ENDOMORPHISM RINGS OF FORMAL $A_0$-MODULES

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ABSTRACT. Let $A_0$ be the valuation ring of a finite extension $K_0$ of $Q_p$ and $A \supset A_0$ be a complete discrete valuation ring with the perfect residue field. We consider the endomorphism rings of $n$-dimensional formal $A_0$-modules $\Gamma$ over $A$ of finite $A_0$-height with reduction absolutely simple up to isogeny. Especially we prove commutativity of $\text{End}_{A_0}(\Gamma)$. Given an arbitrary finite unramified extension $K_1$ of $K_0$, a variety of examples (different dimensions and different $A_0$-heights) is constructed whose absolute endomorphism rings are isomorphic to the valuation ring of $K_1$.

Let $K_0$ be a finite extension of $Q_p$ and $A_0$ the valuation ring of $K_0$; let $K \supset K_0$ be a complete discrete valuation field with the perfect residue field $k$ of characteristic $p > 0$; let $A$ be the valuation ring of $K$.

It is known that the fraction field of the endomorphism ring of a one-dimensional formal group of height $h$ over $A$ is a finite extension of $Q_p$ of degree dividing $h$ (cf. Lubin [7]).

In Theorem 1 and Proposition 2, we prove a higher-dimensional analogue of the above fact: if an $n$-dimensional formal $A_0$-module $\Gamma$ over $A$ satisfies an assumption that the reduction $\Gamma_k = \Gamma \otimes_A k$ of $\Gamma$ is an absolutely simple formal $A_0$-module up to isogeny and of finite $A_0$-height $h \geq n$ ($h$ is relatively prime to $n$), then we prove that the fraction field $\Lambda$ of the endomorphism ring of $\Gamma$ over $A$ as a formal $A_0$-module is a finite extension field of $K_0$ of degree dividing $h$ such that $e(\Lambda/K_0)$ divides $e(K/K_0)$ and that $f(\Lambda/K_0)$ divides $f(K/K_0)$ if $f(K/K_0)$ is finite.

In the corollary of Theorem 2, we give examples: for any positive integers $h$ and $n$ with $h \geq n + 1$ ($h \geq 1$ if $n = 1$) and for any positive divisor $g$ of $h$, there exists an $n$-dimensional formal $A_0$-module over $A_0$ of $A_0$-height $h$ whose absolute $A_0$-endomorphism ring is the valuation ring of the unramified extension of $K_0$ of degree $g$ (cf. Cox [1] and Yamasaki [11]).

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1. Notations

In this paper, a field means a commutative field and we use the following notations.

- \( p \) = a prime number.
- \( \mathbb{Z}_p \) = the ring of \( p \)-adic integers.
- \( \mathbb{Q}_p \) = the field of \( p \)-adic numbers.
- \( K_0 \) = a finite extension of \( \mathbb{Q}_p \) with the residue field \( k_0 \).
- \( A_0 \) = the valuation ring of \( K_0 \).
- \( K \) (\( \supset K_0 \)) = a complete discrete valuation field with the perfect residue field \( k \) of characteristic \( p > 0 \).
- \( \pi \) = a prime element of \( K \).
- \( A \) = the valuation ring of \( K \).
- \( \overline{k} \) = an algebraic closure of \( k \).
- \( [M : C] \) = the dimension of a vector space \( M \) over a field \( C \).

For an extension \( E/E' \) of complete discrete valuation fields,
- \( e(E/E') \) = the ramification index of \( E \) over \( E' \).
- \( f(E/E') \) = the residue degree of \( E \) over \( E' \).
- \( M_{t}(T) \) = the total matrix ring of degree \( t \) over a ring \( T \).
- \( (a, b) \) = the (positive) greatest common divisor of integers \( a \) and \( b \).

2. Endomorphism rings

In this paper, a formal group means a formal group law.

