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CONTINUOUS SPATIAL SEMIGROUPS OF $*$-ENDOMORPHISMS OF $\mathfrak{B}(\mathfrak{H})$

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ABSTRACT. To each continuous semigroup of $*$-endomorphisms α of $\mathfrak{B}(\mathfrak{H})$ with an intertwining semigroup of isometries there is associated a $*$-representation π of the domain $\mathfrak{D}(\delta)$ of the generator of α. It is shown that the Arveson index $d_*(\alpha)$ is the number of times the representation π contains the identity representation of $\mathfrak{D}(\delta)$. This result is obtained from an analysis of the relation between two semigroups of isometries, U and S, satisfying the condition $S(t)^* U(t) = e^{-\lambda t} I$ for $t \geq 0$ and $\lambda > 0$.

I. INTRODUCTION

Let $\mathfrak{H}$ be a separable Hilbert space, and let $\alpha = \{\alpha_t : t \geq 0\}$ be a one-parameter semigroup of $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H})$. Note a $*$-endomorphism of $\mathfrak{B}(\mathfrak{H})$ is automatically normal. We assume that $\alpha_t(I) = I$ for all t and that the mapping $t \to \alpha_t(A)$ is continuous in the weak operator topology. We assume also that $\alpha_t(\mathfrak{B}(\mathfrak{H})) \neq \mathfrak{B}(\mathfrak{H})$ for some (hence for all) positive t. Such a semigroup α is called an E_0-semigroup of $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H})$.

We say a strongly continuous one-parameter semigroup $U = \{U(t) : t \geq 0\}$ of isometries is an intertwining semigroup for the E_0-semigroup α if $U(t)A = \alpha_t(A)U(t)$. In [P2] it was shown that there exist E_0-semigroups which have no intertwining semigroups. When an intertwining semigroup exists one can make the following construction.

Suppose α is an E_0-semigroup of $\mathfrak{B}(\mathfrak{H})$ and $U = \{U(t) : t \geq 0\}$ is a strongly continuous semigroup of intertwining isometries. Let $-d$ be the generator of U so $df = \lim_{t\to 0^+}(f - U(t)f)/t$ for all $f \in \mathfrak{D}(d)$, and $\mathfrak{D}(d)$ is precisely all vectors $f \in \mathfrak{H}$ so that the limit exists in the sense of strong convergence (in fact, if the quotient $(f - U(t_n)f)/t_n$ remains bounded for a sequence $t_n \to 0$ then $f \in \mathfrak{D}(d)$). Let $\delta(A) = \lim_{t\to 0^+}(\alpha_t(A) - A)/t$ for all $A \in \mathfrak{D}(\delta)$ where $\mathfrak{D}(\delta)$ is the set of all $A \in \mathfrak{B}(\mathfrak{H})$ so that the limit exists in the sense of weak operator convergence (again, if the quotient $(\alpha_{t_n}(A) - A)/t_n$ remains bounded for a sequence $t_n \to 0^+$ then the quotient converges σ-strongly as $t \to 0^+$).

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We have that $\mathfrak{D}(\delta)$ is a σ-weakly dense $*$-subalgebra of $\mathfrak{B}(\mathfrak{H})$ and δ is a $*$-derivation of $\mathfrak{D}(\delta)$ into $\mathfrak{B}(\mathfrak{H})$.

The generators δ and d are related as follows. Since $U(t)A = \alpha_t(A)U(t)$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ we find by differentiation of this equation that if $A \in \mathfrak{D}(\delta)$ then $A\mathfrak{D}(d) \subset \mathfrak{D}(d)$, $A\mathfrak{D}(d^*) \subset \mathfrak{D}(d^*)$ and for $f \in \mathfrak{D}(d^*)$ and $A \in \mathfrak{D}(\delta)$ we have $d^*Af = Ad^*f + \delta(A)f$.

We define a nonnegative inner product $\langle\ ,\ \rangle$ on the domain of d^* via the relation

$$\langle f, g\rangle = \tfrac{1}{2}(d^*f, g) + \tfrac{1}{2}(f, d^*g) \quad \text{for } f, g \in \mathfrak{D}(d^*). \tag{1}$$

As is well known from the theory of the extensions of unbounded hermitian operators (see [DS, Lemma 10, p. 1227]), since d is a closed skew hermitian operator every vector $f \in \mathfrak{D}(d^*)$ has a unique decomposition $f = f_0 + f_+ + f_-$ where $f_0 \in \mathfrak{D}(d)$, $f_+, f_- \in \mathfrak{D}(d^*)$ and $d^*f_+ = f_+$ and $d^*f_- = -f_-$. Since $-d$ is the generator of a semigroup of isometries it follows that the equation $d^*f = -f$ has no solutions other than $f = 0$. We denote by $\mathfrak{D}_+ = \{f \in \mathfrak{D}(d^*) : d^*f = f\}$. Then each vector $f \in \mathfrak{D}(d^*)$ has a unique decomposition $f = f_0 + f_+$ with $f_0 \in \mathfrak{D}(d)$ and $f_+ \in \mathfrak{D}_+$. Let E_+ denote the projection of $\mathfrak{H}$ onto $\mathfrak{D}_+$. In the decomposition $f = f_0 + f_+$ the vector f_+ can be obtained from the formula $f_+ = \frac{1}{2}E_+(f + d^*f)$. If $f, g \in \mathfrak{D}(d^*)$ and $f = f_0 + f_+$ and $g = g_0 + g_+$ are the unique decompositions of f and g then one finds $\langle f, g\rangle = (f_+, g_+)$. It follows that if $f \in \mathfrak{D}(d^*)$ then $f \in \mathfrak{D}(d)$ if and only if $\langle f, f\rangle = 0$.

Let $\mathscr{K}$ be the quotient of $\mathfrak{D}(d^*) \bmod \mathfrak{D}(d)$. For $f \in \mathfrak{D}(d^*)$ we denote by $[f]$ the equivalence class in $\mathscr{K}$ containing f. For $f, g \in \mathfrak{D}(d^*)$ we have $[f] = [g]$ if and only if $f - g \in \mathfrak{D}(d)$. Since the inner product $\langle f, g\rangle$ is zero if either f or g lies in $\mathfrak{D}(d)$ it follows that the inner product $([f], [g]) = \langle f, g\rangle$ is well defined on $\mathscr{K}$. Note the mapping $\theta[f] = \frac{1}{2}E_+(f + d^*f)$ is an isometry of $\mathscr{K}$ onto $\mathfrak{D}_+$. Note also that $\Theta[\Theta[f]] = \Theta[f]$ for all $f \in \mathfrak{D}(d^*)$. Since the range of Θ is closed it follows that $\mathscr{K}$ is complete. Hence, $\mathscr{K}$ is a Hilbert space with its inner product $([f], [g]) = \langle f, g\rangle$.

We define a $*$-representation of $\mathfrak{D}(\delta)$ on $\mathscr{K}$ as follows. For $A \in \mathfrak{D}(\delta)$ and $f \in \mathfrak{D}(d^*)$ we define $\pi(A)[f] = [Af]$. This gives us a well-defined operator on $\mathscr{K}$ since $A\mathfrak{D}(d) \subset \mathfrak{D}(d)$ and $A\mathfrak{D}(d^*) \subset \mathfrak{D}(d^*)$. Since $d^*(Af) = Ad^*f + \delta(A)f$ for $A \in \mathfrak{D}(\delta)$ and $f \in \mathfrak{D}(d^*)$ and since δ is a $*$-derivation one checks that $([f], \pi(A)[g]) = \langle f, Ag\rangle = \langle A^*f, g\rangle = (\pi(A^*)[f], [g])$ for $A \in \mathfrak{D}(\delta)$ and $f, g \in \mathfrak{D}(d^*)$ so π is a $*$-representation. One checks that $\pi(A)$ is bounded (see [P1] for details). We denote this representation by $\pi_{(\alpha, U)}$ where we include the symbols α and U to remind us of the dependence of π on α and U.

