COMPOSITE RIBBON NUMBER ONE KNOTS
HAVE TWO-BRIDGE SUMMANDS

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Abstract. A composite ribbon knot which can be sliced with a single band move has a two-bridge summand.

A knot $K$ in the 3-sphere is said to be of ribbon number $r$ if $K$ is a ribbon knot which cannot be “sliced” in fewer than $r$ band moves; i.e. $r$ is the minimal number of saddle points on a ribbon disc in the 4-ball having as boundary the knot $K$. A natural question to ask is if a ribbon number one knot is prime. The answer, of course, is no, as the square knot shows. However, the answer is “nearly” yes. Recall that if a knot $K$ is prime then in every decomposition of $K$ into a connected sum the structure of one of the summands is trivial, i.e. there must be an unknotted summand, or, put another way, a summand with bridge number one. We show for a composite ribbon number one knot, the structure of one of the summands, while possibly nontrivial, must still be “elementary”. As advertised in the title, we show

Theorem. A composite ribbon number one knot has a two-bridge summand.

The theorem above is a direct corollary of Theorem 1.3 below regarding band sums. The proof of 1.3 uses the combinatorial techniques introduced by M. Scharlemann in [Sc1] and [Sc2] to minimize the intersection of planar surfaces in a tangle exterior. A knowledge of these techniques is essential to the understanding of this paper.

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1. THE MAIN THEOREM AND PRELIMINARY ARGUMENTS

Divide a 3-ball into four quadrants by two discs, $D_v$ and $D_h$, one vertical and the other horizontal. Label the quadrants by the points of the compass NE, NW, SW, SE. Let $N$ be the manifold obtained by attaching two 1-handles to the 3-ball, one connecting NE to SE, the other connecting NW to SW. The
1-handles are to be thought of as being attached at points on the boundary of a disc $D_\perp$ in the 3-ball, perpendicular to both $D_v$ and $D_h$. The disc $D_\perp$ divides the boundary of the 3-ball into two-hemispheres, call that part of the two hemispheres lying in $\partial N$ the front face and back face of $\partial N$ respectively. Similarly, the disc $D_h$ divides the boundary of the 3-ball into two hemispheres, call that part of these two hemispheres lying in $\partial N$ the top face and bottom face of $\partial N$ respectively.

1.1. Let $A_m$ be an imbedded family of simple closed curves in $\partial N$ consisting of circles $a_1, \ldots, a_m$ (labelled west to east) parallel to $\partial D_v$. Let $B_n$ be an imbedded family of simple closed curves in $\partial N$ consisting of circles $b_1, \ldots, b_n$ (labelled south to north) parallel to $\partial D_h$, together with two circles $b_+$ and $b_-$ which are meridians of the 1-handles (possibly of the same 1-handle); see Figure 1.

1.2. Imbed into disjoint 3-balls in $S^3$ two copies $\gamma_0, \gamma_1$ of the unknot, and let $b: I \times I \rightarrow S^3$ be an imbedding such that $b^{-1}(\gamma_i) = I \times \{i\}, \ i = 1, 0$. The band sum $\gamma_0 \# b \gamma_1$ of $\gamma_0$ and $\gamma_1$ is obtained by joining $\gamma_0 - b(I \times \{0\})$ to $\gamma_1 - b(1 \times \{1\})$ by the arcs $b(\partial I \times I)$.

1.3 Theorem. Suppose $\gamma_0$ and $\gamma_1$ are unknots in the 3-sphere and a certain band sum $K = \gamma_0 \# b \gamma_1$ is composite. Then $K$ has a two-bridge summand.

1.4. Let $K$ be as in 1.3. As in [Sc1] we obtain an imbedding of a genus two handlebody $N$ into $S^3$ and planar surfaces $P(Q)$ in $\text{closure}(S^3 - N)$ with boundary $A_m$ ($B_n$). The surface $P$ arises from a 2-sphere $S_p$ separating $\gamma_0$.
from $\gamma_1$, the surface $Q$ from a 2-sphere $S_Q$ decomposing $K$ into a nontrivial connected sum. That is, $S_Q$ meets $K$ in two points and separates $S^3$ into 3-balls $B_1$ and $B_2$; each of which meets $K$ in a nontrivial knotted arc. As both $\gamma_0$ and $\gamma_1$ are unknotted, $n > 0$.

