A TOPOLOGICAL PERSISTENCE THEOREM FOR
NORMALLY HYPERBOLIC MANIFOLDS VIA THE CONLEY INDEX

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Abstract. We prove that the cohomology ring of a normally hyperbolic manifold of a diffeomorphism \( f \) persists under perturbation of \( f \). We do not make any quantitative assumptions on the expansion and contraction rates of \( Df \) on the normal and the tangent bundles of \( N \).

1. Introduction

In this paper, we shall apply C. Conley’s homotopy index theory for invariant sets of flows to discrete dynamical systems. In particular, we prove a homotopy version of the persistence theorem for normally hyperbolic invariant manifolds.

Let \( M \) be a smooth i.e. \( C^2 \) manifold and let \( f : M \to M \) be a diffeomorphism which restricts to a diffeomorphism of a compact \( C^2 \) submanifold \( N \) onto itself. One calls \( N \) normally hyperbolic with respect to \( f \) if it satisfies a certain nondegeneracy condition on a normal bundle of \( N \) in \( M \). In this paper, we will use the following definition:

Definition 1. \( N \) is called normally hyperbolic with respect to \( f \) if the tangent bundle of \( M \) over \( N \) splits into smooth subbundles

\[
TM|_N = E^+ \oplus TN \oplus E^-
\]

which are invariant under \( Df : TM \to TM \) such that \( Df \) contracts \( E^- \) and expands \( E^+ \). Here we say that a vector bundle isomorphism \( \psi : E \to E \) contracts \( E \) if for every \( \xi \in E \), the sequence \( \psi^n\xi \) converges to the zero section of \( E \). We say that \( \psi \) expands \( E \) if \( \psi^{-1} \) contracts \( E \).

N. Fenichel [2] examined the question under which conditions a normally hyperbolic manifold persists under small perturbations of \( f \). In fact, he proves that if \( Df \) contracts \( E^- \) and expands \( E^+ \) stronger than any vector in \( TN \), then every \( C^1 \)-small perturbation of \( f \) has a normally hyperbolic manifold \( C^1 \)-diffeomorphic to \( N \) (see also [6, Theorem 4.1] for more persisting properties of \( N \)). Here, the expansion and contraction rates have to be measured by some metric on \( M \). In order to illustrate the importance of this quantitative
hypothesis, consider on \( N \) an attracting fixed point such that the rate of approach in the direction of \( TN \) is greater than in the normal direction. At such a point, a cusp may develop under perturbation (see [2, 8, and 4, pp. 239, 251]). There are also examples (see Jarnik and Kurzweil, [7]), where a normally hyperbolic manifold changes under arbitrarily small perturbations into an invariant set which is not even a topological manifold. In this paper, we shall prove that if \( N \) is normally hyperbolic in the sense of Definition 2, i.e. without any additional quantitative conditions, then the cohomology of \( N \) persists under \( C^0 \) small perturbations of \( f \):

**Theorem 1.** Let \( f: M \to M \) be a diffeomorphism and let \( N \) be a smooth compact normally hyperbolic invariant submanifold with respect to \( f \). Let \( \{f_\lambda\}_{\lambda \in \mathbb{R}} \) be a family of homeomorphisms of \( M \), continuous in the compact-open topology, with \( f_0 = f \). Then for \( \lambda \) small enough, \( f_\lambda \) has an isolated invariant set \( T_\lambda \) whose cohomology ring contains \( H^*(N) \) as a subring. In fact, for every neighborhood \( U \) admitting a retraction \( r: U \to N \), we have \( T_\lambda \subset U \) for \( \lambda \) small enough, and

\[
(r|T_\lambda^*): H^*(N) \to H^*(T_\lambda)
\]

is injective. Here, coefficients are arbitrary if \( N \) and \( E^+ \) are oriented and \( f|N \) and \( Df|E^+ \) are orientation preserving and in \( \mathbb{Z}_2 \) otherwise.

