TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 321, Number 1, September 1990

ON THE EXISTENCE OF CENTRAL SEQUENCES IN SUBFACTORS

DIETMAR H. BISCH

ABSTRACT. We prove a relative version of [Co1, Theorem 2.1] for a pair of type II_1-factors $N \subset M$. This gives a list of necessary and sufficient conditions for the existence of nontrivial central sequences of M contained in the subfactor N. As an immediate application we obtain a result by Bédos [Be, Theorem A], showing that if N has property Γ and G is an amenable group acting freely on N via some action σ, then the crossed product $N \times_\sigma G$ has property Γ. We also include a proof of a relative Mc Duff-type theorem (see [McD, Theorems 1, 2 and 3]), which gives necessary and sufficient conditions implying that the pair $N \subset M$ is stable.

INTRODUCTION

The property Γ for a factor of type II_1 was introduced by Murray-von Neumann (see [MvN]) to distinguish two different classes of factors. It describes an asymptotic commutativity property of the algebra. A stronger property was later considered by Mc Duff (see [McD]) in order to construct more examples of factors. Both concepts turned out to be essential. Connes used them in his fundamental papers (see [Co1, Co2]), not only to prove the uniqueness of the hyperfinite II_1-factor R, but also to classify the automorphisms of R. He gives some surprising alternative characterizations of these properties in [Co1, Theorem 2.1].

In this paper we study necessary and sufficient conditions for the existence of nontrivial central sequences in a type II_1-factor M that are contained in a subfactor $N \subset M$. Our work is motivated by Problem 3 in [Jo] and the related generating problem for pairs of hyperfinite factors with finite index (see [Po3, Oc]): Jones asks in [Jo] for conditions implying that the pair $N \subset M$ is stable, i.e. isomorphic to the pair $N \otimes R \subset M \otimes R$. If $N \subset M$ are hyperfinite and have the generating property, then the pair $N \subset M$ is stable. A necessary condition for stability and hence for the generating property as well, is the existence of nontrivial central sequences of M contained in N.

In the first section we prove a relative version of Connes' Theorem 2.1 in [Co1]. We show that the existence of central sequences for the ambient factor M that are actually contained in N is equivalent to the existence of a singular state φ of M that is invariant under a finitely generated subgroup of $\operatorname{Int} M$

Received by the editors June 7, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L35; Secondary 46L10.

and factors through the conditional expectation E_N from M onto N. This is equivalent to saying that there is no nonzero compact operator of $B(L^2(M, \tau))$ contained in the C^*-algebra generated by M, M' and e_N, where e_N denotes the orthogonal projection from $L^2(M, \tau)$ onto $L^2(N, \tau)$. The proof of our theorem closely follows the ideas of Connes' proof.

In the second section we give some applications to crossed products. We show that if N is a separable II_1-factor with property Γ and G is an amenable group acting freely on N, then the crossed product $N \times_\sigma G$ also has property Γ. Popa proves this result in [Po2] for $G = \mathbf{Z}$ and conjectures it for a general amenable group, which was shown to be true by Bédos [Be, Theorem A]. Bédos uses a technique involving the decomposition of the crossed product [Be, Proposition 3]. We derive the result as an immediate application of the main theorem in §1.

In the third section we prove a relative version of Mc Duff's theorem (see [McD, Theorems 1, 2 and 3]. This result, showing that $N \subset M$ is stable if and only if N contains noncommuting central sequences of M, was probably noticed by specialists, but no detailed proof seems to exist in the literature.

Notation. M denotes a separable II_1-factor acting on the Hilbert space $L^2(M, \tau)$, where τ is the normal faithful normalized trace on M. $N \subset M$ is a subfactor and $E_N \colon M \mapsto N$ the unique conditional expectation with $\tau \circ E_N = \tau$. e_N denotes the orthogonal projection from $L^2(M, \tau)$ onto $L^2(N, \tau) \subset L^2(M, \tau)$, and $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in M$, is the Hilbert norm as usual. Furthermore J will be the canonical involution in $L^2(M, \tau)$, i.e. $Jx = x^*$ for all $x \in M$. We denote by $\otimes$ the algebraic tensor product and by $\bar{\otimes}$ the von Neumann algebra tensor product. $B(L^2(M, \tau))$ is the algebra of bounded linear operators on $L^2(M, \tau)$.

Acknowledgment. I am very grateful to Professor Sorin Popa, who suggested this problem to me.

1. Central sequences in subfactors

We recall that M has *property* Γ if for given elements $x_1, \dots, x_n \in M$ and $\varepsilon > 0$ there is a unitary $u \in M$ with $\tau(u) = 0$ such that $\|[x_i, u]\|_2 \le \varepsilon$, $1 \le i \le n$. If $N \subset M$ is a subfactor, we are interested in conditions that asssure that the unitary u is actually contained in N. We obtain the following theorem:

Theorem 1.1. *Let M be a separable II_1-factor with subfactor $N \subset M$. Then the following conditions are equivalent*:

1°. *For any elements $x_1, \dots, x_n \in M$ and $\varepsilon > 0$ there is a unitary $u \in N$ with $\tau(u) = 0$ and $\|[x_i, u]\|_2 \le \varepsilon$ for all $i = 1, \dots, n$.*

2°. *For any finitely generated group $G \subset \operatorname{Int} M$ there is a nonnormal G-invariant state $\varphi \in M^*$ with $\varphi \circ E_N = \varphi$.*

$2'^{\circ}$. *Let $N \subset M_0 \subset M$ be a $\|\cdot\|_2$-dense $*$-subalgebra of M.*
For any elements $x_1, \dots, x_n \in M_0$ there is a nonnormal state $\varphi \in M^*$ with $\varphi(xx_i) = \varphi(x_i x)$ for all $x \in M$, $1 \le i \le n$, and $\varphi \circ E_N = \varphi$.