Assume that $m + n$ is minimal with respect to the conditions on the surfaces $S_P$ and $S_Q$, and that $S_Q$ decomposes $K$ into summands $K_1$ and $K_2$, each with bridge number at least three. Further assume that $P$ and $Q$ are in general position so that $P \cap Q$ consists of arcs and circles, and that the number of components of $P \cap Q$ is minimal. In particular, any circle left in $P \cap Q$ is essential in both $P$ and $Q$.

Construct as follows semioriented graphs $\Gamma_P$ and $\Gamma_Q$ in the 2-spheres $S_P$ and $S_Q$. Regard each $a_i$ ($b_s$) as a fat vertex in $\Gamma_P$ ($\Gamma_Q$). Regard $b_+$ and $b_-$ as valence zero vertices in $\Gamma_Q$. Finally, regard the arc components of $P \cap Q$ as edges of the graphs $\Gamma_P$ and $\Gamma_Q$. For $1 \leq i \leq m$, $1 \leq s \leq n$ there are two points of intersection of $a_i$ with $b_s$, each of which label $(i, s)$. The end of an edge in $\Gamma_P$ ($\Gamma_Q$) is assigned the second (first) coordinate of the label above for the corresponding point in $A_m \cap B_n$. In $S_P(S_Q - \{b_+, b_\})$ there are $m(n)$ vertices of $\Gamma_P(\Gamma_Q - \{b_+, b_\})$, each of valence $2n(2m)$. Figure 2 shows how the labels sit around the vertices.

1.5. Orient edges from higher labels to lower; those edges running between identical labels are called level. Define a circuit in $\Gamma_P$ ($\Gamma_Q$) to be a subgraph which is an imbedded circle. In $\Gamma_P$ choose a point $x$ in $S_P - \Gamma_P$ and define the interior of a circuit as the complementary region not containing $x$. In $\Gamma_Q$ define the interior of a circuit as the complementary region not containing $b_-$. Further define interior vertex, chord, spoke, loop, base of loop, cycle, unicyle, semicycle, label sequence, interior label, sink, and source as in [SC]. A level circuit is a circuit with all edges level. A level circuit is pure if every edge has the same label, otherwise a level circuit is mixed. The label of the edges of a pure

![Figure 2](https://www.ams.org/journal-terms-of-use)
level circuit is the height of that circuit. A good circuit, cycle, semicycle, or loop in \( \Gamma_Q \) is one whose interior does not contain \( b_+ \). A circuit, cycle, semicycle, or loop in \( \Gamma_Q \) whose interior does contain \( b_+ \) is bad.

1.6. Recall the following elementary facts, cf. [Sc1, 2.5]:

**Proposition.** (1) No chord of an innermost (good) cycle in \( \Gamma_p \) (\( \Gamma_Q \)) is oriented.
(2) Any chord of an innermost (good) semicycle in \( \Gamma_p \) (\( \Gamma_Q \)) is a level loop.
(3) If an innermost (good) cycle or semicycle in \( \Gamma_p \) (\( \Gamma_Q \)) has an interior vertex it must have an interior source or sink.
(4) Any (good) loop in \( \Gamma_p \) (\( \Gamma_Q \)) which has interior vertices either has an interior source or sink or else there is a cycle interior to the loop.

2. PRELIMINARY COMBINATORICS

2.1 **Proposition.** If \( \alpha \) is a level edge in a (good) circuit in \( \Gamma_p \) (\( \Gamma_Q \)) then the labels which precede and follow the labels of \( \alpha \) in the label sequence are equal to each other.

**Proof.** See [Sc1, 4.2].

2.2 **Proposition.** A (good) unicycle in \( \Gamma_p \) (\( \Gamma_Q \)) has interior vertices.

**Proof.** See [Sc1, 4.5].

2.3 **Proposition.** A good semicycle in \( \Gamma_Q \) has interior vertices.

**Proof.** See [Sc1, 4.7].

2.4 **Proposition.** A cycle in \( \Gamma_p \) has interior vertices.

**Proof.** See [Sc2, 5.6].

2.5 **Proposition.** A good loop in \( \Gamma_Q \) has interior vertices.

**Proof.** Suppose there were an innermost good level loop \( \alpha \) based at the vertex \( b_j \) in \( \Gamma_Q \) without interior vertices. The two ends of \( \alpha \) either have label 1 or label \( m \), without loss of generality, say \( m \). Let \( D \) be the interior of \( \alpha \) and denote \( \partial D \cap \partial N \) by \( \alpha' \). The arc \( \alpha' \) is an arc in \( b_j \) with endpoints the two labels \( m \) and meets no other label. Form a new planar surface \( P' \) by attaching the band \( B \) given by a regular neighborhood of \( \alpha' \) in \( \partial N \), see Figure 3.