The notion of an isolated invariant set will be defined in the following section. In analogy to [3, Theorem 2], one also has a global persistence result for certain normally hyperbolic manifolds \( N \), which does not assume that the perturbation is small. Instead, one has to assume, in addition, that \( N \) is a retract of \( M \) by a retraction which commutes with \( f \) up to homotopy. This means that there exists a continuous map

\[
r: M \to N
\]

such that \( r|N \) is homotopic to the identity and so that \( f|N \circ r \) and \( r \circ f \) are homotopic as maps from \( M \) to \( N \).

**Theorem 2.** Let \( N \subset M \) be a compact normally hyperbolic invariant manifold of the diffeomorphism \( f \) and let \( r: M \to N \) be a retraction commuting with \( f \) up to homotopy. Then for every invariant set \( T \) of a homeomorphism \( f^t \) which is related to \((N, f)\) by continuation, \( r \) induces injective homomorphisms

\[
(r|T)^*: H^*(N) \to H^*(T)
\]

with coefficients as in Theorem 1.

The notion of continuation will be defined in the following section. The proof of Theorems 1 and 2 uses the index theory for topological flows developed by C. Conley (see [1]). By a suspension procedure (§2) we construct a flow on a bundle \( M_f \) over \( S^1 \) with fibre \( M \) whose time 1 map equals \( f \). In §3, we show that a normally hyperbolic submanifold \( N \) of \( M \) for \( f \) corresponds to a
normally hyperbolic submanifold $N_f$ of $M_f$ of the flow. In this situation, we can apply Theorem 2 of [3] to continue $N_f$ under a perturbation of the flow. In §4, we show that this proves the perturbation result for the map $f$ on $N$.

2. MAPS AND FLOWS

We will denote by $\mathcal{F}(M)$ the space of all homeomorphisms from a locally compact Hausdorff space $M$ onto itself. In some applications, one also considers an operation $\rho$ of a compact topological group $G$ on $M$, i.e. a continuous map

\begin{equation}
\rho: G \times M \to M, \quad (g, x) \mapsto gy
\end{equation}

such that $(g \circ h)x = g(hx)$. In such a situation, we may restrict ourselves to the subset $\mathcal{F}_\rho \subset \mathcal{F}(M)$ of homeomorphisms commuting with $\rho(g)$ for all $g \in G$, i.e. of equivariant homeomorphisms. Let $\mathcal{F}_\rho$ be equipped with any Hausdorff topology so that the inverse map

\begin{equation}
\mathcal{F} \to \mathcal{F}: f \mapsto f^{-1}
\end{equation}

is continuous, for example, we can take the compact open topology. In analogy to [3, Definition 1], we can introduce the notion of an isolated invariant set

**Definition 2.1.** For any $\rho$-invariant subset $U$ of $M$, and for $f \in \mathcal{F}_\rho$, define

\begin{align*}
T_-^f(U) &= \{ x \in U | f^n(x) \subset U \text{ for all } n \geq 0 \}, \\
T_+^f(U) &= \bigcap_{n \geq 0} f^n(U).
\end{align*}

The maximal invariant set is defined by

\begin{equation}
T^f(U) = T_-^f(U) \cap T_+^f(U).
\end{equation}

We call $U$ isolating, if the closure of $T^f(U)$ is contained in the interior of $U$. In this case, $T^f(U)$ is called an isolated invariant set.

Next we define the notion of continuation. Define the set

\begin{equation}
\mathcal{T} = \{(T, f) | f = \mathcal{F}_\rho \text{ and } T \text{ is a nonempty compact isolated invariant set of } f \text{ in } M \}.
\end{equation}

On $\mathcal{T}$, consider the topology generated by the open sets

\begin{equation}
\theta_U = \{(T, f) | U \text{ is isolated with respect to } f \text{ with nonempty maximal invariant set } T = T^f(U) \},
\end{equation}

where $U$ is any open ($\rho$-invariant) subset of $M$.

**Definition 2.2.** Two elements $(T, f)$ and $(T', f)$ of $\mathcal{T}$ are called related by continuation, if they are connected to $\mathcal{T}$ by a continuous path.