3°. *For any operators $x_1, \dots, x_n \in M$ there is a sequence $(\xi_k)_{k\in\mathbf{N}} \subset L^2(N, \tau)$, $\|\xi_k\|_2 = 1$, $|\langle \xi_k, 1\rangle| \not\to 1$ as $k \to \infty$, such that $\|[x_i, \xi_k]\|_2 \to 0$.*

4°. *The C^*-algebra $C^*(M, M', e_N)$ generated by M, M', e_N in $B(L^2(M, \tau))$ does not contain any nonzero compact operator, i.e.*

$$C^*(M, M', e_N) \cap \mathscr{K}(L^2(M, \tau)) = \{0\}.$$

The proof of this theorem will use the following lemmata.

Lemma 1.2. *Let $N \subset M$ be II_1-factors as in Theorem 1.1, satisfying 2°. Then for any elements $x_1, \dots, x_n \in M$ and $\varepsilon > 0$ there is a nonzero projection $e \in N$ with $\tau(e) \le \varepsilon$ and*

$$\|[x_i, e]\|_2 \le \varepsilon \|e\|_2, \qquad 1 \le i \le n.$$

Proof. We may assume that the given elements x_i are in fact unitaries, denoted by u_i, $1 \le i \le n$. By hypothesis there is a singular state $\varphi \in M^*$ with

$$\varphi = \varphi \circ \mathrm{Ad}\, u_i, \qquad 1 \le i \le n,$$
$$\varphi = \varphi \circ E_N.$$

Then there is a projection $f \in N$ with $\varphi(f) = 1$, $\tau(f) < \varepsilon$, $0 < \varepsilon < \frac{1}{2}$ (see [Ta1]). Set

$$V := \{\, \psi \text{ state on } M \mid \psi(f) \ge 1 - \varepsilon\} \subset M^*,$$
$$W := \{\, (\psi - \psi \circ E_N, \psi - \psi \circ \mathrm{Ad}\, u_1, \dots, \psi - \psi \circ \mathrm{Ad}\, u_n) \mid \psi \in V \cap M_*\}.$$

Identifying as usual $(M^n)^* \cong (M^*)^n$, $((M_*)^n)^* \cong M^n$, we conclude that zero belongs to the $\sigma((M^*)^{n+1}, M^{n+1})$-closure of W, because φ is in the $\sigma(M^*, M)$-closure of $V \cap M_*$. Zero is of course in $(M_*)^{n+1}$, and since W is a (nonempty) convex subset of $(M_*)^{n+1}$, a separation argument shows that zero is in fact in the norm-closure of W in $(M_*)^{n+1}$. Thus there is a $\psi \in V \cap M_*$ such that

$$\|\psi - \psi \circ E_N\|, \quad \|\psi - \psi \circ \mathrm{Ad}\, u_i\| \le \varepsilon, \qquad 1 \le i \le n.$$

The fact $\psi(f) \ge 1 - \varepsilon$ implies that ψ is basically supported on fMf, i.e. if we set

$$\tilde{\psi}(x) := \frac{\psi(fxf)}{\psi(f)}, \qquad x \in M,$$

we get a normal state $\tilde{\psi}$ with $\|\psi - \tilde{\psi}\| \le 3\varepsilon^{1/2}$ (as in [Co1, Lemma 2.4]). Define

$$\psi'(x) := \tilde{\psi}(E_N(x)),$$

then ψ' is a positive normal state with

$$\|\psi - \psi'\| \leq \|(\tilde{\psi} - \psi) \circ E_N\| + \|\psi - \psi \circ E_N\| \leq 3\varepsilon^{1/2} + \varepsilon .$$

Note that $\psi' \circ E_N = \psi'$, $\operatorname{supp} \psi' \leq f$. Since ψ' is positive and normal, there is a unique $\xi \in L^1(M, \tau)$, $\xi \geq 0$ with $\psi'(x) = \tau(x\xi)$, $x \in M$. E_N extends to a continuous $N - N$-bimodule map from $L^1(M, \tau)$ to $L^1(N, \tau)$, hence

$$\begin{aligned} \tau(x\xi) &= \psi'(x) = \psi'(E_N(x)) = \tau(E_N(x)\xi) \\ &= \tau(E_N(x)E_N(\xi)) = \tau(xE_N(\xi)), \qquad \forall x \in M . \end{aligned}$$

But this shows that $\xi \in L^1(N, \tau)$. Thus $h := \xi^{1/2}$ is in $L^2(N, \tau)^+$, $\|h\|_2 = 1$, and

$$\psi'(x) = \tau(x\xi) = \omega_h(x) = \langle xh, h \rangle .$$

We have $\operatorname{supp} h \leq f$ and by the Powers-Størmer inequality

$$\begin{aligned} \|u_i h u_i^* - h\|_2^2 &\leq \|\psi' - \psi' \circ \operatorname{Ad} u_i\| \\ &\leq \|\psi' - \psi\| + \|\psi - \psi \circ \operatorname{Ad} u_i\| + \|(\psi - \psi') \circ \operatorname{Ad} u_i\| \\ &\leq 9\varepsilon^{1/2}, \qquad 1 \leq i \leq n . \end{aligned}$$