In [P1] one of the authors defined the index of α to be the multiplicity of $\pi_{(\alpha, U)}$. The difficulty with this definition of index is that it seems to depend on the intertwining semigroup U. To overcome this difficulty one of the authors and D. W. Robinson found another definition of the index in [PR].

A better definition of the index was found by W. Arveson. We describe his construction briefly as follows (see [A1, A2] for details). For an E_0-semigroup α on $\mathfrak{B}(\mathfrak{H})$, let U_α denote the set of all intertwining semigroups $U = \{U(t): t \geq 0\}$ of bounded operators on $\mathfrak{B}(\mathfrak{H})$. If U_α is nonempty Arveson has shown, [A1], that for a pair U, $\mathscr{S}$ of intertwining semigroups in U_α, there is a complex number $c(\mathscr{S}, U)$ such that

$$S(t)^* U(t) = e^{tc(\mathscr{S}, U)} I. \tag{2}$$

Let $\tilde{\mathfrak{H}}(\alpha)$ be the vector space of all functions $f: U_\alpha \to \mathbf{C}$ which are finitely nonzero and satisfy $\sum_{x \in U_\alpha} f(x) = 0$. Then $\tilde{\mathfrak{H}}(\alpha)$ becomes a pre-Hilbert space under the positive semidefinite inner product $(f, g) = \sum_{x,y} \overline{f(x)} g(y) c(x, y)$. Let d_α be the dimension of the Hilbert space completion $\tilde{\mathfrak{H}}(\alpha)$ of $\mathfrak{H}(\alpha)$. This is Arveson's index for E_0-semigroups of $\mathfrak{B}(\mathfrak{H})$.

In this paper we will show the connection between Arveson's index d_α and the representation $\pi_{(\alpha, U)}$. We will show that d_α is precisely the number of times the identity representation of $\mathfrak{D}(\delta)$ occurs in the representation $\pi_{(\alpha, U)}$. This explains the mystery of why Arveson's index turns out to be equal to the multiplicity of $\pi_{(\alpha, U)}$ in all known examples of E_0-semigroups with intertwining semigroups.

The key to this result is understanding pairs U, $\mathscr{S}$ of strongly continuous semigroups which satisfy (2). First we make some elementary observations to show that the problem may be reduced to the study of semigroups of isometries. Indeed, note that for $U \in U_\alpha$, ΓU also lies in U_α, where Γ is the scalar-valued semigroup of operators $\{\Gamma_t = e^{-tc(U, U)/2} I: t \geq 0\}$. Then ΓU is a semigroup of isometries. It is also easy to show that by a further rescaling of the pair U, $\mathscr{S}$, if necessary, one may assume that they are semigroups of isometries satisfying the commutation relations

$$S(t)^* U(t) = e^{-\lambda t} I, \qquad t \geq 0, \tag{3}$$

for $0 \leq \lambda < \infty$.

In the next section we provide a characterization of a pair of semigroups of isometries U, $\mathscr{S}$, satisfying (3) and in the third section we apply these results to obtain the connection between the Arveson index and the number of copies of the standard representation of $\mathfrak{D}(\delta)$ in $\pi_{(\alpha, U)}$.

II. Structure of intertwining semigroups of isometries

In what follows we shall always assume that $\mathscr{S} = \{S(t): t \geq 0\}$ and $U = \{U(t): t \geq 0\}$ are strongly continuous semigroups of isometries in $\mathfrak{B}(\mathfrak{H})$ which satisfy the commutation relations (3). We shall also use the notation introduced in the preceding section. In addition, we shall denote by $-D$ the infinitesimal generator of the semigroup $\mathscr{S}$. The following series of results shows that (3) imposes a very strong connection between d and D.

Lemma 2.1. *Let $-D$ be the infinitesimal generator of the semigroup $\mathscr{S}$. Then $\mathfrak{D}(D) \subset \mathfrak{D}(d^*)$, and for $f \in \mathfrak{D}(D)$, $Df = -d^* f + \lambda f$.*

Proof. If $f \in \mathfrak{D}(D)$ and $g \in \mathfrak{D}(d)$, then

$$e^{-\lambda t}(f, g) = (U(t)^* S(t) f, g) = (S(t) f, U(t) g).$$

Taking d/dt of this equation at $t = 0$ yields

$$-\lambda(f, g) = (Df, g) + (f, -dg),$$

which shows that the linear mapping on $\mathfrak{D}(d)$ given by $g \to (f, dg)$ is continuous, so $f \in \mathfrak{D}(d^*)$. Moreover, $(-d^* f, g) = (f, -dg) = ((D - \lambda I) f, g)$. Since $\mathfrak{D}(d)$ is dense in $\mathfrak{H}$, this last equation shows that $(D - \lambda I) f = -d^* f$. □

Lemma 2.2. $\mathfrak{D}(d) \cap \mathfrak{D}(D) = \{0\}$.

Proof. Let $f \in \mathfrak{D}(d) \cap \mathfrak{D}(D)$. Since $S(t)$ is an isometry, for $t \geq 0$, $\|f\|^2 = (S(t)f, S(t)f)$. Taking d/dt of this equation at $t = 0$ gives

$$0 = (Df, f) + (f, Df) = (df - \lambda f, f) + (f, df - \lambda f),$$

or $2\lambda \|f\|^2 = 2\,\mathrm{Re}(f, df)$. But this implies that $f = 0$, since d is skew-symmetric. □

Now suppose $f, g \in \mathfrak{D}(D)$. If $[f] = [g]$, it follows that $f - g$ lies in $\mathfrak{D}(d)$; but then $f - g \in \mathfrak{D}(d) \cap \mathfrak{D}(D)$, so the preceding lemma implies $f = g$. Hence there is a one-one linear mapping V_0 from $\mathfrak{D}(D)$ to $\mathscr{K}$ given by $V_0 f = [f]$. It is easy to show that V_0 is continuous; in fact, for any $f, g \in \mathfrak{D}(D)$,

$$\begin{aligned} 0 &= (Df, g) + (f, Dg) \\ &= (-d^* f + \lambda f, g) + (f, -d^* g + \lambda g) \\ &= 2\lambda(f, g) - 2\langle f, g\rangle \\ &= 2\lambda(f, g) - 2([f], [g])_{\mathscr{K}}, \end{aligned}$$

so that $(1/\sqrt{\lambda})V_0$ is an isometry on $\mathfrak{D}(D)$. We denote by V the isometric extension of $(1/\sqrt{\lambda})V_0$ to $\mathfrak{H}$.

If $f \in \mathfrak{D}(d^*)$ and also $Vf = \sqrt{\lambda}[f]$, then for $g \in \mathfrak{D}(D)$,

$$\begin{aligned} (-d^* f + \lambda f, g) + (f, Dg) &= 2\lambda(f, g) - 2\langle f, g\rangle \\ &= 2\lambda(f, g) - 2([f], [g])_{\mathscr{K}} \\ &= 2\lambda(f, g) - 2\lambda(Vf, Vg) \\ &= 0. \end{aligned}$$

Since D is maximal skew-symmetric, we conclude that $f \in \mathfrak{D}(D)$. Hence we have the following characterization of the domain of D.