We want to define a topological index for isolated invariant sets of maps $f$ on $M$ which is invariant under continuation, i.e. which only depends on the
path components in $\mathcal{F}$. Therefore, we use the index theory for flows developed by C. Conley. A flow on a topological space $\Gamma$ is defined as a continuous map
\begin{equation}
\chi: \Gamma \times \mathbb{R}_+ \supset U_\Gamma \to \Gamma: (x, t) \rightarrow \chi(x, t) = x \cdot t,
\end{equation}
where $U_\Gamma$ is a neighborhood of $\Gamma \times \{0\}$ in $\Gamma \times \mathbb{R}_+$ with the following property: If $(x, t)$ and $(x \cdot t, x) \in U_\Gamma$, then $(x, t + s) \in U_\Gamma$ and
\begin{equation}
(x \cdot t) \cdot s = x \cdot (t + s).
\end{equation}
Again, we can assume that the flow is equivariant with respect to some operation $\rho$ of a compact group $G$ on $\Gamma$, i.e. that $U_\Gamma$ is $\rho$-invariant and that $\rho(x) \cdot t = \rho(x \cdot t)$ for $(x, t) \in U_\Gamma$.

Moreover, we can define isolated invariant sets (see, for example [3, Definition I]),

**Definition 2.3.** For any ($\rho$-invariant) subset $U$ of $\Gamma$, define
\begin{align}
U_{-\infty} &= \{x \in U | x \cdot \mathbb{R}_+ \subset U\}, \\
U_{\infty} &= \bigcap_{t \geq 0} U \cdot t
\end{align}
and the maximal invariant set
\begin{equation}
S(U) = U_{\infty} \cap U_{-\infty}.
\end{equation}

We are concerned with the following family of flows:

**Definition 2.4.** Consider the quotient
\begin{equation}
M_\mathcal{F} = p(M \times \mathcal{F}_\rho \times (-1, 1)),
\end{equation}
where $p$ is the projection map corresponding to the equivalence relation generated by
\begin{equation}
(x, f, t - 1) \sim (f(x), f, t)
\end{equation}
for $t \in (0, 1)$. On $M_\mathcal{F}$, consider the flow $\chi$ defined by the map
\begin{equation}
\chi^\tau(x, f, t) = (x, f, t - \tau)
\end{equation}
for $\tau, t \in [0, 1]$.

We can define an operator $\rho$ on $M_\mathcal{F}$ commuting with the flow by
\begin{equation}
\rho(g)(x, f, t) = (\rho(g) x, f, t).
\end{equation}
Clearly, the flow $\chi$ on $M_\mathcal{F}$ restricts to a continuous flow $\chi_f$ on every leaf
\begin{equation}
M_f := p_f(M \times (-1, 1)),
\end{equation}
where $p_f$ is the restriction of $p$ to $M \times (-1, 1) \cong M \times \{f\} \times (-1, 1)$. The flow $\chi_f$ is also called the suspension of the map $f$. 
Proposition 1. The flow invariant map

\[ M_{\mathcal{F}} \rightarrow \mathcal{F}(M): (x, f, t) \mapsto f \]

defined a (\( \rho \)-invariant) local product parametrization in the sense of [3, Definition 3].

Proof. It suffices to show that the map

\[ (2.19) \quad M \times \mathcal{F}_{\rho} \times \mathbb{R} \rightarrow M \times \mathcal{F}_{\rho} \times \mathbb{R}, \]

\[ (x, f, t) \mapsto (f(x), f, t + 1) \]

is a homeomorphism. In fact, in this case, we have \( M_{\mathcal{F}} = (M \times \mathcal{F}_{\rho} \times \mathbb{R})/\mathbb{Z} \), where the operation of \( \mathbb{Z} \) is generated by the map (2.19). Since \( M \times \mathcal{F}_{\rho} \times \mathbb{R} \) is a product space, this proves that \( M \) has the required local product structure.

Obviously, the map (2.19) is bijective and continuous, \( f \in \mathcal{F}_{\rho} \) is a homeomorphism and the evaluation map \( \mathcal{F}_{\rho} \times M \rightarrow M \) is continuous. But also the inverse

\[ (2.20) \quad (x, f, t) \mapsto (f^{-1}(x), f, t - 1), \]

is continuous, since the map (2.1) was assumed to be continuous on \( \mathcal{F}_{\rho} \).

