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REALIZATION OF THE LEVEL ONE STANDARD $\tilde{C}_{2k+1}$-MODULES

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ABSTRACT. In this paper we study the level one standard (or irreducible integrable highest weight) modules for the affine symplectic Lie algebras. In particular, we give concrete realizations of all level one standard modules for the affine symplectic Lie algebras of even rank.

INTRODUCTION

In recent years explicit realizations of nontrivial representations of affine Lie algebras have attracted the attention of many researchers because of their surprising connections with different areas of mathematics and physics. Among these realizations the 'principal realizations' and the 'untwisted (homogeneous) realizations' have proved to be especially useful. In 1978 Lepowsky and Wilson [15] constructed the basic modules—level one standard modules (i.e. irreducible integrable highest weight modules)—for $A_1^{(1)}$ in the principal realization. In [10] this construction was generalized to the level one standard modules for all simply-laced affine Lie algebras. Frenkel and Kac [5] and Segal [26] have given explicit construction of these modules in the untwisted realization. The constructions of these modules in the general realization are given in [9, 12]. Since then some higher level standard modules for certain simply-laced affine Lie algebras have also been constructed (e.g. [14, 17, 18, 19, 21, 22, 25]). However, less is known regarding explicit constructions of standard modules for non-simply-laced affine Lie algebras—that is, affine Lie algebras of type 'B', 'C', 'F' and 'G'. In the last section of [10] Kac, Kazhdan, Lepowsky and Wilson constructed reducible level one modules for the non-simply-laced affine Lie algebras in the principal realization by taking fixed points of Dynkin-diagram-induced automorphisms. Recently, in the untwisted realization the analogous representations (again reducible) have been constructed in [3 and 6].The level one standard modules for the affine orthogonal Lie algebras $B_n^{(1)}$ have been explicitly realized in [4, 8, 13 and 20] from different view points. In [20] Mandia has constructed the level one standard modules for the affine Lie algebras $F_4^{(1)}$ and $G_2^{(1)}$. In [23] the author gave explicit constructions of the level one standard modules for the affine symplectic Lie algebras of rank 3 and 4.

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In this paper we give explicit realization of the level one standard modules for all affine symplectic Lie algebras of even rank in the principal realization. Following the patterns of many of the papers mentioned above, we first start from the fact that the structure of a standard module L is determined by the structure of the vacuum space $\Omega(L)$ for the principal Heisenberg subalgebra $\underline{s}$ in L (see §1). To study $\Omega(L)$ we make use of the Z-operators introduced by Lepowsky and Wilson [16, 18]. We recall necessary facts about these operators in §1. In §2 we prove several generating function identities for these operators. In §3 we use these identities to determine (Theorem 3.15) a suitable spanning set for $\Omega(L)$. We then show (Theorem 3.16) using the generalized Rogers-Ramanujan identities, that our spanning set is in fact a basis for $\Omega(L)$, when L is a level one standard module for any affine symplectic Lie algebra of even rank.

1. Preliminaries

In this section we will recall some notations and facts from [23]. There will be minor changes in the notations which will be self-explanatory. For more details see [23].

Consider the (complex) simple Lie algebra $\underline{sl}(2n, \mathbb{C})$, $n \geq 2$. The associated affine Lie algebra $\underline{sl}(2n, \mathbb{C})^\wedge$ has a basis (see [10, 25]),

$$(1.1) \quad \{c, B(j), \underline{X}(m, i) \mid i, j \in \mathbb{Z},\ j \not\equiv 0 \bmod(2n),\ m = 1, \dots, 2n-1\},$$

where c is a central element (suitably normalized), $B(j) = E^j \otimes t^j$ and $\underline{X}(m, i) = D^m E^i \otimes t^i$. Here t is an indeterminate, $D = \operatorname{diag}(\omega, \dots, \omega^{2n-1}, 1)$, where ω is a primitive $(2n)$th root of unit and

$$E = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \underline{sl}(2n, \mathbb{C}).$$

Consider the order two automorphism

$$(1.2) \qquad \theta : x \mapsto -yx^t y^{-1}$$

of $\underline{sl}(2n, \mathbb{C})$, where $y = ((-1)^{i+1}\delta_{i, 2n-j+1})_{i,j=1}^{2n}$ and x^t denotes the transpose of x. Observe that θ is induced by the usual order two graph automorphism of the Dynkin diagram of $\underline{sl}(2n, \mathbb{C})$. Extend θ to an automorphism of $\underline{sl}(2n, \mathbb{C})^\wedge$, again denoted by θ, by defining

$$(1.3) \qquad \begin{cases} \theta(x \otimes t^i) = \theta(x) \otimes t^i, & \text{for all } x \in \underline{sl}(2n, \mathbb{C}), \text{ and} \\ \theta(c) = c. \end{cases}$$

Observe that

$$(1.4) \qquad \theta(B(j)) = \begin{cases} B(j), & \text{for all } j \text{ odd}, \\ -B(j), & \text{for all even } j \not\equiv 0 \pmod{2n}, \end{cases}$$

and

$$\theta(\underline{X}(m, i)) = (-1)^{i+1}\omega^{m(1-i)}\underline{X}(2n-m, i) \tag{1.5}$$

for all $i \in \mathbb{Z}$ and $m = 1, 2, \dots, 2n-1$. Then it is known that (see [23, Proposition 2.1]) the set

$$\{c, B(j), X(m, i) \mid i, j \in \mathbb{Z},\ j \text{ odd},\ m = 1, 2, \dots, n\} \tag{1.6}$$

where

$$X(m, i) = \tfrac{1}{2}\{\underline{X}(m, i) + \theta\underline{X}(m, i)\} \tag{1.7}$$

forms a basis of the affine symplectic Lie algebra $C_n^{(1)} = \underline{sp}(2n, \mathbb{C})^\wedge$. Denote

$$\underline{s}^{\pm} = \coprod_{\substack{j>0 \\ j \text{ odd}}} \mathbb{C}B(\pm j) \text{ and } \underline{s} = \underline{s}^- \oplus \mathbb{C}c \oplus \underline{s}^+. \tag{1.8}$$

Then $\underline{s}$ is a Heisenberg subalgebra of $\underline{sp}(2n, \mathbb{C})^\wedge$, called the *principal Heisenberg subalgebra.*

Let $\{E_i, H_i, F_i \mid 0 \le i \le 2n-1\}$ be the canonical set of generators of the affine Lie algebra $\underline{sl}(2n, \mathbb{C})^\wedge$. Observe that $c = \sum_{i=0}^{2n-1} H_i$. Set

$$\begin{aligned} &e_0 = E_0, && h_0 = H_0, && f_0 = F_0, \\ &e_n = E_n, && h_n = H_n, && f_n = F_n, \\ &e_i = E_i + E_{2n-i}, && h_i = H_i + H_{2n-i}, && f_i = F_i + F_{2n-i}, \end{aligned} \tag{1.9}$$

for $i = 1, 2, \dots, n-1$. Then $\{e_i, h_i, f_i \mid 0 \le i \le n\}$ forms a set of canonical generators (see [10, 23]) for the affine symplectic Lie algebra $C_n^{(1)} = \underline{sp}(2n, \mathbb{C})^\wedge$.

Let d be the derivation of $\underline{sl}(2n, \mathbb{C})^\wedge$ given by

$$d(E_i) = E_i, \quad d(F_i) = -F_i \quad \text{and} \quad d(H_i) = 0, \tag{1.10}$$

for $i = 0, 1, \dots, 2n-1$. Observe that

$$\begin{gathered} d(B(j)) = jB(j), \quad d(X(m, i)) = iX(m, i), \\ d(\underline{X}(m, i)) = i\underline{X}(m, i) \quad \text{and} \quad d(c) = 0. \end{gathered} \tag{1.11}$$

Note that d restricted to $\underline{sp}(2n, \mathbb{C})^\wedge$, again denoted by d, is a derivation of $\underline{sp}(2n, \mathbb{C})^\wedge$ with

$$d(e_i) = e_i, \quad d(f_i) = -f_i \quad \text{and} \quad d(h_i) = 0, \tag{1.12}$$

for $i = 0, 1, \dots, n$. Form the semidirect product Lie algebras

$$\underline{sl}(2n, \mathbb{C})^\sim = \underline{sl}(2n, \mathbb{C})^\wedge \oplus \mathbb{C}d \tag{1.13}$$

and

$$\tilde{\mathfrak{g}} = \underline{sp}(2n, \mathbb{C})^\sim = \underline{sp}(2n, \mathbb{C})^\wedge \oplus \mathbb{C}d. \tag{1.14}$$

Denote

$$\tilde{\mathfrak{g}}_i = \{x \in \tilde{\mathfrak{g}} \mid [d, x] = ix\}, \qquad i \in \mathbb{Z}.$$

Then

$$\tilde{\mathfrak{g}} = \coprod_{i \in \mathbb{Z}} \tilde{\mathfrak{g}}_i \tag{1.15}$$

which gives a $\mathbb{Z}$-gradation (called the principal gradation) for $\tilde{\mathfrak{g}}$, in the sense that $[\tilde{\mathfrak{g}}_i, \tilde{\mathfrak{g}}_j] \subseteq \tilde{\mathfrak{g}}_{i+j}$. This induces naturally a $\mathbb{Z}$-gradation in the universal enveloping algebra $\mathscr{U}(\tilde{\mathfrak{g}})$,

$$\mathscr{U}(\tilde{\mathfrak{g}}) = \coprod_{i \in \mathbb{Z}} \mathscr{U}(\tilde{\mathfrak{g}})_i. \tag{1.16}$$

For $\mathfrak{h} = \operatorname{span}\{h_1, h_2, \dots, h_n\}$, set

$$\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d \subseteq \tilde{\mathfrak{g}}. \tag{1.17}$$

Then $\tilde{\mathfrak{h}}$ is an abelian subalgebra of $\tilde{\mathfrak{g}}$ which is spanned by $\{h_0, h_1, \dots, h_n, d\}$.