Lemma 2.3. *Let D be the infinitesimal generator of the semigroup $\mathscr{S}$ of isometries. Then there is an isometry V of $\mathfrak{H}$ into $\mathscr{K}$ such that $\mathfrak{D}(D)$ consists precisely of those $f \in \mathfrak{D}(d^*)$ such that $\langle f, g\rangle = \sqrt{\lambda}\langle Vf, g)$, for all $g \in \mathfrak{D}(d^*)$.*

Example. As an illustration of the results above, we present an example which served as our motivation for the study of intertwining semigroups of isometries. We let $\mathfrak{H}$ be the Hilbert space $\mathscr{K}_0 \otimes L^2(0, \infty)$ of $\mathscr{K}_0$-valued functions $f(x)$, $x \in (0, \infty)$, which satisfy $\int_0^\infty (f(x), f(x))\, dx < \infty$. We let $U = \{U(t) : t \geq 0\}$ be the isometric flow of rightward shifts on $\mathscr{K}_0 \otimes L^2(0, \infty)$

given by $(U(t)f)(x) = f(x-t)$ for $x \geq t$, and $(U(t)f)(x) = 0$ for $x < t$. It is well known (see [Y1], for example) that the infinitesimal generator $-d$ of U is the linear operator on $\mathfrak{H}$ whose domain $\mathfrak{D}(d)$ consists of those functions $f(x)$, each of whose components is absolutely continuous, such that $f(0) = (0)$, and $(f'(x)) \in \mathscr{K}_0 \otimes L^2(0, \infty)$. d is the operator $df = f'$ on its domain. The adjoint operator d^* satisfies $d^*f = -f'$ on its domain consisting of functions as above, but with no boundary conditions imposed at $x = 0$. The deficiency space $\mathfrak{D}_+$ consists of those functions $f \in \mathfrak{D}(d^*)$ which satisfy $f(x) = e^{-x}f(0)$. Note that, as in the introduction, there is a natural isometry from $\mathfrak{D}_+$ to $\mathscr{K}_0$ which is given by $V(e^{-x}f(x)) = (1/\sqrt{2})f(0)$.

Let $\mathscr{S}$ be another semigroup of isometries on $\mathfrak{H}$ such that the pair $U, \mathscr{S}$ satisfies (3). Let $-D$ be the infinitesimal generator of $\mathscr{S}$. By the results above, there exists an isometry V of $\mathfrak{H}$ into $\mathscr{K}_0$ such that $\mathfrak{D}(D)$ consists of those elements $f \in \mathfrak{D}(d^*)$ whose image $Vf \in \mathscr{K}_0$ satisfies $\sqrt{\lambda}Vf = (1/\sqrt{2})f(0)$.

By (3) we have $e^{-\lambda t}(f, g) = (U(t)f, S(t)g)$ for all $f, g \in \mathfrak{H}$. By differentiating this equation in this special case where $f \in \mathfrak{D}(d)$, $g \in \mathfrak{D}(D)$, it is not difficult to show that $(S(t)g)(x) = e^{-\lambda t}g(x-t)$ for $x > t$; so that $S(t)$ acts like the (dissipative) translation $e^{-\lambda t}U(t)$ away from the boundary. The $\{S(t) \colon t \geq 0\}$ are isometries, however—hence, norm preserving. Roughly speaking, the "wave function" that is dissipated in translation by $S(t)$ is reintroduced through the boundary.

We should note that this example is not much of a restriction from the consideration of the general case. This follows from the well-known result, [Sz], that any strongly continuous one-parameter semigroup of isometries $\{T(t) \colon t \geq 0\}$ is unitarily equivalent to the orthogonal direct sum of a unitary semigroup and c copies of the rightward shift on $\mathscr{K}_c \otimes L^2(0, \infty)$. If fact, it follows from our results (viz., the existence of an isometry from $\mathfrak{H}$ to $\mathscr{K}$ determining the domain of D) that for a pair $U, \mathscr{S}$ of semigroups to satisfy (3), c must be ∞.

Using the characterization of D in the preceding lemmas, we may also obtain an explicit expression for its adjoint.

Lemma 2.4. *$\mathfrak{D}(D^*)$ coincides with $\mathfrak{D}(d^*)$. For $f \in \mathfrak{D}(d^*)$, $D^*f = d^*f + \lambda f - \sqrt{\lambda}V^*(E_+(f + d^*f))$, where V is the isometry from $\mathfrak{H}$ to $\mathscr{K}$ which determines the domain of D.*

Proof. Let $f \in \mathfrak{D}(d^*)$ and $g \in \mathfrak{D}(D)$. Using the preceding lemma, we calculate

$$\begin{aligned}
(f, Dg) &= (f, -d^*g) + \lambda(f, g) \\
&= (-d^*f, g) + (f, -d^*g) + (d^*f, g) + \lambda(f, g) \\
&= -2\langle f, g\rangle + (d^*f, g) + \lambda(f, g) \\
&= -2\sqrt{\lambda}\langle f, Vg\rangle + (d^*(f + \lambda f, g) \\
&= -2\sqrt{\lambda}(\tfrac{1}{2}E_+(f + d^*f), Vg) + (d^*f + \lambda f, g) \\
&= (-V^*E_+\sqrt{\lambda}(f + d^*f) + d^*f + \lambda f, g).
\end{aligned}$$

This shows that the mapping $g \to (f, Dg)$ is continuous on $\mathfrak{D}(D)$, for $f \in \mathfrak{D}(d^*)$, so that $\mathfrak{D}(d^*) \subset \mathfrak{D}(D^*)$. By the symmetry of (3) in U and $\mathscr{S}$, however, we also have $\mathfrak{D}(D^*) \subset \mathfrak{D}(d^*)$, so that $\mathfrak{D}(D^*) = \mathfrak{D}(d^*)$. Hence the calculation above gives the expression for D^* on its domain. □

Next we wish to show that the converse of the results above holds as well. Let V be an isometry from $\mathfrak{H}$ to $\mathscr{K}$, and let D be the operator $-d^* + \lambda I$ on its domain

$$\mathfrak{D}(D) = \{f \in \mathfrak{D}(d^*) \colon \langle f, g \rangle = \sqrt{\lambda}\langle Vf, g\rangle, \text{ for all } g \in \mathfrak{D}(d^*)\}.$$

We shall show that D is the infinitesimal generator of a strongly continuous one-parameter semigroup of isometries $\mathscr{S} = \{S(t) \colon t \geq 0\}$ which satisfies (3). The following construction provides a means of showing that D has a dense domain. For $s > 0$, define I_s to be the operator $I_s = \int_0^\infty e^{-s\gamma} U(\gamma) U(\gamma)^* d\gamma$ on $\mathfrak{H}$. Observe that I_s is a selfadjoint operator on $\mathfrak{H}$. Moreover, for $t > 0$ one can show (see [D1, Proposition 1]) that

$$U(t) I_s = e^{st} I_s U(t). \tag{4}$$

From (4) we observe that I_s maps $\mathfrak{D}(d)$ into itself. In fact:

Lemma 2.5. *For $s > 0$, I_s maps $\mathfrak{D}(d^*)$ into $\mathfrak{D}(d)$.*

Proof. Let $f \in \mathfrak{D}(d^*)$. As before we may write $f = f_0 + f_+$, where $f_0 \in \mathfrak{D}(d)$, $f_+ \in \mathfrak{D}_+$. We have $I_s f_0 \in \mathfrak{D}(d)$, from the comment made above. Recalling that $U(\gamma)^* f_+ = e^{-\gamma} f_+$, we have

$$\begin{aligned}
(U(t) - I) f_+ &= \int_0^\infty e^{-(s+1)\gamma} U(t+\gamma) f_+ \, d\gamma - \int_0^\infty e^{-(s+1)\gamma} U(\gamma) f_+ \, d\gamma \\
&= e^{(s+1)t} \int_0^\infty e^{-(s+1)\gamma} U(\gamma) f_+ \, d\gamma - \int_0^\infty e^{-(s+1)\gamma} U(\gamma) f_+ \, d\gamma \\
&\quad - e^{(s+1)t} \int_0^t e^{-(s+1)\gamma} U(\gamma) f_+ \, d\gamma \\
&= (e^{(s+1)t} - 1) \int_0^\infty e^{-(s+1)\gamma} U(\gamma) f_+ \, d\gamma \\
&\quad - e^{(s+1)t} \int_0^t e^{-(s+1)\gamma} U(\gamma) f_+ \, d\gamma.
\end{aligned}$$

Dividing by t and taking the limit as $t \to 0^+$, we find that $I_s f_+ \in \mathfrak{D}(d)$ and $d I_s f_+ = (s+1) I_s f - f$. □

Definition 2.1. For $s > 0$, let R_s be the operator $I - sI_s$ on $\mathfrak{H}$.