An \( n \)-dimensional formal \( A_0 \)-module over a commutative \( A_0 \)-algebra \( S \) is an \( n \)-dimensional commutative formal group over \( S \) such that there is an endomorphism \([a] \) of a formal group over \( S \) for each \( a \in A_0 \) whose Jacobian matrix is \( aI_n \) (\( I_n \) = the unit matrix of degree \( n \)) and that \( a \rightarrow [a] \) is a ring homomorphism. If a formal \( A_0 \)-module is of height \( H \) as a formal group, we define the \( A_0 \)-height of \( \Gamma \) as the number \( H/[K_0 : \mathbb{Q}_p] \) (cf. [2], [3, III. 4.3, 5.5] and [5, V. 29.7.2]). A formal \( A_0 \)-homomorphism over \( S \) between formal \( A_0 \)-modules is a homomorphism over \( S \) of the formal groups which commutes with \([a] \) for all \( a \in A_0 \). We write \( \text{End}_{S,A_0}(\Psi) \) the formal \( A_0 \)-endomorphism ring of a formal \( A_0 \)-module \( \Psi \) over \( S \). An isogeny over \( S \) between formal \( A_0 \)-modules is a formal \( A_0 \)-homomorphism that, as a homomorphism of formal groups, is an isogeny.

Let \( \Gamma \) be an \( n \)-dimensional formal \( A_0 \)-module over \( A \) of finite \( A_0 \)-height. Let \( \Gamma_k = \Gamma \otimes_A k \) be the formal \( A_0 \)-module over \( k \) obtained by reducing the coefficients of \( \Gamma \) modulo the maximal ideal of \( A \).

We put \( \Lambda = Q_p \otimes_{\mathbb{Z}_p} \text{End}_{A,A_0}(\Gamma) \). \( \Lambda \) is a \( K_0 \)-algebra. As \( \Gamma \) is of finite \( A_0 \)-height, we identify \( \Lambda \) with its image in \( Q_p \otimes_{\mathbb{Z}_p} \text{End}_{k,A_0}(\Gamma_k) \) through reduction (cf. [5, IV.21.8.19]).

Let \( \text{END}_{S,A_0}(\Gamma) \), the absolute \( A_0 \)-endomorphism ring of \( \Gamma \), be the union of \( \text{End}_{B,A_0}(\Gamma \otimes_A B) \) where \( B \) runs over all the valuation rings of finite extensions of \( K \). (\( \Gamma \otimes_A B \) is the scalar extension of \( \Gamma \) to \( B \).)
We assume,

(*) \( \Gamma_k \) is an \( n \)-dimensional formal \( A_0 \)-module such that the scalar extension \( \Gamma_{\overline{k}} = \Gamma_k \otimes_k \overline{k} \) of \( \Gamma_k \) to \( \overline{k} \) is simple as a formal \( A_0 \)-module up to isogeny (i.e. \( \Gamma_{\overline{k}} \) is absolutely simple up to isogeny) and of finite \( A_0 \)-height \( h \) (\( \geq n \)).

Remark 1. (i) \((h, n) = 1\) by [5, V.29.8.3] (cf. [3, III.4, Corollary 2 of Proposition 8]).

(ii) For examples of \( \Gamma \) and \( \Gamma_k \) satisfying (*), see §4 and [9, Proposition 5].

We put \( D = Q_p \otimes_{Z_p} \text{End}_{\overline{k}, A_0}(\Gamma_{\overline{k}}) \). By assumption (*), \( D \) is a division algebra over \( K_0 \). We put \( \Omega = \Lambda \otimes_{K_0} K \).

The following simple proof of Proposition 1 is due to the referee.

**Proposition 1.** Under our assumption (*), \( D \) is a central division algebra over \( K_0 \) of dimension \( h^2 \).

**Proof.** By [8, II] (or [5, V.28.5.9]), \( \Gamma_{\overline{k}} \) is isogeneous to a product \( \Gamma_1^t \times \Gamma_2^t \times \cdots \times \Gamma_m^t \), where each \( \Gamma_i \) is simple up to isogeny of finite height \( H_i \), \( \Gamma_i \) is not isogeneous to \( \Gamma_j \) for \( i \neq j \), and the decomposition is unique up to isogeny. All this is taking place in the category of formal groups, not of formal \( A_0 \)-modules. \( \Delta \) denotes the scalar extension \( Q_p \otimes_{Z_p} \text{End}_{\overline{k}, A_0}(\Gamma_{\overline{k}}) \) of the endomorphism ring of \( \Gamma_{\overline{k}} \) as a formal group. It follows that \( \Delta \cong \bigoplus M_{t_i}(\Delta_i) \), where each \( \Delta_i \) is a central division algebra over \( Q_p \) of dimension \( H_i^2 \). From the definition of \( \Gamma \) as a formal \( A_0 \)-module, we see that \( K_0 \) injects into \( \Delta \) and that \( D \) is the commutant of \( K_0 \) in \( \Delta \). Since \( D \) is a division algebra, the center of \( \Delta \) must be a field. Thus \( \Gamma \) is isogeneous to \( \Gamma_1^t \) and \( \Delta \cong M_{t_1}(\Delta_1) \). From standard theorems about central division algebras, we see that \( D \) is a central division algebra over \( K_0 \) (double commutant theorem) and that

\[
[D : Q_p][K_0 : Q_p] = [\Delta : Q_p] = t_1^2 H_1^2.
\]

