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SEMIALGEBRAIC EXPANSIONS OF C

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ABSTRACT. We prove no nontrivial expansion of the field of complex numbers can be obtained from a reduct of the field of real numbers.

1. INTRODUCTION

Recently conjectures of Zilber have focused attention on expansions of algebraically closed fields. In particular can an algebraically closed field have a nontrivial strongly minimal expansion? Here we will examine a natural class of expansions of $\mathbf{C}$ and show in the strongest way possible that none violates Zilber's conjecture.

Definition. We say that $S \subseteq \mathbf{R}^n$ is *semialgebraic* if there is a formula

$$\varphi(v_1, \ldots, v_n, w_1, \ldots, w_m)$$

in the language of ordered rings and $a_1, \ldots, a_m \in \mathbf{R}$ such that $S = \{\overline{x} \in \mathbf{R}^n : \mathbf{R} \models \varphi(\overline{x}, \overline{a})\}$. If $S \subseteq \mathbf{R}^{2n}$ is semialgebraic we say that

$$\widetilde{S} = \left\{ \overline{z} \in \mathbf{C}^n : \exists \overline{a} \in S \bigwedge_{j=1}^{n} z_j = a_{2j-1} + a_{2j} i \right\}.$$

We say that $A \subseteq \mathbf{C}^n$ is *constructible* if there is $\varphi(v_1, \ldots, v_n, w_1, \ldots, w_m)$, a formula in the language of rings, and $a_1, \ldots, a_m \in \mathbf{C}$ such that $A = \{\overline{x} \in \mathbf{C}^n : \mathbf{C} \models \varphi(\overline{x}, \overline{a})\}$.

Usually in the definitions of constructible and semialgebraic we restrict our attention to quantifier free formulas. By Tarski's elimination of quantifiers for algebraically closed fields and real closed field these formulations are equivalent.

For each semialgebraic set $S \subseteq \mathbf{R}^{2n}$ we will consider the expansion $\mathfrak{A}_s$ of $\mathbf{C}$ in the language $\mathscr{L}_n$ where we add to the language of rings a new n-ary predicate symbol P_n which we interpret in $\mathfrak{A}_s$ as $\widetilde{S}$.

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Definition. For any such structure $\mathfrak{A}_s$ we say $X \subseteq \mathbf{C}^m$ is *definable* if there is an $\mathscr{L}_n$-formula $\varphi(v_1, \ldots, v_n, w_1, \ldots, w_k)$ and $a_1, \ldots, a_k \in \mathbf{C}$ such that $X = \{\overline{x} \in \mathbf{C}^m : \mathfrak{A}_s \models \varphi(\overline{x}, \overline{a})\}$. Let $\mathrm{Def}(\mathfrak{A}_s)$ denote the collection of all definable subsets of $\mathfrak{A}_s$.

The following facts are obvious:

Proposition 1.1. *Let $S \subseteq \mathbf{R}^{2n}$ be semialgebraic.*
(1) *Every constructible set is in* $\mathrm{Def}(\mathfrak{A}_s)$.
(2) *Every set in* $\mathrm{Def}(\mathfrak{A}_s)$ *is semialgebraic.*
(3) $\mathbf{R} \in \mathrm{Def}(\mathfrak{A}_s)$ *if and only if every semialgebraic set is in* $\mathrm{Def}(\mathfrak{A}_s)$.
If $S = \varnothing$, then $\mathrm{Def}(\mathfrak{A}_s)$ *is just the constructible sets, while if* $S\{(x, 0) : x \in \mathbf{R}\}$, *then* $\mathrm{Def}(\mathfrak{A}_s)$ *is all of the semialgebraic sets.*

Our main theorem states that these are the only possibilities.

Theorem 1.2. *If $S \subseteq \mathbf{R}^n$ is semialgebraic, then either $\widetilde{S}$ is constructible or* $\mathbf{R} \in \mathrm{Def}(\mathfrak{A}_s)$.

Thus Zilber's conjecture holds for semialgebraic expansions since they are either trivial or unstable.

Recently Hrushovski has refuted the general case of Zilber's conjecture and its seems likely that his methods will extend to produce a strongly minimal expansion of $\mathbf{C}$. Our results provide a counterpoint to Hrushovski. The proof of Theorem 1.2 makes use of the analysis of definable sets in $\mathscr{O}$-minimal theories and elementary algebraic geometry and real algebraic geometry. In §2, we review some preliminaries on $\mathscr{O}$-minimality and real algebraic geometry. [H and B-C-R] are the standard references on algebraic and real algebraic geometry. [Po and Di] give more model theoretic treatments of these subjects.

I would like to thank Ali Nesin and Lou van den Dries for several stimulating discussions on this topic.

2. Preliminaries

A. $\mathscr{O}$-minimality. We begin with some basics on $\mathscr{O}$-minimal theories. These results come from [Dr1, P-S and K-P-S]. Let $\mathscr{L}$ be a language containing a binary relation symbol $<$.

Definition. A complete $\mathscr{L}$ theory T is said to be *$\mathscr{O}$-minimal* if for every $M \models T$, $<$ is a linear order of M and every $\mathscr{L}$-definable subset of M is a finite union of points and intervals (Throughout this section "definable" means "definable with parameters".)

Using Tarski's quantifier elimination it is easy to see that the theory of real closed fields in $\mathscr{O}$-minimal. Below for simplicity we assume that M is an $\mathscr{O}$-minimal expansion of $\mathbf{R}$.

Theorem 2.1 (Monotonicity Theorem). *If X is a definable subset of M and $f : X \to M$ is definable, then for each $m \in \omega$ we can find open intervals*

$I_1, \ldots, I_n$ and a finite set Y such that $X = I_1 \cup \cdots \cup I_n \cup Y$ and f is C^m on each I_j.

In fact if M is the field of real numbers every definable function is piecewise analytic [B-C-R].

Remarkably $\mathcal{O}$-minimality imposes strong constraints of subsets of M^n.