The surface \( P' \) as boundary components \( a_1, \ldots, a_{m-1} \) and two other components \( a_+, a_- \), which (up to isotopy) are meridians of a 1-handle. By Figure 4, \( \partial D \) separates \( a_+ \) from \( a_- \) in \( P' \).

Compressing \( P' \) via \( D \) gives two planar surfaces \( P_+, P_- \) each of which has a single boundary component on a 1-handle and any other boundary component parallel to \( D_h \). Now it is easy to construct a simple closed curve on \( \partial N \) intersecting the boundary of \( P_+ \) in a single point, and hence intersecting in one point the sphere \( S \) in \( M \) obtained by attaching a disc to each component of \( \partial P_+ \) in \( N \).
2.6 **Proposition.** If $\Gamma_Q$ contains a good loop then $\Gamma_p$ contains a unicycle.

*Proof.* See [Sc$_1$, 5.2].

2.7 **Proposition.** If $\Gamma_Q$ contains a good loop, then $\Gamma_Q$ does not contain both a source and a sink.

*Proof.* See [Sc$_1$, 5.4].

2.8 **Proposition.** Good loops do not exist in $\Gamma_Q$.

*Proof.* In the presence of 2.1–2.7, the path threading argument of [Sc$_1$, §6] and the minimality of $m + n$ show that there cannot be an innermost good semicycle in $\Gamma_Q$. So no good unicycle exists in $\Gamma_Q$. Similarly, no good level loop can exist in $\Gamma_Q$, as applying the path threading argument of [Sc$_1$, §6] to the interior of a good level loop in $\Gamma_Q$ produces a good semicycle interior to that loop.
3. Pure level circuits in $\Gamma_p$

3.1 Lemma. A pure level circuit in $\Gamma_p$ without oriented chords has interior vertices.

3.2 Proof of 3.1. Suppose $\Omega$ is an innermost counterexample and that $\Omega$ has height $k$.

3.3 Proposition. $\Omega$ has no chords.

3.4 Proof of 3.3. Suppose $\alpha_0$ is a level chord of $\Omega$ with label $l$. The ends of $\alpha_0$ are on distinct vertices of $\Omega$ as a level loop would form a pure level circuit without oriented chords interior to $\Omega$. By Figure 2, the other label $l$ at either of these vertices is an interior label of $\Omega$, and so must be incident to another level chord $\alpha_1$, with label $l$. Continuing, one forms a pure level circuit without oriented chords interior to $\Omega$.

3.5 Continuation of 3.2. As $\Omega$ has no chords, $\Omega$ is of height 1 or $n$. Without loss of generality, say $n$. Moreover, the interior of $\Omega$ is a disc, denote this disc by $D$.

Claim. $\partial D$ lies entirely on one side of $b_n$.

Proof of claim. If $n > 1$, this is immediate. If $n = 1$, the claim follows from a slight modification of the argument in [Sc, 4.2].

Let $D$ be the interior of $\Omega$ and let $Q'$ be the planar surface obtained by attaching to $Q$ the band $B$ in $\partial N$ which contains the "tops" of the vertices $a_1, \ldots, a_n$, see Figure 5.

The surface $Q'$ has boundary $b_1, \ldots, b_{n-1}, b_+, b_-$, and two new boundary components $b_+$ and $b_-$ which are (up to isotopy) meridians of the $1$-handles. Call any one of the circles $b_+, b_-, b_+$ or $b_-$ a $1$-handle boundary of $Q'$.

Let $S_{Q'}$ be the two-sphere in $S^3$ obtained by capping off the planar surface $Q'$ with discs in $N$. The surface $S_{Q'}$ is a Conway sphere for the composite knot $K$, decomposing $S^3$ into tangles $T_1$, in which $D$ is a proper disc, and

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Figure 5}
\end{figure}
The distinguished arcs of \( T_2 \) are separated by the disc \( D \) in \( N \) bounded by \( b_n \). One of the distinguished arcs of \( T_2 \) is unknotted and the other distinguished arc is knotted according to one of the summands, say \( K_2 \), of \( K \). It follows that the other summand \( K_1 \) of \( K \) is obtained by attaching an untangle, i.e. a trivial (3-ball, arc pair) to the tangle \( T_1 \).