Thus we have

\[
[D : K_0] = \left( \frac{t_1 H_1}{[K_0 : Q_p]} \right)^2 = h^2,
\]

where the last equality follows from the definition of \( A_0 \)-height.

Let \( L \) be the tangent space (or the Lie algebra) of the scalar extension \( \Gamma \otimes \Lambda K \) of \( \Gamma \) to \( K \). Then \( L \) is an \( n \)-dimensional vector space over \( K \), a faithful \( \Lambda \)-module and a bimodule over \( \Lambda \) and \( K \). \( L \) is thus a nontrivial module over \( \Omega \) (cf. [5, II.14.2]).

Theorem 1 is a higher-dimensional analogue of [7, Theorem 2.3.2] (or [5, IV. 23.2.6]).
Theorem 1. Under our assumption (*), \( \Lambda \) is a finite extension field of \( K_0 \).

Proof. By Proposition 1, the \( K_0 \)-subalgebra \( \Lambda \) of \( D \) is a division algebra over \( K_0 \) of dimension dividing \( h^2 \). Let \( Z \) be the center of \( \Lambda \). Then we have \([\Lambda : Z] = h^2\) with \( h' \) dividing \( h \). \( Z \otimes_{K_0} K \) is a finite direct sum of finite extensions \( K_i \) (\( \supset Z \)) of \( K \). Hence every minimal left ideal of \( \Lambda \otimes_Z K_i \) has dimension over \( K \) divisible by \( h' \), and so does every minimal ideal of \( \Omega \equiv \Lambda \otimes_Z (Z \otimes_{K_0} K) \). \( \Omega \) is a semisimple \( K \)-algebra. Since \( L \) is a nontrivial \( \Omega \)-module of dimension \( n \) over \( K \), \( h' \) divides \( n \).

On the other hand, \((h, n) = 1\) and \( h' \) divides \( h \). Thus \( h' = 1 \) and \( \Lambda \) is a field.

Remark 2. If \( \Gamma_k \) is absolutely simple up to isogeny and of \( A_0 \)-height \( \infty \), then we have \( \dim \Gamma = \dim \Gamma_k = \dim \Gamma_{\overline{k}} = 1 \) (cf. \([5, V.29.8.3]\)) and so \( \text{End}_{A_0} A_0(\Gamma) \) is commutative.

Proposition 2. Under our assumption (*), \([\Lambda : K_0] \) divides \( h \). Furthermore \( e(\Lambda/K_0) \) divides \( e(K/K_0) \) and \( f(\Lambda/K_0) \) divides \( f(K/K_0) \) if \( f(K/K_0) \) is finite.

Proof. By Theorem 1, \( \Lambda \) (\( \supset K_0 \)) is a subfield of \( D \). Therefore, by Proposition 1, \([\Lambda : K_0] \) divides \( h \).

Let \( F \) be a minimal ideal of \( \Omega \). By Theorem 1, \( F \) is a composite of \( K \) and \( \Lambda \) over \( K_0 \). We have

\[
e(F/\Lambda)e(\Lambda/K_0) = e(F/K)e(K/K_0).
\]

Then \( e(\Lambda/K_0)/e(\Lambda/K_0), e(K/K_0) \) divides \( e(F/K) \) and so \([F : K] \). Therefore \( e(\Lambda/K_0)/e(\Lambda/K_0), e(K/K_0) \) divides \( n = [L : K] \), since \( \Omega \) is semisimple and \( L \) is a nontrivial \( \Omega \)-module.

On the other hand, \( e(\Lambda/K_0) \) divides \([\Lambda : K_0] \) and so \( h \). Hence, by \((h, n) = 1\), \( e(\Lambda/K_0) \) divides \( e(K/K_0) \).