Let $\lambda \in \tilde{\mathfrak{h}}^*$. A $\tilde{\mathfrak{g}}$-module V generated by a vector $v_\lambda \neq 0$ such that $e_i \cdot v_\lambda = 0$ for $i = 0, 1, \dots, n$ and $h \cdot v_\lambda = \lambda(h) v_\lambda$ for all $h \in \tilde{\mathfrak{h}}$, is called a *highest weight module* with *highest weight* λ. Such a vector v_λ is called a *highest weight vector* and is unique up to a scalar multiple. The scalar $\lambda(c)$ is said to be the *level* of V. A highest weight $\tilde{\mathfrak{g}}$-module with highest weight λ and corresponding highest weight vector v_λ is called a *standard* (or *integrable highest weight*) $\tilde{\mathfrak{g}}$-module if there is an integer $r \geq 1$ such that $f_i^r \cdot v_\lambda = 0$, $0 \leq i \leq n$, which in turn implies that λ is *dominant integral*, that is, $\lambda(h_i) \in \mathbb{N}$, for $0 \leq i \leq n$ (see [7, 11]). For each dominant integral $\lambda \in \tilde{\mathfrak{h}}^*$, there is a unique (up to isomorphism) standard $\tilde{\mathfrak{g}}$-module $L(\lambda)$ and it is irreducible (see [7, 11]). For convenience we will restrict our attention to $L(\lambda)$ when $\lambda(d) = 0$, so that $\lambda \in \operatorname{span}\{h_i^* \mid 0 \leq i \leq n\} \subseteq \tilde{\mathfrak{h}}^*$ where $h_i^*(h_j) = \delta_{i,j}$ and $h_i^*(d) = 0$, $0 \leq i, j \leq n$.

A standard $C_n^{(1)}$-module by definition is the restriction to $C_n^{(1)} = \underline{sp}(2n, \mathbb{C})^\frown$ of a standard $\underline{sp}(2n, \mathbb{C})^\sim$-module. The standard modules for $\underline{sl}(2n, \mathbb{C})^\frown$ or $\underline{sl}(2n, \mathbb{C})^\sim$ are defined in an analogous way.

For the standard $\tilde{\mathfrak{g}}$-module $L = L(\lambda)$ (with $\lambda(d) = 0$) denote by $L_i \subseteq L(\lambda)$ the eigenspace of d with eigenvalue $i \in \mathbb{Z}$. Then

$$L = L(\lambda) = \coprod_{i \leq 0} L_i \tag{1.18}$$

with $L_0 = \mathbb{C}v_\lambda$, $L_i = \mathscr{U}(\tilde{\mathfrak{g}})_i v_\lambda$ and $\dim L_i < \infty$. Note that for $i \leq 0$, $v \in L_i$, we have $d \cdot v = iv$. Hence we call (see [18]) the set of elements in L_i to be the set of *homogeneous elements of degree* i. In particular, we say that $T \in \operatorname{End}(L(\lambda))$ is *homogeneous of degree* $z \in \mathbb{C}$ if $[d, T] = zT$. Observe that $T \in \operatorname{End}(L(\lambda))$ has degree $z \in \mathbb{C}$ if and only if $TL_i \subseteq L_{i+z}$ for all $i \in \mathbb{Z}$. Define the *principal character* $\chi(L(\lambda))$ of $L(\lambda)$ by

$$\chi(L(\lambda)) = \sum_{i \geq 0} (\dim L_{-i}) q^i \tag{1.19}$$

where q is an indeterminate. Then $\chi(L(\lambda))$ has a known product expansion (for example, see [22, Formula 1.1]).

In this paper we will focus our attention to the level one standard (or integrable highest weight) $\tilde{\mathfrak{g}}$-modules $L(\lambda)$ with $\lambda(d) = 0$. To be more precise, we will study the standard $\tilde{\mathfrak{g}}$-modules $L(h_i^*)$, $i = 0, 1, \dots, n$. Let us write $t = [n/2] + 1$, where $[\cdot]$ denotes the greatest integer. From here on $L(\lambda)$ will always denote a level one standard $\tilde{\mathfrak{g}}$-module with $\lambda(d) = 0$. By direct computation (cf. [24]) it can be easily shown that for $i = 0, 1, \dots, t-1$, and $i \neq n/2$, we have

$$\chi(L(h_i^*)) = \chi(L(h_{n-i}^*)) = F \prod_{\substack{k>0 \\ k \not\equiv 0, \pm(i+1) \bmod (n+2)}} (1 - q^{2k})^{-1} \tag{1.20}$$

and for $i = n/2$ (i.e. n even), we have

$$\chi(L(h_i^*)) = F \prod_{\substack{k>0 \\ k \not\equiv 0, (i+1) \bmod (n+2)}} (1 - q^{2k})^{-1} \cdot \prod_{\substack{k>0 \\ k \equiv (i+1) \bmod (n+2)}} (1 - q^{2k}), \tag{1.21}$$

where

$$F = \prod_{k>0} (1 - q^{2k-1})^{-1}. \tag{1.22}$$

It is important to observe that $\chi(L(h_i^*))$ differs from the product side of the generalized Rogers-Ramanujan identities (with 'q' replaced by 'q^2') due to Gordon, Andrews and Bressoud (see [1, 2]) by a simple factor.

Recall the subalgebra $\underline{s}^+$ (see (1.8)). Let $\underline{p}$ denote the subalgebra $\underline{s}^+ \oplus \mathbb{C}c$. For the standard $\tilde{\mathfrak{g}}$-module $L = L(\lambda)$ denote by $\Omega(L) = \Omega(L(\lambda))$ (or by Ω if there is no confusion) its vacuum space with respect to the Heisenberg subalgebra $\underline{s}$,

$$\Omega(L(\lambda)) = \{v \in L(\lambda) \mid \underline{s}^+ \cdot v = 0\}, \tag{1.23}$$

which is graded. Then the map (cf. [16, 18]),

$$\begin{aligned} f\colon U(\underline{s}) \otimes_{\mathscr{U}(\underline{p})} \Omega(L(\lambda)) &\to L(\lambda) \\ u \otimes w &\mapsto u \cdot w \end{aligned} \tag{1.24}$$

for all $u \in \mathscr{U}(\underline{s})$, $w \in \Omega(L(\lambda))$, is an $\underline{s}$-module isomorphism. In particular

$$\chi(L(\lambda)) = F \cdot \chi(\Omega(L(\lambda))) \tag{1.25}$$

where

$$F = \chi(\mathscr{U}(\underline{s}^-)) = \prod_{k>0} (1 - q^{2k-1})^{-1}.$$

Hence, in order to give concrete realizations of the level one standard $\tilde{\mathfrak{g}}$-modules $L(h_i^*)$, $0 \le i \le n$, it is enough to give explicit constructions of the corresponding vacuum spaces $\Omega(L(h_i^*))$, $0 \le i \le n$. This is exactly what we will do in this paper. It follows from (1.20), (1.21) and (1.25) that for $i = 0, 1, \dots, n$, $i \neq n/2$. We have

$$\chi(\Omega(L(h_i^*))) = \chi(\Omega(L(h_{n-i}^*))) = \prod_{\substack{k>0 \\ k \not\equiv 0, \pm(i+1) \bmod (n+2)}} (1 - q^{2k})^{-1} \tag{1.26}$$

and for $i = n/2$ (i.e. n even), we have

$$(1.27)\quad \chi(\Omega(L(h_i^*))) = \prod_{\substack{k>0 \\ k\not\equiv 0,\,(i+1)\bmod(n+2)}} (1-q^{2k})^{-1} \cdot \prod_{\substack{k>0 \\ k\equiv (i+1)\bmod(n+2)}} (1-q^{2k}).$$

For a formal indeterminate ζ, denote by $\mathrm{End}(L(\lambda))\{\zeta\}$ the $\mathbb{C}$-vector space of formal Laurent series in ζ with coefficients in $\mathrm{End}(L(\lambda))$. In $\mathrm{End}(L(\lambda))\{\zeta\}$ define, for $k \in \mathbb{Z}$,

$$(1.28)\qquad \underline{E}^{\pm}(k, \zeta) = \exp\left(\pm \sum_{\substack{j>0 \\ j\not\equiv 0 \bmod(2n)}} (\omega^{\mp kj} - 1)B(\pm j)\zeta^{\pm j}/j\right),$$

$$(1.29)\qquad E^{\pm}(k, \zeta) = \exp\left(\pm \sum_{\substack{j>0 \\ j \text{ odd}}} (\omega^{\mp kj} - 1)B(\pm j)\zeta^{\pm j}/j\right),$$

$$(1.30)\qquad \underline{X}(m, \zeta) = \sum_{i\in\mathbb{Z}} \underline{X}(m, i)\zeta^i, \qquad m = 1, 2, \dots, 2n-1,$$

$$(1.31)\qquad X(m, \zeta) = \sum_{i\in\mathbb{Z}} X(m, i)\zeta^i, \qquad m = 1, 2, \dots, n,$$

$$(1.32)\qquad \delta(\zeta) = \sum_{i\in\mathbb{Z}} \zeta^i \quad \text{and} \quad D\delta(\zeta) = \sum_{i\in\mathbb{Z}} i\zeta^i,$$

where 'exp' means the formal exponential series. Observe that

$$(1.33)\qquad \underline{E}^{\pm}(k, \zeta) = E^{\pm}(k, \zeta)\exp(\pm P^{\pm}(k, \zeta))$$

where

$$(1.34)\qquad P^{\pm}(k, \zeta) = \sum_{\substack{j>0 \\ j \text{ even},\, j\not\equiv 0 \bmod(2n)}} (\omega^{\mp kj} - 1)B(\pm j)\zeta^{\pm j}/j.$$