Definition. (a) $X \subseteq M$ is a *cell* if and only if it is a singleton or an interval.

(b) $X \subseteq M^{n+1}$ is a *cell* if and only if

(i) there is a cell $Y \subseteq M^n$ and $f: Y \to M$ a continuous definable function such that $X = \{(\overline{y}, f(\overline{y})): \overline{y} \in Y\}$ or

(ii) there is a cell $Y \subseteq M^n$ and f, $g: Y \to M$ are continuous functions such that for all $\overline{y} \in Y$, $f(\overline{y}) < g(\overline{y})$ and $X = \{(\overline{y}, x): \overline{y} \in Y \wedge f(\overline{y}) < x < g(\overline{y})\}$. [Here we allow the possibility that $f = -\infty$ or $g = +\infty$.]

We associate to each cell a dimension. Singletons have dimension 0. Intervals have dimension 1. If Y is a cell of dimension n and $f: Y \to M$ is definable, then the graph of f has a dimension n. If f and g are definable functions from Y to M such that for all $\overline{y} \in Y$, $f(\overline{y}) < g(\overline{y})$ and Y has dimension n, then $\{(\overline{y}, x): \overline{y} \in Y \text{ and } f(\overline{y}) < x < g(\overline{y})\}$ has dimension $n+1$. If $C_1, \ldots, C_m$ are cells we let the dimension of the $C_1 \cup \cdots \cup C_m$ be the maximum of the dimension of the C_i. Let $\dim(X)$ be the dimension of X.

Theorem 2.2 (Cell decomposition). *If $X \subseteq M^n$ is definable, then X is a finite disjoint union of cells.*

We will use the following consequences of 2.2.

Corollary 2.3 (Uniform boundedness). *Suppose $X \subseteq M^n$ is definable and $k < n$. For $\overline{a} \in M^k$ let $X_{\overline{a}} = \{\overline{y} \in M^{n-k}: (\overline{a}, \overline{y}) \in X\}$. There is a number N such that for all $\overline{a} \in M^k$ if $X_{\overline{a}}$ is finite then $|X_{\overline{a}}| < N$.*

Corollary 2.4. *Suppose X is a definable subset of M^n, Y is a definable subset of X and $\dim(X) = \dim(Y)$. Then there is a definable open set $U \subseteq M^n$ such that $\dim(U \cap X) = \dim(X)$ and $U \cap X \subseteq Y$.*

B. Real algebraic geometry.

Definition. A prime ideal $I \subseteq \mathbf{R}[X_1, \ldots, X_n]$ is said to be *real* if and only if whenever $f_1, \ldots, f_m \in \mathbb{R}[\overline{X}]$, $a_1, \ldots, a_m \in \mathbf{R}^+$ and $\sum a_i f_i^2 \in I$, then all $f_i \in I$.

$V \subseteq \mathbf{R}^n$ is a real variety if and only if $I(V) = \{f \in \mathbf{R}[\overline{X}]: \text{ for all } \overline{a} \in V\ f(\overline{a}) = 0\}$ is a real ideal.

The next result gives an algebraic treatment of dimension.

Theorem 2.5. (a) *If V is a real irreducible variety, then the dimension of V is transcendence degree of the coordinate ring $\mathbf{R}[\overline{X}]/I(V)$.*

(b) *If $X \subseteq \mathbf{R}^n$ is semialgebraic and V is its Zariski closure then $\dim(X) = \dim(V)$.*

In §4 we will also use the fact that the dimension of a real irreducible variety is its Krull dimension.

C. Generics. Let $K = \mathbf{R}$ or $\mathbf{C}$. For $X \subseteq K^n$, let $\dim_K(X)$ be its dimension. Let $S_n(K)$ be the space of n-types over K (alternatively we could use the real spectrum or the Zariski spectrum.)

Definition. If $\varphi(v_1, \dots, v_n, \overline{w})$ is a formula in the language of fields and $\overline{a} \in K^{|\overline{w}|}$, let $\dim_K(\varphi(\overline{v}, \overline{a})) = \dim_K(\{\overline{b} \in K^n : \varphi(\overline{b}, \overline{a})\})$.

If $p(v_1, \dots, v_n) \in S_n(K)$, let $\dim_K(p) = \min\{\dim_K(\varphi(\overline{v}, \overline{a})) : \varphi(\overline{v}, \overline{a}) \in p\}$. (For $K = \mathbf{C}$, this is the Morely rank of p.)

If $A = \{\overline{b} \in K^n : K \models \varphi(\overline{b}, \overline{a})\}$ and F is a $|K|^+$-saturated elementary extension of K, we say that $x_1, \dots, x_n$ is a *generic point* of A if and only if $F \models \varphi(\overline{x}, \overline{a})$ and if $p(\overline{v})$ is the type of $\overline{x}$ over K, then $\dim_K(A) = \dim_K(p)$.

Generics are useful fictions when computing dimensions.

Proposition 2.6. *If $(x_1, \dots, x_n)$ is a generic of $A \subseteq K^n$, then $\dim_K(A)$ is equal to the transcendence degree of $K(x_1, \dots, x_n)$ over K.*

3. $S \subseteq \mathbf{R}^2$

For $S \subseteq \mathbf{R}^2$ we can prove a somewhat stronger result.

Theorem 3.1. *Suppose $S \subseteq \mathbf{R}^2$ is definable in an $\mathscr{O}$-minimal expansion of $\mathbf{R}$. Then $\widetilde{S}$ is constructible or $\mathbf{R} \in \mathrm{Def}(\mathfrak{A}_s)$.*

In particular by [D-Dr] if $S \subseteq \mathbf{R}^2$ is subanalytic and $\widetilde{S}$ is not constructible, then $\mathbf{R}$ is definable in $\mathfrak{A}_s$.

