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THE GENERALIZED LUSTERNIK-SCHNIRELMANN CATEGORY OF A PRODUCT SPACE

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ABSTRACT. We continue to study the notions of $\mathscr{A}$-category and strong $\mathscr{A}$-category which we introduced in [2]. We give a characterization of them in terms of homotopy colimits and then use it to prove some product theorems in this context.

In [2] we introduced new homotopy invariants by generalizing the notions of Lusternik-Schnirelmann category and strong category as follows:

For any given class of spaces $\mathscr{A}$ we define the *$\mathscr{A}$-category* $\mathscr{A}\text{-cat}(X)$ of a space X to be the smallest integer k for which there exists a numerable covering $(X_1, \dots, X_k)$ of X such that each inclusion $X_j \subset X$ factors through some space $A_j \in \mathscr{A}$ up to homotopy. If no such covering exists we set $\mathscr{A}\text{-cat}(X) := \infty$. $\mathscr{A}\text{-cat}(X)$ is an invariant of the homotopy type of X [2, 1.4].

If we replace the condition that each $X_j \subset X$ factors through some space in $\mathscr{A}$ up to homotopy by asking each X_j to have the homotopy type of some space in $\mathscr{A}$, then the number $\mathscr{A}\text{-gcat}(X)$ thus obtained is not a homotopy invariant [3]. So we define *the strong $\mathscr{A}$-category* $\mathscr{A}\text{-Cat}(X)$ of X to be the minimum of $\mathscr{A}\text{-gcat}(X')$ for all spaces X' having the homotopy type of X.

If $\mathscr{A}$ consists only of the one-point-space then $\mathscr{A}\text{-cat}(X)$ is just *Lusternik-Schnirelmann's category* $\operatorname{cat}(X)$ of X and $\mathscr{A}\text{-Cat}(X)$ is *the strong category* $\operatorname{Cat}(X)$ introduced by Ganea in [5]. Other interesting examples are *the q-connective (strong) category* $\operatorname{cat}_q(X)$ $(\operatorname{Cat}_q(X))$ of X obtained by taking $\mathscr{A}$ to be the class of q-connected CW-complexes, $q \geq 0$, and the *q-dimensional (strong) category* $\operatorname{cat}^q(X)$ $(\operatorname{Cat}^q(X))$ of X obtained by taking for $\mathscr{A}$ the class of all CW-complexes of dimension $\leq q$ [2, 1.2].

In this paper we shall prove the following product theorems, which generalize the well known ones for the classical case [7]. Unlike [7] however, we do not restrict ourselves to polyhedra but rather exploit the linear structure given by the numeration of the coverings.

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Theorem 1. *Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be classes of spaces such that*

(a) $A \times B \in \mathscr{C}$ *for all* $A \in \mathscr{A}$, $B \in \mathscr{B}$,

(b) $\mathscr{C}$ *contains all finite wedges of elements in* $\mathscr{C}$.

Then, for any two path-connected spaces X *and* Y,

$$\mathscr{C}\text{-}\mathrm{cat}(X \times Y) \le \mathscr{A}\text{-}\mathrm{cat}(X) + \mathscr{B}\text{-}\mathrm{cat}(Y) - 1$$

and

$$\mathscr{C}\text{-}\mathrm{Cat}(x \times Y) \le \mathscr{A}\text{-}\mathrm{Cat}(X) + \mathscr{B}\text{-}\mathrm{Cat}(Y) - 1.$$

Theorem 2. *Let* X *and* Y *be pathwise connected spaces and let* $\mathscr{A}$ *be a class of spaces which contains all finite wedges of elements in it. Then*

$$\mathscr{A}\text{-}\mathrm{Cat}(X \times Y) \le \mathscr{A}\text{-}\mathrm{cat}(X) + \max\{\mathrm{Cat}(Y), 2\} - 1.$$

