\right)^{2\beta} j^{-\beta-1} \le C''\sigma(I)j^{-\beta-1}.$$

where the second inequality uses that $l(J_\alpha)\le l(I)$ for $J_\alpha\in\mathscr{J}$. Hence, we may add up the contributions from each I_j and obtain the desired estimate for the contribution from cubes in $\mathscr{J}_1$.

Contribution from cubes in $\mathscr{J}_2$. We will consider the contribution to $\partial^\beta_{x_0}\chi_{T(J_\alpha)}$ from each spacelike face of $T(J_\alpha)$ separately. To obtain some cancellation, we group the faces which lie in a hyperplane $\{x_0 = \text{constant}\}$.

To carry this out, choose J and J' in $\mathscr{B}$ such that J and J' have a face in common, say lying in the hyperplane $\{x_0 = a\}$. If $J\in\mathscr{B}_k$, then $J'\in\mathscr{B}_{k'}$ with $|k-k'|\le 1$. Observe that if a face of $J_\alpha\in\mathscr{J}_2$ satisfies $\bar{J}_\alpha\cap\{x_0=a\}\ne\varnothing$ then either $J_\alpha\subset J$ and $J_\alpha\cap J'=\varnothing$ or $J_\alpha\subset J'$ and $J_\alpha\cap J=\varnothing$. The contribution from faces lying in the hyperplane $\{x_0=a\}$ is estimated by

$$C\int_{T(I)}\left|\sum_{\substack{J_\alpha\supset J\\ J_\alpha\cap J'=\varnothing}} c_\alpha - \sum_{\substack{J_\alpha\supset J'\\ J_\alpha\cap J=\varnothing}} c_\alpha\right| x_n^{2\beta-1}|x_0-a|^{-\beta}\,dX \tag{6.11}$$

$$= C' l(I)^{n+1}4^{-\beta k} \le C''\sigma(I)4^{-\beta k}$$

where we have used (6.7) and that $\delta(J;I)\ge 2^k l(I)$ for $k\ge 2$. Observe that each face of $J_\alpha\cap\{x_0 = \text{constant}\}$ meets a face of some $J\in\mathscr{B}$. Hence, summing the estimate (6.11) on pairs of adjacent cubes in $\mathscr{B}$, we obtain

$$\left|\int_{T(I)} x_n^{2\beta-1}\sum_{J_\alpha\in\mathscr{J}_2} c_\alpha\partial^\beta_{x_0}\chi_{T(J_\alpha)}(X)\,dX\right| \le C\sigma(I)\sum_{k\ge 1}4^{-\beta k} = C'\sigma(I)$$

where we have used that there are at most six cubes in each $\mathscr{B}_k$.

Contribution from cubes in $\mathscr{J}_3$. This is easy since there are not very many cubes in $\mathscr{J}_3$ and they are far away. To be precise, let

$$J_{j,k} = \tilde{I}^k = j(2^k l(I))^2 e_0 \quad \text{for } k\ge 1 \text{ and } |j|\ge 2.$$

Observe that all cubes in $\mathscr{J}_3$ arise as some $J_{j,k}$. Thus since $\delta(J_{j,k};I)\ge \frac{1}{2}\sqrt{j}2^k l(I)$,

$$\int_{T(I)} |\partial^\beta_{x_0}\chi_{T(J_{j,k})}(X)|\,dX \le l(I)^{n+1}2^{-k}|j|^{-3/2}$$

and we may sum on $|j|\ge 1$ and $k\ge 2$. □

From Proposition 6.4, we can easily prove the following corollary.

Corollary 6.12. *Let* $f \in C_c(\Sigma_T)$ *and* $0 < \beta \leq 1$, *then there exists a Lipschitz function* v *defined on* $(-\infty, T) \times \Omega$ *and a function* $g \in L^\infty(\Sigma_T)$ *which satisfy* $f(P) = v(P) + g(P)$, *for* $P \in \Sigma_T$, $v(X) = 0$ *for* $x_0 < 0$,

$$\|g\|_{L^\infty(\Sigma_T)} \leq C\|f\|_{\widetilde{BMO}(\Sigma_T)}$$

and

$$d\mu(X) = (|\nabla v(X)| + \delta(X ; \Sigma_T)^{2\beta-1}|\partial^\beta_{x_0} v(X)|)\, dX$$

is a Carleson measure on Ω_T *with norm* $\|\mu\|_c \leq C\|f\|_{\widetilde{BMO}(\Sigma_T)}$.