Lemma 3.2. *If $S \subseteq \mathbf{R}^2$ is infinite, coinfinite, and definable in an $\mathscr{O}$-minimal expansion of $\mathbf{R}$, then there is a $Y \subseteq \mathbf{C}$ such that $Y \in \mathrm{Def}(\mathfrak{A}_s)$ and if $\widehat{Y} = \{(x, y) \in \mathbf{R}^2 : x + iy \in Y\}$, $\widehat{Y}$ is two dimensional and bounded.*

Proof.

Case 1. $\dim(S) = \dim(\mathbf{R}^2 - S) = 2$.

By 2.4 $\mathbf{R}^2 - S$ contains an open set. In general if $z \in \mathbf{C}$ and $r \in \mathbf{R}^+$, let $B_r(z) = \{x \in \mathbf{C} : \|x - z\| < r\}$. We let $B_r = B_r(0)$. There is $z \in \mathbf{C}$ and $r \in \mathbf{R}^+$ such that $B_r(z) \cap \widetilde{S} = \varnothing$. Let $Y = \{\frac{r}{x-z} : x \in \widetilde{S}\}$. Then $Y = \mathrm{Def}(\mathfrak{A}_s)$, $\dim(\widehat{Y}) = 2$ and $Y \subset B_1$.

Case 2. $\dim(S) = 1$ or $\dim(\mathbf{R}^2 - S) = 1$.

Assume $\dim(S) = 1$ (replace S by $\mathbf{R}^2 - S$ if necessary). Once again $\mathbf{R}^2 - S$ contains an open set say $B_r(z) \cap S = \varnothing$. Let $Y_1 = \{\frac{r}{x-z} : x \in \widetilde{S}\}$. Then $Y_1 \subseteq B_1$ and $\dim(\widehat{Y}_1) = 1$. Translating Y_1 if necessary, we may assume that $(0, 0) \in \widehat{Y}$ and $\widehat{Y}_1$ is smooth at $(0, 0)$. [Recall that at all but a finite set of points $\widehat{Y}_1$ is locally the graph of a C^m function.]

Let $Y = \{x + iy: x, y \in Y_1\}$. Clearly Y is bounded.

Claim. $\dim \widehat{Y} = 2$.

If $\dim \widehat{Y} = 1$ then there will be points x arbitrarily close to 0 with open neighborhoods $B_r(x)$ such that $B_r(x) \cap Y = B_r(x) \cap Y_1$. But then if $Y_1^x = \{x + iy: y \in Y_1\}$, $Y_1^x \cap B_r(x) \subseteq Y_1$. But for x near 0, $\widehat{Y}_1^x$ is nearly normal to $\widehat{Y}_1$, a contradiction. □

The idea for Case 2 was pointed out to me by Ali Nesin. It simplifies an earlier algebraic argument which did not work in the general $\mathscr{O}$-minimal setting.

Definition. If $X \subseteq \mathbf{R}^2$ let ∂X, the boundary of X, be the points in the intersection of closures of the interiors of X and $\mathbf{R}^2 - X$.

It is clear that if X is definable in a fixed $\mathscr{O}$-minimal expansion of $\mathbf{R}$ then so is ∂X. Moreover, by cell decomposition if X is two dimensional and bounded then $\dim(\partial X) = 1$.

Lemma 3.3. *If $S \subseteq \mathbf{R}^2$ is infinite, coinfinite and definable in an $\mathscr{O}$-minimal expansion of $\mathbf{R}$, then there is $Y \in \mathrm{Def}(\mathfrak{A}_s)$ such that $Y \subseteq B_1$, $\dim(Y) = 2$, and there is $\overline{x} \in \partial\widehat{Y}$ such that $\partial\widehat{Y}$ is smooth at $\overline{x}$ and $\overline{Y}$ is locally convex at $\overline{x}$.*

Proof. By Lemma 3.2 we may assume that S is two dimensional and bounded. Using cell decomposition and the fact that definable functions are piecewise C^n, it is easy to see that for almost all $\overline{x} \in \partial S$, ∂S is smooth at $\overline{x}$ and either S is locally convex at $\overline{x}$ or $\mathbf{R}^2 - S$ is locally convex at $\overline{x}$.

Unfortunately it is possible that at no point is S locally convex. Suppose $\overline{x}$ is a smooth point on ∂S such that $\mathbf{R}^2 - S$ is locally convex at $\overline{x}$. For $r > 0$, $B_r(\overline{x}) = \{\overline{z} \in \mathbf{R}^2: \|\overline{z} - \overline{x}\| < r\}$. Choose r such that $B_r(\overline{x}) \cap (\mathbf{R}^2 - S)$ is convex. Let $\overline{y} \in S$ be such that the line l through $\overline{y}$ and $\overline{x}$ is normal to ∂S at $\overline{x}$, and $s = \|\overline{y} - \overline{x}\| < \frac{r}{2}$. Then $B_s(\overline{y}) \subseteq B_r(\overline{x})$ and if t is the line tangent to ∂S at $\overline{x}$, $B_s(\overline{y}) \cap t = \varnothing$. Thus $B_s(\overline{y}) \subseteq S$.

Without loss of generality we may assume $s = 1$, $\overline{y} = (0, 0)$, and $\overline{x} = (1, 0)$. Let $Y = \{\frac{1}{z}: z \notin \widetilde{S}\}$. Then $Y \subseteq B_1$ and $\overline{x} \in \partial Y$. Since ∂S is smooth at $\overline{x}$ and $z \mapsto \frac{1}{z}$ is smooth on its domain $\overline{x}$ is a smooth point on $\partial\widehat{Y}$. Since $\overline{x}$ is on the boundary of B_1, $\partial\widehat{Y}$ is locally convex at $\overline{x}$. □

Proof of 3.1. If S is finite or cofinite then $\widetilde{S}$ is constructible. If S is infinite and coinfinite then by Lemma 3.2 we may assume that S is bounded and two dimensional, and that there is $\overline{x} \in \partial S$ such that ∂S is smooth at $\overline{x}$ and S is locally convex at $\overline{x}$. By applying affine transformations we may without loss of generality assume that $\overline{x} = (0, 0)$ and $S \subseteq B_1$.