Now define the elements

$$(1.35)\qquad Z(m, \zeta) = E^{-}(m, \zeta)X(m, \zeta)E^{+}(m, \zeta)$$

for $m = 1, 2, \dots, n$, in $\mathrm{End}(L(\lambda))\{\zeta\}$ (with $\lambda(c) = 1$). These elements are well-defined since the grading of $L(\lambda)$ is truncated from above. Let

$$(1.36)\qquad Z(m, \zeta) = \sum_{i\in\mathbb{Z}} Z(m, i)\zeta^i$$

for $m = 1, 2, \dots, n$, where $Z(m, i) \in \mathrm{End}(L(\lambda))$ is the homogeneous component of degree i of $Z(m, \zeta)$. Denote by $\underline{Z} = \underline{Z}(L(\lambda))$ the subalgebra (see

[16, 18]) of $\mathrm{End}(L(\lambda))$ generated by

$$\{Z(m, i) \mid i \in \mathbb{Z},\ m = 1, 2, \dots, n\}. \tag{1.37}$$

Then (see [16, 18]) the algebra $\underline{Z}$ centralizes the action of the Heisenberg subalgebra $\underline{s}$ on $L(\lambda)$. In particular, the algebra $\underline{Z}$ preserves the vacuum space $\Omega = \Omega(L(\lambda))$ of $L(\lambda)$ with respect to $\underline{s}$. Furthermore, (see [16, 18]) we have

$$\Omega = \underline{Z} \cdot v_\lambda = \mathrm{span}_{\mathbb{C}}\{Z(m_1, i_1) \cdots Z(m_k, i_k) \cdot v_\lambda\} \tag{1.38}$$

where $i_j \in \mathbb{Z}$ and $1 \le m_j \le n$, for each j. For two commuting indeterminates ζ_1 and ζ_2, and $m, l = 1, 2, \dots, n$, denote

$$F(m, l; \zeta_1, \zeta_2) = \left\{ \frac{\left(1 - \frac{\zeta_1}{\zeta_2}\right)\left(1 + \frac{\omega^l \zeta_1}{\zeta_2}\right)\left(1 + \frac{\zeta_1}{\omega^m \zeta_2}\right)\left(1 - \frac{\omega^l \zeta_1}{\omega^m \zeta_2}\right)}{\left(1 + \frac{\zeta_1}{\zeta_2}\right)\left(1 - \frac{\omega^l \zeta_1}{\zeta_2}\right)\left(1 - \frac{\zeta_1}{\omega^m \zeta_2}\right)\left(1 + \frac{\omega^l \zeta_1}{\omega^m \zeta_2}\right)} \right\}^{\frac{1}{2}}. \tag{1.39}$$

Theorem 1.1 [23, Theorem 2.4]. *Let $L(\lambda)$ be a level one standard $\tilde{\mathfrak{g}}$-module with $\lambda(d) = 0$, and ζ_1, ζ_2 be two commuting indeterminates. Then on $L(\lambda)$, for $m, l = 1, 2, \dots, n$, we have*

$$\begin{aligned} &F(m, l; \zeta_1, \zeta_2) Z(m, \zeta_1) Z(l, \zeta_2) - F(l, m; \zeta_2, \zeta_1) Z(l, \zeta_2) Z(m, \zeta_1) \\ &= \begin{cases} -\frac{1}{2}\omega^m D\delta(-\zeta_1/\zeta_2) + \frac{1}{2}\delta(\omega^m \zeta_1/\zeta_2) Z(2m, \zeta_2) \\ \quad -\frac{1}{2}\delta(\omega^{-m}\zeta_1/\zeta_2) Z(2m, \omega^m \zeta_2), \quad \text{if } m = l \text{ and } m + l \le n; \\ \frac{1}{2}\delta(\omega^l \zeta_1/\zeta_2) Z(m + l, \zeta_2) - \frac{1}{2}\delta(\omega^{-m}\zeta_1/\zeta_2) Z(m + l, \omega^m \zeta_2) \\ \quad -\frac{1}{2}\omega^m \delta(-\omega^{l-m}\zeta_1/\zeta_2) Z(l - m, \zeta_2) \\ \quad +\frac{1}{2}\omega^m \delta(-\zeta_1/\zeta_2) Z(l - m, \omega^{-m}\zeta_2), \\ \qquad\qquad \text{if } m \ne l,\ m < l \text{ and } m + l \le n \text{ and} \\ D\delta(-\zeta_1/\zeta_2), \quad \text{if } m = l = n. \quad \square \end{cases} \end{aligned} \tag{1.40}$$

Now define the numbers $1 = a_0, a_1, a_2, \dots$ by the expansion

$$F(m, l;\ \omega^{-l}\zeta_1, \zeta_2) = \sum_{j \ge 0} a_j (\zeta_1/\zeta_2)^j \tag{1.41}$$

where $F(m, l; \omega^{-l}\zeta_1, \zeta_2)$ is given in (1.39). Using (1.32), (1.36), (1.41) and equating coefficients of all the monomials $\zeta_1^r \zeta_2^s$ $(r, s \in \mathbb{Z})$ we have the following corollary of Theorem 1.1.

Corollary 1.2 [23, Theorem 2.5]. *On the level one standard $\tilde{\mathfrak{g}}$-module $L(\lambda)$ (with*

$\lambda(d) = 0$), *for all* $r, s \in \mathbb{Z}$ *and* $m, l = 1, 2, \dots, n$, *we have*

$$(1.42)\quad \sum_{j\geq 0} a_j(\omega^{lj} Z(m, r-j)Z(l, s+j) - \omega^{mj} Z(l, s-j)Z(m, r+j))$$
$$= \begin{cases} -\frac{1}{2}\omega^m r(-1)^r \delta_{r+s,0} + \frac{1}{2}(\omega^{mr} - \omega^{ms})Z(2m, r+s), \\ \qquad\qquad \text{if } m = l \text{ and } m+l \leq n; \\ \frac{1}{2}(\omega^{lr} - \omega^{ms})Z(m+l, r+s) \\ +\frac{1}{2}\omega^m(\omega^{-ms} - \omega^{lr})\omega^{-mr}(-1)^r Z(l-m, r+s), \\ \qquad\qquad \text{if } m \neq l,\ m < l, \text{ and } m+l \leq n; \\ r(-1)^r \delta_{r+s,0}, \quad \text{if } m = l = n; \end{cases}$$

where $a_0, a_1, a_2, \dots$ *are defined by* (1.41). □

Observe that for each $r, s \in \mathbb{Z}$, the sum in (1.42) is locally finite on $L(\lambda)$ since the grading of $L(\lambda)$ is truncated from above. Hence locally, if necessary, we can compose the identities in (1.42). Now it is clear from Corollary 1.2 that for any $v \in L(\lambda)$ $Z(m, i)\cdot v$, $m = 2, 3, \dots, n$, $i \in \mathbb{Z}$, can be expressed as a linear combination of vectors of the form $Z(1, i_1)\cdots Z(1, i_k)\cdot v$, where $i_j \in \mathbb{Z}$. Hence it follows from (1.38) that

$$(1.43)\qquad \Omega = \operatorname{span}_{\mathbb{C}}\{Z(1, i_1)\cdots Z(1, i_k)\cdot v_\lambda\}$$

where $i_j \in \mathbb{Z}$ for each j.

From here on, for convenience, we will denote the operators, $Z(1, \zeta)$, $\underline{X}(1, \zeta)$, $X(1, \zeta)$, $Z(1, i)$, $\underline{X}(1, i)$, $X(1, i)$ and $P^{\pm}(1, \zeta)$, by $Z(\zeta)$, $\underline{X}(\zeta)$, $X(\zeta)$, $Z(i)$, $\underline{X}(i)$, $X(i)$ and $P^{\pm}(\zeta)$ respectively.

2. Generating function identities

Let $\underline{L}(H_i^*)$ denote the standard $\underline{sl}(2n, \mathbb{C})^{\sim}$-module with highest weight H_i^* (where $H_i^*(H_j) = \delta_{ij}$, $H_i^*(d) = 0$, $0 \leq i$, $j \leq 2n-1$), $i = 0, 1, \dots, n$, and highest weight vector v_0. Since $\tilde{\mathfrak{g}} = \underline{sp}(2n, \mathbb{C})^{\sim} \subset \underline{sl}(2n, \mathbb{C})^{\sim}$, by restriction $\underline{L}(H_i^*)$ is a $\tilde{\mathfrak{g}}$-module. Let V denote the $\tilde{\mathfrak{g}}$-module generated by v_0. Note that $H_i^*(h_j) = \delta_{i,j}$, $0 \leq i$, $j \leq n$, (see (1.9)). Clearly V is a highest weight $\tilde{\mathfrak{g}}$-module with highest weight h_i^* and highest weight vector v_0. Furthermore, since $F_i^m \cdot v_0 = 0$, for some integer m, and since $[F_i, F_{2n-i}] = 0$, for $i = 1, 2, \dots, n-1$, it follows from (1.9) that $f_i^r \cdot v_0 = 0$, $i = 0, 1, \dots, n$, for some integer r. Hence V is a standard $\tilde{\mathfrak{g}}$-module with highest weight $\lambda = h_i^*$ and highest weight vector $v_\lambda = v_0$. Since standard $\tilde{\mathfrak{g}}$-modules are unique (up to isomorphism) from here on we can and do assume that the standard $\tilde{\mathfrak{g}}$-module $L(\lambda)$ with highest weight $\lambda = h_i^*$, $i = 0, 1, \dots, n$, is contained in the standard $\underline{sl}(2n, \mathbb{C})^{\sim}$-module $\underline{L}(H_i^*)$, $i = 0, 1, \dots, n$, respectively.