3.6 Proposition. The tangle \( T_1 \) is an untangle.

3.7 Proof of 3.1 from 3.6. By assumption the bridge number of the summand \( K_1 \) is at least three. From 3.5 and 3.6 we conclude that the summand \( K_1 \) can be expressed as the union of two untangles. It follows that \( K_1 \) has bridge number at most two, a contradiction.

3.8 Proposition. The distinguished arcs of \( T_1 \) are separated by the disc \( D \).

3.9 Proof of 3.6 from 3.8. It remains to show that there are no local knots in the distinguished arcs of \( T_1 \). Compressing the surface \( Q' \) via the disc \( D \) one obtains two planar surfaces \( Q_* \) and \( Q^* \). The planar surface \( Q_* \) (\( Q^* \)) in closure \( (M - N) \) has exactly two boundary components which are meridians of the 1-handles (possibly of the same 1-handle) and at most \( n - 1 \) other boundary components, each of which is parallel to \( \partial D_h \). So the two-sphere \( S_{Q_*} \) (\( S_{Q^*} \)) formed by capping of the surface \( Q_* \) (\( Q^* \)) with discs in \( N \) meets \( K \) in two points and separates \( S^3 \) into 3-balls \( B_1 \) and \( B_2 \). One of these balls, say \( B_1 \), is contained in the tangle \( T_1 \) and contains one of the two distinguished arcs (the other distinguished arc) of \( T_1 \). A local knot in either distinguished arc of \( T_1 \) would therefore contradict the minimality of \( m + n \).

3.10 Proof of 3.8. Proposition 3.8 follows immediately from

3.11 Proposition. No three of the 1-handle boundaries of \( Q' \) lie in the same component of \( Q' - \partial D \).

3.12 Proof of 3.11. Assume at least three of the four 1-handle boundaries of \( Q' \) lie in the same component of \( Q' - \partial D \). There are two cases to consider.

Case 1. The circuit \( \Omega \) is a level loop.

Proof of Case 1. By 2.8 the level loop \( \alpha \) in \( \Gamma_Q \) corresponding to \( \Omega \) is bad, and hence \( \alpha \) separates \( b_+ \) from \( b_- \) in \( \Gamma_Q \). As in the proof of 2.5 it follows that \( \partial D \) separates \( \{ b_+, b_* \} \) from \( \{ b_-, b_# \} \), see Figure 6.

Case 2. The circuit \( \Omega \) has more than one edge.

Proof of Case 2. It is essential to what follows to have a good picture of the family \( l \) of loops in \( \Gamma_Q \) corresponding to the edges of \( \Omega \) in \( \Gamma_p \) and how their endpoints are distributed about \( b_n \). A loop in \( l \) will be called an \( l \)-loop and a label on \( b_n \) incident to an \( l \)-loop an \( l \)-label. Both labels \( n \) on a vertex \( a_i \) of \( \Omega \) in \( \Gamma_p \) are incident to edges of \( \Omega \), so both labels \( i \) on the vertex \( b_n \) in \( \Gamma_Q \) are \( l \)-labels. In particular, there is an equal number of \( l \)-labels on the front and back of \( b_n \).
By 3.7, \( l \) is a family of bad loops based at a common vertex, so every \( l \)-loop separates \( b_+ \) from \( b_- \) in \( \Gamma_Q \) and the \( l \)-loops are linearly ordered by "interior".

3.13 **Definition.** A loop \( \alpha \) in \( \Gamma_n \) is a lobe if both ends of \( \alpha \) lie on either the front face or the back face of \( b_n \). This is a slightly more general definition of a lobe than that of [BS, 6.1]

3.14 **Proposition.** The inner- and outermost \( l \)-loops are lobes.

3.15 **Proof of 3.14.** Consider a nonlobe \( \alpha \) in \( l \). As \( \Omega \) is not a level loop, the labels \( i, j \) at the endpoints of \( \alpha \) are not equal, say \( i < j \). From Figure 7 we see that exactly one of the other \( l \)-labels \( i, j \) on \( b_n \) not incident to \( \alpha \) is an interior label of \( \alpha \) while the other is an exterior label.

Hence there are members of \( l \) interior and exterior to \( \alpha \).

3.16 **Continuation of 3.12.** Now refer to certain intervals on \( b_n \) as "points of the compass". In particular, call that interval meeting no label and having endpoints the two labels \( n(1) \) north (south) and the interval incident to the inner (outer) most component of \( Q - \{ l \cup b_n \} \) as east (west). Call the four complementary
intervals on $b_n$ the NW, SW, SE, or NE *quadrant* of $b_n$ as appropriate. Define NW, SW, SE, or NE (*l*-) *labels* in the obvious manner.