Let $0 < t < 1$ be such that $B_1 \cap S$ is convex. Let $Y = \{z \in \widetilde{S}: \frac{z}{t} \in \widetilde{S}\}$.

Claim 1. If $0 < s < t$, then $sY \subseteq Y$.

If $z \in Y$, then $\frac{z}{t} \in \widetilde{S}$. Thus $\|\frac{z}{t}\| < 1$, so $\|z\| < t$. Hence $z \in \widetilde{S} \cap \widetilde{B}_t$. Thus by convexity $sz \in \widetilde{S}$ and $\frac{s}{t}z \in \widetilde{S}$. Therefore $sz \in Y$.

Let $M = \{\alpha \in \mathbf{C}: \alpha Y \subseteq Y\}$. Since $Y \in \operatorname{Def}(\mathfrak{A}_s)$, $M \in \operatorname{Def}(\mathfrak{A}_s)$. By Claim 1 the interval $(0, t)$ is a subset of M.

Claim 2. If l is any line through $(0, 0)$ that is not tangent to ∂S at $(0, 0)$, then $l \cap \widehat{Y} \neq \varnothing$. In fact $l \cap \widehat{Y}$ contains points arbitrarily close to $(0, 0)$.

Since l is not tangent to $(0, 0)$ there is a point (u, v) on l such that $(u, v) \in B_t \cap S$. But then by convexity $t(u + iv) \in \widetilde{S}$, so $u + iv \in Y$ and $\widehat{Y} \cap l \neq \varnothing$.

Claim 3. $M \subseteq \mathbf{R}$.

Let $\alpha \in \mathbf{C} - \mathbf{R}$, say $\alpha = re^{i\theta}$, where θ is not an integral multiple of π. Multiplying Y by α causes us to rotate Y by θ and expand or contract by r. Thus by Claim 2 we can find elements $X \in Y$ such that $\alpha x = a + bi$ for (a, b) on the line tangent to $\widehat{Y}$ at $(0, 0)$. These points can be found arbitrarily close to $(0, 0)$. So $\alpha Y \nsubseteq Y$.

Thus $(0, t) \subseteq M \subseteq \mathbf{R}$ and $M \in \operatorname{Def}(\mathfrak{A}_s)$. But then

$$\mathbf{R} = \left\{x \in \mathbf{C}: x = 0 \vee x = \pm 1 \vee \pm tx \in M \vee \pm\frac{1}{tx} \in M\right\} \in \operatorname{Def}(\mathfrak{A}_s). \quad \square$$

Once we have proved Theorem 3.1 our program of finding the reals merges with efforts to verify Zilber's conjecture for $\mathfrak{A}_s$.

Definition. Fix a language $\mathscr{L}$. An $\mathscr{L}$-structure $\mathfrak{A}$ with universe A is *minimal* if for every $\mathscr{L}$-formula $\varphi(\overline{v}, \overline{w})$ and every $\overline{a} \in A$ $\{\overline{b} \in A: \mathfrak{A} \models \varphi(\overline{b}, \overline{a})\}$ is finite or cofinite. We say $\mathfrak{A}$ is *strongly minimal* if every elementary extension $\mathfrak{B} \succ \mathfrak{A}$ is minimal.

An easy argument shows that if $\mathscr{L}$ is countable and $\mathfrak{A}$ is a minimal uncountable $\mathscr{L}$-structure, then $\mathfrak{A}$ is strongly minimal. We have argued that in a nonminimal semialgebraic expansion of $\mathbf{C}$ we can define $\mathbf{R}$. Thus in any nonstrongly minimal semialgebraic expansion we can define $\mathbf{R}$.

4. Zariski closures

We are now ready to begin the proof of Theorem 1.2. The goal of this section is to reduce to the case where there is an irreducible affine variety V such that $\widetilde{S}$ is a "large" subset of V.

We proceed by induction on n. Let $S \subseteq \mathbf{R}^{2n}$ be a semialgebraic. We may assume that if $m < n$, $Y \subseteq \mathbf{C}^m$ and $Y \in \operatorname{Def}(\mathfrak{A}_s)$ then Y is constructible. Further we may assume that if $x \subseteq \mathbf{R}^{2n}$, $\widetilde{X} \in \operatorname{Def}(\mathfrak{A}_s)$ and $\dim(X) < \dim(S)$, then X is constructible.

For $\overline{a} \in \mathbf{C}^{n-1}$ let $\widetilde{S}_{\overline{a}} = \{y \in \mathbf{C}: (\overline{a}, y) \in \widetilde{S}\}$. By our induction hypothesis each $\widetilde{S}_{\overline{a}}$ is constructible and hence finite or cofinite. By 2.3 there is a number $n \in \omega$ such that for all $\overline{a} \in \mathbf{C}^{n-1}$, $|\widetilde{S}_{\overline{a}}| \leq n$ or $|\mathbf{C} - \widetilde{S}_{\overline{a}}| \leq n$. Thus $F = \{\overline{a}: \widetilde{S}_{\overline{a}} \text{ is finite}\}$ is in $\operatorname{Def}(\mathfrak{A}_s)$ and hence constructible. Let $F' = \{\overline{a} \in F: |\widetilde{S}_{\overline{a}}| \neq \varnothing\}$. Clearly F' is constructible as well. We decompose $\widetilde{S}$ into $\widetilde{S}_F = \{(\overline{a}, y) \in$

$S: \bar{a} \in F\}$ and $\widetilde{S}_c = \{(\bar{a}, y) \in S: \bar{a} \notin F\}$. Clearly

$$\dim S = \max(\dim(S_F), \dim(S_c)).$$

But $\dim(S_F) = \dim(F')$ and $\dim(S_c) = \dim(\mathbf{C}^{n-1} - F) + 2$. Thus S has the same dimension as a constructible set. In particular $\dim(S)$ is even. Say $\dim(S) = 2m$.