1. Category and homotopy colimits

Let K be a simplicial complex and let K also denote the poset of its simplices ordered by opposite inclusion. For any functor $F: K \to \mathrm{Top}$, *the homotopy colimit* h-colim F of F may be defined as the quotient space of the disjoint union

$$\coprod_{\sigma \in K} (F(\sigma) \times \Delta_\sigma)$$

obtained by identifying $(x, \varepsilon_\tau^\sigma t)$ with $(f_\sigma^\tau x, t)$ for all $x \in F(\tau)$, $t \in \Delta_\sigma$, where Δ_σ is the geometric simplex spanned by σ, $\varepsilon_\tau^\sigma : \Delta_\sigma \to \Delta_\tau$ is the face map induced by $\sigma \subset \tau$, and $f_\sigma^\tau : F(\tau) \to F(\sigma)$ is the image of $\tau \le \sigma$ under the functor F. The images $F(v)$ of the vertices of K are called the *vertex-spaces of* F *or of* h-colim F.

A *k-fold mapping cylinder* is the homotopy colimit of a functor $F: \Delta^{k-1} \to \mathrm{Top}$, where Δ^{k-1} is the standard $(k-1)$-simplex [2, 5.1].

Given a covering $U = (X_0, \dots, X_{k-1})$ of a space X we associate to it the functor $U: \Delta^{k-1} \to X$ such that

$$U(\sigma) = \bigcap_{i \in \sigma} X_i \quad \text{and} \quad U(\tau \le \sigma) : U(\tau) \subset U(\sigma).$$

The homotopy colimit of U is known as the classifying space BU of U and the canonical map $BU \to X$ is a homotopy equivalence if U is, for example, numerable [4].

On the other hand, for any functor $F: K \to \mathrm{Top}$, there is a canonical map

$$p: h\text{-colim } F \to K.$$

The inverse images under p of the open stars $\mathrm{St}(v)$ of the vertices v of K form a numerable covering of h-colim F and the canonical contraction of $\mathrm{St}(v)$ to v lifts canonically to a fiberwise strong deformation retraction of $p^{-1}\,\mathrm{St}(v)$ to $p^{-1}(v) = F(v)$.

The discussion above shows in particular that

Proposition 3 [2, 5.2]. *For every class of spaces $\mathscr{A}$, $\mathscr{A}$-Cat$(X) \le k$ if and only if X has the homotopy type of a k-fold mapping cylinder with vertex-spaces in $\mathscr{A}$.* □

For every class of spaces $\mathscr{C}$, let $h\mathscr{C}$ denote the class of spaces having the homotopy type of some space in $\mathscr{C}$. The "if" part of the proposition above can be improved as follows:

Proposition 4. *Let $\mathscr{C}$ be a class of spaces and let $F : K \to \mathrm{Top}$ be a functor with vertex-spaces in $h\mathscr{C}$. Assume that one of the following conditions holds:*

(a) *$h\mathscr{C}$ contains all finite disjoint unions of elements in $h\mathscr{C}$, or*
(b) *h-colim F is pathwise connected and $h\mathscr{C}$ contains all finite wedges of elements in $h\mathscr{C}$. Then*

$$\mathscr{C}\text{-}\mathrm{Cat}(h\text{-colim } F) \le 1 + \dim K.$$

Proof. Let $k = 1 + \dim K$. Since at most k open stars have a nonempty intersection one can construct a partition $V_1, \dots, V_k$ of the vertices of K such that, for each $j = 1, \dots, k$, the family $(\mathrm{St}(v) | v \in V_j)$ is disjoint. Hence

$$U_j := p^{-1}\left(\bigcup_{v \in V_j} \mathrm{St}(v)\right) = \bigsqcup_{v \in V_j} p^{-1}\,\mathrm{St}(v).$$

Since $F(v)$ is a strong deformation retract of $p^{-1}\,\mathrm{St}(v)$, $p^{-1}\,\mathrm{St}(v) \in h\mathscr{C}$. If condition (a) holds we are done. If condition (b) holds we now apply the following lemma. □