We now can define the notion of continuation in \( M \). Following [1, Section 4] and [3, Section 2], we consider the set

\[ (2.21) \quad \mathcal{S} = \{(S, f) | f \in \mathcal{F}_{\rho}(M) \text{ and } S \text{ is a nonempty compact isolated invariant (and } \rho\text{-invariant) set in } M_f \}. \]

The topology is generated by the open sets

\[ (2.22) \quad \Sigma_U = \{(S, f) | U \cap M^f \text{ is isolating in } M_f \text{ with maximal invariant set } S \}, \]

where \( U \) is any open (\( \rho \)-invariant) subset of \( M_{\mathcal{F}} \).

Proposition 2. The map

\[ (2.23) \quad \Psi: \mathcal{F} \rightarrow \mathcal{S}: (T, f) \mapsto (T_f, f) \]

is a homeomorphism.

Proof. First, note that every nonempty invariant set of \( \chi_f \) on \( M_f \) is of the form \( T_f \), where \( T \) is an invariant set of \( f \). We now show that for every open set \( U \subset M \) and for every \( f \in \mathcal{F} \), the set \( U_f \) is isolating for the flow \( \chi_f \) if and only if \( U \) is isolating for \( f \). In fact, it is easy to see that \( x \in T^+_f(U) \) if and only if \( p_f(x, t) \in U^{f}_{\infty} \) for all \( t \in (-1, 1) \). Similarly, \( x \in T^-_f(U, f) \) if and only if \( p_f(x, t) \in U^{-}_{\infty} \) for all \( t \in I \).

This shows that \( T \) is isolated in \( U \) if and only if \( T_f \) is isolated in \( U^f \). Moreover, if \( \tilde{U} \) is any isolating neighborhood of \( T^f \) in \( M_f \), then it contains a neighborhood of the form \( U^f \) for some neighborhood \( U \) of \( T \) in \( M \). Hence the map \( \Psi \) of Proposition 2 is bijective.
In order to show that it is also a homeomorphism, note that the topology of \( \mathcal{T} \times M \) is generated by sets \( U = U_m \times U_{\mathcal{T}} \), where \( U_{\mathcal{T}} \) is open in \( \mathcal{T} \) and \( U_m \) is open in \( M \). Similarly, the topology of \( M_{\mathcal{T}} \) is generated by sets \( p(U_m \times U_{\mathcal{T}} \times I) \), where, in addition, \( I \subset (-1, 1) \) is open. However, since we are only interested in neighborhoods \( U \) of \((S, f) \in \mathcal{T} \) for which \( S = Sf(U) \) is nonempty, we can restrict ourselves to those open sets where \( I = (-1, 1) \). Then by the above,

\[
\Sigma_{p(U_m \times U_{\mathcal{T}} \times I)} = \{(S^f, f) | (S, f) \in \Theta U_m \times U_{\mathcal{T}} \} = \Psi(\theta(U_m \times U_{\mathcal{T}})),
\]

which completes the proof of Proposition 2. □

We can use the map \( \Psi \) and the Conley index for flows (see [1 and 3]) to define a topological index on \( T \).

**Definition 2.5.** For \((T, f) \in \mathcal{T} \), define

\[
I^*_\rho(T, f) := H^*_\rho(X, A)
\]

where \((X, A)\) is any \((\rho\text{-invariant})\) index pair for \( \mathcal{T} \subset M_f \) and \( H^*_\rho \) denotes the equivariant Alexander-Spanier cohomology with values in some ring with unit. Moreover, with \( \pi: M_f \to S^1 \) given by \( \pi(x, t) = t \) and for the standard generator \( e \) of \( H^1(S^1) \), define

\[
\theta_{(T, f)}: I^*_\rho(T, f) \to I^*_\rho(T, f): \alpha \mapsto \alpha \cup (\pi|_x)^* e.
\]

The continuation invariance of the index \( I_\rho = (I^*_\rho, \theta) \) on \( \mathcal{T} \) then follows immediately from Proposition 2 and from Theorem 1 of [3].