Proof. We assume that $T+r_0^2$, the general case follows from the same argument, however one obtains a constant that depends on $\max(T, r_0^2)$. Using a Lipschitz partition of unity on $\partial\Omega$, $1 = \sum \psi_i$, we may assume that f is supported in $(0, T) \times Z \cap \partial\Omega$, for Z a coordinate cylinder. We extend f by an even reflection by setting

$$\tilde{f}(p_0, p) = \begin{cases} f(p_0, p), & 0 < p_0 < T, \\ f(T - p_0, p), & T < p_0 < 2T, \\ 0, & \text{otherwise.} \end{cases}$$

Letting ϕ be the Lipschitz function whose graph is $\partial\Omega$ near Z, π the coordinate projection and $d\sigma(X') = \sqrt{1 + |\nabla'\phi(x')|^2}\, dX'$ we observe that

$$\|\tilde{f} \circ \pi^{-1}\|_{BMO(\mathbf{R}\times\mathbf{R}^{n-1} ; \sigma)} \approx \|f\|_{\widetilde{BMO}(\Sigma_T)}.$$

Applying Proposition 6.4, we obtain a function $h \in \mathbf{R} \times \mathbf{R}^{n-1}$ and an extension w of $\tilde{f} \circ \pi^{-1} - h$ to the upper halfspace. We let $\pi \colon \{x_n > \phi(x')\} \to \{x_n > 0\}$ also denote the map $\pi(X', x_n) = (X', x_n - \phi(x'))$. We see that $v \equiv w \circ \pi$ and $g = h \circ \pi$ have the desired properties. The restriction on the support of w in Proposition 6.4 guarantees that $\operatorname{supp} v$ only approaches Σ_T near $\operatorname{supp} f$, thus it is clear that v satisfies the Carleson measure condition. □

In order to use this last result, we will need an estimate for parabolic maximal functions involving fractional derivatives. This is the subject of the next lemma. To prove our estimate, we will use a strong Hardy-Littlewood maximal operator. For a function f defined on Σ_T, we let

$$M(f)(P) = \sup_{s, r>0} s^{-1} r^{1-n} \int_{p_0-s}^{p_0} \int_{i_r(p)} |f(Q)|\, dQ, \qquad P \in \Sigma_T.$$

It is well known that M is bounded on $L^p(\Sigma_T)$ for $1 < p \leq \infty$.

Lemma 6.13. *Let* u *be caloric in* Ω_T *and zero initially, then for* $\frac{1}{2} < \beta \leq 1$ *and* $\frac{1}{1+\beta} < p < \infty$ *we have*

$$\int_{\Sigma_T} N_\alpha(\delta(\cdot ; \Sigma_T)^{2\beta-1}\partial^\beta_{x_0} u)(Q)^p\, dQ \leq C \int_{\Sigma_T} N_\alpha(\nabla u)(Q)^p\, dQ.$$

The constant C *depends on* α, p, β, T, m *and* $\{Z_i\}$.

Proof. Let u be caloric in Ω_T and zero initially. We begin by claiming that for $1 \geq \beta > \frac{1}{2}$ and $\frac{1}{1+\beta} < q \leq 1$, there exists $r_0 > 0$ and α large such that

$$(6.14)\quad \begin{aligned} &N_{\alpha/2}(\delta(\cdot\,;\Sigma_T)^{2\beta-1}\partial^{\beta}_{x_0}u)(P) \\ &\qquad \leq C\left(M(N_\alpha(\nabla u)^q)(P)^{1/q} + T^{1-\beta}r_0^{2\beta-1}\sup_{Y\in\Omega_{r_0,T}}|\partial_{y_0}u(Y)|\right)\end{aligned}$$

where $\Omega_{r_0,T} = \{Y \in \Omega_T : \delta(Y;\Sigma_T) > r_0\}$.