Let $\widetilde{S}'_c = \{(\bar{a}, y) \in \widetilde{S}: \bar{a} \notin F\}$. Clearly $\widetilde{S}$ is definable from $\widetilde{S}_F$ and $\widetilde{S}'_c$. If both $\widetilde{S}_F$ and $\widetilde{S}'_c$ are constructible so is $\widetilde{S}$, thus we may assume that one, say $\widetilde{S}_F$, is not. Note that by our choice of $\widetilde{S}$, $\dim(S_F) = \dim(S) = 2m$. Thus we may without loss of generality assume that for all $\bar{a} \in \mathbf{C}^{n-1}$, $\widetilde{S}_{\bar{a}}$ is finite.

Consider $D = \{\bar{a}: |S_{\bar{a}}| \neq \varnothing\}$. $\widetilde{D}$ is constructible and $\dim_{\mathbf{C}}(\widetilde{D}) = m$. We can write $D = D_0 \cup \cdots \cup D_k$ where each D_i is a Zariski open subset of an irreducible affine variety. If $\dim_{\mathbf{C}}(D_i) < m$, we can throw out D_i by considering $\widetilde{S} - \{(\bar{a}, y) \in \widetilde{S}: \bar{a} \in D_i\}$. Thus without loss of generality we may assume that for all i, $\dim_{\mathbf{C}}(D_i) = m$. For each i let $\widetilde{S}_i = \{(\bar{a}, \bar{y}) \in \widetilde{S}: \bar{a} \in D_i\}$. If each $\widetilde{S}_i$ is constructible then $\widetilde{S}$ is constructible. Thus we may consider each $\widetilde{S}_i$ individually. Thus without loss of generality we may assume D is a Zariski open subset of an irreducible affine variety.

Let $(x_1, \dots, x_{n-1})$ be a generic point of D. Then $\{x_1, \dots, x_{n-1}\}$ must have transcendence degree m over $\mathbf{C}$. Without loss of generality we may assume that $\{x_1, \dots, x_m\}$ are algebraically independent. Thus if $Y = \{\bar{a} \in \mathbf{C}^m: \exists y_1, \dots, y_{n-m-1}(\bar{a}, \bar{y}) \in D\}$, $\dim_{\mathbf{C}}(Y) = m$. Consider $Y_\infty = \{\bar{a} \in \mathbf{C}^m: \{\bar{y}: (\bar{a}, \bar{y}) \in D\}$ is infinite$\}$, then Y_∞ is constructible and $\dim_{\mathbf{C}}(Y_\infty) < m$. Let $D_\infty = \{(\bar{a}, \bar{y}) \in D: \bar{a} \in Y_\infty\}$. Clearly D_∞ cannot contain a generic point of D so $\dim_{\mathbf{C}}(D_\infty) < m$. So D_∞ and $\widetilde{S}_\infty = \{(\bar{a}, y) \in \widetilde{S}: \bar{a} \in D_\infty\}$ are constructible. Thus by replacing $\widetilde{S}$ with $\widetilde{S} - \widetilde{S}_\infty$ we may assume $Y_\infty = \varnothing$. Thus without loss of generality we may assume that for all $\bar{a} \in \mathbf{C}^m$, $\{\bar{y}: (\bar{a}, y) \in \widetilde{S}\}$ is finite.

Claim. $\{\bar{a} \in \mathbf{R}^m: \exists \bar{y} \in \mathbf{C}^{n-m}(\bar{a}, \bar{y}) \in \widetilde{S}\}$ has real dimension m.

We claim that if $X \subseteq \mathbf{C}^m$ is constructible and $\dim_{\mathbf{C}}(X) = m$ then

$$\dim(\mathbf{R}^m \cap X) = m.$$

Since $\{\bar{a} \in \mathbf{C}^m: \exists \bar{y}(\bar{a}, \bar{y}) \in \widetilde{S}\}$ is constructible and m-dimensional this will suffice.

We prove the claim by induction on m. If $m = 1$, then X is cofinite so $X \cap \mathbf{R}$ is cofinite. If $m = k+1$, then $Y = \{\bar{x} \in \mathbf{C}^k: \{y: (\bar{x}, y) \in X\}$ is infinite$\}$ is a k-dimensional constructible subset of $\mathbf{C}^k$ so $Y \cap \mathbf{R}^k$ is k-dimensional. Clearly for $\bar{x} \in Y \cap \mathbf{R}^k$, $\{y \in \mathbf{R}: (\bar{x}, y) \in X\}$ is cofinite, so $\dim(X \cap \mathbf{R}) = k+1$. $\square$

We would now like to show that $\widetilde{S}$ has a Zariski closure V in $\mathbf{C}_n$ with $\dim_{\mathbf{C}}(V) = m$. We first work in $\mathbf{R}^n$. Consider R_0, R_1 such that:

$$(x_1, \ldots, x_m, y_1, \ldots, y_{n-m}) \in R_0$$
$$\Leftrightarrow \exists z_1, \ldots, z_{n-m} S(x_1, 0, x_2, 0, \ldots, y_1, z_1, \ldots, y_{n-m} z_{n-m})$$

and

$$(x_1, \ldots, x_m, z_1, \ldots, z_{n-m}) \in R_1$$
$$\Leftrightarrow \exists y_1, \ldots, y_{n-m} S(x_1, 0, x_2, 0, \ldots, y_1, z_1, \ldots, y_{n-m}, z_{n-m}).$$

If $\overline{a} \in \mathbf{R}^m$ and $(\overline{a}, \overline{w}) \in \widetilde{S}$ then there are $\overline{y}$ and $\overline{z}$ such that $(\overline{a}, \overline{y}) \in R_0$, $(\overline{a}, \overline{z}) \in R_1$ and for $j = 1, \ldots, n-m$, $w_j = y_j = iz_j$.