**Theorem 3.** \( I^*_\rho(T, f) \) does not depend on the choice of the index pair \((X, A)\). Moreover, if \((T, f)\) is related to \((T', f')\) by continuation, then there exists an isomorphism \( i: I^*_\rho(T, f) \to I^*_\rho(T', f') \) such that

\[
\theta_{(T, f)} \circ i = i \circ \theta_{(T', f')}.
\]

## 3. Suspending Normally Hyperbolic Manifolds

In this section, we prove the following proposition.

**Proposition 3.** If \( N \subset M \) is a normally hyperbolic invariant submanifold with respect to a smooth map \( f \), then \( N_f \) is a normally hyperbolic invariant submanifold of \( M_f \) with respect to the suspended flow in the sense of [3, Proposition 1].

Note therefore that, by Definition 1, \( f \) is in fact a diffeomorphism from some neighborhood \( O \) of \( N \subset M \) onto its image. Therefore,

\[
O_f := p_f(O \times (-1, 1)) \subset M_f
\]

is a smooth open manifold with a smooth submanifold \( N_f = p_f(N \times (-1, 1)) \). Now consider on \( N_f \) the vector bundles

\[
E^\pm_f = p_{DF}(E^\pm \times (-1, 1)),
\]
where \( p_{Df} \) identifies \((\xi \cdot f, t - 1)\) with \((f \cdot \xi, t) = (Df \cdot \xi, t)\) for \( t \in (0, 1) \). This is well defined, since \( f \cdot \) leaves \( E^\pm \) invariant. Moreover, (3.2) constitutes a decomposition of the normal bundle of \( N_f \) in \( O_f \). The flow on \( O_f \) is generated by the vector field

(3.3) \[ \eta := (p_f)_t \frac{\partial}{\partial t}, \]

which is also well defined on \( O_f \). In order to show that \( N_f \) satisfies the conditions of Proposition 1 of [3], we have to find a metric \( \bar{g} \) on \( O_f \) so that for every \( v \in N \), the linear operator

(3.4) \[ L: T_v(M) \to T_v M: \xi \mapsto \xi \cdot \nabla_{\bar{g}} \eta \]

leaves the bundles \( E^\pm f \) invariant and satisfies

(3.5) \[ \langle \xi, L\xi \rangle \geq m|\xi|^2 \quad \text{for} \quad \xi \in E^+_f, \]

(3.6) \[ \langle \xi, L\xi \rangle \leq -m|\xi|^2 \quad \text{for} \quad \xi \in E^-_f \]

for some constant \( m > 0 \). In (3.4), \( \xi \cdot \nabla_{\bar{g}} \) denotes the covariant derivative with respect to the metric \( \bar{g} \) in the direction of \( \xi \).

The first step towards the construction of the metric \( \bar{g} \) is

**Lemma 3.1.** Let \( \psi \) be a contracting bundle map on the vector bundle \( E \). For any metric \( g_0 \) on \( E \), define \( g_k = (\psi^*)^k g_0 \) for \( K \in \mathbb{N} \). Then the series

(3.7) \[ g = \sum_{k=0}^{\infty} g_k \]

converges to a metric on \( E \) with

(3.8) \[ |\xi|^2_{g_k} - |f_\cdot \xi|^2_{g_k} = |\xi|^2_{g_0}. \]

**Proof.** Set \( B_\alpha = \{ \xi \in E^- | g_0(\xi, \xi) \leq g \alpha^2 \} \). Since \( B_1 \) is compact, the contracting property of \( Df \) on \( E^- \) implies that there exists an \( n \in \mathbb{N} \) with

(3.9) \[ f^n_\cdot B_1 \subset B_{1/2}. \]

Consequently, we have

(3.10) \[ |f_\cdot^{k \cdot n} \xi|_{g_0} \leq 2^{-k} |\xi|_{g_0}, \]

and hence for some \( c \) independent of \( x \in N \):

(3.11) \[ [(f^{k \cdot n})^* \gamma]_j \leq c 2^{-k}. \]

Hence the sum of (3.7) converges and (3.8) holds. Moreover, \( g \) is positive as a sum of positive metrics. \( \square \)

We apply Lemma 3.1 to the contracting bundle maps \( Df \) on \( E^- \) and \((Df)^{-1} \) on \( E^+ \) to obtain metrics \( \bar{g}^\pm \) on \( E^\pm \) with