It is known (see [10, 22]) that for $w \in \underline{L}(H_i^*)$, $i = 0, 1, \dots, 2n-1$,

$$(2.1)\qquad \underline{X}(m, \zeta)\cdot w = c_i^{(m)} \underline{E}^-(-m, \omega^{-m}\zeta)\underline{E}^+(-m, \omega^{-m}\zeta)\cdot w$$

where

$$c_i^{(m)} = \omega^{(i+1)m}/(\omega^m - 1) \tag{2.2}$$

for $m = 1, 2, \dots, 2n-1$. Therefore, for $v \in L(h_i^*) \subset \underline{L}(H_i^*)$, $i = 0, 1, \dots, n$, (see [23, Equation 2.16], (1.7), (1.33) and (1.34)) we have

$$\begin{aligned} X(\zeta)\cdot v = X(1,\zeta)\cdot v = \tfrac{1}{2}[X(\zeta)+\theta X(\zeta)]\cdot v = aE^-(-1,\omega^{-m}\zeta) \\ \times[\omega^i \exp(P^-(\zeta))\exp(-P^+(\zeta)) \\ + \omega^{-i}\exp(-P^-(\zeta))\exp(P^+(\zeta))]E^+(-1,\omega^{-m}\zeta)\cdot v \end{aligned} \tag{2.3}$$

where $a = \frac{1}{2}\omega/(\omega-1)$. Hence it follows from (1.35) that (see [23, Equation 2.17]) we have

(2.4)

$$Z(\zeta)\cdot v = a[\omega^i \exp(P^-(\zeta))\exp(-P^+(\zeta)) + \omega^{-i}\exp(-P^-(\zeta))\exp(P^+(\zeta))]\cdot v.$$

Now looking at the homogeneous components it follows from (1.34), (1.36) and (2.4) that on $L(h_i^*)$, $i = 0, 1, \dots, n$, we have

$$Z(j) = 0, \quad \text{for all } j \text{ odd}. \tag{2.5}$$

Hence it follows from (1.43) that

$$\Omega(L(\lambda)) = \operatorname{span}_{\mathbb{C}}\{Z(2i_1)\cdots Z(2i_k)\cdot v_\lambda \mid k \geq 0,\ i_j \in \mathbb{Z}\}. \tag{2.6}$$

Observe that for two commuting indeterminates ζ_1 and ζ_2 we have on $L(h_i^*)$, $i = 0, 1, \dots, n$, (see [23, Equation 2.20])

$$[P^+(\zeta_1), P^-(\zeta_2)] = \frac{1}{2}\log\left[\frac{(1-\omega^2\zeta_1^2/\zeta_2^2)(1-\omega^{-2}\zeta_1^2/\zeta_2^2)}{(1-\zeta_1^2/\zeta_2^2)^2}\right] \tag{2.7}$$

where 'log' denotes the formal logarithmic series. Hence it follows from (2.7) and the Campbell-Baker-Hausdorff formula that on $L(h_i^*)$, $i = 0, 1, \dots, n$, we have

$$\begin{aligned} \exp(\pm P^+(\zeta_1))\exp(\pm P^-(\zeta_2)) = \exp(\pm P^-(\zeta_2))\exp(\pm P^+(\zeta_1)) \\ \cdot(1-\omega^2\zeta_1^2/\zeta_2^2)^{\frac{1}{2}}(1-\omega^{-2}\zeta_1^2/\zeta_2^2)^{\frac{1}{2}}(1-\zeta_1^2/\zeta_2^2)^{-1} \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \exp(\pm P^+(\zeta_1))\exp(\mp P^-(\zeta_2)) = \exp(\mp P^-(\zeta_2))\exp(\pm P^+(\zeta_1)) \\ \cdot(1-\omega^2\zeta_1^2/\zeta_2^2)^{-\frac{1}{2}}(1-\omega^{-2}\zeta_1^2/\zeta_2^2)^{-\frac{1}{2}}(1-\zeta_1^2/\zeta_2^2). \end{aligned} \tag{2.9}$$

Now for two commuting indeterminates ζ_1 and ζ_2 let $F(\zeta_1, \zeta_2)$ denote the formal series $F(1, 1;\ \zeta_1, \zeta_2)$, (see (1.39)) so that

$$F(\zeta_1, \zeta_2) = \frac{(1-\zeta_1/\zeta_2)(1+\omega\zeta_1/\zeta_2)^{\frac{1}{2}}(1+\omega^{-1}\zeta_1/\zeta_2)^{\frac{1}{2}}}{(1+\zeta_1/\zeta_2)(1-\omega\zeta_1/\zeta_2)^{\frac{1}{2}}(1-\omega^{-1}\zeta_1/\zeta_2)^{\frac{1}{2}}}. \tag{2.10}$$

For a commuting set of indeterminates $\zeta_1, \zeta_2, \dots, \zeta_p$, $p \geq 2$, we define

$$(2.11) \qquad \circ Z(\zeta_1)\cdots Z(\zeta_p)\circ = \prod_{1\leq j<k\leq p} F(\zeta_j, \zeta_k) Z(\zeta_1)\cdots Z(\zeta_p),$$

$$(2.12) \qquad P(\zeta_1, \dots, \zeta_p) = \prod_{1\leq j<k\leq p} (1+\zeta_j/\zeta_k)^2(1-\omega\zeta_j/\zeta_k)(1-\omega^{-1}\zeta_j/\zeta_k)$$

with $P(\zeta_{\sigma(1)}, \dots, \zeta_{\sigma(p)}) = P(\zeta_1, \dots, \zeta_p)$ for $\sigma \in S_p$ and define

(2.13)

$$Z(\zeta_1, \dots, \zeta_p) = P(\zeta_1, \dots, \zeta_p)\circ Z(\zeta_1)\cdots Z(\zeta_p)\circ$$

$$= \prod_{1\leq j<k\leq p} (1-\zeta_j^2/\zeta_k^2)(1-\omega^2\zeta_j^2/\zeta_k^2)^{\frac{1}{2}}(1-\omega^{-2}\zeta_j^2/\zeta_k^2)^{\frac{1}{2}} \cdot Z(\zeta_1)\cdots Z(\zeta_p).$$

For any two commuting indeterminates ζ_1 and ζ_2 we define the generalized bracket

$$(2.14) \qquad [\![Z(\zeta_1), Z(\zeta_2)]\!] = \circ Z(\zeta_1)Z(\zeta_2)\circ - \circ Z(\zeta_2)Z(\zeta_1)\circ .$$

Then it follows from Theorem 1.1 that

$$(2.15) \qquad \begin{aligned} [\![Z(\zeta_1), Z(\zeta_2)]\!] = &-\tfrac{1}{2}\omega D\delta(-\zeta_1/\zeta_2) + \tfrac{1}{2}Z(2, \zeta_2)\delta(\omega\zeta_1/\zeta_2) \\ &- \tfrac{1}{2}Z(2, \omega\zeta_2)\delta(\omega^{-1}\zeta_1/\zeta_2). \end{aligned}$$

Theorem 2.1. *Let $\zeta_1, \zeta_2, \dots, \zeta_p$ be a commuting set of indeterminates. Then for every permutation $\sigma \in S_p$, we have*

$$(2.16) \qquad Z(\zeta_{\sigma(1)}, \dots, \zeta_{\sigma(p)}) = Z(\zeta_1, \dots, \zeta_p).$$

Proof. First observe that (see [22, Lemma 2.7]) we have

$$\begin{aligned} &(1+\zeta_1/\zeta_2)^2 D\delta(-\zeta_1/\zeta_2) = 0, \\ &(1-\omega\zeta_1/\zeta_2)\delta(\omega\zeta_1/\zeta_2) = 0, \text{ and} \\ &(1-\omega^{-1}\zeta_1/\zeta_2)\delta(\omega^{-1}\zeta_1/\zeta_2) = 0. \end{aligned}$$

Hence for $r = 1, 2, \dots, p-1$, Equation 2.15 implies

$$\begin{aligned} P(\zeta_1, \dots, \zeta_p) &\prod_{\substack{1\leq j<k\leq p \\ (j,k)\neq(r,r+1)}} F(\zeta_j, \zeta_k) \\ &\cdot Z(\zeta_1)\cdots Z(\zeta_{r-1})[\![Z(\zeta_r), Z(\zeta_{r+1})]\!]Z(\zeta_{r+2})\cdots Z(\zeta_p) = 0, \end{aligned}$$

and the result follows. □

Let $A(\zeta)$ and $C(\xi)$ be the following Laurent series in commuting indeterminates ζ and ξ with coefficients in $\mathrm{End}(L(h_i^*))$, $i = 0, 1, \dots, n$:

$$(2.17) \qquad A(\zeta) = \sum_{j\in\mathbb{Z}} A(j)\zeta^j, \quad C(\xi) = \sum_{j\in\mathbb{Z}} C(j)\xi^j,$$

with $[d, A(j)] = jA(j)$ and $[d, C(j)] = jC(j)$ for $j \in \mathbb{Z}$. Then

$$A(\zeta)C(\xi) = \sum_{j,k\in\mathbb{Z}} A(j)C(k)\zeta^j\xi^k \tag{2.18}$$

is a well-defined Laurent series in two indeterminates ζ and ξ, with coefficients in End $L(h_i^*)$. However, if we set $\zeta = \xi$ in (2.18), the product

$$A(\zeta)C(\zeta) = \sum_{k\in\mathbb{Z}} \Big(\sum_{i+j=k} A(i)C(j) \Big) \zeta^k \tag{2.19}$$

is not defined in general (see [21]). Whenever the product (2.19) is defined, we write,

$$A(\zeta)C(\zeta) = \lim_{\xi\to\zeta} A(\zeta)C(\xi). \tag{2.20}$$

Now for products of type (2.18) involving more than two commuting indeterminates extend this definition of limit inductively (see [21]).