Assuming (6.14) for the moment we establish the proposition. We choose q so that $\frac{1}{1+\beta} < q < p$. Using the boundedness of $f \to M(|f|^q)^{1/q}$ on $L^p(\Sigma_T)$, we obtain

$$\begin{aligned} &\int_{\Sigma_T} N_{\alpha/2}(\delta(\cdot\,;\Sigma_T)^{2\beta-1}\partial^{\beta}_{x_0}u)(Q)^p\,dQ \\ &\qquad \leq C\left(\int_{\Sigma_T} N_\alpha(\nabla u)(Q)^p\,dQ + |\Sigma_T|\sup_{\Omega_{r_0,T}}|\partial_{y_0}u(Y)|^p\right)\end{aligned}$$

for α as in (6.14). Now using the argument in Lemma 5.9 for the second inequality below we have

$$\sup_{\Omega_{r_0,T}}|\partial_{y_0}u| \leq \frac{C}{r_0}\sup_{\Omega_{r_0/2,T}}|\nabla u| \leq Cr_0\left(\frac{1}{r_0^{n+2}}\int_{\Sigma_T} N_\alpha(\nabla u)(Q)^p\,dQ\right)^{\frac{1}{p}}.$$

Combining these last two inequalities and recalling that parabolic maximal functions defined using different cone openings α have comparable L^p-norms, we see that we are done with the proof of the Proposition.

To prove the pointwise estimate (6.14), we make a second claim: Let $X \in \Gamma(P,\alpha/2)$ and let v be caloric in Ω_T with zero initial values, then for $\frac{1}{2} < \beta \leq 1$ and $\frac{1}{1+\beta} < q \leq 1$ we have

$$(6.15)\qquad \delta(X;\Sigma_T)^{2\beta}|\partial^{\beta}_{x_0}v(X)| \leq CM(N_\alpha(v)^q)(P)^{\frac{1}{q}}.$$

Assuming this claim for a moment, we prove (6.14).

To do this, suppose that $P = (p_0, p)$ with $p \in Z_i$, a coordinate cylinder, and let e_n be the direction of the x_n-axis in the coordinate system associated to Z_i. Choose $r_0 > 0$ so that if $X \in \Gamma(P,\alpha/2)$, then $X + re_n \in \Gamma(P,\alpha/2)$ when $r > 0$ and $\delta(X + re_n;\Sigma_T) < r_0$. We may do this when α is large. We assume that $\delta(X;\Sigma_T) < r_0$, otherwise (6.14) is trivial, and let $h > 0$ be the smallest number so that $\delta(X + he_n;\Sigma_T) = r_0$. We have

$$\begin{aligned}|\partial^{\beta}_{x_0}u(X)| &\leq \int_0^h |\partial^{\beta}_{x_0}\partial_{x_n}u(X+\rho e_n)|\,d\rho + |\partial^{\beta}_{x_0}u(X+he_n)| \\ &\leq C\left(M(N_\alpha(\nabla u)^q)(P)^{\frac{1}{q}}\int_{\delta(X;\Sigma_T)}^{\infty}\rho^{-2\beta}\,d\rho + \sup_{Y\in\Omega_{r_0,T}}|\partial^{\beta}_{y_0}u(Y)|\right)\end{aligned}$$

where the second inequality uses the claim applied to $\partial_{x_n} u$ and that $\delta(X+\rho e_n;\Sigma_T) \approx \delta(X;\Sigma_T)+\rho$ for $0<\rho<h$. We evaluate the integral and multiply by $\delta(X;\Sigma_T)^{2\beta-1} \le r_0^{2\beta-1}$, obtaining

$$\delta(X;\Sigma_T)^{2\beta-1}|\partial^{\beta}_{x_0}u(X)| \le C\left(M(N_\alpha(\nabla u)^q)(P)^{\frac{1}{q}} + r_0^{2\beta-1}\sup_{Y\in\Omega_{r_0,T}}|\partial^{\beta}_{y_0}u(Y)|\right).$$

Finally, observing that

$$|\partial^{\beta}_{y_0}u(Y)| \le CT^{1-\beta}\sup_{0<s<T}|\partial_s u(s,y)|$$

we have established (6.14).