Remark. Returning to the example cited above, one may check easily that for $f \in \mathscr{K}_0 \otimes L^2(0, \infty)$, $(R_s f)(x) = e^{-sx} f(x)$.

Proposition 2.6. *The following properties hold for the operators* R_s, $s > 0$:

(a) $R_s(\mathfrak{D}(d)) \subset \mathfrak{D}(d)$ *and* $R_s(\mathfrak{D}(d^*)) \subset \mathfrak{D}(d^*)$,

(b) *for* $f \in \mathfrak{D}_+$, $d^* R_s f = (s+1) R_s f$,

(c) *for* $f, g \in \mathfrak{D}(d^*)$, $\langle f, g\rangle = \langle f, R_s g\rangle$, *and*

(d) *there are positive constants* $c(s)$ *such that* $c(s) \to 0$ *as* $s \to \infty$ *and* $\|R_s f\| \le c(s)\|f\|$ *for all* $f \in \mathfrak{D}_+$

Proof. The inclusions in (a) follow immediately from the results about I_s in the preceding lemma. From the proof of this lemma we have $-dI_s f_+ = (s+1)I_s f_+ - f_+$, and (b) follows directly from this. To show (c) recall from the previous lemma that $I_s g \in \mathfrak{D}(d)$ if $g \in \mathfrak{D}(d^*)$, so $\langle f, R_s g\rangle = \langle f, g\rangle - s\langle f, I_s g\rangle = \langle f, g\rangle$.

Finally, we show (d). First we observe that $\gamma > 0$, and a unit vector $f_+ \in \mathfrak{D}_+$, $\|f_+ - U(\gamma)f_+\|^2 = 2\|f_+\|^2 - 2\operatorname{Re}(U(\gamma)^* f_+, f_+) = 2\|f\|^2(1 - e^{-\gamma})$. Hence

$$\begin{aligned}
\|R_s f_+\| &= \left\| f_+ - s\int_0^\infty e^{-s\gamma} U(\gamma)U(\gamma)^* f_+ \, d\gamma \right\| \\
&= \left\| f_+ - s\int_0^\infty e^{-(s+1)\gamma} U(\gamma) f_+ \, d\gamma \right\| \\
&= \left\| s\int_0^\infty e^{-(s+1)\gamma} [s^{-1}(s+1)f_+ - U(\gamma)f_+] \, d\gamma \right\| \\
&\le s\int_r^\infty e^{-(s+1)\gamma} \|s^{-1}(s+1)f_+ - U(\gamma)f_+\| \, d\gamma \\
&\quad + s\int_r^\infty e^{-(s+1)\gamma} \|s^{-1}(s+1)f_+ - U(\gamma)f_+\| \, d\gamma \\
&= s\int_0^r e^{-(s+1)\gamma} [(s^{-1}(s+1) - 1) + \|f_+ - U(\gamma)f_+\|] \, d\gamma \\
&\quad + 3s\int_r^\infty e^{-(s+1)\gamma} \, d\gamma \\
&= \int_0^r e^{-(s+1)\gamma} \, d\gamma + \sqrt{2}s\int_0^r e^{-(s+1)\gamma}\sqrt{1 - e^{-\gamma}} \, d\gamma \\
&\quad + 3s\int_0^\infty e^{-(s+1)\gamma} \, d\gamma .
\end{aligned}$$

Choosing $r = 1$, the first and third terms vanish as $s \to \infty$. As for the second term, note that (for $s > 1$)

$$\sqrt{2}s\int_0^1 e^{-(s+1)\gamma}\sqrt{1 - e^{-\gamma}} \, d\gamma \le \sqrt{2}s\int_0^{s^{-3/4}} \sqrt{\gamma} \, d\gamma + \sqrt{2}s\int_{s^{-3/4}}^1 e^{-(s+1)\gamma} \, d\gamma ,$$

and that each of these terms tends to 0 as $s \to \infty$. This gives (d). □

Lemma 2.7. *Let* V *be an isometry from* $\mathfrak{H}$ *to* $\mathscr{K}$, *and let* D *be the operator defined by* $-d^* + \lambda I$ *on its domain* $\mathfrak{D}(D)$ *consisting of those* $f \in \mathfrak{D}(d^*)$ *such that*

$\langle f, g\rangle = \sqrt{\lambda}\langle Vf, g\rangle$, *for all* $g \in \mathfrak{D}(d^*)$. *Then* D *is a maximal skew-symmetric operator.*

Proof. Fix s sufficiently large so that $c(s)\sqrt{\lambda} < 1$. For $f \in \mathfrak{D}(d)$ and $n \in \mathbf{N}$, let $f_n = (R_s V_0)^n f$, where $V_0 = \sqrt{\lambda}V$. Then $\|f_n\| \le (\sqrt{\lambda}c(s))^n$, so that $\sum_{n=1}^{\infty} f_n$ converges to a vector F in $\mathfrak{H}$. From Proposition 2.6(a), $f_n \in \mathfrak{D}(d^*)$; in fact, from Proposition 2.6(b), $d^* f_n = (s+1)f_n$, so that $F \in \mathfrak{D}(d^*)$, and $d^*F = -df + (s+1)\sum_{n=1}^{\infty} f_n$. We now can show that $f \in \mathfrak{D}(D)$. To see this, recall that $\langle f, g\rangle = 0$ for all $g \in \mathfrak{D}(d^*)$, since $f \in \mathfrak{D}(d)$, so by Proposition 2.6(c),

$$\begin{aligned}\langle F, g\rangle &= \sum_{n=0}^{\infty}\langle (R_s V_0)^n f, g\rangle = \sum_{n=1}^{\infty}\langle (R_s V_0)^n f, g\rangle \\ &= \sum_{n=1}^{\infty}\langle R_s V_0 (R_s V_0)^{n-1} f, g\rangle \\ &= \sum_{n=1}^{\infty}\langle V_0 (R_s V_0)^{n-1} f, g\rangle = \sqrt{\lambda}\langle Vf, g\rangle .\end{aligned}$$

Now for $k = s + 1 + \lambda$,

$$(D-kI)F = (-d^*F + \lambda F) - kF = df - (s+1)\sum_{n=1}^{\infty} f_n + \lambda F - kF = df - (s+1)f .$$

From this calculation we conclude $(d - KI)\mathfrak{D}(D) \supset (d - (s+1))\mathfrak{D}(d)$. Since d is maximal skew-symmetric, $(d - (s+1)I)\mathfrak{D}(d) = \mathfrak{H}$, by the Lumer-Phillips Theorem, [Y1]. Hence $(D - kI)\mathfrak{D}(D) = \mathfrak{H}$, so that another application of the Lumer-Phillips Theorem implies that D is also maximal skew-symmetric. □

Lemma 2.8. *Let* $\mathcal{S} = \{S(t) \colon t \ge 0\}$ *be the strongly continuous one-parameter semigroup of isometries generated by the operator* $-D$. *Then* U, $\mathcal{S}$ *satisfy* (3).