The next corollary follows from Theorem 2.1 and definition of limit by an argument similar to Corollary 5.8 in [21].

Corollary 2.2. *For $p \geq 2$ the limit*

$$\lim_{\zeta_l\to\omega^{2b_l}\zeta} Z(\zeta_1, \zeta_2, \dots, \zeta_p)$$

exists, where $b_1 = 0$, $b_2 = 1, \dots, b_{t-1} = t-2$ and for $k > t-1$, $b_k = b_{k'}$, where $k = k' \bmod (t-1)$, $k' < t-1$. □

Now define $Z^{[0]}(\zeta) = 1$, $Z^{[1]}(\zeta) = Z(\zeta)$, and for $p \geq 2$,

$$Z^{[p]}(\zeta) = \lim_{\zeta_l\to\omega^{2b_l}\zeta} Z(\zeta_1, \zeta_2, \dots, \zeta_p). \tag{2.21}$$

Theorem 2.3. *On a level one standard $\tilde{\mathfrak{g}}$-module $L(h_i^*)$, $i = 0, 1, \dots, n$, we have*

$$Z^{[t]}(\zeta) = bZ^{[t-2]}(\omega^2\zeta) \tag{2.22}$$

where 'b' is a nonzero constant independent of i, given by

$$b = -\frac{1}{2}(1+\omega)^2 \prod_{1\leq j\leq t-2} (1-\omega^{4j+2})(1-\omega^{4j-2})(1-\omega^{-4j})^2.$$

Proof. It follows from (2.4) and (2.13) that on the $\tilde{\mathfrak{g}}$-module $L(h_i^*)$, $i = 0, 1, \dots, n$, we have

$$\begin{aligned} Z(\zeta_1, \zeta_2, \dots, \zeta_t) &= a^t \prod_{1\leq j<k\leq t} (1-\zeta_j^2/\zeta_k^2)(1-\omega^2\zeta_j^2/\zeta_k^2)^{\frac{1}{2}}(1-\omega^{-2}\zeta_j^2/\zeta_k^2)^{\frac{1}{2}} \\ &\quad \cdot \prod_{l=1}^{t} [\omega^i \exp(P^-(\zeta_l))\exp(-P^+(\zeta_l)) + \omega^{-i}\exp(-P^-(\zeta_l))\exp(P^+(\zeta_l))] \\ &= a^t \prod_{1\leq j<k\leq t} (1-\zeta_j^2/\zeta_k^2)(1-\omega^2\zeta_j^2/\zeta_k^2)^{\frac{1}{2}}(1-\omega^{-2}\zeta_j^2/\zeta_k^2)^{\frac{1}{2}} \end{aligned}$$

(continues)

(continued)

$$\cdot [\omega^i \exp(P^-(\zeta_1)) \exp(-P^+(\zeta_1)) + \omega^{-i} \exp(-P^-(\zeta_1)) \exp(P^+(\zeta_1))]$$

$$\cdot \prod_{l=2}^{t-1} [\omega^i \exp(P^-(\zeta_l)) \exp(-P^+(\zeta_l)) + \omega^{-i} \exp(-P^-(\zeta_l)) \exp(P^+(\zeta_l))]$$

$$\cdot [\omega^i \exp(P^-(\zeta_t)) \exp(-P^+(\zeta_t)) + \omega^{-i} \exp(-P^-(\zeta_t)) \exp(P^+(\zeta_t))]$$

$$= a^t \prod_{1 \le j < k \le t} (1 - \zeta_j^2/\zeta_k^2)(1 - \omega^2 \zeta_j^2/\zeta_k^2)^{\frac{1}{2}} (1 - \omega^{-2} \zeta_j^2/\zeta_k^2)^{\frac{1}{2}}$$

$$\cdot [\omega^i \exp(P^-(\zeta_1)) \exp(-P^+(\zeta_1)) + \omega^{-i} \exp(-P^-(\zeta_1)) \exp(P^+(\zeta_1))]$$

$$\cdot \Bigg[\sum_{r=0}^{t-2} \omega^{(t-2r-2)i} \sum_{2 \le l_1 < \cdots < l_r \le t-1} \exp(P^-(\zeta_2)) \exp(-P^+(\zeta_2))$$

$$\cdots \exp(-P^-(\zeta_{l_1})) \exp(P^+(\zeta_{l_1})) \cdots$$

$$\exp(-P^-(\zeta_{l_r})) \exp(P^+(\zeta_{l_r})) \cdots \exp(P^-(\zeta_{t-1})) \exp(-P^+(\zeta_{t-1})) \Bigg]$$

$$\cdot [\omega^i \exp(P^-(\zeta_t)) \exp(-P^+(\zeta_t)) + \omega^{-i} \exp(-P^-(\zeta_t)) \exp(P^+(\zeta_t))]$$

$$= A_1 + A_2 + A_3 + A_4, \text{ say}$$

where

$$A_1 = a^t \prod_{1 \le j < k \le t} (1 - \zeta_j^2/\zeta_k^2)(1 - \omega^2 \zeta_j^2/\zeta_k^2)^{\frac{1}{2}} (1 - \omega^{-2} \zeta_j^2/\zeta_k^2)^{\frac{1}{2}}$$

$$\cdot \omega^{2i} \exp(P^-(\zeta_1)) \exp(-P^+(\zeta_1))$$

$$\cdot \Bigg[\sum_{r=0}^{t-2} \omega^{t-2r-2)i} \sum_{2 \le l_1 < \cdots < l_r \le t-1} \exp(P^-(\zeta_2)) \exp(-P^+(\zeta_2))$$

$$\cdots \exp(-P^-(\zeta_{l_1})) \exp(P^+(\zeta_{l_1})) \cdots$$

$$\exp(-P^-(\zeta_{l_r})) \exp(P^+(\zeta_{l_r})) \cdots \exp(P^-(\zeta_{t-1})) \exp(-P^+(\zeta_{t-1})) \Bigg]$$

$$\cdot \exp(P^-(\zeta_t)) \exp(-P^+(\zeta_t)),$$

$$A_2 = a^t \prod_{1 \le j < k \le t} (1 - \zeta_j^2/\zeta_k^2)(1 - \omega^2 \zeta_j^2/\zeta_k^2)^{\frac{1}{2}} (1 - \omega^{-2} \zeta_j^2/\zeta_k^2)^{\frac{1}{2}}$$

$$\cdot \omega^{-2i} \exp(-P^-(\zeta_1)) \exp(P^+(\zeta_1))$$

$$\cdot \Bigg[\sum_{r=0}^{t-2} \omega^{(t-2r-2)i} \sum_{2 \le l_1 < \cdots < l_r \le t-1} \exp(P^-(\zeta_2)) \exp(-P^+(\zeta_2))$$

$$\cdots \exp(P^-(\zeta_{t-1})) \exp(-P^+(\zeta_{t-1})) \Bigg]$$

(continues)

(continued)

$$\cdots \exp(-P^-(\zeta_{l_1}))\exp(P^+(\zeta_{l_1}))$$
$$\cdots \exp(-P^-(\zeta_{l_r}))\exp(P^+(\zeta_{l_r}))$$
$$\cdot \exp(-P^-(\zeta_t))\exp(P^+(\zeta_t)),$$

$$A_3 = a^t \prod_{1\le j<k\le t} (1-\zeta_j^2/\zeta_k^2)(1-\omega^2\zeta_j^2/\zeta_k^2)^{\frac{1}{2}}(1-\omega^{-2}\zeta_j^2/\zeta_k^2)^{\frac{1}{2}}$$
$$\cdot \exp(P^-(\zeta_1))\exp(-P^+(\zeta_1))$$
$$\cdot\Bigg[\sum_{r=0}^{t-2}\omega^{(t-2r-2)i}\sum_{2\le l_1<\cdots<l_r\le t-1}\exp(P^-(\zeta_2))\exp(-P^+(\zeta_2))$$
$$\cdots \exp(-P^-(\zeta_{l_1}))\exp(P^+(\zeta_{l_1}))$$
$$\cdots \exp(-P^-(\zeta_{l_r}))\exp(P^+(\zeta_{l_r}))$$
$$\cdots \exp(P^-(\zeta_{t-1}))\exp(-P^+(\zeta_{t-1}))\Bigg]$$
$$\cdot \exp(-P^-(\zeta_t))\exp(P^+(\zeta_t)),$$

and

$$A_4 = a^t \prod_{1\le j<k\le t} (1-\zeta_j^2/\zeta_k^2)(1-\omega^2\zeta_j^2/\zeta_k^2)^{\frac{1}{2}}(1-\omega^{-2}\zeta_j^2/\zeta_k^2)^{\frac{1}{2}}$$
$$\cdot \exp(-P^-(\zeta_1))\exp(P^+(\zeta_1))$$
$$\cdot\Bigg[\sum_{r=0}^{t-2}\omega^{(t-2r-2)i}\sum_{2\le l_1<\cdots<l_r\le t-1}\exp(P^-(\zeta_2))\exp(-P^+(\zeta_2))$$
$$\cdots \exp(-P^-(\zeta_{l_1}))\exp(P^+(\zeta_{l_1}))$$
$$\cdots \exp(-P^-(\zeta_{l_r}))\exp(P^+(\zeta_{l_r}))$$
$$\cdots \exp(P^-(\zeta_{t-1}))\exp(-P^+(\zeta_{t-1}))\Bigg]$$
$$\cdot \exp(P^-(\zeta_t))\exp(-P^+(\zeta_t)).$$