(3.12) \[ \langle \xi, \xi \rangle_{\bar{g}^\pm} - \langle f_\cdot \xi, f_\cdot \xi \rangle_{\bar{g}^\pm} = \mp \langle \xi, \xi \rangle_{g_0}. \]
We can extend these metrics to a Riemannian metric \( g \) on \( M \) such that
\[
\langle \xi^-, \xi^+ \rangle_g = 0 \quad \text{for} \quad \xi^\pm \in E^\pm.
\]
Now \( \bar{g} \) is defined as the unique metric on \( M_f \) satisfying
\[
\begin{align*}
(3.13) & \quad i_t^* \bar{g} = tg + (1 - t)f^* g, \\
(3.14) & \quad |\eta|_{\bar{g}} \equiv 1, \\
(3.15) & \quad \langle \eta, i_t^* \xi \rangle_{\bar{g}} = 0 \quad \text{for} \quad \xi \in TM
\end{align*}
\]
where \( i_t: M \to M_f \) is given by \( i_t(x) = p_f(x, t) \). Note that \( i_1 = i_0 \circ f \), so that condition (3.13) is compatible for \( t = 0, 1 \).

To calculate the linear operator \( L \) of (3.4), note that for any \((x, t) \in M_f\) there exists a neighborhood which is naturally isomorphic by the flow \( f^t \) to \( I \times U \), where \( I \subset \mathbb{R} \) and \( U \subset M \) are open sets. Choose any chart of \( M \) at \( x \). Let the index \( O \) denote the direction of \( t \), so that \( \eta \) corresponds to the constant vector field \( e^O \) on the chart. With \( i, j, \) and \( k \) running from \( 0 \) to \( \dim M \), the covariant derivative of \( e^O \) can now be expressed in terms of Christoffel symbols (see [5])
\[
(3.16) \quad (L\xi)^i = (\xi \cdot \nabla_\bar{g} e^O)^i = \Gamma_{jk}^i \xi^j (e^O)^k = \Gamma_{ok}^i \xi^k.
\]
By [5], we have therefore
\[
\bar{g}_{ij}(L\xi)^i = \Gamma_{i,ok}^j \xi^k
\]
with
\[
(3.17) \quad T_{ok,j} = \frac{\partial}{\partial x^j} \bar{g}_{ko} + \frac{\partial}{\partial x^0} \bar{g}_{jk} - \frac{\partial}{\partial x^j} \bar{g}_{ok}
\]
whereas \( \Gamma_{ok,j} = 0 \) for \( k = 0 \) or \( j = 0 \). Hence we have for \( \xi, \zeta \in T_{(x, t)} M \):
\[
(3.18) \quad \langle \xi, L\zeta \rangle_{\bar{g}} = \langle \pi_* \xi, \pi_* \zeta \rangle_g - \langle f_* \pi_* \xi, f_* \pi_* \zeta \rangle_g
\]
where \( \pi: U \times I \to U \) is the projection. This immediately shows that \( L \) leaves the mutually orthogonal subspaces \( E^\pm \) and \( T N_f \) invariant. Moreover, choosing \( \zeta = \xi \in E^+ \) or \( E^- \) and using (3.8) proves (3.5) and (3.6).

4. Proof of the Continuation Results

In this section we will complete the proof of Theorems 1 and 2. Let us first consider the situation of Theorem 2. Since it can be done without much additional work, we will always assume that \( f \in \mathcal{F}_\rho \) for some operation \( \rho \) of a compact group \( G \). Theorem 2 is obtained by considering \( G = \{1\} \).

**Theorem 4.** Let \( \rho, M, f, N \) be as in Proposition 3. Let \( r: M \to N \) be a \( \rho \)-equivariant retraction such that there is a \( \rho \)-equivariant homotopy between \( f|_N \circ r \) and \( r \circ f \). Then if \( T \) is a \( \rho \)-invariant isolated invariant set of a map \( f \in \mathcal{F}_\rho \) and \( (T, f') \) is related to \( (N, f) \) by continuation,
\[
(4.1) \quad (r|_T)^*: \mathcal{H}_{\rho}^*(T) \to \mathcal{H}_{\rho}^*(N)
\]
is injective. Here \( H^*_\rho \) denotes the \( \rho \)-equivariant cohomology with coefficients as in Theorem 1.