Using Connes' trick [Co1, Theorem 1.2.2], for $\delta = 6n(3\varepsilon^{1/4})^{1/8} < 1$ (i.e. ε small enough), we can find a $t \in \mathbf{R}$, $t > 0$, such that the spectral projection $E_t(h) \in N$ ($E_t(h)$ denotes as usual the spectral projection $\chi_{(t, \infty)}(h)$) is nonzero and satisfies

$$\|E_t(u_i h u_i^*) - E_t(h)\|_2 \leq \delta \|E_t(h)\|_2, \qquad 1 \leq i \leq n .$$

Putting $e = E_t(h)$, we get

$$\|[u_i, e]\|_2 \leq \delta \|e\|_2 ,$$

since $E_t(u_i h u_i^*) = u_i e u_i^*$, $1 \leq i \leq n$. We have clearly

$$\tau(e) \leq \tau(\operatorname{supp} h) \leq \tau(f) < \varepsilon ,$$

and $e \in N$ since $h \in L^2(N, \tau)$. Q.E.D.

We use the following relative version of [Co1, Lemma 2.6]—the proof is the same.

Lemma 1.3. *Let M be a II_1-factor, $N \subset M$ a subfactor, ω a free ultrafilter on $\mathbf{N}$, N^ω and M^ω the usual ultrapower algebras and $u_1, \dots, u_n$ unitary operators in M. Then if the commutant of $u_1, \dots, u_n$ in N^ω is finite dimensional, we can find unitary operators $u_{n+1}, \dots, u_q$ in M such that the commutant of $u_1, \dots, u_q$ in N^ω is trivial.*

Similar to [Co1, Lemma 2.5], we need the next lemma for the proof of the implication 2° (resp. $2'^\circ$) $\Rightarrow 1^\circ$.

Lemma 1.4. *Let $N \subset M_0 \subset M$ satisfy statement $2'^\circ$ in Theorem* 1.1. *Then the reduced algebras $N_p \subset pM_0p \subset M_p$ also satisfy $2'^\circ$, where p denotes a nontrivial projection in N.*

Proof. We use Connes' argument: since N is a II_1-factor there is a projection $f \in N$ with $f \leq p$, $\tau(f) = \frac{1}{k}$, $k \in \mathbf{N}$. It is now easy to construct a type I_k-subfactor P of N with minimal projection f. P is generated by two unitaries $v_1, v_2 \in N$.

Let $x_1, \dots, x_n \in pM_0p \subset M_0$. Then by hypothesis there is a singular state $\varphi \in M^*$ with $\varphi(x_ix) = \varphi(xx_i)$, $1 \leq i \leq n$, $\varphi(v_ix) = \varphi(xv_i)$, $i = 1, 2$, $\forall x \in M$, and $\varphi = \varphi \circ E_N$. The restriction of φ to M_p is the desired singular functional. Q.E.D.

Lemma 1.4 shows in particular that if $N \subset M$ satisfy 2°, then the reduced algebras $N_p \subset M_p$ also satisfy 2°.

Proof of Theorem 1.1 (see [Co1, Proof of Theorem 2.1]).

We prove the following chain of implications: $1^\circ \Rightarrow 4^\circ \Rightarrow 3^\circ \Rightarrow 2^\circ \Rightarrow 1^\circ$, $2^\circ \Rightarrow 2'^\circ \Rightarrow 1^\circ \Rightarrow 2^\circ$.

$\underline{1^\circ \Rightarrow 4^\circ}$: The proof of this implication is based on Connes' idea.

$1.^o$ implies the existence of a nontrivial central sequence of unitaries $(u_i)_{i\in\mathbf{N}} \subset N$ with $\tau(u_i) = 0$, $i \in \mathbf{N}$, $\|[x, u_i]\|_2 \to 0$ as $i \to \infty$ for all $x \in M$. Since M is a factor, $C^*(M, M', e_N)$ is irreducible and therefore we have either

$$\mathscr{K}(H) \cap C^*(M, M', e_N) = \{0\} \quad or \quad \mathscr{K}(H) \subset C^*(M, M', e_N).$$

If 4° does not hold, the one-dimensional projection $\pi : H \mapsto \operatorname{span} \hat{1}$, $\xi \mapsto \langle \xi, \hat{1}\rangle \hat{1}$, where $\hat{1}$ is as usual the cyclic and separating vector in $L^2(M, \tau)$, is contained in $C^*(M, M', e_N)$. Hence there are words $a_1^{(1)} \cdots a_{n_1}^{(1)}, \dots, a_1^{(k)} \cdots a_{n_k}^{(k)} \in C^*(M, M', e_N)$ with $a_i^{(j)} \in M \cup M' \cup \{e_N\}$ such that

$$\left\| \pi - \sum_{i=1}^{k} a_1^{(i)} \cdots a_{n_i}^{(i)} \right\| < \frac{\varepsilon}{2}. \tag{1}$$

The elements in M commute with those in M', but not necessarily with e_N. We suppose that every word is written in a form where all the elements in M that occur in the word are moved as far to the left as possible. Note that u_i commutes with M' and e_N. We can assume that every word contains an element in M (otherwise the process described below is not necessary). Let $a_{s_i}^{(i)}$ denote the first element of M that occurs in the word $a_1^{(i)} \cdots a_{n_i}^{(i)}$ (from left to right); then

$$\left\| \sum_{i=1}^{k} a_1^{(i)} \cdots [a_{s_i}^{(i)}, u_j] a_{s_i+1}^{(i)} \cdots a_{n_i}^{(i)} \hat{1} \right\|_2 \to 0 \qquad as\ j \to \infty,$$

i.e.