Now using equations (2.8), (2.9) and then taking limit as $\zeta_l \to \omega^{2b_l}\zeta$, it can be easily checked that

$$\lim_{\zeta_l\to\omega^{2b_l}\zeta} A_1 = \lim_{\zeta_l\to\omega^{2b_l}\zeta} A_2 = 0$$

and

$$\lim_{\zeta_l \to \omega^{2b_l}\zeta} A_3 = \lim_{\zeta_l \to \omega^{2b_l}\zeta} A_4 = a^2(1-\omega^2)(1-\omega^{-2})$$
$$\cdot \prod_{1\le j\le t-2} (1-\omega^{4j+2})(1-\omega^{4j-2})(1-\omega^{-4j})^2$$
$$\cdot \prod_{1\le j<k\le t-2} (1-\omega^{4(j-k)})(1-\omega^{4(j-k)+2})^{\frac{1}{2}}(1-\omega^{4(j-k)-2})^{\frac{1}{2}}$$
$$\cdot a^{t-2}\Bigg[\sum_{r=0}^{t-2} \omega^{(t-2r-2)i} \sum_{1\le m_1<\cdots<m_r\le t-2} \cdot \exp(P^-(\omega^2\zeta))\exp(-P^+(\omega^2\zeta))$$
$$\cdots \exp(-P^-(\omega^{2m_1}\zeta))\exp(P^+(\omega^{2m_1}\zeta))$$
$$\cdots \exp(-P^-(\omega^{2m_r}\zeta))\exp(P^+(\omega^{2m_r}\zeta))$$
$$\cdots \exp(P^-(\omega^{2(t-2)}\zeta))\exp(-P^+(\omega^{2(t-2)}\zeta))\Bigg]$$
$$= -\frac{1}{4}(1+\omega)^2 \prod_{1\le j\le t-2} (1-\omega^{4j+2})(1-\omega^{4j-2})(1-\omega^{-4j})^2 Z^{[t-2]}(\omega^2\zeta),$$

since by (2.4) and (2.13) we have

$$Z^{[t-2]}(\omega^2\zeta) = \lim_{\zeta_l \to \omega^{2b_l+2}\zeta} Z(\zeta_1, \zeta_2, \ldots, \zeta_{t-2})$$
$$= \lim_{\zeta_l \to \omega^{2l}\zeta} \prod_{1\le j<k\le t-2} (1-\zeta_j^2/\zeta_k^2)(1-\omega^2\zeta_j^2/\zeta_k^2)^{\frac{1}{2}}$$
$$\cdot (1-\omega^{-2}\zeta_j^2/\zeta_k^2)^{\frac{1}{2}} Z(\zeta_1)Z(\zeta_2)\cdots Z(\zeta_{t-2})$$
$$= a^{t-2} \lim_{\zeta_l \to \omega^{2l}\zeta} \prod_{1\le j<k\le t-2} (1-\zeta_j^2/\zeta_k^2)(1-\omega^2\zeta_j^2/\zeta_k^2)^{\frac{1}{2}}(1-\omega^{-2}\zeta_j^2/\zeta_k^2)^{\frac{1}{2}}$$
$$\cdot \Bigg[\sum_{r=0}^{t-2} \omega^{(t-2r-2)i} \sum_{1\le m_1<\cdots<m_r\le t-2} \exp(P^-(\zeta_1))\exp(-P^+(\zeta_1))$$
$$\cdots \exp(-P^-(\zeta_{m_1}))\exp(P^+(\zeta_{m_1}))$$
$$\cdots \exp(-P^-(\zeta_{m_r}))\exp(P^+(\zeta_{m_r}))\cdots \exp(P^-(\zeta_{t-2}))\exp(-P^+(\zeta_{t-2}))\Bigg]$$
$$= a^{t-2} \prod_{1\le j<k\le t-2} (1-\omega^{4(j-k)})(1-\omega^{4(j-k)+2})^{\frac{1}{2}}(1-\omega^{4(j-k)-2})^{\frac{1}{2}}$$
$$\cdot \Bigg[\sum_{r=0}^{t-2} \omega^{(t-2r-2)i} \sum_{1\le m_1<\cdots<m_r\le t-2} \exp(P^-(\omega^2\zeta))\exp(-P^+(\omega^2\zeta))$$
$$\cdots \exp(-P^-(\omega^{2m_1}\zeta))\exp(P^+(\omega^{2m_1}\zeta))$$
$$\cdots \exp(-P^-(\omega^{2m_r}\zeta))\exp(P^+(\omega^{2m_r}\zeta))$$
$$\cdots \exp(P^-(\omega^{2(t-2)}\zeta))\exp(-P^+(\omega^{2(t-2)}\zeta))\Bigg].$$

Hence we have

$$\begin{aligned} Z^{[t]}(\zeta) &= \lim_{\zeta_l \to \omega^{2b_l}\zeta} Z(\zeta_1, \zeta_2, \dots, \zeta_t) \\ &= \lim_{\zeta_l \to \omega^{2b_l}\zeta} (A_1 + A_2 + A_3 + A_4) \\ &= -\frac{1}{2}(1+\omega)^2 \prod_{1 \le j \le t-2} (1-\omega^{4j+2})(1-\omega^{4j-2})(1-\omega^{-4j})^2 Z^{[t-2]}(\omega^2\zeta) \end{aligned}$$

as claimed. □

3. Bases for level one standard $\underline{sp}(4k+2, \mathbb{C})^{\sim}$-modules

As in §2, let $L = L(\lambda)$ denote the level one standard $\underline{sp}(2n, \mathbb{C})^{\sim}$-module with highest weight $\lambda = h_i^*$, $0 \le i \le n$, and highest weight vector v_λ and let $\Omega(L)$ denote the corresponding vacuum space. For any sequence of integers $\mu = (m_1, m_2, \dots, m_p)$, $p > 0$ define the elements $Z(\mu) = Z(m_1, m_2, \dots, m_p)$ in End $L(\lambda)$ by the equation

$$Z(\zeta_1, \dots, \zeta_p) = \sum Z(m_1, \dots, m_p)\zeta_1^{m_1} \cdots \zeta_p^{m_p}, \tag{3.1}$$

where the summations ranges over all integers $m_1, \dots, m_p$. Note that for $p = 0$ we have the unique sequence $\mu = \varnothing$ (empty sequence) and in this case we define $Z(\varnothing) = 1$. It follows immediately from equation (2.13) that

$$Z(m_1, m_2, \dots, m_p) = 0 \tag{3.2}$$

unless $m_1, m_2, \dots, m_p \in 2\mathbb{Z}$ (even integers). By Theorem 2.1 we also have

$$Z(m_1, \dots, m_p) = Z(m_{\sigma(1)}, \dots, m_{\sigma(p)}) \tag{3.3}$$

for any permutation $\sigma \in S_p$.

For any sequence $\mu = (m_1, \dots, m_p) \in \mathbb{Z}^p$, $p > 0$, define $l(\mu) = p$, $|\mu| = m_1 + \cdots + m_p$, and write $\mu(j) = m_j$, $1 \le j \le p$. Also define $l(\varnothing) = 0$. For two sequences of integers $\mu = (m_1, \dots, m_p)$ and $\nu = (n_1, \dots, n_p)$, $p > 0$, define (cf. [25]) $\mu \ge_T \nu$ if and only if

$$m_p \ge n_p;\ m_{p-1} + m_p \ge n_{p-1} + n_p;\ \cdots;\ m_1 + \cdots + m_p \ge n_1 + \cdots + n_p.$$

Then clearly for any sequence $\theta = (r_1, \dots, r_q)$, $q \ge 0$, of integers we have

$$\mu \ge_T \nu \Rightarrow \mu \circ \theta \ge_T \nu \circ \theta \quad \text{and} \quad \theta \circ \mu \ge_T \theta \circ \nu, \tag{3.4}$$

where the composition is defined by juxtaposition, i.e.

$$\mu \circ \theta = (m_1, \dots, m_p, r_1, \dots, r_q).$$

Lemma 3.1. *Let us define the coefficients $a(\mu)$ and $b(\mu)$ by the formal identities:*

$$\prod_{1 \le j < k \le p} (1 - \zeta_j^2/\zeta_k^2)(1 - \omega^2\zeta_j^2/\zeta_k^2)^{\frac{1}{2}}(1 - \omega^{-2}\zeta_j^2/\zeta_k^2)^{\frac{1}{2}} = \sum_{\mu \in \mathbb{Z}^p} a(\mu)\zeta_1^{\mu(1)} \cdots \zeta_p^{\mu(p)},$$

and

$$\prod_{1\le j<k\le p}(1-\zeta_j^2/\zeta_k^2)^{-1}(1-\omega^2\zeta_j^2/\zeta_k^2)^{-\frac{1}{2}}(1-\omega^{-2}\zeta_j^2/\zeta_k^2)^{-\frac{1}{2}} = \sum_{\mu\in\mathbb{Z}^p} b(\mu)\zeta_1^{\mu(1)}\cdots\zeta_p^{\mu(p)}.$$

Then $a(\mathbf{O}) = 1 = b(\mathbf{O})$ *and* $a(\mu) = 0 = b(\mu)$ *unless* $\mu \le_T \mathbf{O}$, *where* $\mathbf{O} = (0, \dots, 0)$. *Furthermore,*

$$Z(m_1, \dots, m_p) = \sum_{\mu\in\mathbb{Z}^p} a(\mu)Z(m_1-\mu(1))\cdots Z(m_p-\mu(p)), \tag{1}$$

and

$$Z(m_1)\cdots Z(m_p) = \sum_{\mu\in\mathbb{Z}^p} b(\mu)Z(m_1-\mu(1), \dots, m_p-\mu(p)). \tag{2}$$

Proof. For (1) compare the coefficient of $\zeta_1^{m_1}\zeta_2^{m_2}\cdots\zeta_p^{m_p}$ in equation (3.1). For (2) multiply equation (3.1) by

$$\prod_{1\le j<k\le p}(1-\zeta_j^2/\zeta_k^2)^{-1}(1-\omega^2\zeta_j^2/\zeta_k^2)^{-\frac{1}{2}}(1-\omega^{-2}\zeta_j^2/\zeta_k^2)^{-\frac{1}{2}}$$

and then compare the coefficients of $\zeta_1^{m_1}\zeta_2^{m_2}\cdots\zeta_p^{m_p}$. □

The next corollary follows immediately from Lemma 3.1.