For $\overline{a} \in \mathbf{R}^m$, $\{\overline{y} \in \mathbf{R}^{n-m} : (\overline{a}, \overline{y}) \in R_0\}$ and $\{\overline{z} : (\overline{a}, \overline{z}) \in R_1\}$ are finite. Since $\{\overline{a} \in \mathbf{R}^m : \exists \overline{y} \in \mathbf{C}^{n-m} (\overline{a}, \overline{y}) \in \widetilde{S}\}$ has dimension m, R_0 an R_1 are m-dimensional semialgebraic sets in $\mathbf{R}^n$. Let W_0 and W_1 be the real Zariski closures in $\mathbf{R}^n$ of R_0 and R_1 respectively. W_0 and W_1 both have dimension m. For $i = 0, 1$ let $W_i = W_{i1} \cup \cdots \cup W_{il_i}$, where each W_{ij} is a real irreducible variety. Let

$$W_{ij} = \left\{ \overline{x} \in \mathbf{R}^n : \bigwedge_{k=1}^{t_{ij}} f_{ijk}(\overline{x}) = 0 \right\},$$

where $f_{ij1}, \ldots, f_{ijt_{ij}} \in \mathbf{R}[X_1, \ldots, X_n]$ and $\langle f_{ij1}, \ldots, f_{ijt_{ij}} \rangle$ is a real prime ideal

For each ij let

$$W_{ij}^* = \left\{ \overline{z} \in \mathbf{C}^n : \bigwedge_{k=1}^{t_{ij}} f_{ijk}(\overline{z}) = 0 \right\}.$$

Note that W_{ij}^* is constructible and $W_{ij} \subseteq W_{ij}^*$.

Lemma 4.1. $\dim(W_{ij}) = \dim_{\mathbf{C}}(W_{ij}^*)$.

Proof. Suppose $I = \langle f_1, \ldots, f_k \rangle$ is a real prime ideal in $\mathbf{R}[X_1, \ldots, X_n]$. Let $V = \{\overline{x} \in \mathbf{R}^n : \forall f \in I f(\overline{x}) = 0\}$ and $V^* = \{\overline{x} \in \mathbf{C}^m : \forall f \in I f(\overline{x}) = 0\}$. We will show $\dim(V) = \dim_{\mathbf{C}}(V^*)$. Let V_1^* be an irreducible component of V^* of maximal dimension. We will see that $\dim(V) = \dim_{\mathbf{C}}(V_1^*)$ which suffices.

In general if A is a commutative Noetherian ring we define the Krull dimension of a prime ideal P to be the largest l such that there is a proper chain of prime ideals $P = P_0 \subset P_1 \subset \cdots \subset P_l$.

Facts. (1) [C-R] If $W \subseteq \mathbf{R}^n$ is a real irreducible variety, $\dim(W)$ is the Krull dimension of $I(W)$ in $\mathbf{R}[X_1, \ldots, X_n]$.

(2) If $W \subseteq \mathbf{C}^n$ is an irreducible variety, $\dim_{\mathbf{C}}(W)$ is the Krull dimension of $I(W)$ in $\mathbf{C}[X_1, \ldots, X_n]$.

[If F is a field and $X \subseteq F^n$, $I(F) = \{g(\overline{x}) \in F[\overline{X}] : \forall \overline{a} \in X g(\overline{a}) = 0\}$.]

Since $\mathbf{C}[\overline{X}]$ is an integral extension of $\mathbf{R}[\overline{X}]$, we may apply the Cohen-Seidenberg going up and going down lemmas (see [A-M]). Let $J = I(V_1^*)$. Clearly $J \supseteq I$. We claim that $I = J \cap \mathbf{R}[\overline{X}]$. If not then $I_1 = J \cap \mathbf{R}[\overline{X}]$ is a prime ideal and there is $J_0 \subseteq J$ a prime ideal in $\mathbf{C}[\overline{X}]$ with $J_0 \cap \mathbf{R}[\overline{X}] = I$. But then $V(J_0) = \{\overline{x} \in \mathbf{C}^n : \forall f \in J_0 f(\overline{x}) = 0\}$ is an irreducible component of V^* and $\dim(V(J_0)) > \dim(V_1^*)$ a contradiction. Thus $I = J \cap \mathbf{R}[\overline{X}]$.

If $J = J_0 \subset J_1 \subset \cdots \subset J_l$ is a chain of prime ideals in $\mathbf{C}[\overline{X}]$, then $I \subset J_1 \cap \mathbf{R}[\overline{X}] \subset \cdots \subset J_l \cap \mathbf{R}[\overline{X}]$ is a chain of prime ideals in $\mathbf{R}[\overline{X}]$. Thus $\dim_{\mathbf{R}}(V) \geq \dim_{\mathbf{C}}(V_1^*)$. On the other hand if $I \subset I_1 \subset \cdots \subset I_l$ is a chain of prime ideals in $\mathbf{R}[\overline{X}]$, this will lift to $J \subset J_1 \subset \cdots \subset J_l$, a chain of prime ideals in $\mathbf{C}[\overline{X}]$. Thus I and J have the same Krull dimension. Hence $\dim(V) = \dim_{\mathbf{C}}(V^*)$. □

Below we use the following claim.

Claim. If $X \subseteq \mathbf{C}^m$ is constructible, then $\dim(X \cap \mathbf{R}^m) \leq \dim_{\mathbf{C}}(X)$.

A generic for $X \cap \mathbf{R}^m$ has transcendence degree $\dim(X \cap \mathbf{R}^m)$. This point will still have transcendence degree $\dim(X \cap \mathbf{R}^m)$ over $\mathbf{C}$.