$$\left\| \sum_{i=1}^{k} a_1^{(i)} \cdots a_{s_i}^{(i)} u_j a_{s_i+1}^{(i)} \cdots a_{n_i}^{(i)} \hat{1} - u_j \sum_{i=1}^{k} a_1^{(i)} \cdots a_{n_i}^{(i)} \hat{1} \right\|_2 \to 0 \qquad as\ j \to \infty,$$

which implies that

$$(2) \qquad \left\| \sum_{i=1}^{k} a_1^{(i)} \cdots a_{s_i}^{(i)} u_j a_{s_i+1}^{(i)} \cdots a_{n_i}^{(i)} \hat{1} \right\|_2 \to \left\| \sum_{i=1}^{k} a_1^{(i)} \cdots a_{n_i}^{(i)} \hat{1} \right\|_2 \qquad as\ j \to \infty.$$

(1) shows that

$$1 - \frac{\varepsilon}{2} \le \left\| \sum_{i=1}^{k} a_1^{(i)} \cdots a_{n_i}^{(i)} \hat{1} \right\|_2 \le 1 + \frac{\varepsilon}{2}.$$

This together with (2) implies that we can fix a $j_0 \ge 1$ such that

$$\left\| \sum_{i=1}^{k} a_1^{(i)} \cdots a_{s_i}^{(i)} u_j a_{s_i+1}^{(i)} \cdots a_{n_i}^{(i)} \hat{1} \right\|_2 \ge 1 - \varepsilon, \quad \forall j \ge j_0.$$

The idea is now to move u_j through the word to the right. u_j commutes with all $a_{s_i}^{(i)}$, $a_{s_i+1}^{(i)}, \ldots$ until it hits the next element $a_{r_i}^{(i)}$ in M. Then we repeat the process described above, and we can find an index j_1 such that for all $j \ge j_1 \ge j_0$, we have

$$\left\| \sum_{i=1}^{k} a_1^{(i)} \cdots a_{r_i}^{(i)} u_j a_{r_i+1}^{(i)} \cdots a_{n_i}^{(i)} \hat{1} \right\|_2 \ge 1 - \varepsilon - \frac{\varepsilon}{2}.$$

This procedure stops after at most $n = \max\{n_1, \ldots, n_k\}$ steps, so we get an index $j_n \ge 1$ such that for all $j \ge j_n$, we have

$$(3) \qquad \left\| \sum_{i=1}^{k} a_1^{(i)} \cdots a_{n_i}^{(i)} u_j \hat{1} \right\|_2 \ge 1 - \varepsilon \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n > \varepsilon,$$

if $0 < \varepsilon < \frac{1}{3}$. On the other hand we get from (1)

$$\frac{\varepsilon}{2} > \left\| \pi(\hat{u}_j) - \sum_{i=1}^{k} a_1^{(i)} \cdots a_{n_i}^{(i)} u_j \hat{1} \right\|_2 = \left\| \sum_{i=1}^{k} a_1^{(i)} \cdots a_{n_i}^{(i)} u_j \hat{1} \right\|_2,$$

which contradicts (3) for all $j \ge j_n$. This shows that $C^*(M, M', e_N)$ cannot contain any nonzero compact operator.

<u>$4^\circ \Rightarrow 3^\circ$</u>: Suppose 3° does not hold. This implies that there are unitaries $u_1, \ldots, u_n$ in M such that there is no sequence $(\xi_k)_{k \in \mathbf{N}} \subset L^2(N, \tau)$, $\langle \xi_k, \hat{1} \rangle = 0$, $\|\xi_k\|_2 = 1$ with $\|(u_i - J u_i^* J)\xi_k\|_2 \to 0$, $1 \le i \le n$.

We show that $C^*(M, M', e_N)$ must contain the orthogonal projection from $L^2(M, \tau)$ onto $\mathbf{C}\hat{1}$. Consider

$$T := \sum_{i=1}^{n} e_N u_i J u_i J e_N.$$

Then $T \in C^*(M, M', e_N)$ and adjoining u_i^*, $1 \le i \le n$, to the $u_1, \dots, u_n$ if necessary, we may assume T is selfadjoint. Then $\|T\| = n$ and $T\hat{1} = n\hat{1}$. $T - n\cdot 1$ is invertible as an operator from the complement of $\hat{1}$ in $L^2(N, \tau)$ into itself. Indeed, because if we assume $T - n \cdot 1$ is not invertible, then there is a sequence $(\xi_k)_{k\in\mathbf{N}} \subset L^2(N, \tau)$, $\langle \xi_k, \hat{1}\rangle = 0$, $\|\xi_k\|_2 = 1$ with

$$\|(T - n1)\xi_k\|_2 \to 0 \qquad as\ k \to \infty.$$

This would imply $\|T\xi_k\|_2 \to n$. But

$$\|T\xi_k\|_2 \le \left\| \sum_{i=1}^{n} u_i J u_i J \xi_k \right\|_2 \le n,$$

and hence $\|\sum_{i=1}^n u_i J u_i J \xi_k\|_2 \to n$. The strict convexity of $L^2(M, \tau)$ implies then that $\|(u_i - Ju_i^* J)\xi_k\|_2 \to 0$, $1 \le i \le n$, which contradicts the choice of the $u_1, \dots, u_n$. Therefore $T - n \cdot 1$ is invertible on the orthogonal complement of $\mathbf{C}\hat{1}$ in $L^2(N, \tau)$, hence n is an isolated simple point of the spectrum of T, so the corresponding spectral projection is one-dimensional and in $C^*(M, M', e_N)$, which contradicts 4°.