For the residue degrees, the same argument holds if \( f(K/K_0) \) is finite.

Corollary (of Theorem 1 and Proposition 2). Under our assumption (*), the absolute \( A_0 \)-endomorphism ring of \( \Gamma \) is commutative. Its fraction field is an extension of \( K_0 \) of degree dividing \( h \) and has the ramification index dividing \( e(K/K_0) \).

Proof. Let \( K^* \) be the composite of \( K \) and the fraction field of the Witt vector ring over \( \overline{k} \). Let \( A^* \) be the valuation ring of \( K^* \). We remark \( e(K^*_v/K_0) = e(K/K_0) \). By \([10, Theorem 3.2]\) (or \([5, IV.23.2.2]\)) and \([5, IV.21.1.4, Remarks (ii)]\), \( \text{End}_{A^*, A_0}(\Gamma) \) is contained in \( \text{End}_{A^*, A_0}(\Gamma \otimes_A A^*) \). Hence our result follows from Theorem 1 and Proposition 2.

3. A LEMMA

Let \( K' \) (\( \subset K \)) be the composite of \( K_0 \) and the fraction field (in \( K \)) of the Witt vector ring over \( k \). Then \( K \) is a totally ramified finite extension of \( K' \) and
Let $A'$ be the valuation ring of $K'$. Let $\tau'$ be the Frobenius of $K'$ over $K_0$ (i.e. the $K_0$-automorphism of $K'$ satisfying $a^{\tau'} \equiv a^{p^{e(K'/K_0)}} \pmod{\text{maximal ideal of } A'}$ for all $a \in A'$).

In §§3 and 4, we assume that there exists an extension $\tau$ of $\tau'$ to an automorphism of $K$.

We write $R$ the $A$-module of formal power series

$$x = a_h t^h + a_{h+1} t^{h+1} + \cdots + a_n t^n + \cdots$$

in an indeterminate $t$, with coefficients $a_n \in A$, where the exponent $h$ is arbitrary. The $A$-module $R$ is made into a ring by the multiplication law

$$(a t^i)(b t^j) = (ab t^{i+j})$$

for all $a, b \in A$.

We also write $A_t[[t]]$ the subring of $R$ with only terms of nonnegative exponents (cf. Hilbert-Witt ring and localized Hilbert-Witt ring in [3, III.4.1]).

**Lemma** (a generalization of the claim in [3, III.5, Proof of Theorem 3]). Let $x = \sum a_i t^i \in A_t[[t]]$ with $a_i \in A$ be such that $v(a_i) > 0$ for all $0 \leq i \leq s-1$ and $v(a_s) = 0$, where $v$ is the normalized discrete valuation of $K$ with $v(\pi) = 1$.

Suppose that $u = b_0 + b_1 t + b_2 t^2 + \cdots + b_{s-1} t^{s-1}$ with $b_i \in A$ belongs to the left $A_t[[t]]$-ideal $A_t[[t]]x$ generated by $x$. Then we have $u = 0$, i.e. $b_i = 0$ for all $0 \leq i \leq s-1$.

**Proof.** We remark that $v$ is invariant under $\tau$. We take $\sum c_i t^i \in A_t[[t]]$ such that

$$\sum b_i t^i = \left( \sum c_i t^i \right) \left( \sum a_j t^j \right) = \sum c_i a_j t^{i+j}.$$

Then we have, for all integers $h \geq 0$,

$$0 = c_{s+h} a_0^{i+h} + c_{s+h-1} a_1^{i+h-1} + \cdots + c_0 a_{s+h}.$$

Hence we have, for all integers $h \geq 1$,

$$v(c_h) = v(c_h) + v(a_s^{i+h}) = v(c_h a_s^{i+h})$$

$$= v(-c_{s+h} a_0^{i+h} - c_{s+h-1} a_1^{i+h-1} - \cdots - c_{h+1} a_{s+1}^{i+h-1} - c_{h-1} a_{s+1}^{i+h-1} - \cdots - c_0 a_s^{i+h})$$

$$\geq \min \{ v(c_{s+h}) + v(a_0), v(c_{s+h-1}) + v(a_1), \ldots, v(c_{h+1}) + v(a_s-1), \ldots, v(c_{h-1}) + v(a_{s+h}) \}$$

$$\geq \min \{ v(c_0), v(c_1), \ldots, v(c_{h-1}), v(c_{h+1}) + 1, \ldots, v(c_{s+h}) + 1 \}$$

and, for $h = 0$,

$$v(c_0) \geq \min \{ v(c_1), v(c_2), \ldots, v(c_s) \} + 1.$$ 

Therefore if $v(c_h') \geq q$ for all $0 \leq h' \leq h$ and $v(c_h') \geq q - 1$ for all $h'' \geq h + 1$, then $v(c_{h+1}) \geq q$. Also if $v(c_h) \geq q$ for all integers $h \geq 0$, then $v(c_0) \geq q + 1$. 