By 2.8, no *l*-loop joins NW to NE or SW to SE. Hence any nonlobe in *l* joins NE to SW or NW to SE. However, it is impossible to draw both types of nonlobes on the planar surface $P$, cf. [Sc$_2$, 3.2]. Without loss of generality then, assume all nonlobes in *l* join NE to SW, and hence that any member of *l* with a NW or SE label is a lobe. Proposition 3.14 implies that the number of *l*-labels in each quadrant is nonzero. The family *l* is as illustrated below in Figure 8.

To see how $\partial D$ sits on $Q'$, add the band $B$ and recall that on $B$ the curve $\partial D$ is parallel to the core of $B$. Figure 9 and the convention above imply that the four 1-handle vertices of $Q'$ correspond to the four cardinal directions N, S, E, and W on $b_n$. By the Jordan curve theorem, two 1-handle vertices are in the same component of $Q' - \partial D$ if and only if there is an arc joining the corresponding cardinal directions in $b_n$ which contains an even number of *l*-labels.
3.17 **Proposition.** The curve $\partial D$ separates $b_+$ from $b_*$ in $Q'$ if and only if $\partial D$ separates $b_-$ from $b_\#$ in $Q'$.

3.18 **Corollary to 3.17.** No 1-handle boundary in $Q'$ is separated from the other three by $\partial D$.

3.19 **Proof of 3.17.** There are an equal number of $l$-labels on the front and back face of $b_n$. As a nonlobe has one end on the front and one end on the back of $b_n$, the front and back face of $b_n$ have an equal number of $l$-labels corresponding to nonlobes, hence equal numbers corresponding to lobes. From Figure 8, it follows that this number is exactly twice the number of NW (SE) $l$-labels. Hence there are an equal number of NW and SE $l$-labels. The curve $\partial D$ separates $b_+$ from $b_\#$ ($b_-$ from $b_*$) in $Q'$ if and only if this number is odd.

3.20 **Continuation of 3.12.** By 3.18, all four of the 1-handle boundaries of $Q'$ must lie in the same component of $Q' - \partial D$, so there must be an even number of $l$-labels in each quadrant. It follows that for some $k > 0$ there are $4k$ $l$-labels in all and so the circuit $\Omega$ has exactly $2k$ edges.

3.21 **The final contradiction.** We will contradict 3.20 by counting the edges of $\Omega$ another way. Consider traversing $\partial D$ in $Q'$ by beginning on and traveling in the direction of the arrow of the innermost loop in $l$ with a NW label. Call an $l$-label where we pass from $Q$ to $B$ ($B$ to $Q$) as we traverse $\partial D$ an exit (entry) point, see Figure 10.

Map an exit (entry) point to the next exit (entry) point encountered as $\partial D$ is traversed in the manner above to obtain a cyclic permutation of the $4k$ $l$-labels. As $\partial D$ runs parallel to the core of $b$, an exit (entry) point moves a fixed number $r$ of $l$-labels counter-clockwise (clockwise) about $b_n$ under this permutation. From Figure 9, $r$ equals twice the number of NW $l$-labels. As this latter number is even, $r = 4j$ for some $j > 0$. 

![Figure 10](https://www.ams.org/journal-terms-of-use)
The number of edges in $\Omega$ is just the cardinality of an orbit of an exit (entry) point under this permutation. It follows that $\Omega$ has exactly
\[
\frac{4k}{\text{g.c.d.}(4k, 4j)} \leq k \text{ edges}.
\]
This contradicts 3.20.

4. X-CIRCUITS

4.1 Lemma. There is either a cycle or a pure level circuit in $\Gamma_p$ without chords and without interior vertices.

Choose orientations for the planar surfaces $P$ and $Q$, and let these induce orientations on their respective boundaries $A_m$ and $B_n$. Call two components of $A_m$ ($B_n$) parallel if their orientations are parallel in $\partial N$; if not call them antiparallel. There are two points of intersection of $a_j$ with $b_s$, one on the front face and one on the back face of $N$. Label them respectively $f(i, s)$ and $r(i, s)$. As in [Sc1, 2.2] we have the following “parity rule”.