$\underline{3^\circ \Rightarrow 2^\circ}$: Given $u_1, \dots, u_n$ unitaries in M, we want to show that there is a nonnormal state $\varphi \in M^*$ with $\varphi \circ \mathrm{Ad}\, u_i = \varphi$, $1 \le i \le n$, $\varphi \circ E_N = \varphi$. The proof is the same as Connes', but for the convenience of the reader we briefly sketch the argument. Fix a free ultrafilter ω of $\mathbf{N}$. By Lemma 1.3 we only have to consider the following two cases:

(a) $\{u_1, \dots, u_n\}' \cap N^\omega = \mathbf{C}$. Let $(\xi_k)_{k\in\mathbf{N}} \subset L^2(N, \tau)$ be as in 3°, i.e. $\|\xi_k\|_2 = 1$, $\langle \xi_k, \hat{1}\rangle = 0$, $k \in \mathbf{N}$, $\|[u_i, \xi_k]\|_2 \to 0$ as $k \to \infty$, $1 \le i \le n$. As in [Co1], using the hypothesis on the relative commutant, there is an $\varepsilon > 0$ and a subsequence $(\eta_k)_{k\in\mathbf{N}}$ of the sequence $(\xi_k)_{k\in\mathbf{N}}$ such that $\tau(E_k(|\eta_k|^2)|\eta_k|^2) \ge \varepsilon$ for all $k \in \mathbf{N}$. Put $\varphi_k = \tau(\,\cdot\,|\eta_k|^2)$ and $e_k = E_k(|\eta_k|^2)$. The φ_k are states on M with $\varphi_k \circ E_N = \varphi_k$ (since $\eta_k \in L^2(N, \tau)$) and $\|[\varphi_k, u_i]\| \to 0$ as $k \to \infty$, $1 \le i \le n$. Akemann's theorem [Ak, Theorem 2.3] shows that $(\varphi_k)_{k\in\mathbf{N}}$ is not weakly relatively compact in M_* since $\varphi_k(e_k) \ge \varepsilon$ for all $k \in \mathbf{N}$, but $e_k \to 0$ strongly. Any nonnormal φ in the weak closure of $\{\varphi_k\}_{k\in\mathbf{N}}$ in M^* will do.

(b) $\{u_1, \dots, u_n\}' \cap N^\omega$ is infinite-dimensional. Thus this commutant contains an infinite-dimensional abelian von Neumann subalgebra and hence contains nonzero projections e_k with $\tau_\omega(e_k) \le \frac{1}{k}$, $k \in \mathbf{N}$. Each e_k is represented by a sequence of projections $(f_k)_{k\in\mathbf{N}} \subset N$ with $f_k \ne 0$, $\tau(f_k) \le \frac{1}{k}$, $\|[u_i, f_k]\|_2 \le \frac{1}{k}\|f_k\|_2$ $\forall i, k$. Let $\varphi_k = \tau(\,\cdot\,\frac{f_k}{\tau(f_k)})$; then as in (a) we have $\varphi_k = \varphi_k \circ E_N$, and we can find a weak limit of the φ_k with the desired properties.

$\underline{2^\circ \Rightarrow 1^\circ}$: We prove this implication using a maximality argument, slightly different from the one in [Co1], similar to the ones in [CoFW and Po2]. Let

$u_1, \dots, u_n$ be unitaries in M and $\varepsilon > 0$. Consider

$$R := \{ f \mid f \text{ projection in } N \text{ such that: } (1)\ \tau(f) \le \tfrac{1}{2} \text{ and } (2)\ \|(1-f)u_i f - f u_i (1-f)\|_2 \le \varepsilon \|f\|_2,\ 1 \le i \le n \}.$$

R is clearly inductively ordered and nonempty, so we can take a maximal element $f \in R$. We show that $\tau(f) = \frac{1}{2}$. If not, there is a $\delta > 0$ with $\tau(f)+\delta \le \frac{1}{2}$ $(\delta < \varepsilon)$. Lemmas 1.4 and 1.2 applied to $N_{1-f} \subset M_{1-f}$, $(1-f)u_i(1-f)$, $1 \le i \le n$, and $\delta > 0$, give a nonzero projection $e \in N_{1-f}$ with $\tau(e) \le \delta$ and

$$\|[(1-f)u_i(1-f), e]\|_2 \le \delta \|e\|_2, \qquad 1 \le i \le n.$$

Put $f_0 := f + e$, which is a projection in N, strictly larger than f, and satisfies $\tau(f_0) \le \tau(f) + \delta \le \frac{1}{2}$. We compute

$$\begin{aligned}\|[u_i, f_0]\|_2^2 &= \|(1-f-e)u_i f - f u_i (1-f-e) + (1-f)u_i e - e u_i (1-f)\|_2^2, \\ &= \|(1-f-e)u_i f - f u_i(1-f-e)\|_2^2 + \|[(1-f)u_i(1-f), e]\|_2^2, \\ &\le \|(1-f)u_i f - f u_i (1-f)\|_2^2 + \|[(1-f)u_i(1-f), e]\|_2^2, \\ &= \varepsilon^2 \|f\|_2^2 + \delta^2 \|e\|_2^2 \le \varepsilon^2 \|f+e\|_2^2 = \varepsilon^2 \|f_0\|_2^2.\end{aligned}$$

Therefore

$$\|(1-f_0)u_i f_0 - f_0 u_i (1-f_0)\|_2 \le \varepsilon \|f_0\|_2, \qquad 1 \le i \le n,$$

which contradicts the maximality of f.