Corollary 3.2. *For* $\mu = (m_1, m_2, \dots, m_p) \in \mathbb{Z}^p$, *we have*

$$Z(\mu) = Z(m_1)\cdots Z(m_p) + \sum_{\nu>_T\mu} a(\mu-\nu)Z(\nu(1))\cdots Z(\nu(p)), \tag{1}$$

$$Z(m_1)\cdots Z(m_p) = Z(\mu) + \sum_{\nu>_T\mu} b(\mu-\nu)Z(\nu). \quad \square \tag{2}$$

Corollary 3.3. *We have*

$$\begin{aligned} &Z(m_1, \dots, m_p)Z(n_1, \dots, n_r) \\ &\quad = Z(m_1, \dots, m_p, n_1, \dots, n_r) + \sum_{\nu>_T(m_1,\dots,m_p,n_1,\dots,n_r)} c(\nu)Z(\nu) \end{aligned}$$

for some scalars $c(\nu)$.

Proof. It is clear from Corollary 3.2 (1) and (3.4) that

$$\begin{aligned} Z(m_1, \dots, m_p)Z(n_1, \dots, n_r) &= Z(m_1)\cdots Z(m_p)Z(n_1)\cdots Z(n_r) \\ &\quad + \sum_{\nu>_T(m_1,\dots,m_p,n_1,\dots,n_r)} d(\nu)Z(\nu(1))\cdots Z(\nu(p+r)) \end{aligned}$$

for some scalars $d(\nu)$. Now the result follows from Corollary 3.2 (2). □

For any two sequences μ and ν, we say $\mu < \nu$ if and only if $l(\mu) > l(\nu)$ or $\mu <_T \nu$. Then it follows from (3.4) that

$$\mu < \nu \Rightarrow \mu\circ\theta < \nu\circ\theta \quad \text{and} \quad \theta\circ\mu < \theta\circ\nu, \tag{3.5}$$

for any sequence θ. Let $\mathscr{P}$ denote the set of all sequences $\mu = (2m_1, \dots, 2m_p) \in (2\mathbb{Z})^p$, $p \geq 0$, such that $m_1 \leq m_2 \leq \cdots \leq m_p < 0$. For any standard $\tilde{\mathfrak{g}}$-module $L = L(\lambda)$ with highest weight v_λ, where $\lambda = h_i^*$, $0 \leq i \leq n$, define

$$\Omega(L)_{(\mu)} = \sum_{\substack{\nu > \mu \\ \nu \in \mathscr{P}}} \mathbb{C} Z(\nu) v_\lambda, \tag{3.6}$$

for all $\mu \in \mathscr{P}$, $\mu \neq \varnothing$ and define $\Omega(L)_{(\varnothing)} = \{0\}$. Clearly, we have

$$\Omega(L)_{(\mu)} \supseteq \Omega(L)_{(\nu)}, \quad \text{for } \mu \leq \nu;\ \mu, \nu \in \mathscr{P}. \tag{3.7}$$

Proposition 3.4. $\Omega = \Omega(L) = \bigcup_{\mu \in \mathscr{P}} \Omega(L)_{(\mu)}$.

Proof. It follows from (2.6), (3.3) and Corollary 3.2 that

$$\Omega = \operatorname{span}_{\mathbb{C}}\{Z(\mu)v_\lambda \mid \mu = (m_1, \dots, m_p) \in (2\mathbb{Z})^p,\ p \geq 0,\ m_1 \leq \cdots \leq m_p\}.$$

Let

$$V = \operatorname{span}_{\mathbb{C}}\{Z(\nu)v_\lambda \mid \nu \in \mathscr{P}\}.$$

Clearly $V \subseteq \Omega$. To show $\Omega \subseteq V$ we use induction on the ordering of the sequences $\mu = (m_1, \dots, m_p) \in (2\mathbb{Z})^p$, $p \geq 0$. Fix $\mu = (m_1, \dots, m_p) \in (2\mathbb{Z})^p$, $m_1 \leq \cdots \leq m_p$. Assume that $Z(\mu')v_\lambda \in V$ for all $\mu' > \mu$. We want to show that $Z(\mu)v_\lambda \in V$. Suppose $\mu = (m_1, \dots, m_r, m_{r+1}, \dots, m_p)$, where $m_1 \leq \cdots \leq m_r < 0$ and $0 \leq m_{r+1} \leq \cdots \leq m_p$. By Corollary 3.3 we have

$$Z(\mu)v_\lambda = Z(m_1, \dots, m_r)Z(m_{r+1}, \dots, m_p)v_\lambda + \sum_{\mu' > \mu} c(\mu')Z(\mu')v_\lambda.$$

Hence by assumption, it is enough to show that

$$Z(m_1, \dots, m_r)Z(m_{r+1}, \dots, m_p)v_\lambda \in V.$$

If $r > 0$, then this follows from the induction hypothesis and Corollary 3.3 since $(m_{r+1}, \dots, m_p) > \mu$. Now suppose $r = 0$. Then we have $0 \leq m_1 \leq \cdots \leq m_p$ and by Corollary 3.2,

$$Z(\mu)v_\lambda = Z(m_1) \cdots Z(m_p)v_\lambda + \sum_{\mu' > \mu} d(\mu')Z(\mu')v_\lambda,$$

where $d(\mu')$ are some scalars. But since v_λ is a highest weight vector and $0 \leq m_1 \leq \cdots \leq m_p$, so $Z(m_1) \cdots Z(m_p)v_\lambda$ is a scalar multiple of v_λ. Hence it again follows by the induction hypothesis that $Z(\mu)v_\lambda \in V$. □

For $p \geq 1$ and $r \in \mathbb{Z}$ denote by $(p;\ r)$ the unique sequence of integers (see [21]),

$$(p;\ r) = (m_1, \dots, m_p)$$

such that

$$r = m_1 + \cdots + m_p, \quad m_1 \leq \cdots \leq m_p \quad \text{and}$$
$$0 \leq m_p - m_1 \leq 1$$

Let $2(p;\ r)$ denote the corresponding unique sequence of even integers $(2m_1, \dots, 2m_p)$. The following lemma is clear (see [21, Lemma 8.2]).

Lemma 3.5. *Let* $\mu = (2n_1, \dots, 2n_p) \in \mathscr{P}$, $p > 0$, *and* $|\mu| = 2r$. *Then we have*

$$2(p;\ r) \leq \mu, \tag{1}$$

$$\text{if } \mu \neq 2(p, r), \text{ then } n_1 \leq -2 + n_p. \quad \square \tag{2}$$

For $p \geq 2$, let

$$Z^{[p]}(\zeta) = \sum_{r \in \mathbb{Z}} Z^{[p]}(r)\zeta^r. \tag{3.8}$$

It follows from (2.5), (2.13) and (2.21) that $Z^{[p]}(r) = 0$ unless $r \in 2\mathbb{Z}$. The next proposition follows immediately from (2.13), (2.21) and Lemma 3.5.

Proposition 3.6. *For* $r \in \mathbb{Z}$ *and* $p \geq 2$, *we have*

$$Z^{[p]}(2r) = eZ(2(p;\ r)) + \sum_{\nu > 2(p;\ r)} e_\nu Z(\nu),$$

for some nonzero scalar e *and scalars* e_ν*'s and* $\nu \in \mathscr{P}$. $\square$

For a sequence $\mu = (2m_1, \dots, 2m_p) \in (2\mathbb{Z})^p$, $m_1 \leq \cdots \leq m_p$, we say that μ satisfies the *difference two condition* if for every $r \in \{1, \dots, p - t + 1\}$ we have $m_r \leq -2 + m_{r+t-1}$. Observe that for $s \in \mathbb{Z}$ the sequence $2(t; s)$ does not satisfy the difference two condition.

Theorem 3.7. *If* $\mu \in \mathscr{P}$ *does not satisfy the difference two condition, then* $Z(\mu)v_\lambda \in \Omega(L)_\mu$ *for* $\lambda = h_i^*$, $i = 0, 1, \dots, n$.

Proof. Since $\mu \in \mathscr{P}$ does not satisfy the difference two condition, so it must be of the form

$$\mu = (2m_1, \dots, 2m_s,\ 2(t;\ r),\ 2m_{s+t+1}, \dots, 2m_p).$$

Let

$$2(t; r) = (2m_{s+1}, \dots, 2m_{s+t}).$$

Then Theorem 2.3 along with equation (3.8) implies that for any $v \in L = L(h_i^*)$, $i = 0, 1, \dots, n$,

$$Z^{[t]}(2r)v = \sum_{l(\nu) < t} b_\nu Z(\nu)v,$$

for some scalars b_ν. This together with Proposition 3.6 then implies that

$$Z(2(t; r))v = \sum_{\nu > 2(t; r)} a_\nu Z(\nu)v, \tag{3.9}$$

for some scalars a_ν. Now set $v = Z(2m_{s+t+1}, \dots, 2m_p)v_\lambda$ and multiply (3.9) from the left by $Z(2m_1, \dots, 2m_s)$. Now thanks to Corollary 3.3 and equations (3.2), (3.3), (3.5) we have the desired result. $\square$

It is clear from (2.4) that

$$Z(0)v_\lambda = a(\omega^i + \omega^{-i})v_\lambda, \tag{3.10}$$

for $\lambda = h_i^*$, $0 \le i \le n$, where $a = \frac{1}{2}\omega/(\omega - 1)$. For $r \ge 0$, write

$$\mu(r) = (-2, \dots, -2) \in (2\mathbb{Z})^r . \tag{3.11}$$

For $p \ge r$, write

$$\mu_p(r) = (-2, \dots, -2, 0, \dots, 0) \in (2\mathbb{Z})^p \tag{3.12}$$

with r entries equal to -2. The next lemma follows from (3.10) and Corollary 3.2.