Using induction on $h$ and $q$, we have $v(c_h) \geq q$ for all integers $h \geq 0$ and $q \geq 1$. Hence we have $c_h = 0$ for all integers $h \geq 0$ and therefore $u = 0$.

Corollary. Let $x$ be as in the lemma. Suppose that $a_0 \neq 0$. If $u = b_0 + b_1 t + b_2 t^2 + \cdots + b_{s-1} t^{s-1}$ with $b_i \in A$ belongs to $R_x$, then $u = 0$.

4. Examples

Let $\tau$ be as in §3. $\sigma = \tau$ and $q = p^{f(K_0/Q_p)}$ satisfy the assumption (F) in [6, §2]. Let $n$ and $m$ be positive integers ($m \geq 0$ if $n = 1$) and $d$ an integer with $0 \leq d \leq m+n-1$. Let $\Gamma_{n,m,d}$ be the $n$-dimensional commutative formal group over $A$ obtained by the following special element $u_{n,m,d}$ as was done in [6]. $u_{n,m,d}$ commutes with $\text{diag}(a, a, \ldots, a)$ for all $a \in A_0$. Hence $\Gamma_{n,m,d}$ is an $n$-dimensional formal $A_0$-module over $A$. By [5, IV.21.1.4, Remarks (ii)], $\text{End}_{A,A_0}(\Gamma_{n,m,d})$ coincides with the endomorphism ring $\text{End}_{A,A_0}(\Gamma_{n,m,d})$ of $\Gamma_{n,m,d}$ as a formal group over $A$. $u_{1,m,d} = \pi - t^{m+1}(1+t^d)$, and for $n \geq 2$,

$$
\begin{pmatrix}
\pi & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \pi \\
-t^{m+1}(1+t^d) & \cdots & \pi
\end{pmatrix}
$$

We have the following generalization of [11, Theorem 2] for $K$.

Theorem 2. Suppose that there exists an extension $\tau$ of $\tau'$ to an automorphism of $K$. Then we have

$$
\text{End}_{A,A_0}(\Gamma_{n,m,d}) \cong \{ \text{diag}(a^{r^{n-1}}, a^{r^{n-2}}, \ldots, a) | a^{r^d} = a^{r^{m+n}} = a \in A \}.
$$

Therefore $\text{End}_{A,A_0}(\Gamma_{n,m,d})$ is isomorphic to the valuation ring of invariants of $\tau^{(m+n,d)}$ in $A$.

Proof. For a ring $T$, $T^n$ denotes the left free $T$-module of the $n$-dimensional row vectors over $T$.

Let $\{ \bar{e}_1, \ldots, \bar{e}_n \}$ be the images of the canonical basis $\{ e_1, \ldots, e_n \}$ under the composition of the inclusion $A^n \to R^n$ and the canonical surjection $R^n \to R^n/R^n u_{n,m,d}$.

First we assume $n \geq 2$. The left $R$-module $R^n u_{n,m,d}$ is generated by $te_2 - \pi e_1$, $te_3 - \pi e_2$, $\ldots$, $te_n - \pi e_{n-1}$ and $\pi e_n - t^{m+1}(1+t^d)e_1$. Then we have the relations $te_2 = \pi e_1$, $t e_3 = \pi t e_1$, $\ldots$, $t^{n-1} e_n = \pi t^{n-2} \pi t^{n-3} \cdots \pi t \pi e_1$ and the annihilator of $\bar{e}_1$ is the left $R$-ideal $R(\pi^{r^{n-1}}, \pi^{r^{n-2}}, \ldots, \pi^{r^{m+n}}(1+t^d))$. 


of $R$. Especially
\[ R^n / R^n u_{n,m,d} \cong R / R(\pi^{n-1} \cdots \pi \pi - t^{m+n}(1 + t^d)) \]
is a monogenic left $R$-module (cf. [3, III.5.5]).