Put $u := 2(1-f) - 1$, then $\tau(u) = 0$, u is a unitary in N and $\|[u_i, u]\|_2 \le 2 \cdot \frac{\varepsilon}{2} = \varepsilon$, $1 \le i \le n$.

$2^\circ \Leftrightarrow 2'^\circ$: $2^\circ \Rightarrow 2'^\circ$ is trivial. Conversely, we show $2'^\circ \Rightarrow 1^\circ$, which is equivalent to 2°. Using Lemma 1.4 and Lemma 1.2, the same maximality argument used above works, because the projections in the set R are in N and N is contained in M_0. We get therefore statement 1° with M_0 in place of M. This is clearly enough to show 1° since u is bounded. Q.E.D.

2. Applications to crossed products

Throughout this section N denotes a separable II_1-factor and G a countable (discrete) amenable group acting freely on N via $\sigma : G \mapsto \operatorname{Aut} N$. Applying Theorem 1.1 to the pair of factors $N \subset N \times_\sigma G$, we get a simple proof of the following result due to Bédos [Be, Theorem A]:

Theorem 2.1. *Let N be a separable II_1-factor with property Γ and G an amenable group acting freely on N via some action σ. Then given any $x_1, \dots, x_n \in N \times_\sigma G$ and $\varepsilon > 0$, there is a unitary $u \in N$ with $\tau(u) = 0$ and $\|[u, x_i]\|_2 < \varepsilon$ for all $i = 1, \dots, n$. In particular, there are nontrivial central sequences of $N \times_\sigma G$ contained in N and $N \times_\sigma G$ has property Γ.*

Proof. Let $M := N \times_\sigma G$ and denote by u_g the unitaries implementing the action of G on N, i.e. $u_g x u_g^* = \sigma_g(x)$, $x \in N$. Set $M_0 := \{\sum_{\text{finite}} x_g u_g,\ x_g \in N\}$; then with $N \subset M_0 \subset M$ we are in the situation of statement $2'^\circ$ of

Theorem 1.1. Given unitaries $u_1, \dots, u_n \in N$ and $u_{g_1}, \dots, u_{g_k}$, we have to construct a nonnormal state $\varphi \in M^*$ with $\varphi(xu_i) = \varphi(u_i x)$, $\varphi(xu_{g_j}) = \varphi(u_{g_j} x)$, $\forall x \in M$, $\forall i, j$ and $\varphi \circ E_N = \varphi$.

Let $(F_n)_{n \in \mathbf{N}}$ be an increasing sequence of finite subset of G with $\bigcup_{n=1}^{\infty} F_n = G$. Since G is amenable, given F_n and $\varepsilon_n > 0$, there is a finite set $K_n \subset G$ with $|K_n F_n \setminus K_n| < \varepsilon_n |K_n|$. We choose $\varepsilon_n \searrow 0$. Given $u_1, \dots, u_n$, $\sigma_g(u_i)$, $g \in K_j$, $1 \le i \le n$, unitaries in N, there is a singular state $\varphi_j \in N^*$ which is invariant under these unitaries [Co1, Theorem 2.1]. We put

$$\varphi_{K_j}(x) = \frac{1}{|K_j|} \sum_{g \in K_j} \varphi_j \circ E_N(u_g x u_g^{\,*}), \qquad x \in M.$$

By construction we have $\varphi_{K_j}(xu_i) = \varphi_{K_j}(u_i x)$, $\forall x \in M$, $1 \le i \le n$, and the φ_{K_j} are clearly singular states on M which factor through E_N. The choice of the sets $(K_n)_{n \in \mathbf{N}}$ implies that

$$\|[\varphi_{K_j}, u_g]\| \to 0 \quad \text{as } j \to \infty, \qquad \forall g \in G.$$

Let φ be a $\sigma(M^*, M)$-limit point of $\{\varphi_{K_j}\}_{j \in \mathbf{N}}$; then by [Ta2, Chapter III, Proposition 5.8] φ is still singular and has, by the construction of the φ_{K_j}, the desired invariance properties. Applying $2'^{\circ}$ of Theorem 1.1 gives the result. Q.E.D.

We also mention here a remark, essentially showing that the set of subfactors of a given II_1-factor M, that contain a fixed subfactor N and have nontrivial central sequences in N, is inductively ordered.

Proposition 2.2. *Let N, M_n, M be II_1-factors with M separable such that $N \subset M_n \subset M$ for all $n \in \mathbf{N}$ and $M_n \nearrow M$ (i.e. $\overline{\bigcup_{n=1}^{\infty} M_n}^{w} = M$). If there are nontrivial central sequences for M_n contained in N for all $n \in \mathbf{N}$, then there are nontrivial central sequences for M contained in N.*

Proof. The hypothesis imply that 1° of Theorem 1.1 holds for all pairs $N \subset M_n$, $n \in \mathbf{N}$. We will check condition 1° for $N \subset M$. Given $x_1, \dots, x_n \in M$ and $\varepsilon > 0$, there is an $n_0 \ge 1$ and $x_1^0, \dots, x_n^0 \in M_{n_0}$ such that $\|x_i - x_i^0\|_2 < \frac{\varepsilon}{2}$, $1 \le i \le n$. But for $x_1^0, \dots, x_n^0$ there is a unitary $u \in N$ with $\tau(u) = 0$ and $\|[x_i^0, u]\|_2 < \frac{\varepsilon}{2}$, thus $\|[x_i, u]\|_2 < \varepsilon$, $1 \le i \le n$. Q.E.D.