Lemma 5. *Let X be a path-connected space and let $(Y_1, Y_2, X_1, \dots, X_k)$ be a numerable covering of X such that $Y_1 \cap Y_2 = \varnothing$. Then there is a space X' of the homotopy type of X and a numerable covering $(W, X_1', \dots, X_k')$ of X' such that $W \simeq Y_1 \vee Y_2$ and $X_i' \simeq X_i$ for all $i = 1, \dots, k$.*

Proof. Replacing X by the classifying space of the given covering if necessary we may assume that the covering is closed. Hence, since X is path-connected, there is a path $\omega : [0, 1] \to X$ such that $\omega(0) \in Y_1$, $\omega(1) \in Y_2$ and $\omega(0, 1) \in X - (Y_1 \cup Y_2)$. Now just take $X' := X \times [0, 1]$, $X_i' = X_i \times [0, 1]$, $i = 1, \dots, k$, and

$$W := (Y_1 \times I) \cup \{(\omega(t), t) | 0 \le t \le 1\} \cup (Y_2 \times I). \quad \square$$

Corollary 6. *Let $\mathscr{C}$ be a class of spaces and let X be a space such that $\mathscr{C}$-Cat$(X) = k$. Assume that either*

(a) *$h\mathscr{C}$ contains all finite disjoint unions of elements in $h\mathscr{C}$, or*
(b) *X is pathwise connected and $h\mathscr{C}$ contains all finite wedges of elements in $h\mathscr{C}$.*

Then every numerable covering $U = (X_1, \dots, X_k)$ of X with $X_i \in h\mathscr{C}$ has a nonempty intersection.

Proof. Assume $X_1 \cap \cdots \cap X_k = \varnothing$. Then the classifying space BU of U is the homotopy colimit of the restriction of the functor U to the boundary of Δ^{k-1}. Hence, by Proposition 4, $\mathscr{C}$-Cat$(X) = \mathscr{C}$-Cat$(BU) \leq k-1$. □

We are now ready to prove Theorem 1. It is a particular case of the following more general Theorems 7 and $7'$.

Theorem 7. *Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be classes of spaces and X and Y be spaces such that $A \times B \in h\mathscr{C}$ for all $A \in \mathscr{A}$, $B \in \mathscr{B}$ and such that one of the following conditions holds*:

(a) *$h\mathscr{C}$ contains all finite disjoint unions of elements of $h\mathscr{C}$, or*
(b) *X and Y are path-connected and $h\mathscr{C}$ contains all finite wedges of elements in $h\mathscr{C}$.*

Then

$$\mathscr{C}\text{-Cat}(X \times Y) \leq \mathscr{A}\text{-Cat}(X) + \mathscr{B}\text{-Cat}(Y) - 1.$$

Proof. Let $k = \mathscr{A}$-Cat(X) and $n = \mathscr{B}$-Cat(Y). By Proposition 3, we may assume that X and Y are the homotopy colimits of functors $U : \Delta^{k-1} \to \mathrm{Top}$ and $V : \Delta^{n-1} \to \mathrm{Top}$ with vertex-spaces in $\mathscr{A}$ and $\mathscr{B}$ respectively. Consider the canonical simplicial subdivision K of $\Delta^{k-1} \times \Delta^{n-1}$ and define a functor $F : K \to \mathrm{Top}$ by

$$F(\rho) = U(\sigma) \times V(\tau)$$

where σ and τ are the smallest simplices of Δ^{k-1} and Δ^{n-1} respectively such that $\rho \subset \sigma \times \tau$. Then $X \times Y$ is homeomorphic to h-colim F and the result follows from Proposition 4. □