Proof. Define $\varphi(t) = e^{\lambda t}(U(t)f, S(t)g)$ for some fixed $f \in \mathfrak{D}(d)$, $g \in \mathfrak{D}(D)$. Then $U(t)f \in \mathfrak{D}(d)$, $S(t)g \in \mathfrak{D}(D)$, so

$$\begin{aligned}(d/dt)\varphi(t) &= e^{\lambda t}[\lambda(U(t)f, S(t)g) + (-dU(t)f, g) + (U(t)f, -DS(t)g)] \\ &= e^{\lambda t}[\lambda(U(t)f, S(t)g) - (dU(t)f, S(t)g) \\ &\qquad + (U(t)f, d^*S(t)g) - \lambda(U(t)f, S(t)g)] \\ &= e^{\lambda t}[-(dU(t)f, S(t)g) + (U(t)f, d^*S(t)g)] \\ &= 0,\end{aligned}$$

for any $t > 0$. Hence $\varphi(t)$ is the constant function (f, g). Since this holds for all $f \in \mathfrak{D}(d)$, and $g \in \mathfrak{D}(D)$ we conclude, by continuity, that $e^{\lambda t}S(t)^*U(t) = 1$, for all $t \ge 0$. But this is just (3). □

Finally we may assemble all of our results to give the following structure theorem for pairs of semigroups of isometries satisfying the commutation relations (3).

Theorem 2.9. *Suppose $U = \{U(t): t \geq 0\}$ and $\mathscr{S} = \{S(t): t \geq 0\}$ are strongly continuous one-parameter semigroups of isometries, satisfying $S(t)^* U(t) = e^{-\lambda t} I$, $t \geq 0$, for some fixed $\lambda > 0$. Let $-d$, $-D$ be the infinitesimal generators of U, $\mathscr{S}$, respectively. Let $\langle f, g\rangle = \frac{1}{2}(d^* f, g) + \frac{1}{2}(f, d^* g)$, $f, g \in \mathfrak{D}(d^*)$, be the positive semidefinite inner product on $\mathfrak{D}(d^*)$; and let $\mathfrak{D}_+$ be the deficiency space of $-d$. Then there is an isometry V mapping $\mathfrak{H}$ into $\mathfrak{D}_+$ so that $\mathfrak{D}(D) = \{f \in \mathfrak{D}(d^*): \langle f, g\rangle = \sqrt{\lambda}\langle Vf, g\rangle$, for all $g \in \mathfrak{D}(d^*)\}$. Furthermore, $\mathfrak{D}(D^*) = \mathfrak{D}(d^*)$, and $D^* f = d^* f + \lambda f - \sqrt{\lambda} V^* E_+(f + d^* f)$.*

Conversely, if U and d are as above, and if V is an isometry from $\mathfrak{H}$ to $\mathscr{K}$, then the operator $D = -d^* + \lambda I$, on its domain $\{f \in \mathfrak{D}(d^*): \langle f, g\rangle = \sqrt{\lambda}\langle Vf, g\rangle$, for all $g \in \mathfrak{D}(d^*)\}$ is a maximal skew-symmetric operator. The one-parameter semigroup $\mathscr{S} = \{S(t): t \geq 0\}$ generated by $-D$ satisfies $S(t)^* U(t) = e^{-\lambda t} I$, for all $t \geq 0$.

Remark. Of course, the situation analyzed in the preceding theorem is only one case of the more general problem of considering pairs of semigroups of bounded operators, $\{U(t): t \geq 0\}$, and $\{S(t): t \geq 0\}$, which approximate each other in some sense. In the case considered here, note that the identity (3) implies that $\|U(t) - S(t)\| = 0(t^\alpha)$, for $0 < \alpha < \frac{1}{2}$, as $t \to 0^+$. This follows from the calculation $\|(U(t) - S(t))f\|^2 = 2(1 - e^{-\lambda t})\|f\|^2$. In [R1] D. W. Robinson showed (cf. the paper of Bratteli, Herman, and Robinson also [BHR]), among other results, that for C_0-semigroups of operators on a Hilbert space $\mathfrak{H}$ satisfying $\|U(t) - S(t)\| = 0(t^\alpha)$, as $t \to 0^+$, for $0 < \alpha \leq 1$, then there is a bounded perturbation of D which is similar to d, i.e.,

$$d = \Lambda^{-1}(D + P)\Lambda$$

for some $P \in \mathfrak{B}(\mathfrak{H})$, where $\Lambda = (1/\varepsilon)\int_0^\varepsilon U(s)V(s)^* \, ds$, for sufficiently small ε. (Note that Λ transports $\mathfrak{D}(D)$ onto $\mathfrak{D}(d)$.) Of course, the special identity (3) satisfied by the semigroups in this paper allows us to determine further connections between the infinitesimal generators of U and $\mathscr{S}$.

Finally, suppose $\mathscr{S}_1$ and $\mathscr{S}_2$ are semigroups of isometries on $\mathfrak{H}$, each of which satisfy (3) when paired with U. A natural question to ask is, under what conditions does the pair $\mathscr{S}_1$, $\mathscr{S}_2$ satisfy (3)?

Theorem 2.10. *Suppose $\mathscr{S}_1$ and $\mathscr{S}_2$ are strongly continuous one-parameter semigroups of isometries such that $S_i(t)^* U(t) = e^{-\lambda_i t} I$, for $i = 1, 2$. Let $-D_i$ be the infinitesimal generators of the $\mathscr{S}_i$, and let V_i be the isometries from $\mathfrak{H}$ to $\mathscr{K}$ satisfying $\mathfrak{D}(D_i) = \{f \in \mathfrak{D}(d^*): \langle f, g\rangle = \sqrt{\lambda_i}\langle V_i f, g\rangle$, for all $g \in \mathfrak{D}(d^*)\}$. Then the pair $\mathscr{S}_1$, $\mathscr{S}_2$ satisfies $S_2(t)^* S_1(t) = e^{-\lambda t} I$ for some positive λ if and only if $V_2^* V_1$ is a scalar multiple of the identity.*

Proof. First suppose that for some $\lambda > 0$, $S_2(t)^* S_1(t) = e^{-\lambda t} I$. Then for $f \in \mathfrak{D}(D_1)$, $g \in \mathfrak{D}(D_2)$, $e^{-\lambda t}(f, g) = (S_1(t)f, S_2(t)g)$. Taking d/dt of this equation at $t = 0$ gives

$$\begin{aligned} -\lambda(f, g) &= (-D_1 f, g) + (f, -D_2 g) \\ &= ((d^* - \lambda_1 I)f, g) + (f, (d^* - \lambda_2 I)g), \end{aligned}$$

or $(\lambda_1 + \lambda_2 - \lambda)(f, g) = \langle f, g\rangle$, for all $f \in \mathfrak{D}(D_1)$, $g \in \mathfrak{D}(D_2)$. Applying the theorem, there are isometries V_i from $\mathfrak{H}$ into $\mathscr{K}$ (which determine the domains of D_i) such that $(\lambda_1 + \lambda_2 - \lambda)(f, g) = \sqrt{\lambda_1}\langle V_1 f, g\rangle = \sqrt{\lambda_1\lambda_2}\langle V_1 f, V_2 g\rangle$ for all $f \in \mathfrak{D}(D_1)$, $g \in \mathfrak{D}(D_2)$. Since this equation is continuous in f and g, however, we may assume that f, g are arbitrary elements of $\mathfrak{H}$. Moreover, since $V_1 f, V_2 g \in \mathscr{K}$ (i.e., $d^* V_1 f = V_1 f$, $d^* V_2 g = V_2 g$), we have $\langle V_1 f, V_2 g\rangle = (V_1 f, V_2 g)$, so that $(V_2^* V_1 f, g) = ((\lambda_1 + \lambda_2 - \lambda)/\sqrt{\lambda_1\lambda_2})(f, g)$. Hence $V_2^* V_1$ is the scalar operator $[(\lambda_1 + \lambda_2 - \lambda)/\sqrt{\lambda_1\lambda_2}]I$.