Thus we are left with proving (6.15). Before giving the argument, we recall an interior estimate for caloric functions: Let v be caloric in $J_{2r}(X)$ and suppose $0<p<\infty$, then

$$\sup_{Y\in J_r(X)}(r^2|\partial_{y_0}u(Y)|+|v(Y)|) \le \left(r^{-n-2}\int_{J_{2r}(X)}|v(Y)|^p\,dY\right)^{\frac{1}{p}}. \tag{6.16}$$

When p is in the range $1\le p<\infty$, this estimate is well known. The case $0<p<1$ follows from the case of $p\ge 1$ using the techniques of Dahlberg and Kenig in [FSt, pp. 1004–1005]. Turning to the proof of (6.15), we have $X\in\Gamma(P,\alpha/2)$ and we choose $r=c\delta(X;\Sigma_T)$ so that $\delta(J_{2r}(X);\Sigma_T)\ge\frac{1}{2}\delta(X;\Sigma_T)$ and $J_{2r}(X)\subset\Gamma(P,\frac{3\alpha}{4})$. Define $X_k=X-kr^2e_0$ and $P_k=P-kr^2e_0$. Ignoring the gamma function in the definition of $\partial^{\beta}_{x_0}$, we have

$$\begin{aligned}|\partial^{\beta}_{x_0}v(X)| &= \left|\int_{-\infty}^{x_0}\partial_t v(t,x)(x_0-t)^{-\beta}\,dt\right|\\ &\le \int_{x_0-r^2}^{x_0}|\partial_t v(t,x)|(x_0-t)^{-\beta}\,dt + r^{-2\beta}|v(x_0-r^2,x)|\\ &\quad+\beta\sum_{k=1}^{\infty}\int_{x_0-(k+1)r^2}^{x_0-kr^2}|v(t,x)|(x_0-t)^{-\beta-1}\,dt\end{aligned} \tag{6.17}$$

where the second expression is obtained by splitting the integral at x_0-r^2 and then integrating by parts on $(-\infty,x_0-r^2)$. Using (6.16), we may bound each of the first two terms on the right of (6.17) by

$$Cr^{-2\beta}\left(r^{-n-2}\int_{J_{2r}(X)}|v(Y)|^q\,dY\right)^{\frac{1}{q}}$$

while the kth term of the sum is bounded by

$$C(k+1)^{-\beta-1}r^{-2\beta}\left(r^{-n-2}\int_{J_{2r}(X_k)}|v(Y)|^q\,dY\right)^{\frac{1}{q}}.$$

Thus multiplying by $\delta(X;\Sigma_T)^{2\beta} \approx r^{2\beta}$, we have
(6.18)

$$\begin{aligned}\delta(X;\Sigma_T)^{2\beta q}|\partial_{x_0} v(X)|^q &\le C\left(\sum_{k=0}^{\infty}(k+1)^{-\beta-1}\left(r^{-n-2}\int_{J_{2r}(X_k)}|v(Y)|^q\,dY\right)^{\frac{1}{q}}\right)^q\\ &\le C'\sum_{k=0}^{\infty}(k+1)^{-q(\beta+1)}r^{-n-2}\int_{J_{2r}(X_k)}|v(Y)|^q\,dY\\ &\le C''\sum_{k=0}^{\infty}(k+1)^{-q(\beta+1)}r^{-n-1}\int_{I_{Cr}(P_k)}N_\alpha(v)(Q)^q\,dQ\end{aligned}$$

where the second inequality is Minkowski's inequality applied to the sum; this is valid since $\frac{1}{q} \ge 1$. The last expression in (6.18) may be easily estimated by the strong maximal function:

$$\begin{aligned}&r^{-n-1}\sum_{k=0}^{\infty}(k+1)^{-q(\beta+1)}\int_{I_{Cr}(P_k)}N_\alpha(v)(Q)^q\,dQ\\ &\le Cr^{-n-1}\int_{-\infty}^{p_0}\int_{i_{Cr}(P)}N_\alpha(v)(Q)^q\left(1+\frac{|p_0-q_0|}{r^2}\right)^{-q(\beta+1)}dQ\\ &\le C'\sum_{k=0}^{\infty}2^{-kq(\beta+1)}r^{-n-1}\int_{p_0-2^kr^2}^{p_0}\int_{i_{Cr}(P)}N_\alpha(v)(Q)^q\,dQ\\ &\le C''M(N_\alpha(v)^q)(P)\sum_{k=0}^{\infty}2^{-k(q(\beta+1)-1)}.\end{aligned} \tag{6.19}$$