**Proof.** By Proposition 3, \( N_f \) is a normally hyperbolic submanifold of \( O_f \subset M_f \). It is therefore by Proposition 1 of [3] a \( \ast \)-hyperbolic invariant set of the flow \( \chi_f \) in \( M_f \) (see [3, Definition 4]). We now construct a homotopy retraction from \( M_f \) to \( N_f \). Let us therefore restrict our attention to a continuous family \( f_t, \tau \in \mathbb{R} \), with \( f_0 = f \). Set

\[
M_R = p_R(M \times \mathbb{R} \times (-1, 1))
\]

where \( p_R \) is uniquely defined by

\[
p_R(x, \tau, t) = p_R(f_t(x), \tau, t + 1),
\]

for \( t \in (-1, 0) \). There is a continuous map

\[
H: \mathbb{R} \times M \times [0, 1] \to M: (\tau, x, t) \mapsto H_{\tau}(x, t)
\]

such that \( H_{\tau}(x, 0) = r(f_t(x)) \) and \( H_{\tau}(x, 1) = f(r(x)) \). Define for \( t \in [0, 1] \)

\[
r_{\tau}(p_R(x, \tau, t)) = p_f(H_{\tau}(x, t), t).
\]

This defines a continuous map \( r_{\tau}: M_R \to N_f \), since

\[
r_{\tau}(p_R(x, \tau, 0)) = p_f(H_{\tau}(x, 0), 0) = p_f(r(f_t(x)), 0)
\]

\[
= p_f(f(r(f_t(x))), 1) = p_f(H_{\tau}(f_t(x), 1), 1)
\]

\[
= r_{\tau}(f_t(x), \tau, 1).
\]

The following lemma summarizes the properties of this construction.

**Lemma 4.1.** If \( f: X \to X \) is a homeomorphism, then there exists an exact sequence

\[
H^*(X_f) \to H^*(X) \xrightarrow{f^* - \text{id}} H^*(X) \xrightarrow{\delta} H^{*+1}(X_f).
\]

Moreover, if \( r: X \to Y \) is a map which homotopy commutes with \( f: X \to X \), \( Y \to Y \), and \( r_f: X_f \to Y_f \) is defined by (4.5) restricted to the set \( \{\tau = 0\} \), then the following diagram commutes

\[
\begin{array}{ccc}
H^*(X_f) & \xrightarrow{r^*} & H^*(X) \\
\downarrow r_{\tau}^* & & \downarrow r^* \\
H^*(Y_f) & \xrightarrow{f^* - \text{id}} & H^*(Y) \\
\downarrow r_{\tau}^* & & \downarrow r_{\tau}^* \\
H^*(X_f) & \xrightarrow{r^* - \text{id}} & H^*(X) \\
\downarrow r_{\tau}^* & & \downarrow r_{\tau}^* \\
H^*(X_f) & \xrightarrow{f^* - \text{id}} & H^*(X) \xrightarrow{\delta} H^{*+1}(X_f).
\end{array}
\]

**Proof.** We consider the Mayer Vietoris sequence (see [9]) for

\[
X_f = p_f(X \times [0, \frac{1}{2}]) \cup p_f(X \times [\frac{1}{2}, 1])
\]

which is given by

\[
\delta: H^*(X_f) \to H^*(p_f(X \times [0, \frac{1}{2}])) \oplus H^*(p_f(X \times [\frac{1}{2}, 1])
\]

\[
\xrightarrow{\xi_f + \xi_{\tau}^*} H^*(p_f(X \times \{0\})) \oplus H^*(p_f(X \times \{\frac{1}{2}\})) \xrightarrow{\delta}.
\]
Applying the homotopy equivalence $\pi_X \circ p_f^{-1}$, we obtain the exact sequence

$$H^*(X_f) \to (H^*(X))^2 \xrightarrow{\theta} (H^*(X))^2 \to H^{*+1}(X_f)$$

with $\theta(\alpha, \beta) = (f^*\alpha + \beta, \alpha + \beta)$. Now define the isomorphism $j_1(x, y) = (x - y, y)$ and $j_2(\gamma, \delta) = (\gamma, \delta - \gamma)$ of $(H^*(X))^2$. Since

$$j_1 \circ \theta \circ j_2(\alpha, \beta) = (f^*\alpha - \alpha, \beta),$$

we can eliminate the factor $H^*(X)$ in (4.11) and obtain the sequence (4.7).