Lemma 3.8. *For $p \ge r$, $\lambda = h_i^*$, $i = 0, 1, \dots, n$, we have*

$$Z(\mu_p(r))v_\lambda = (a(\omega^i + \omega^{-i}))^{p-r} Z(\mu(r))v_\lambda + \sum_{\nu > \mu(r)} b_\nu Z(\nu)v_\lambda ,$$

for some scalars b_ν.

Remark 3.9. For $\lambda = h_{n-i}^*$, we have from equation (3.10),

$$Z(0)v_\lambda = a(\omega^{n-i} + \omega^{-(n-i)})v_\lambda = -a(\omega^i + \omega^{-i})v_\lambda$$

since $\omega^n = -1 = \omega^{-n}$.

Lemma 3.10. *For $p \ge r$, $\lambda = h_i^*$ or h_{n-i}^*, $i = 0, 1, \dots, t-1$, there is a polynomial $f_{t,p,r}(x)$ of degree $p - r$ which is independent of i such that the coefficient of $\zeta_1^{-2}\zeta_2^{-2}\cdots\zeta_r^{-2}$ in $Z(\zeta_1, \dots, \zeta_p)v_\lambda$ is of the form*

$$f_{t,p,r}(\pm a(\omega^i + \omega^{-i}))Z(\mu(r))v_\lambda + \sum_{\nu > \mu(r)} d_\nu Z(\nu)v_\lambda$$

for some scalars d_ν. Here the sign is $+$ or $-$ depending on $\lambda = h_i^*$ or $\lambda = h_{n-i}^*$, $i = 0, 1, \dots, t-1$.

Proof. It follows from Theorem 2.1 and equations (3.1), (3.2) that

$$Z(\zeta_1, \dots, \zeta_p) = \sum_{\sigma \in S_p} \Big(\sum_{m_1 \le \cdots \le m_p} Z(2m_1, \dots, 2m_p) \Big) \zeta_1^{2m_{\sigma(1)}} \cdots \zeta_p^{2m_{\sigma(p)}} .$$

Hence the coefficient of $\zeta_1^{-2}\cdots\zeta_r^{-2}$ in $Z(\zeta_1, \dots, \zeta_p)v_\lambda$ is a linear combination of terms of the form $Z(\mu)v_\lambda$ where $\mu = (2m_1, \dots, 2m_s)$, $s \le p$, $m_1 \le \cdots \le m_s$ and $m_1 + \cdots + m_s = -r$. Since the coefficient of $Z(\mu(r))v_\lambda$ is nonzero and also if $\mu >_T \mu_s(r)$ then $Z(\mu)v_\lambda \in \Omega(L)_{(\mu(r))}$, therefore the result follows from Lemma 3.8, Corollary 3.2 and Remark 3.9. □

Proposition 3.11. *For $r \le t$, $\lambda = h_i^*$ or h_{n-i}^*, $i = 0, 1, \dots, t-1$, there is a polynomial $g_{t,r}(x)$ of degree $t - r$, independent of i, such that*

$$g_{t,r}(\pm a(\omega^i + \omega^{-i}))Z(\mu(r))v_\lambda \in \Omega(L)_{(\mu(r))} .$$

Here the sign is $+$ or $-$ depending on $\lambda = h_i^*$ or h_{n-i}^*, $0 \le i \le t-1$.

Proof. From Theorem 2.3, we have

$$\lim_{\zeta_l \to \omega^{2b_l}\zeta} [Z(\zeta_1, \dots, \zeta_t) - bZ(\zeta_1, \dots, \zeta_{t-2})]v_\lambda = 0 \tag{3.13}$$

where 'b' is a nonzero scalar independent of i. In (3.13), first collecting the coefficients of $\zeta_1^{-2}\cdots\zeta_r^{-2}$ by using Lemma 3.10 and then taking the limit we get

$$(3.14)\quad [f_{t,t,r}(\pm a(\omega^i+\omega^{-i}))-bf_{t,t-2,r}(\pm a(\omega^i+\omega^{-i}))]\cdot Z(\mu(r))v_\lambda \in \Omega(L)_{\mu(r)}.$$

Now setting $g_{t,r}(x) = f_{t,t,r}(x)-bf_{t,t-2,r}(x)$, the result follows from (3.14). □

The following lemma is an immediate consequence of the character formulas (1.26) and (1.27).

Lemma 3.12. *If $r \le i$, $\lambda = h_i^*$ or h_{n-i}^*, $i = 0, 1, \dots, t-1$, then $Z(\mu(r))v_\lambda \notin \Omega(L)_{(\mu(r))}$.*

Proposition 3.13. *For $\lambda = h_i^*$ or h_{n-i}^*, $i = 0, 1, \dots, t-1$, we have $Z(\mu(i+1))v_\lambda \in \Omega(L)_{(\mu(i+1))}$.*

Proof. First let us consider the case when $\lambda = h_i^*$, $i = 0, 1, \dots, t-1$. If $r \le t$, then by Proposition 3.11 and Lemma 3.12, we have

$$g_{t,r}(a(\omega^i+\omega^{-i})) = 0 \quad \text{for } i = r, r+1, \dots, t-1.$$

But since $g_{t,r}$ has degree $t-r$, this implies that for $i < r$

$$(3.15)\qquad g_{t,r}(a(\omega^i+\omega^{-i})) \ne 0.$$

Now setting $r = i+1$ in (3.15), we have $g_{t,i+1}(a(\omega^i+\omega^{-i})) \ne 0$. Hence by Proposition 3.11, we have $Z(\mu(i+1))v_\lambda \in \Omega(L)_{\mu(i+1)}$ as desired. The case $\lambda = h_{n-i}^*$, $0 \le i \le t-1$ follows similarly by replacing 'a' with '$-a$' in the above argument. □

We say that a sequence $\mu = (2m_1, \dots, 2m_p) \in \mathscr{P}$ satisfies the *initial condition* if at most i $(0 \le i \le t-1)$ elements m_s are equal to -1.

Theorem 3.14. *For $\lambda = h_i^*$ or h_{n-i}^*, $0 \le i \le t-1$, if $\mu \in \mathscr{P}$ does not satisfy the initial condition, then $Z(\mu)v_\lambda \in \Omega(L)_{(\mu)}$.*

Proof. Since $\mu \in \mathscr{P}$ does not satisfy the initial condition, so it must be of the form

$$\mu = (2m_1, \dots, 2m_{p-i-1}, 2m_{p-i}, \dots, 2m_p)$$

where

$$m_{p-i} = \cdots = m_p = -1.$$

Set $\mu' = (2m_1, \dots, 2m_{p-i-1})$. Then $\mu = \mu' \circ \mu(i+1)$. By Corollary 3.3, we have

$$Z(\mu)v_\lambda = Z(\mu')Z(\mu(i+1))v_\lambda + \sum_{\nu>\mu} e_\nu Z(\nu)v_\lambda,$$

for some scalars e_ν. Now the desired result follows thanks to Proposition 3.13, Corollary 3.3 and equation (3.5). □

For $\lambda = h_i^*$ or h_{n-i}^*, $0 \le i \le t-1$, denote by $\mathscr{C}_\lambda$ the set of all $\mu \in \mathscr{P}$ such that μ satisfies the difference two condition and the initial condition.

Theorem 3.15. *For $\lambda = h_i^*$ or h_{n-i}^*, $0 \le i \le t-1$, let $L(\lambda)$ denote the standard $\tilde{\mathfrak{g}}$-module with highest weight λ and highest weight vector v_λ. Let $\Omega(L)$ denote the vacuum space of $L(\lambda)$. The set of vectors $\{Z(\mu)v_\lambda \mid \mu \in \mathscr{C}_\lambda\}$ spans $\Omega(L)$.*

Proof. By Proposition 3.4, we have that the set of vectors $\{Z(\mu)v_\lambda \mid \mu \in \mathscr{P}\}$ span the space $\Omega(L)$. Now using Theorems 3.7 and 3.14, the desired result follows by induction on the ordering of $\mathscr{P}$. □

Observe that for $\mu \in \mathscr{C}_\lambda$, the vector $Z(\mu)v_\lambda \in \Omega(L)_{|\mu|}$. Hence in the case when $n = 2k+1$, the following theorem is an immediate consequence of Theorem 3.15 and the character formula (1.26) of $\Omega(L)$ together with the generalized Rogers-Ramanujan identities (see [1, 2]) due to Gordon, Andrews and Bressoud.

Theorem 3.16. *For $\lambda = h_i^*$ or h_{n-i}^*, $0 \le i \le k$, let $L(\lambda)$ denote the standard $\underline{sp}(4k+2, \mathbb{C})^{\sim}$-module with highest weight λ and highest weight vector v_λ. Let $\Omega(L)$ denote the vacuum space of $L(\lambda)$. The set of vectors $\{Z(\mu)v_\lambda \mid \mu \in \mathscr{C}_\lambda\}$ is a basis of $\Omega(L)$.* □

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