This last sum converges precisely when $q > \frac{1}{1+\beta}$. Combining (6.18) and (6.19) yields (6.15). □

For the proof of Theorem 6.1, we recall the pairing between $\tilde{H}^1(\Sigma_T)$ and $\widetilde{BMO}(\Sigma_T)$ defined in (1.14).

Proof of Theorem 6.1. We show that for $f \in C_c(\Sigma_+)$ and $u \in H^1_N(\Omega_T)$ that

$$\left|\int_{\Sigma_T}\partial_\nu u(P)R(f)(P)\,dP\right| \le C_T\|f\|_{\widetilde{BMO}(\Sigma_T)}\|N_\alpha(\nabla u)\|_{L^1(\Sigma_T)}. \tag{6.20}$$

Since $C_c(\Sigma_T)$ is dense in $\widetilde{VMO}(\Sigma_T)$ and $\tilde{H}^1(\Sigma_T)$ is the dual of $\widetilde{VMO}(\Sigma_T)$, we have $\partial_\nu u \in \tilde{H}^1(\Sigma_T)$ and $\|\partial_\nu u\|_{\tilde{H}^1(\Sigma_T)} \le C\|N_\alpha(\nabla u)\|_{L^1(\Sigma_T)}$. The rest of Theorem 6.1 is contained in Corollary 5.12.

Let $f = v + g$ be the extension of f given by Corollary 6.12. Since we clearly have

$$\left|\int_{\Sigma_T}\partial_\nu u(P)R(g)(P)\,dP\right| \le C\|N_\alpha(\nabla u)\|_{L^1(\Sigma_T)}\|f\|_{\widetilde{BMO}(\Sigma_T)},$$

thus to establish (6.20), we only need to study

$$\int_{\Sigma_T} \partial_\nu u(P) R(v)(P)\, dP.$$

To establish this, we recall the main property of Carleson measures:

$$\int_{\Omega_T} |v(Y)|\, d\mu(Y) \le C\|\mu\|_c \|N_\alpha(v)\|_{L^1(\Sigma_T)}. \tag{6.21}$$

We use the divergence theorem to obtain

$$\int_{\Sigma_T} \partial_\nu u(P) R(v)(P)\, dP = -\int_{\Omega_T} \nabla u(Y) \nabla R(v)(Y) + \partial_{y_0} u(Y) R(v)(Y)\, dY. \tag{6.22}$$

Next we note that since u is zero at $x_0 = 0$ and $R(v) = 0$ at $x_0 = T$, we may write $\partial_{x_0} = \partial_{x_0}^{1/4} \partial_{x_0}^{3/4}$, integrate one-quarter of a derivative by parts in the last term of (6.22) and obtain

$$\begin{aligned}
&\left| \int_{\Sigma_T} \partial_\nu u(P) R(v)(P)\, dP \right| \\
&\quad \le C \int_{\Omega_T} |\nabla u(Y) \nabla R(v)(Y)| + \delta(Y; S)^{-\frac{1}{2}} |\partial_{y_0}^{\frac{3}{4}} u(Y)| \delta(Y; S)^{\frac{1}{2}} |\partial_{y_0}^{\frac{1}{4}} R(v)(Y)|\, dY \\
&\quad \le C\|f\|_{\widetilde{BMO}(\Sigma_T)} \int_{\Sigma_T} N_\alpha(\nabla u)(P)\, dP
\end{aligned}$$

where the second inequality uses Lemma 6.13 and (6.21). □

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Department of Mathematics, University of Chicago, Chicago Illinois 60637

Current address: Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506