Conversely, if the isometries $V_i : \mathfrak{H} \to \mathscr{K}$ determining the domains of $\mathfrak{D}(D_i)$ satisfy $V_2^* V_1 = kI$, then one can trace backwards through the preceding argument to show that there exists a positive λ such that

$$-\lambda(f, g) = (-D_1 f, g) + (f, -D_2 g),$$

Arguing as in the proof of Lemma 2.8, we see that $e^{-\lambda t}(f, g) = (S_1(t)f, S_2(t)g)$, for all $f \in \mathfrak{D}(D_1)$, $g \in \mathfrak{D}(D_2)$. By continuity this equation actually holds for all $f, g \in \mathfrak{H}$. This establishes the converse. □

III. Connection between Arveson's index and multiplicity

In this section we consider the case of an E_0-semigroup α of $\mathfrak{B}(\mathfrak{H})$ with an intertwining semigroup U of isometries $U(t)$. We suppose δ is the generator of α, and $-d$ is the generator of U and $\langle f, g\rangle = \frac{1}{2}(d^* f, g) + \frac{1}{2}(f, d^* g)$ for all $f, g \in \mathfrak{D}(d^*)$. Let $\mathscr{K}$ denote the quotient space $\mathfrak{D}(d^*) \bmod \mathfrak{D}(d)$ and let $\pi_{(\alpha, U)}$ be the $*$-representation of $\mathfrak{D}(\delta)$ on $\mathscr{K}$ constructed in §I.

Note $\mathfrak{D}(\delta)$ is a $*$-algebra acting on $\mathfrak{H}$. We mean by the identity representation of $\mathfrak{D}(\delta)$ on $\mathfrak{H}$ simply the action of $\mathfrak{D}(\delta)$ on $\mathfrak{H}$. Since $\mathfrak{D}(\delta)$ is σ-weakly dense in $\mathfrak{B}(\mathfrak{H})$ we have the identity representation of $\mathfrak{D}(\delta)$ is irreducible. Given any $*$-representation π of $\mathfrak{D}(\delta)$ one can ask if this representation contains the standard representation and we can ask how many copies of the standard representation π contains. To make this notion more precise assume π is a $*$-representation of $\mathfrak{D}(\delta)$ on a Hilbert space $\mathfrak{H}_1$. Let $\mathscr{I}(\pi)$ be the space of all bounded linear operators V mapping $\mathfrak{H}$ into $\mathfrak{H}_1$ so that $\pi(A)V = VA$ for all $A \in \mathfrak{D}(\delta)$. Suppose $V_1, V_2 \in \mathscr{I}(\pi)$. Taking adjoints we find $V_1^* \pi(A^*) = A^* V_1^*$ for all $A \in \mathfrak{D}(\delta)$. Since $\mathfrak{D}(\delta)$ is a $*$-algebra we have $V_1^* V_2 A = V_1^* \pi(A) V_2 = A V_1^* V_2$. Since $V_1^* V_2$ commutes with all $A \in \mathfrak{D}(\delta)$ and $\mathfrak{D}(\delta)$ is σ-weakly dense in $\mathfrak{B}(\mathfrak{H})$ we have $V_1^* V_2 = \lambda I$ with λ a complex number. Then there is a natural nondegenerate inner product on $\mathscr{I}(\pi)$ given by $(V_1, V_2)I = V_1^* V_2$. We define the number of copies of the standard representation contained in π to be the dimension of $\mathscr{I}(\pi)$ as a Hilbert space. While we admit that the term "number of copies" is a little misleading since this "number" is really a cardinal, we feel the phrase is useful because it immediately calls to mind what is meant. In fact, if $\mathcal{I}(\pi)$ is of dimension n then there are isometries V_i for $i = 1, \ldots, n$ so that $V_i^* V_j = 0$ for $i \neq j$ and $V_i A = \pi(A) V_i$ for all $A \in \mathfrak{D}(\delta)$ for $i = 1, \ldots, n$. So we see that π does indeed contain n copies of the standard representation of $\mathfrak{D}(\delta)$.

In this section we will show that the Arveson index d_α of α is the number of copies of the standard representation of $\mathfrak{D}(\delta)$ contained in the representation $\pi_{(\alpha, U)}$. We begin with the following lemma.

Lemma 3.1. *Suppose α is an E_0-semigroup of $\mathfrak{B}(\mathfrak{H})$ and δ is the generator of α so $\delta(A) = \lim_{t\to 0^+}(\alpha_t(A) - A)/t$ and the domain $\mathfrak{D}(\delta)$ is the set of all $A \in \mathfrak{B}(\mathfrak{H})$ so that the limit exists in the sense of σ-weak convergence. Suppose $U = \{U(t)\colon t \geq 0\}$ is a strongly continuous one-parameter semigroup of isometries and $-d$ is the generator of U so $df = \lim_{t\to 0^+}(f - U(t)f)/t$ where the domain $\mathfrak{D}(d)$ is the set of $f \in \mathfrak{H}$ so that the limit exists in the sense of strong convergence. Suppose for each $A \in \mathfrak{D}(\delta)$ we have $A\mathfrak{D}(d) \subset \mathfrak{D}(d)$ and $dAf = Adf - \delta(A)f$ for $f \in \mathfrak{D}(d)$. Then U intertwines α (i.e., $U(t)A = \alpha_t(A)U(t)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$).*

Proof. Suppose the hypothesis and notation of the lemma are valid. Suppose $f \in \mathfrak{D}(d)$ and $A \in \mathfrak{D}(\delta)$. Let $f_t = \alpha_t(A)U(t)f$. Note $f_t \in \mathfrak{D}(d)$ for all $t \geq 0$. We have

$$h^{-1}(f_{t+h} - f_t) = h^{-1}(\alpha_{t+h}(A) - \alpha_t(A))U(t+h)f + h^{-1}\alpha_t(A)(U(t+h) - U(t))f.$$

Hence, we see that f_t is strongly differentiable in t and

$$(d/dt)f_t = \delta(\alpha_t(A))U(t)f - \alpha_t(A)\, dU(t)f.$$

Since $\alpha_t(A) \in \mathfrak{D}(\delta)$ we have from the hypothesis of the lemma that

$$d\alpha_t(A)U(t)f = \alpha_t(A)\, dU(t)f - \delta(\alpha_t(A))U(t)f.$$

Hence, we have $f_t \in \mathfrak{D}(d)$ and $(d/dt)f_t = -df_t$ for all $t \geq 0$. Let $g_t = U(t)Af$. We have $g_t \in \mathfrak{D}(d)$ and $(d/dt)g_t = -dg_t$ for all $t \geq 0$. Let $h_t = f_t - g_t$. Then $h_t \in \mathfrak{D}(d)$ and $(d/dt)h_t = -dh_t$ for all $t \geq 0$. Now we have $(d/dt)\|h_t\|^2 = (d/dt)(h_t, h_t) = -(dh_t, h_t) - (h_t, dh_t) = 0$. Hence, $\|h_t\|$ is constant in t. Since $h_0 = 0$ we have $h_t = 0$ for all $t \geq 0$. Hence, we have $U(t)Af = \alpha_t(A)U(t)f$ for all $f \in \mathfrak{D}(d)$, $A \in \mathfrak{D}(\delta)$, and $t \geq 0$. Since $U(t)$, A, and $\alpha_t(A)$ are bounded this equation extends to all $f \in \mathfrak{H}$ by continuity. Since $\mathfrak{D}(\delta)$ is σ-weakly dense in $\mathfrak{B}(\mathfrak{H})$ and α_t is σ-weakly continuous it follows that this relation extends to all $A \in \mathfrak{B}(\mathfrak{H})$. □