We suppose $C = (c_{ij}) \in M_n(A)$ be such that

\[ f_{n,m,d}^{-1}(C f_{n,m,d}) \in \text{End}_{A,A_0}(\Gamma_{n,m,d}), \]

where $f_{n,m,d}$ is the transformer of $\Gamma_{n,m,d}$. The left $A_1[[t]]$-module
\[ (A_1[[t]])^n u_{n,m,d} C \text{ is contained in } (A_1[[t]])^n u_{n,m,d} \text{ by } [6, \text{Theorem 3}]. \]
Then $C$ gives an $R$-endomorphism of $R^n / R^n u_{n,m,d}$ which stabilizes $\sum_{1 \leq i \leq n} A \bar{e}_i$. $(\sum_{1 \leq i \leq n} A \bar{e}_i$ is a bimodule over $\text{End}_{A,A_0}(\Gamma_{n,m,d})$ and $A$.) Therefore, we have

\[ t(c_{21} \bar{e}_1 + c_{22} \bar{e}_2 + \cdots + c_{2n} \bar{e}_n) = \pi(c_{11} \bar{e}_1 + \cdots + c_{1n} \bar{e}_n), \]

\[ \vdots \]

\[ t(c_{n1} \bar{e}_1 + c_{n2} \bar{e}_2 + \cdots + c_{nn} \bar{e}_n) = \pi(c_{n-11} \bar{e}_1 + \cdots + c_{n-1n} \bar{e}_n), \]

\[ \pi(c_{11} \bar{e}_1 + c_{n2} \bar{e}_2 + \cdots + c_{nn} \bar{e}_n) = t^{m+1}(1 + t^d)(c_{11} \bar{e}_1 + \cdots + c_{1n} \bar{e}_n). \]

By representing $\bar{e}_i$'s with $\bar{e}_1$, we have

\[
\begin{cases}
(t(c_{21} + c_{22}(t^{-1} \pi) + \cdots + c_{2n}(t^{-(n-1)} \pi \pi^{n-2} \cdots \pi \pi)) \\
- \pi(c_{11} + \cdots + c_{1n}(t^{-(n-1)} \pi \pi^{n-2} \cdots \pi \pi)) \bar{e}_1 = 0,
\end{cases}
\]

\[(**)
\]

\[
\begin{cases}
(t(c_{n1} + c_{n2}(t^{-1} \pi) + \cdots + c_{nn}(t^{-(n-1)} \pi \pi^{n-2} \cdots \pi \pi)) \\
- \pi(c_{n-11} + \cdots + c_{n-1n}(t^{-(n-1)} \pi \pi^{n-2} \cdots \pi \pi)) \bar{e}_1 = 0,
\end{cases}
\]

and

\[(***) \quad \{\pi(c_{n1} + c_{n2}(t^{-1} \pi) + \cdots + c_{nn}(t^{-(n-1)} \pi \pi^{n-2} \cdots \pi \pi)) \\
- t^{m+1}(1 + t^d)(c_{11} + \cdots + c_{1n}(t^{-(n-1)} \pi \pi^{n-2} \cdots \pi \pi)) \bar{e}_1 = 0.\]

We multiply (**) by $t^{n-1}$ from the left.

Since $n \leq m + n - 1$, by the corollary of the lemma we have $c_{ii} = 0$ ($2 \leq i \leq n$), $c_{ik} = c_{i+1,k+1}^r$ ($1 \leq i, k \leq n - 1$), and $c_{in} = 0$ ($1 \leq i \leq n - 1$). Hence we have $c_{ij} = 0$ if $i \neq j$ and $c_{ii} = c_{nn}^{r^{n-1}}$ for $1 \leq i \leq n - 1$ and so

\[ C = \text{diag}(c_{nn}^{r^{n-1}}, c_{nn}^{r^{n-2}}, \ldots, c_{nn}). \]

Since the annihilator of $\bar{e}_1$ is $R(\pi^{n-1} \cdots \pi \pi - t^{m+n}(1 + t^d))$, from (***) we have

\[ \{c_{nn}^{r^{-(m+1)}}(1 + t^d) - t^{m+1}(1 + t^d)c_{11} \bar{e}_1 = 0. \]
Then, by dividing the above equation by $t^{m+1}$, we have

$$c^{r_{m+n}}_{nn} + c^{r_{m+1}}_{nn} t^d - c^{r_{n+1}}_{nn} - c^{r_{n+d-1}}_{nn} t^d \tilde{e}_1 = 0.$$ 

From $0 \leq d \leq m + n - 1$, by the corollary of the lemma we have $c^{r_{m+n}}_{nn} = c^d_{nn}$.