Let

$$A_0 = \left\{ (\overline{x}, \overline{y}, \overline{z}, \overline{w}) : \bigvee_{j=0}^{L_0} (\overline{x}, \overline{z}) \in W_{0j}^* \wedge \bigvee_{j=0}^{l_1} (\overline{x}, \overline{w}) \in W_{1j}^* \wedge \bigwedge_{j=1}^{n-m} y_j = z_j + i w_j \right\}.$$

A_0 is a constructible set. Let $A = \{(\overline{x}, \overline{y}) : \exists \overline{z}, \overline{w} (\overline{x}, \overline{y}, \overline{z}, \overline{w}) \in A_0\}$. Let $A_1 = \{\overline{x} : \exists \overline{y} (\overline{x}, \overline{y}) \in A\}$. By construction $\{\overline{x} \in \mathbf{R}^m : \exists \overline{y} (\overline{x}, \overline{y}) \in \widetilde{S}\}$ is a subset of A_1, so $\dim_{\mathbf{C}}(A) \geq \dim_{\mathbf{C}}(A_1) \geq m$. If $(x_1, \ldots, x_m, \overline{y}, \overline{z}, \overline{w})$ is a generic point of A_0, then $x_1, \ldots, x_m$ are algebraically independent over $\mathbf{C}$. Since all i, j, $\dim_{\mathbf{C}}(W_{ij}^*) \leq m$, $\overline{z}$ and $\overline{w}$ are algebraic over $x_1, \ldots, x_m$. Clearly each y_i is algebraic over z_i an d w_i. Hence $\dim_{\mathbf{C}}(A_0) \leq m$. Since $\dim_{\mathbf{C}}(A_0) \geq \dim_{\mathbf{C}}(A)$, $\dim_{\mathbf{C}}(A) = m$.

Let $B = \{\overline{x} \in \mathbf{C}^m : \forall \overline{y} (\overline{x}, \overline{y}) \in \widetilde{S} \to (\overline{x}, \overline{y}) \in A\}$. This contains $\{(\overline{x}, \overline{y}) \in \widetilde{S} : \overline{x} \in \mathbf{R}^m\}$, so $\dim_{\mathbf{C}}(B) = m$. Thus $\dim(S - \widehat{A}) < \dim(S)$ so $\widetilde{S} - A$ is constructible. Thus without loss of generality we may assume $\widetilde{S} \subseteq A$.

Thus we have reduced to the case where there is a constructible set $A \supseteq \widetilde{S}$ with $\dim(\widetilde{S}) = \dim(\widehat{A})$. We next take the Zariski closure of A and consider separately the intersection of $\widetilde{S}$ with each irreducible component of complex dimension m.

5. Definability of dimension

Before we finish the proof of Theorem 1.2 we must prove a technical lemma on the definability of dimension.

We generalize the fact that if $A \subseteq \mathbf{C}^n$ is constructible and for all $a \in \mathbf{C}$, A_a denotes $\{\overline{b} \in \mathbf{C}^{n-1} : (a, \overline{b}) \in A\}$, then for $0 \leq k \leq n$, $\{a \in \mathbf{C} : \dim_{\mathbf{C}}(A_a) = k\}$ is constructible.

Setting. For the remainder of this section we assume that: $(*)$ $S \subset \mathbf{R}^n$ is semialgebraic and for all $Y \in \operatorname{Def}(\mathfrak{A}_s)$ if $m < n$ and $Y \subseteq \mathbf{C}^m$, then Y is constructible.

Suppose $x \subset \{1, \ldots, n\}$. Let $i_1 < \cdots < i_k$ list the elements of x and let $j_1 < \cdots < j_{n-k}$ list the complement of x. For all $\overline{a} \in \mathbf{C}^k$ let

$$\widetilde{S}^x_{\overline{a}} = \left\{ \overline{b} \in \mathbf{C}^{n-k} : \exists \overline{c} \in \widetilde{S} \bigwedge_{l \le k} x_{i_l} = a_l \wedge \bigwedge_{l \le n-k} x_{j_l} = b_l \right\}.$$

Lemma 5.1. *Assume we are in the setting* $(*)$. *If* $x \subseteq \{1, \ldots, n\}$ *and* $0 \le k \le n$, *then* $\{\overline{a} \in \mathbf{C}^{|x|} \colon \dim_{\mathbf{C}}(\widetilde{S}^x_{\overline{a}}) \ge k\} \in \operatorname{Def}(\mathfrak{A}_s)$.

Proof. By induction on k. If $k = 0$, this is trivial (for our purposes $\dim_{\mathbf{C}}(\varnothing) = 0$).

Claim. Let W be constructible. Then $\dim_{\mathbf{C}}(W) \ge k+1$ if and only if there is $1 \le i \le n$ such that $\dim(\{a \in \mathbf{C} \colon \dim(W^{\{i\}}_a) \ge k\}) \ge 1$.

Let $(x_1, \ldots, x_n)$ be a generic point of a maximum dimensional irreducible component of the Zariski closure of W. Without loss of generality we may assume that $x_1, \ldots, x_{k+1}$ are algebraically independent. Then for all but finitely many $a \in \mathbf{C}$, $\dim(W^{\{i\}}_a) \ge k$.

For $x \subseteq \{1, \ldots, n\}$ and $1 \le j \le n$ with $j \notin x$, if $\overline{a} \in \mathbf{C}^{|x|}$ and $b \in \mathbf{C}$, let $\overline{a} *_j b = (a_{i_1}, \ldots, a_{i_l}, b a_{i_{l+1}}, \ldots, a_{i_{|x|}})$ where $i_l < j < i_{l+1}$.

Then

$$\dim_{\mathbf{C}}(\widetilde{S}^x_{\overline{a}}) \ge k+1 \Leftrightarrow \bigvee_{j \notin x} \dim_{\mathbf{C}}(\{b \colon \dim_{\mathbf{C}}(\widetilde{S}^{x \cup \{j\}}_{\overline{a} *_j b}) \ge k\}) \ge 1.$$

By our induction hypothesis for each j, $\{\overline{a} *_j b \colon \dim_{\mathbf{C}}(\widetilde{S}^{x \cup \{j\}}_{\overline{a} *_j b}) \ge k\}$ is in $\operatorname{Def}(\mathfrak{A}_s)$ and hence constructible. By 2.3 there is a natural number N_j such that for all $\overline{a} \in \mathbf{C}^{|x|}$ if $|\{b \colon \dim_{\mathbf{C}}(\widetilde{S}^{x \cup \{j\}}_{\overline{a} *_j b})\}| > N_j$ then it is infinite.