Lemma 3.2. *Suppose α is an E_0-semigroup of $\mathfrak{B}(\mathfrak{H})$ with a strongly continuous intertwining semigroup $U = \{U(t)\colon t \geq 0\}$. Let $-d$ be the generator of U and let $\langle f, g\rangle = \frac{1}{2}(d^* f, g) + \frac{1}{2}(f, d^* g)$ for all $f, g \in \mathfrak{D}(d^*)$. Suppose V is an isometry of $\mathfrak{H}$ into $\mathfrak{D}_+$ ($\mathfrak{D}_+ = \{f \in \mathfrak{D}(d^*)\colon d^* f = f\}$) which intertwines the* *identity representation of $\mathfrak{D}(\delta)$ on $\mathfrak{H}$ with the representation $\pi_{(\alpha, U)}$ of $\mathfrak{D}(\delta)$ (i.e., $\langle f, AVg\rangle = \langle f, VAg\rangle$ for all $f \in \mathfrak{D}(d^*)$ and $g \in \mathfrak{H}$). Suppose $\lambda > 0$. Let $\mathfrak{D}(D) = \{f \in \mathfrak{D}(d^*): \langle f, g\rangle = \sqrt{\lambda}\langle Vf, g\rangle$ for all $g \in \mathfrak{D}(d^*)\}$ and for $f \in \mathfrak{D}(D)$ let $Df = -d^*f + \lambda f$. Then $-D$ is the generator of a one-parameter semigroup of isometries $S(t)$ and these isometries intertwine α (i.e. $S(t)A = \alpha_t(A)S(t)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$).*

Proof. Suppose the hypothesis and notation of the lemma are valid. From Theorem 2.9 it follows that $-D$ is the generator of a strongly continuous semigroup of isometries and $\mathfrak{D}(D^*) = \mathfrak{D}(d^*)$ and for $f \in \mathfrak{D}(D^*)$ we have $D^*f = d^*f + \lambda f - \sqrt{\lambda}V^*(I + d^*)f$. Suppose $f \in \mathfrak{D}(D^*)$ and $g \in \mathfrak{D}(D)$ and $A \in \mathfrak{D}(\delta)$. We show that $(D^*f, Ag) - (f, ADg) = -(f, \delta(A)g)$. We have

$$(D^*f, Ag) - (f, ADg) = (d^*f, Ag) + (f, Ad^*g) - \sqrt{\lambda}((I + d^*)f, VAg)$$

Since $Ad^*g = d^*Ag - \delta(A)g$ we have

$$(D^*f, Ag) - (f, ADg) = 2\langle f, Ag\rangle - 2\sqrt{\lambda}\langle f, VAg\rangle - (f, \delta(A)g).$$

Since $g \in \mathfrak{D}(D)$ we have $\langle f, Ag\rangle = \langle A^*f, g\rangle = \sqrt{\lambda}\langle A^*f, Vg\rangle = \sqrt{\lambda}\langle f, AVg\rangle$. Since V intertwines the identity representation of $\mathfrak{D}(\delta)$ and $\pi_{(\alpha, U)}$ we have $\langle f, AVg\rangle = \langle f, VAg\rangle$. Hence, we have

$$(D^*f, Ag) - (f, ADg) = -(f, \delta(A)g).$$

Since this is true for all $f \in \mathfrak{D}(D^*)$ it follows that $Ag \in \mathfrak{D}(D^{**}) = \mathfrak{D}(D)$ and therefore we have $A\mathfrak{D}(D) \subset \mathfrak{D}(D)$ and $DAg = ADg - \delta(A)g$. Then it follows from Lemma 3.1 that S intertwines α (i.e., $S(t)A = \alpha_t(A)S(t)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$). □

Lemma 3.3. *Suppose α is an E_0-semigroup of $\mathfrak{B}(\mathfrak{H})$ and U is a strongly continuous one-parameter semigroup of isometries which intertwines α. Let $-d$ be the generator of U and let $\langle f, g\rangle = \frac{1}{2}(d^*f, g) + \frac{1}{2}(f, d^*g)$ for all $f, g \in \mathfrak{D}(d^*)$. Suppose V is an isometry of $\mathfrak{H}$ into $\mathfrak{D}_+ = \{f \in \mathfrak{D}(d^*): d^*f = f\}$. Suppose $\lambda > 0$. Let $\mathfrak{D}(D) = \{f \in \mathfrak{D}(d^*): \langle f, g\rangle = \sqrt{\lambda}\langle Vf, g\rangle$, for all $g \in \mathfrak{D}(d^*)\}$ and $Df = -d^*f + \lambda f$ for all $f \in \mathfrak{D}(D)$. Let S be the strongly continuous semigroup generated by $-D$ (the fact that $-D$ generates a semigroup of isometries is assured by Theorem 2.9). Suppose S intertwines α. Then V intertwines the identity representation of $\mathfrak{D}(\delta)$ on $\mathfrak{H}$ with the representation $\pi_{(\alpha, U)}$ (i.e., $\langle f, AVg\rangle = \langle f, VAg\rangle$ for all $f \in \mathfrak{D}(d^*)$ and $g \in \mathfrak{H}$).*

Proof. Suppose the hypothesis and notation of the lemma are valid. Since $S(t)A = \alpha_t(A)S(t)$ for all $a \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Suppose $g \in \mathfrak{D}(D)$ and $A \in \mathfrak{D}(\delta)$ where δ is the generator of α. Then we have

$$t^{-1}(I - S)(t))Ag = t^{-1}(A - \alpha_t(A))S(t)g + t^{-1}A(I - S(t))g.$$

Since both terms on the right have limits as $t \to 0^+$ we have $Ag \in \mathfrak{D}(D)$ and $DAg = ADg - \delta(A)g$. Now suppose $f \in \mathfrak{D}(d^*)$ and we continue to assume

$A \in \mathfrak{D}(\delta)$ and $g \in \mathfrak{D}(D)$. Then, since $g \in \mathfrak{D}(D)$ we have $\langle f, Ag\rangle = \langle A^* f, g\rangle = \sqrt{\lambda}\langle A^* f, Vg\rangle = \sqrt{\lambda}\langle f, AVg\rangle$. Since $Ag \in \mathfrak{D}(D)$ we have $\langle f, Ag\rangle = \sqrt{\lambda}\langle f, VAg\rangle$. Hence it follows that $\langle f, AVg\rangle = \langle f, VAg\rangle$ for all $f \in \mathfrak{D}(d^*)$, $A \in \mathfrak{D}(\delta)$, and $g \in \mathfrak{D}(D)$. Since both sides of this equation are continuous in g for fixed f and A it follows that this equation extends to all $g \in \mathfrak{H}$. Hence, V intertwines the identity representation of $\mathfrak{D}(\delta)$ on $\mathfrak{H}$ with the representation $\pi_{(\alpha, U)}$. □

Theorem 3.4. *Suppose α is an E_0-semigroup of $\mathfrak{B}(\mathfrak{H})$ with a strongly continuous semigroup U of intertwining isometries. Let $-d$ be the generator of U and let $\langle f, g\rangle = \frac{1}{2}(d^* f, g) + \frac{1}{2}(f, d^* g)$ for all $f, g \in \mathfrak{D}(d^*)$. Let $\pi_{(\alpha, U)}$ be the $*$-representation of $\mathfrak{D}(\delta)$ coming from this inner product $\langle \cdot, \cdot\rangle$ as described in* §I. *Then the Arveson index of α is the number of times the identity representation of $\mathfrak{D}(\delta)$ occurs in $\pi_{(\alpha, U)}$.*