It is easy to show [2, 5.5] that, for every class of spaces $\mathscr{A}$, $\mathscr{A}$-cat$(X) \leq k$ if and only if X is dominated by a space Z such that $\mathscr{A}$-Cat$(Z) \leq k$. Hence, if the inequality in Theorem 7 holds, then

$$\mathscr{C}\text{-cat}(X \times Y) \leq \mathscr{A}\text{-cat}(X) + \mathscr{B}\text{-cat}(Y) - 1$$

also holds, and this proves Theorem 1. This last inequality holds however under even weaker assumptions. There are in fact analogous results to the ones given above (with weaker assumptions) for $\mathscr{A}$-cat. We state them below without proofs. These are completely analogous to the given ones for $\mathscr{A}$-Cat.

Proposition $3'$. *For every class of spaces $\mathscr{A}$, $\mathscr{A}$-cat$(X) \leq k$ if and only if X has the homotopy type of a k-fold mapping cylinder Z such that the inclusion $Z_v \subset Z$ of each vertex-space Z_v factors through some space in $\mathscr{A}$ up to homotopy.* □

We shall say that a class $\mathscr{C}$ satisfies condition $(*)$ for a space X if the following holds

$(*)$ For each finite family of maps $f_i : C_i \to X$ with $C_i \in \mathscr{C}$, the induced map on the disjoint union of the C_i's,

$$\{f_i\} : \coprod_i C_i \to X$$

factors through some space in $\mathscr{C}$ up to homotopy.

This condition is satisfied if, for example, there is a $\mathscr{C}$-universal map $u : U \to X$ in the sense of [2, 4.1]. Observe that conditions (a) and (b) above imply condition $(*)$ for the corresponding spaces.

Proposition 4$'$. *Let $\mathscr{C}$ be a class of spaces and let $F : K \to \mathrm{Top}$ be a functor such that the inclusion $F(v) \subset h\text{-colim}\, F$ of each vertex-space $F(v)$ factors through some space in $\mathscr{C}$ up to homotopy. Assume further that $\mathscr{C}$ satisfies condition $(*)$ for $h\text{-colim}\, F$. Then*

$$\mathscr{C}\text{-}\mathrm{cat}(h\text{-colim}\, F) \leq 1 + \dim K. \quad \square$$

Corollary 6$'$. *If $\mathscr{C}$-cat$(C) = k$ and $\mathscr{C}$ satisfies condition $(*)$ for X, then every numerable covering $(X_1, \dots, X_k)$ of X such that the inclusion $X_i \subset X$ factors through some space in $\mathscr{C}$ up to homotopy for all $i = 1, \dots, k$, has a nonempty intersection.* $\square$

Theorem 7$'$. *Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be classes of spaces such that $A \times B$ is dominated by some $C \in \mathscr{C}$ for all $A \in \mathscr{A}$, $B \in \mathscr{B}$ and let X and Y be spaces such that $\mathscr{C}$ satisfies condition $(*)$ for $X \times Y$. Then*

$$\mathscr{C}\text{-}\mathrm{cat}(X \times Y) \leq \mathscr{A}\text{-}\mathrm{cat}(X) + \mathscr{B}\text{-}\mathrm{cat}(Y) - 1. \quad \square$$

Applying these results to the particular examples mentioned in the introduction we obtain

Corollary 8. *If X and Y are path-connected then*

$$\mathrm{Cat}(X \times Y) \leq \mathrm{Cat}(X) + \mathrm{Cat}(Y) - 1 \quad (\textit{cf.}\ [7])$$

and

$$\mathrm{Cat}_{\min(p,q)}(X \times Y) \leq \mathrm{Cat}_q(X) + \mathrm{Cat}_p(Y) - 1.$$