Conversely if $C = (c_{ij})$ satisfies the above conditions, then $u_{n,m,d} C = C u_{n,m,d}$ and so $f_{n,m,d}(C f_{n,m,d}) \in \text{End}_{A_0}(\Gamma_{n,m,d})$.

Hence $\text{End}_{A,A_0}(\Gamma_{n,m,d})$ is isomorphic to the invariants of $\tau^{(m+n,d)}$ in $A$.

Finally, for $n = 1$, the analogous argument holds since the annihilator of $\tilde{e}_1$ is $R(\pi - t^{m+1}(1 + t^d))$.

**Remark 3.** (i) The field consisting of the invariants of $\tau^{(m+n,d)}$ in $K$ has been determined more explicitly in [11, Theorem 3].

(ii) Suppose that $e(K/K_0) = 1$, $(n, m) = 1$, and $k$ is algebraically closed for simplicity. By [3, III.5.2, Proof of Theorem 2], we have

$$R/R(\pi^{r_{m+n}}_n \cdots \pi^r \pi - t^{m+n}(1 + t^d)) \cong R/R(\pi^m_0 - t^{m+n}),$$

where $\pi_0$ is a prime element of $K_0$. Hence $\Gamma_{n,m,d} \otimes_A k$ is absolutely simple up to isogeny (cf. Proof of the corollary below).

The following corollary is a higher-dimensional analogue of [1, Theorem 5.2.2] (or [5, IV.23.2.16]).

**Corollary.** For any positive integers $n$ and $h$ with $h \geq n + 1$ ($h \geq 1$ if $n = 1$) and for any positive divisor $g$ of $h$, there exists an $n$-dimensional formal $A_0$-module over $A_0$ of $A_0$-height $h$ whose absolute $A_0$-endomorphism ring is the valuation ring of the unramified extension of $K_0$ of degree $g$.

**Proof.** Let $K = K_0$ and $m = h - n$. Put $d = g$ if $g < h$ and $d = 0$ if $g = h$. Let $K^*_0$ be the completion of the maximal unramified extension of $K_0$ and $A^*_0$ the valuation ring of $K^*_0$. As in the proof of the corollary in §2, we have

$$\text{END}_{*,A_0}(\Gamma_{n,m,d}) \subset \text{End}_{A^*_0}(\Gamma_{n,m,d} \otimes_A A^*_0).$$

We apply Theorem 2 to $\Gamma_{n,m,d} \otimes_A A^*_0$. Thus $\text{END}_{*,A_0}(\Gamma_{n,m,d})$ is contained in

$$\{\text{diag}(a^{r_{n-1}}, a^{r_{n-2}}, \ldots, a) | a^{r_{m,n}} = a \in A^*_0\}.$$ 

The invariants of $\tau^g$ in $A^*_0$ coincide with the valuation ring of the unramified extension of degree $g$ over $K_0$. Especially, any $A_0$-endomorphism of $\Gamma_{n,m,d} \otimes_A A^*_0$ is defined over the valuation ring of the finite extension of $K_0$. Hence the converse inclusion follows.

For the functor $M$ in [2], we have

$$M(\Gamma_{n,m,d} \otimes_A k_0) \cong A_{0r}[[[t]]/\langle A_{0r}[[t]]^n u_{n,m,d} \rangle.$$

as in [4, V.2]. Thus we have

\[ [k_0 \otimes_{A_0} M(\Gamma_{n,m,d} \otimes_{A_0} k_0) : k_0] = m + n \]

and therefore \( \Gamma_{n,m,d} \) is of \( A_0 \)-height \( m + n = h \).

**Remark 4.** If \((n, m) = 1\), then \( \Gamma_{n,m,d} \otimes_{A_0} k_0 \) is absolutely simple up to isogeny as in Remark 3(ii) (cf. [3, III.5.5]).

**References**


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