If we set up the same sequence for $X$ replaced by $Y$, then the maps $R_f^*: H^*(Y_f) \to H^*(X_f)$ and $f^* r^* \oplus f^* r^*: (H^*(Y))^2 \to (H^*(X))^2$ commute with the exact sequences. Moreover, the latter homomorphism commutes with $j_1$ and $j_2$, as one readily verifies. Hence eliminating one factor $H^*(X)$ as before, we obtain the commuting diagram (4.8).

Setting $X = Y = N$ in (4.8), we conclude by the five lemma (see [9]) that $f_f: M_f \to N_f$ induces isomorphisms in cohomology when restricted to $N_f$. Now the proof of [3, Theorem 2] applies to this situation and we can conclude that under the hypothesis of Theorem 3,

$$\delta|N: H^p(N) \to H^p(T_f)$$

is injective. In fact, although Theorem 2 of [3] formally requires $r_f$ to be a retraction, the proof depends only on the fact that $(r_f|N_f)^*$ is injective.

We now have to pass from $T_f$ to the fibre $T$. Note that in order to prove the injectivity of (4.1), it suffices to prove that

$$(4.14) \quad r^*[N] \neq 0,$$

where $[N] \in H^d_{\rho}(N)$, $d = \dim N$, is the fundamental class. In fact, it then follows from Poincaré duality (see [9]) that for every $x \in H^d_{\rho}(N)$, $x \neq 0$, there exists a class $y \in H^d_{\rho}(N)$ with $x \cup y = [N]$. Then $(r^*x) \cup (r^*y) = r^*[N] \neq 0$ implies that $r^*x \neq 0$.

To prove (4.14), we apply (4.8) of Lemma 4.1 to $X = N$ and $Y = T$. Since $f: N \to N$ is a diffeomorphism, we conclude that for coefficients as in Theorem 1, $f^*[N] = [N]$. Therefore, $f^* - \text{id}$, so that $\delta[N] \neq 0$. Hence $r_f^* \delta[N] \neq 0$ by injectivity of (4.13). Since (4.8) is commutative, we also have $\delta \circ f^* \circ r^*[N] \neq 0$, which proves (4.14). This completes the proof of Theorem 3.

In order to obtain a perturbation result, choose a ($\rho$-invariant) neighborhood $U$ of $N$ in $M$ with a ($\rho$-invariant) retraction $r$. For example, we can use the metric $g$ on $M$ defined in $\S 3$ to define a diffeomorphism of the disc bundle $D_\varepsilon \subset E^+ \oplus E^-$ onto $U := \exp_g(D_\varepsilon)$ by means of

$$\exp_g: D_\varepsilon \to N.$$

We can define a retraction $r$ which corresponds to the bundle projection $D_\varepsilon \to N$ under $\exp_g$. It is homotopic to $f^{-1}|_N \circ r \circ f$, since all retractions
$D_\varepsilon \to N$ are homotopic to the projection as is easily verified. Choosing such a homotopy $H$, we can define $r_f$ as in (4.5). Now we apply Theorem 2 of [3] to the flow on the open set $U_f = p_f(U \times (-1, 1)) \subset M_f$.

Moreover, it is known that if $U_f$ is isolating for flow $\chi_f$, then this is true for all $f_\lambda$ for $\lambda$ small enough as in Theorem 1. For example, this is shown in [1, III.3]. In this case, $U$ is also isolating for $f'$ by Proposition 2. We can now apply Theorem 2 to the manifold $U$ and the retraction $r$ in order to conclude that the maximal invariant set $T = T_{f'}(U)$ satisfies the assertion of Theorem 1. This completes the proof of Theorem 1.

REFERENCES


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