4.2 Proposition. If an arc of $P \cap Q$ runs between $f(i, s)$ and $f(j, t)$, then one of the pairs $a_j$ and $a_j$ or $b_s$ and $b_t$ is parallel and the other is antiparallel. The same is true for an arc of $P \cap Q$ which runs between $r(i, s)$ and $r(j, t)$. On the other hand, if an arc of $P \cap Q$ runs between $f(i, s)$ and $r(j, t)$, then either both pairs are parallel or both are antiparallel.

There are two classes of vertices in $\Gamma_p$, each consisting of parallel vertices. A typical vertex in each class is illustrated in Figure 11, where $f(s)$ ($r(t)$) denotes that the label $s$ ($t$) lies on the front (back) face of $N$.

We form a new graph $\Gamma_{p^*}$ from the graph $\Gamma_p$ as follows: Reenumerate the labels of the vertices in the first class given above by changing $f(i)$ to $i$, and $r(j)$ to $2n + 1 - j$, see Figure 12.

Similarly, reenumerate the labels of the vertices of the second class by changing $r(i)$ to $i$, and $f(j)$ to $2n + 1 - j$, see Figure 13.

By construction, therefore, all the vertices in $\Gamma_{p^*}$ are parallel. As before, orient edges from higher labels to lower, and continue to call level those edges running between identical labels. Proposition 4.2 now implies the following.

![Diagram](https://www.ams.org/journal-terms-of-use)
4.3 Proposition. There are no level edges in $\Gamma_p$.

For $x \in \{1, 2, \ldots, 2n\}$ define an $x$-circuit to be a circuit which can be traversed in a direction so that the initial point of every edge has label $x$.

4.4 Proposition. In $\Gamma_p$ there is an $x$-circuit without interior vertices and without chords.

4.5 Proof of 4.4. (cf. [CLGS, 2.6.2]). Choose an $x \in \{1, 2, \ldots, 2n\}$ and construct a path beginning at some vertex $v$, always choosing the label $x$ as the initial point of each edge; by 4.3 this path can be followed until a vertex is repeated, forming an $x$-circuit. Choose an innermost such circuit (varying over all $x$), and denote it by $\Omega$ and suppose that $\Omega$ has interior vertices or chords.

If $\Omega$ has no interior labels but has interior vertices, then for any $y \in \{1, 2, \ldots, 2n\}$ it is possible to find a $y$-circuit interior to $\Omega$. So suppose $\Omega$ has interior labels. Choose a vertex $v$ in $\Omega$, and choose the label on $v$ which is adjacent to $x$ and is either an interior label of $\Omega$ or a label of an edge in the circuit. This label is either $x - 1$ or $x + 1 \pmod{2n}$, without loss of generality, suppose it is $x - 1$. Since all the vertices of $\Gamma_p$ are parallel, for each vertex of $\Omega$ the label $x - 1$ is either an interior label or a label of an edge in the circuit. The circuit $\Omega$ has interior labels, so there is at least one vertex $u$ in $\Omega$ for which the label $x - 1$ is an interior label. Beginning at $u$, we can construct a path in which every edge has initial label $x - 1$; when a vertex is repeated we obtain an $(x - 1)$-circuit, call it $\Omega'$.

The circuit $\Omega'$ must be interior to $\Omega$ as each vertex in $\Omega$ or in its interior either has the label $x - 1$ as an end of an edge of $\Omega$, or as an end of an interior
edge. The circuit $\Omega' \neq \Omega$ as the edge of $\Omega$ incident to $u$ with label at $u$ different from $x$ cannot be part of $\Omega'$ as its label at $u$ is also different from the interior label $x - 1$. This contradicts the fact that $\Omega$ was innermost, and 4.4 follows.

4.6 Proof of 4.1. The $x$-circuit in $\Gamma_p^*$ given by 4.4 has a consistent orientation of its edges so that the initial point of every edge has label $x$ and the terminal point of every edge has label $x - 1$. It follows that in $\Gamma_p$ this circuit is either a cycle or a pure level circuit.

4.7 Proof of 1.3. A contradiction exists between 2.4, 3.1, and 4.1.

5. CONCLUDING REMARKS

5.1. A ribbon disc for $K^2$, the connected sum of a knot $K$ with its mirror image, is obtained by spinning an appropriately knotted arc in $R^3$. If $K$ has bridge number $b$, this ribbon disc has $b - 1$ saddle points. Theorem 1.3 shows that the ribbon number of $K^2$ is $b - 1$ for $b < 4$. So it seems appropriate to conjecture that the ribbon number of $K^2$ always equals the bridge number of $K$ minus 1.

REFERENCES