Hence

$$\dim_{\mathbf{C}}(\widetilde{S}^x_{\overline{a}}) \ge k+1 \Leftrightarrow \bigvee_{j \notin x} |\{b \colon \dim_{\mathbf{C}}(\widetilde{S}^{x \cup \{j\}}_{\overline{a} *_j b}) \ge k\}| > N_j.$$

Thus by induction $\{\overline{a} \colon \dim_{\mathbf{C}}(\widetilde{S}^x_{\overline{a}}) \ge k+1\} \in \operatorname{Def}(\mathfrak{A}_s)$. $\square$ Lemma 5.1 is in the spirit of [Dr 2].

6. Subsets of irreducible varieties

We now consider the case where $V \subseteq \mathbf{C}^n$ is an irreducible variety, $\widetilde{S} \subseteq V$, $\dim_{\mathbf{C}}(V) = m$ and $\dim(S) = 2m$. We consider the case $m = 1$ and $m > 1$ separately.

Lemma 6.1. *If V is an irreducible curve, $S \subseteq \mathbf{R}^{2n}$ is semialgebraic, $\widetilde{S} \subseteq V$, and $\widetilde{S}$ and $V - \widetilde{S}$ are both infinite, then* $\mathbf{R} \in \mathrm{Def}(\mathfrak{A}_s)$.

Proof. Without loss of generality we may assume that $\pi(V) = \{z \in \mathbf{C}: \exists \overline{x}(z, \overline{x}) \in V\}$ is coinfinite. Thus either we can define $\mathbf{R}$ or $\pi(\widetilde{S})$ and $\pi(V - \widetilde{S})$ are both cofinite. Thus either we can define $\mathbf{R}$ or $\dim(S) = \dim(\widehat{V} - S) = 2$. Thus by 2.4 we can find a Euclidean open set U such that $U \cap V \neq \varnothing$ and $U \cap C \subseteq \widetilde{S}$.

Let $V^* \subseteq \mathbf{P}^n$ be the projective completion of the curve. Let $a \in V \cap U$. For the moment assume that V^* is nonsingular. Then by the Riemann-Roch theorem [H, p. 297] there is a rational function ρ on V^* which has its only zero at a. Since V^* is compact, $\rho(V^* - U)$ is bounded away from zero. Thus $\rho(V - \widetilde{S})$ is an infinite, cofinite subset of $\mathbf{C}$ so by 3.1 we can define $\mathbf{R}$.

If V^* has singularities, then we replace V^* by a nonsingular curve $V^{**} \subseteq \mathbf{P}^k$ which is birationally equivalent to V^* [H, p. 28].

Since birational maps are constructible, V^{**} has a infinite, coinfinite subset of $\mathrm{Def}(\mathfrak{A}_s)$. Thus by the above arguments, $\mathbf{R} \in \mathrm{Def}(\mathfrak{A}_s)$. □

Thus if $\dim_{\mathbf{C}}(V) = 1$, either $\mathbf{R} \in \mathrm{Def}(\mathfrak{A}_s)$ or $\widetilde{S}$ or $V - \widetilde{S}$ is finite. In the latter case $\widetilde{S}$ is constructible.

Suppose $\dim_{\mathbf{C}}(V) \geq 2$. We use the following version of Bertini's theorem (see [S]).

Theorem 6.2. *Let $V \subseteq \mathbf{C}^n$ be an irreducible variety with $\dim_{\mathbf{C}}(V) = m \geq 2$. For $(a_0, \ldots, a_n) \in \mathbf{C}^{n+1}$ let $H_{\overline{a}} = \{(x_1, \ldots, x_n) \in \mathbf{C}^n: a_0 + \sum a_i x_i = 0\}$. Then $\{\overline{a} \in \mathbf{C}^{n+1}: H_{\overline{a}} \cap V$ is an irreducible $(m-1)$-dimensional variety$\}$ is an $(n+1)$-dimensional subset of $\mathbf{C}^{n+1}$.*

In other words, a generic hyperplane intersects V in an irreducible $(m-1)$-dimensional variety. Let $a_1, \ldots, a_n \in \mathbf{C}$ be such that for all but finitely many $a_0 \in \mathbf{C}$, $H_{\overline{a}} \cap V$ is an irreducible $(m-1)$-dimensional variety. Thus by a suitable change of coordinates we may assume that for all but finitely many $a \in \mathbf{C}$, $V_a = \{\overline{y} \in \mathbf{C}^{n-1}: (a, \overline{y}) \in V\}$ is an irreducible $(m-1)$-dimensional variety. Let $\widetilde{S}_a = \{\overline{y} \in \mathbf{C}^{n-1}: (a, \overline{y}) \in \widetilde{S}\}$. Recall we are assuming each $\widetilde{S}_a$ is constructible.

Let $X = \{a \in \mathbf{C}: V_a$ is irreducible and $\dim_{\mathbf{C}}(V_a) = \dim_{\mathbf{C}}(\widetilde{S}_a) = m - 1\}$. By 5.1 X is constructible. Since $\dim(S) = 2m$, X must be cofinite. It suffices to show that $S' = \{(a, \overline{y}) \in \widetilde{S}: a \in X\}$ is constructible. But for $a \in X$, $\dim_{\mathbf{C}}(V_a - \widetilde{S}_a) < m - 1$. Let $S'' = \{(a, \overline{y}) \in V: a \in X \wedge (a, \overline{y}) \notin \widetilde{S}\}$. Then $\dim(\widehat{S''}) < 2m$, so S'' is constructible. But then S' is constructible.

This completes the proof of Theorem 1.2.

Note. In unpublished work Hrushovski has shown that if V is an irreducible curve and X is an infinite, coinfinite subset of V then $(\mathbf{C}, +, \cdot, X)$ is not strongly minimal. Thus 5.1 follows from his work by the remarks at the end of §3.

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