Proof. Assume the hypothesis and notation of the theorem are valid. Suppose V_1 and V_2 are linear operators mapping $\mathfrak{H}$ into $\mathfrak{D}_+ = \{f \in \mathfrak{D}(d^*) : d^* f = f\}$ which intertwine the identity representation of $\mathfrak{D}(\delta)$ on $\mathfrak{H}$ with $\pi_{(\alpha, U)}$ so $\langle f, AV_i g\rangle = \langle f, V_i Ag\rangle$ for all $f \in \mathfrak{D}(d^*)$ and $g \in \mathfrak{H}$ and $i = 1, 2$. Then for arbitrary $f, g \in \mathfrak{H}$ and $A \in \mathfrak{B}(\mathfrak{H})$ we have

$$\begin{aligned}(f, V_1^* V_2 Ag) &= (V_1 f, V_2 Ag) = \langle V_1 f, V_2 Ag\rangle = \langle V_1 f, AV_2 g\rangle \\ &= \langle A^* V_1 f, V_2 g\rangle = \langle V_1 A^* f, V_2 g\rangle = (V_1 A^* f, V_2 g) \\ &= (f, AV_1^* V_2 g).\end{aligned}$$

Hence, $V_1^* V_2$ commutes with $\mathfrak{B}(\mathfrak{H})$ and therefore $V_1^* V_2 = \lambda I$. It follows then that the space of operators from $\mathfrak{H}$ to $\mathfrak{D}_+$ which intertwine the identity representation $\mathfrak{D}(\delta)$ on $\mathfrak{H}$ and $\pi_{(\alpha, U)}$ form a Hilbert space which we will call $\mathscr{K}(\alpha, U)$. The dimension of this Hilbert space is the number of times the identity representation of $\mathfrak{D}(\delta)$ occurs in $\pi_{(\alpha, U)}$.

Let $Q(\alpha, U)$ be the set of strongly continuous one parameter semigroups of isometries which intertwine α and have real Arveson covariance with U (i.e. if $S \in Q(\alpha, U)$ then $S(t)^* U(t) = e^{-\lambda t} I$ with $\lambda \geq 0$). We define a mapping $W \to S_W$ from $\mathscr{K}(\alpha, U)$ to $Q(\alpha, U)$ as follows. Suppose $W \in \mathscr{K}(\alpha, U)$. Then $W^* W = \lambda I$ with $\lambda \geq 0$. Let $V = \lambda^{-1/2} W$ if $\lambda > 0$ and $V = 0$ if $\lambda = 0$. If $\lambda > 0$ then V is an isometry. In any case, we have $W = \sqrt{\lambda} V$. Let $\mathfrak{D}(D) = \{f \in \mathfrak{D}(d^*) : \langle f, g\rangle = \sqrt{\lambda}\langle Vf, g\rangle$ for all $g \in \mathfrak{D}(d^*)\}$ and let $Df = -d^* f - \lambda f$ for $f \in \mathfrak{D}(D)$. Then from Lemma 3.2 we have $-D$ is the generator of a semigroup S_W of isometries and $S_W \in Q(\alpha, U)$. From Lemma 3.3 and Theorem 2.9 it follows that the range of the mapping $W \to S_W$ is all of $Q(\alpha, U)$.

Given $W, T \in \mathscr{K}(\alpha, U)$ we compute the Arveson covariance of S_W and S_T. Let $S_W(t)^* U(t) = e^{-\lambda_1 t}$ and $S_T(t)^* U(t) = e^{-\lambda_2 t}$. Suppose $-D_1$ and $-D_2$ are the generators of S_W and S_T and $f \in \mathfrak{D}(D_1)$ and $g \in \mathfrak{D}(D_2)$. Then we have

$$\begin{aligned}(d/dt)&(S_W(t)f, S_T(t)g)\\ &= (d^*S_W(t)f, S_T(t)g) + (S_W(t)f, d^*S_T(t)g)\\ &\quad - (\lambda_1+\lambda_2)(S_W(t)f, S_T(t)g)\\ &= 2\langle S_W(t)f, S_T(t)g\rangle - (\lambda_1+\lambda_2)(S_W(t)f, S_T(t)g)\\ &= 2\langle WS_W(t)f, TS_T(t)g\rangle - (\lambda_1+\lambda_2)(S_W(t)f, S_T(t)g)\\ &= (S_W(t)f, (2W^*T - W^*W - T^*T)S_T(t)g).\end{aligned}$$

Since $2W^*T - W^*W - T^*T$ is a multiple of the identity we have

$$(d/dt)(f, S_1(t)^*S_2(t)g) = c(f, S_1(t)^*S_2(t)g),$$

where $cI = c(S_1, S_2)I = 2W^*T - W^*W - T^*T$. Solving the above differential equation we find $S_1(t)^*S_2(t) = e^{ct}I$.

Recall the definition of $\widetilde{\mathfrak{H}}(\alpha)$ given in §I of this paper. Suppose $S \in U_\alpha$ is a semigroup of isometries which intertwine α. Let $W(t) = e^{zt}S(t)$ for $t \geq 0$ where z is an arbitrary complex number. Let F be the function on U_α given by $F(x) = 1$ if $x = S$, $F(x) = -1$ if $x = W$ and $F(x) = 0$ otherwise. Calculating Arveson's covariance function we have $c(S, S) = 0$, $c(S, W) = z$, $c(W, S) = \overline{z}$, and $c(W, W) = z + \overline{z}$. Hence, the inner product $(F, F) = \sum_{x,y} \overline{F(x)}F(y)c(x, y) = 0$. If $W \in U_\alpha$ there is a complex number z so that the semigroup $S(t) = e^{zt}W(t)$ lies in $Q(\alpha, U)$. Hence, it follows that if F is a finitely nonzero function on U_α there is a finitely nonzero function G with support on $Q(\alpha, U)$ so that F and G differ by a null function (i.e., $((F-G), (F-G)) = 0$). Hence, in defining the Hilbert space $\widetilde{\mathfrak{H}}(\alpha)$ we can restrict our attention to finitely nonzero functions F on $Q(\alpha, U)$.

We will construct a unitary operator J from $\mathscr{K}(\alpha, U)$ to $\widetilde{\mathfrak{H}}(\alpha)$ and thereby show these spaces have the same dimension. For $W \in \mathscr{K}(\alpha, U)$ let F_W be the function on U_α given by $F_W(x) = 1/\sqrt{2}$ if $x = U$ and $F(x) = -1/\sqrt{2}$ if $x = S_W$ and $F_W(x) = 0$ otherwise. Suppose $W, T \in \mathscr{K}(\alpha, U)$. From a previous paragraph we have the Arveson covariances between U, S_W, and S_T are given by

$$\begin{aligned}c(U, S_W)I = -W^*W, \qquad c(U, S_T)I = -T^*T,\\ c(S_W, S_T)I = 2W^*T - W^*W - T^*T.\end{aligned}$$

Then $(F_W, F_T)I = \sum_{x,y\in U_\alpha} \overline{F_W(x)}F_T(y)c(x, y)I = W^*T$. Note that $zF_W - F_{zW}$ is a null function in $\widetilde{\mathfrak{H}}(\alpha)$ for any complex number z. Let $JW = F_W$. We see that J is an isometry of $\mathscr{K}(\alpha, U)$ into $\widetilde{\mathfrak{H}}(\alpha)$. Since the mapping $W \to S_W$ has range all of $Q(\alpha, U)$ and by the remarks at the end of the previous paragraph we see that linear combinations of the F_W are dense in $\widetilde{\mathfrak{H}}(\alpha)$. It follows that range of J is dense in $\widetilde{\mathfrak{H}}(\alpha)$. Hence, J defines an isometry of $\mathscr{K}(\alpha, U)$ onto $\widetilde{\mathfrak{H}}(\alpha)$. Hence, these spaces have the same dimension. □

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