The analogous inequalities for cat *and* cat_k *also hold.* $\square$

Corollary 9. *For all p, $q \geq 0$ and all spaces X and Y,*

$$\mathrm{Cat}^{p+q}(X \times Y) \leq \mathrm{Cat}^p(X) + \mathrm{Cat}^q(Y) - 1.$$

and analogously for cat^k. $\square$

2. THE MIXED PRODUCT THEOREM

Following Takens' ideas [7] we shall now prove

Theorem 10. *Let X and Y be spaces such that Y is path-connected and* $\mathrm{Cat}(Y) > 1$ *and let $\mathscr{C}$ be a class of spaces which satisfies condition $(*)$ for X. Then*

$$\mathscr{C}\text{-}\mathrm{Cat}(X \times Y) \leq \mathscr{C}\text{-}\mathrm{cat}(X) + \mathrm{Cat}(Y) - 1.$$

Proof. Let $k = \mathscr{C}\text{-}\mathrm{cat}(X)$ and $n = \mathrm{Cat}(Y)$, $n \geq 2$. We may assume that Y is an n-fold mapping cylinder whose vertex-spaces are points and we denote by $p : Y \to \Delta^{n-1}$ the canonical map. Let $S_i(j)$ be the closed $(1 - 1/(n+i+1))$-star of the jth vertex of Δ^{n-1}, $0 \leq i \leq k-1$, $0 \leq j \leq n-1$, and let

$$K_{ij} := S_i(j) - \mathrm{int}(S_i(j+1) \cup \cdots \cup S_i(n-1)).$$

Then we have:

(1) The canonical contraction of $S_i(j)$ induces a contraction of K_{ij}.

(2) $(K_{ij} | 0 \leq j \leq n-1)$ is a closed covering of Δ^{n-1} for each i.

(3) $(K_{ij} | i + j = q)$ is a disjoint family for each $q = 0, \dots, k+n-2$.

Now we make K_{ij} a little larger such that (keeping the same notation) the above properties (1)–(3) are still satisfied and in addition

(4) the covering $(K_{ij} | 0 \leq j \leq n-1)$ of Δ^{n-1} is numerable for each i.

Let $Y_{ij} = p^{-1}K_{ij}$. Since $n \geq 2$, there are embeddings $\omega_{ij} : [0, 1] \to Y$ such that the image of ω_{ij} meets Y_{ij} precisely in the point $\omega_{ij}(0)$ and, for each $0 \leq q \leq k+n-2$, the family $(Y_{ij} \cup \omega_{ij}[0, 1] | i + j = q)$ is disjoint except for one common point which is $\omega_{ij}(1)$ for all $i + j = q$. One way of getting such embedded paths is by constructing embeddings $\omega'_{ij} : [0, 1] \to \Delta^{n-1}$ which have the analogous properties for K_{ij} instead of Y_{ij} and taking their images under a section of p, which exists because of Corollary 6.

Let now $(X_0, \dots, X_{k-1})$ be a numerable covering of X such that each inclusion $X_i \subset X$ factors through some space in $\mathscr{C}$ up to homotopy. Since $\mathscr{C}$ satisfies condition $(*)$ for X there is a space $C \in h\mathscr{C}$ and maps $f_i : X_i \to C$ and $\alpha : C \to X$ such that αf_i is homotopic to $X_i \subset X$. Replacing α by its associated fibration we may assume that αf_i is the inclusion $X_i \subset X$.

Let Z_i be the mapping cylinder of $f_i : X_i \to C$ and Z be the mapping cylinder of $\{f_i\} : X_0 \sqcup \cdots \sqcup X_{k-1} \to C$ (in both cases we identify $(x, 1) \in X_i \times I$ with $f_i(x) \in C$). The idea is now to glue Z_i to $X_i \times Y_{ij}$ along $X_i \times 0 \subset Z_i$, obtaining a space which is homotopy equivalent to C, hence an element of $h\mathscr{C}$, and then embed it into a space of the homotopy type of $X \times Y$. More precisely we proceed as follows: Let $CZ = Z \times I / Z \times \{1\}$ be the cone of Z and let W_{ij} be the quotient space of the disjoint union of $X_i \times Y_{ij} \times CZ$ and Z_i obtained by identifying $(x, 0) \in Z_i$ with $(x, \omega_{ij}(0), x, 0, 0) \in X_i \times Y_{ij} \times CZ$ for each $x \in X_i$. Then, since Y_{ij} and CZ are contractible, C is a strong deformation retract of W_{ij}, and there is a closed embedding