3. A relative Mc Duff-type theorem

Mc Duff considers in [McD] separable II_1-factors M which are isomorphic to $M \bar{\otimes} R$, R the hyperfinite II_1-factor. This property is stronger than property Γ and is in fact equivalent to the noncommutativity of the algebra $M' \cap M^{\omega}$, ω a free ultrafilter in $\mathbf{N}$. Using Mc Duff's methods we prove a relative version of her theorem for a pair of (separable) II_1-factors $N \subset M$. The theorem gives a necessary and sufficient condition implying that the pair $N \subset M$ is isomorphic to the pair $N \bar{\otimes} R \subset M \bar{\otimes} R$.

Theorem 3.1. *Let M be a separable II_1-factor with subfactor $N \subset M$. Then the following conditions are equivalent*:

1°. *Given $x_1, \dots, x_n \in M$ and $\varepsilon > 0$, there is a I_2-subfactor $M_{2\times 2}$ in N with matrix units $\{e_{ij}\}_{1\le i,j\le 2}$ such that*

$$\|[e_{ij}, x_k]\|_2 < \varepsilon, \qquad 1 \le k \le n, \quad 1 \le i, j \le 2.$$

2°. *$N \subset M \cong N\bar{\otimes} R \subset M \bar{\otimes} R$, i.e. there is an isomorphism $\Phi : M \mapsto M \bar{\otimes} R$ (onto) with $\Phi(N) = N \bar{\otimes} R$.*

3°. *$M' \cap N^\omega$ is noncommutative, where ω is a free ultrafilter in $\mathbf{N}$.*

Since the proof follows closely Mc Duff's ideas, we avoid details where our proof coincides with Mc Duff's.

Proof. We show $1^\circ \Leftrightarrow 2^\circ$ and $1^\circ \Leftrightarrow 3^\circ$.

$\underline{1^\circ \Rightarrow 2^\circ}$: In Lemmas 3.2 and 3.3 below we prove two results similar to [McD, Theorems 1, 2]. The proofs are essentially the ones given by Mc Duff adapted to the situation $N \subset M$.

Lemma 3.2. *Let $(M_k)_{k=1}^\infty$ be a sequence of I_2-subfactors of N with matrix units $\{e_{ij}^k\}$ such that*

(i) *M_j commutes with M_k, $j \ne k$.*

(ii) *$(e_{ij}^k)_{k=1}^\infty$ is a central sequence in M for all (i, j) fixed.*

Then there is a hyperfinite II_1-factor $R \subset N$ with

$$N \cong R \bar{\otimes} (R' \cap N), \qquad M \cong R \bar{\otimes} (R' \cap M)$$

and thus $N \subset M \cong N \bar{\otimes} R \subset M \bar{\otimes} R$.

Proof. We use the following well-known fact: If $K \subset M$ is a I_k-subfactor, then

$$\|x - E_{K'\cap M}(x)\|_2 \le k^{\frac{3}{2}} \sup_{1\le i,j\le k} \|[e_{ij}, x]\|_2, \tag{4}$$

where $\{e_{ij}\}_{1\le i,j\le k}$ are matrix units in K and $E_{K'\cap M}$ is the conditional expectation $M \mapsto K' \cap M$.

Let $(x_n)_{n=1}^\infty$ (resp. $(y_n)_{n=1}^\infty$) be a $\|\cdot\|_2$-dense sequence in M (resp. N). Using (4) and (ii) of the hypothesis, we can find for given $x_1, \dots, x_k$, $y_1, \dots, y_k$ and $\varepsilon = 2^{-k}$ a matrix-algebra $M_{n_k} \in \{M_k,\ k = 1, 2, \dots\}$ such that

$$\|x_i - E_{M'_{n_k}\cap M}(x_i)\|_2, \ \|y_i - E_{M'_{n_k}\cap N}(y_i)\|_2 < 2^{-k}, \quad 1 \le i \le k. \tag{5}$$

Notice that $M_{n_k} \subset N$. This provides a subsequence $\{M_{n_k}\}_{k=1}^\infty$ of $\{M_k\}_{k=1}^\infty$ and we set

$$R := \overline{\bigotimes}_{k=1}^\infty M_{n_k}.$$

R splits M and N simultaneously. For this it is enough to show that $x_i \in (R \cup (R' \cap M))''$ (resp. $y_i \in (R \cup (R' \cap N))''$) for all $i \in \mathbf{N}$. Similar to [McD, inequality (7)], we get

$$(6)\quad \|x - E_{(\otimes_{i=n+1}^{\infty} M_{n_i})' \cap M}(x)\|_2 \leq \sum_{i=n+1}^{\infty} \|x - E_{M'_{n_i} \cap M}(x)\|_2, \quad x \in M, \ n \in \mathbf{N}.$$

So if we fix x_{i_0} and y_{i_0}, we get from (6) using (5) for $n > i_0, j_0$:

$$\|x_{i_0} - E_{(\otimes_{i=n+1}^{\infty} M_{n_i})' \cap M}(x_{i_0})\|_2 \leq \sum_{i=n+1}^{\infty} 2^{-i} \to 0 \quad as\ n \to \infty,$$

and similarly for y_{i_0}. But $E_{(\otimes_{i=n+1}^{\infty} M_{n_i})' \cap M}(x_{i_0})$ (resp. $E_{(\otimes_{i=n+1}^{\infty} M_{n_i})' \cap N}(y_{i_0})$) is contained in $(\bigotimes_{i=1}^{n} M_{n_i} \cup (R' \cap M))''$ (resp. $(\bigotimes_{i=1}^{n} M_{n_i} \cup (R' \cap N))''$), which shows that $N \cong R \bar{\otimes} (R' \cap N)$, $M \cong R \bar{\otimes} (R' \cap M)$. Analyzing these standard isomorphisms, we get the desired result, since $R \cong R \bar{\otimes} R$. This ends the proof of Lemma 3.2.

As in [McD, Theorem 2], we need a second lemma, which will prove the implication $1^\circ \Rightarrow 2^\circ$ together with Lemma 3.2.

Lemma 3.3. *Let $(M_k)_{k=1}^{\infty}$ be a sequence of I_2-subfactors of N with matrix units $\{e_{ij}^k\}$ forming central sequences in M. Then there is a sequence $(N_k)_{k=1}^{\infty}$ of mutually commuting I_2-subfactors in N with matrix units $\{g_{ij}^k\}$, which also form central sequences in M.*

Proof. Mc Duff's proof of Theorem 2 in [McD] applies completely; her construction never leaves N.

This ends the proof of $1^\circ \Rightarrow 2^\circ$.

$\underline{2^\circ \Rightarrow 1^\circ}$: The desired I_2-subfactor can be constructed in R and then transported to N via the given isomorphism.

$\underline{1^\circ \Rightarrow 3^\circ}$: Using the separability of N we obtain a sequence of I_2-subfactors of N with matrix units $\{e_{ij}^k\}$ such that

$$\|[x, e_{ij}^k]\|_2 \to 0 \quad as\ k \to \infty, \quad \forall x \in M, \ (i, j) \text{ fixed}.$$

But then $(e_{12}^n)(e_{21}^n) \neq (e_{21}^n)(e_{12}^n)$ and the elements (e_{12}^n) and (e_{21}^n) are nontrivial in N^ω.

$\underline{3^\circ \Rightarrow 1^\circ}$: The argument is the same as Mc Duff's. For the sake of completeness we recall briefly the steps.

Let $I_0 = \{(x_n) \in l^\infty(\mathbf{N}, M) \mid \lim_{n\to\infty} \|x_n\|_2 = 0\}$, $N^0 = l^\infty(\mathbf{N}, M)/I_0$. It is easy to see that $M' \cap N^\omega$ is commutative iff $M' \cap N^0$ is commutative. Then one shows that if $M' \cap N^\omega$ is noncommutative, there are no abelian projections in $M' \cap N^\omega$, i.e. $M' \cap N^\omega$ is continuous and contains therefore a I_2-subfactor. This proves 1°, since whenever we have matrix units $\{e_{ij}\}_{1\leq i,j\leq 2}$ in $M' \cap N^\omega$, they lift to matrix units $\{e_{ij}^k\}_{1\leq i,j\leq 2}$ in N with $\{e_{ij}^k\}_{k=1}^{\infty} \in e_{ij}$. In particular we have that $\lim_{k\to\omega} \|[e_{ij}^k, x]\|_2 = 0$, (i, j) fixed, $x \in M$, which implies 1°.

To prove the result about the nonexistence of abelian projections in a noncommutative algebra $M' \cap N^{\omega}$, we take a dense sequence $(x_k)_{k=1}^{\infty}$ in $(M)_1$ and put

$$N_k = \{ x \in (N)_1 \mid \|[x, x_i]\|_2 < 1/k, 1 \leq i \leq k \}.$$

As in [McD, Lemma 6], we get the following result:

Suppose $M' \cap N^0$ is noncommutative. Let $f \in N_{3k}$ be a projection of trace $\lambda \neq 0$. Then there is an $\alpha(\lambda) > 0$ and $y, z \in N_k$ such that $y = fyf$, $z = fzf$ and $\|[y, z]\|_2 > \alpha(\lambda)$. Moreover, $\alpha(\lambda)$ is independent of k.

This completes the proof of the theorem. Q.E.D.

References

[Ak] C. Akemann, *The dual space of an operator algebra*, Trans. Amer. Math. Soc. **126** (1967), 286–302.

[Be] E. Bédos, *On actions of amenable groups on* II_1 *-factors*, Preprint 1988.

[Co1] A. Connes, *Classification of injective factors, Cases* II_1, II_∞, III_λ, $\lambda \neq 1$, Ann. of Math. **104** (1976), 73–115.

[Co2] ——, *Outer conjugacy classes of automorphisms of factors*, Ann. Sci. École Norm. Sup. **8** (1975), 383–420.

[CoFW] A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory Dynamical Systems **1** (1981), 431–450.

[Jo] V. F. R. Jones, *Index of subfactors*, Invent. Math. **72** (1983), 1–25.

[Oc] A. Ocneanu, *Quantized group string algebras and Galois theory for algebras*, preprint.

[Po1] S. Popa, *A short proof of "Injectivity implies Hyperfiniteness" for finite von Neumann-Algebras*, J. Operations Theory **16** (1986), 261–272.

[Po2] ——, *Maximal injective subalgebras in factors associated with free groups* , Adv. in Math. **50** (1983), 27–48.

[Po3] ——, *Classification of subfactors: reduction to commuting squares*, Invent. Math. (to appear).

[McD] D. McDuff, *Central sequences and the hyperfinite factor*, Proc. London Math. Soc. (3) **21** (1970), 443–461.

[MvN] F. Murray and J. von Neumann, *Rings of operators* IV, Ann. of Math. **44** (1943), 716–808.

[Ta1] M. Takesaki, *On the singularity of a positive linear functional* , Proc. Japan Acad. **35** (1959), 365–366.

[Ta2] ——, *Theory of operator algebras* I, Springer, Berlin, Heidelberg, and New York, 1979.

Department of Mathematics, University of California at Los Angeles, 405 Hilgard Avenue, Los Angeles, California 90024-1555