$$W_{ij} \subset X \times Y \times CZ$$

which is just the inclusion on $X_i \times Y_{ij} \times CZ$, and on Z_i is given by

$$(x, t) \mapsto (x, \omega_{ij}(t), x, t, 0) \quad \text{and} \quad c \mapsto (\alpha(c), \omega_{ij}(1), c, 0)$$

for $x \in X_i$, $t \in [0, 1]$, $c \in C$. Let now

$$W_q := \bigcup_{i+j=q} W_{ij}.$$

$(W_q | q = 0, \dots, k+n-2)$ is a numerable covering of $X \times Y \times CZ$ and, since $W_{ij} \cap W_{rs} = C$ if $i+j = r+s$ and $i \neq r$, C is a strong deformation retract of W_q. But $X \times Y \times CZ \simeq X \times Y$, hence $\mathscr{C}\text{-Cat}(X \times Y) \leq k+n-1$. □

Replacing Y by $[0, 1]$ and p by the identity $(n = 2)$ in the proof of Theorem 10 we obtain a proof of

Theorem 11 [2, 5.6]. *If $\mathscr{C}$ satisfies condition $(*)$ for X, then*

$$\mathscr{C}\text{-Cat}(X) \leq \mathscr{C}\text{-cat}(X) + 1. \quad \square$$

Theorem 2 is just a special case of Theorems 10 and 11 together. Both of these theorems apply to the examples we gave in the introduction. In the classical case they are just Takens' mixed product theorem [7].

For the q-connective categories there is actually a better result, namely

Proposition 12.

$$\mathrm{Cat}_{\min(p,q)}(X \times Y) \leq \mathrm{cat}_p X + \max\{\mathrm{Cat}_q Y, 2\} - 1.$$

Proof. Let $r = \min(p, q)$. Observe that the spaces W_{ij} constructed in the proof of Theorem 10 are r-connected if Y_{ij} and C are r-connected. So the proof of Theorem 10 carries over to this situation. □

Remarks. We do not know whether a better mixed product formula holds in general, that is, a formula that looks like the one in Theorem 7 with $\mathscr{A}\text{-cat}(X)$ instead of $\mathscr{A}\text{-Cat}(X)$, assuming $\mathscr{B}\text{-Cat}(Y) > 1$. It is also not known whether Takens' formula is the best possible, that is, whether for example

$$\mathrm{Cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y) - 1$$

also holds for $\mathrm{cat}(Y) > 1$. The trouble in finding counterexamples in both cases is that one has to prove that Cat is greater than 3. In fact this is the main difficulty in finding examples of spaces X such that $\mathrm{Cat}(X) \neq \mathrm{cat}(X)$, for $\mathrm{cat}(X) \geq 3$ [2, 5.10]. For $\mathrm{cat}(X) = 2$ one has the Berstein-Hilton example C_α which is obtained by attaching a 7-cell to S^3 via the element α of order 3 in $\pi_6 S^3$. Then $\mathrm{cat}(C_\alpha) = 2$ but $\mathrm{Cat}(C_\alpha) = 3$ [1; 6, 2.4]. Using the formulas above one can show that

$$\mathrm{cat}(C_\alpha \times C_\alpha) = 3 \quad \text{and} \quad \mathrm{Cat}(C_\alpha \times C_\alpha) \leq 4$$

and one would of course like to have $\mathrm{Cat}(C_\alpha \times C_\alpha) = 4$. Now, Hans-Werner Henn has shown (personal communication) that after localizing away from 2, $\mathrm{Cat}(C_\alpha \times C_\alpha) = 3$. The unlocalized question remains open.

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