

TILED ORDERS OF FINITE GLOBAL DIMENSION

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Dedicated to Professor Hiroyuki Tachikawa on his 60th birthday

ABSTRACT. We define a projective link between maximal ideals, with respect to which an idealizer preserves being of finite global dimension. Let D be a local Dedekind domain with the quotient ring K . We show that for $2 \leq n \leq 5$, every tiled D -order of finite global dimension in $(K)_n$ is obtained by iterating idealizers w.r.t. projective links from a hereditary order. For $n \geq 6$, we give a tiled D -order in $(K)_n$ without this property, which is also a counterexample to Tarsy's conjecture, saying that the maximum finite global dimension of such an order is $n - 1$.

INTRODUCTION

Let D be a local Dedekind domain with a unique maximal ideal πD and the quotient ring K . If a D -order Λ in the full $n \times n$ matrix ring $(K)_n$ contains n orthogonal idempotents then Λ is isomorphic to a subring $(\pi^{\lambda_{ij}} D)$ of $(D)_n$ where $\lambda_{ij} \geq 0$, $\lambda_{ii} = 0$ and $\lambda_{ij} + \lambda_{jk} \geq \lambda_{ik}$ for any $1 \leq i, j, k \leq n$, and so Λ is called a *tiled D -order* in $(K)_n$ (cf. [5]).

In [2] we have introduced an idealizer with respect to a link between maximal ideals in an arbitrary ring and shown that every tiled D -order is obtained by iterating the idealizers w.r.t. links of maximal ideals from a hereditary order. In this paper we define a *projective link* between maximal ideals, with respect to which an idealizer preserves being of finite global dimension (Theorem 1.1) and show that for $2 \leq n \leq 5$, every tiled D -order of finite global dimension in $(K)_n$ is obtained by iterating the idealizers w.r.t. projective links from a hereditary order (Theorem 2.1). This fact is deeply connected with the following conjecture posed by V. A. Jategaonkar [5]:

(J) If Λ is a basic tiled D -order of finite global dimension then there is a vertex i in the quiver of Λ such that $|i^+| = 1$ or $|i^-| = 1$.

As noted in [5], if (J) were true it would be also true that

(T) the maximum finite global dimension of a D -order in $(K)_n$ is $n - 1$, holds for the class of tiled D -orders. This was conjectured by Tarsy [8]. For some classes of tiled D -orders, (T) is settled by Jategaonkar [4, 5] and Kirkman

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and Kuzmanovich [6], and in [6] a counterexample to (J) (but not to (T)) is obtained. We shall give a counterexample to (T) in the class of tiled D -orders in $(K)_n$ where $n \geq 6$ (Example 2.5). When $2 \leq n \leq 5$, (J) holds and using Theorem 2.1, a list of the representatives of isomorphism classes of tiled D -orders of finite global dimension in $(K)_n$ is obtained (Appendix).

We shall note that for the class of tiled D -orders of global dimension two, (J) holds. This gives some connection between these D -orders and maximal D -orders (Proposition 2.8).

Throughout this paper every ring is associative with an identity element and every module is unital.

The right (left) global dimension of a ring R is denoted by $\text{r.gl.dim } R$ ($\text{l.gl.dim } R$). Let R be an order in a simple artinian ring Q . Let I, J be subsets of Q . Then we put $(I:J)_r = \{q \in Q \mid Jq \subset I\}$, $(I:J)_l = \{q \in Q \mid qJ \subset I\}$, $O_r(I) = (I:I)_r$ and $O_l(I) = (I:I)_l$.

1. AN IDEALIZER W.R.T. A LINK

Let S be a ring and A a right ideal of S . Then the *idealizer* of A in S is the subring $\{s \in S \mid sA \subset A\}$ and denoted by $I_S(A)$. A is said to be *generative* if $SA = S$ holds. A subring R of $I_S(A)$ containing A is a *tame subidealizer* if A is a generative right ideal of S and if R/A is a semisimple artinian ring (cf. [3]). We also define similar notions for a left ideal. Let M_a, M_b be maximal ideals of S . Then there is a *link* from M_a to M_b , denoted by $M_a \rightsquigarrow M_b$, if $M_a \cap M_b \supseteq M_b M_a$ and if S/M_a and S/M_b are artinian. When $M_a \rightsquigarrow M_b$ and $M_a \neq M_b$, there exists a right ideal A of S such that

$$M_b/M_b M_a = A/M_b M_a \oplus (M_a \cap M_b)/M_b M_a.$$

The idealizer of A in S is called an *idealizer with respect to a link* $M_a \rightsquigarrow M_b$. It should be noted that if $\Lambda \supset \Gamma$ are successive tiled D -orders then Γ is obtained as an idealizer w.r.t. a link in Λ (cf. [2, §4]).

Utilizing tame subidealizers, we can obtain bounds of global dimension of an idealizer w.r.t. a certain link.

Theorem 1.1. *Let S be a ring and M_a, M_b maximal ideals of S such that $M_a \rightsquigarrow M_b$ and $M_b \not\rightsquigarrow M_a$, and let R be an idealizer w.r.t. $M_a \rightsquigarrow M_b$. Suppose that there exists a ring T such that S is contained in T as a subring and that M_b is a generative right ideal of T (or M_a is a generative left ideal of T). Then*

$$\text{r.gl.dim } S - 1 \leq \text{r.gl.dim } R \leq \text{r.gl.dim } S + 1$$

and

$$\text{l.gl.dim } S - 1 \leq \text{l.gl.dim } R \leq \text{l.gl.dim } S + 1.$$

Proof. It follows from [2, Lemma 1.1] that there are a right ideal A and a left ideal B of S such that $M_b \supset A \supset M_b M_a$, $M_a \supset B \supset M_b M_a$,

$$M_b/M_b M_a = A/M_b M_a \oplus (M_a \cap M_b)/M_b M_a,$$

$$M_a/M_bM_a = B/M_bM_a \oplus (M_a \cap M_b)/M_bM_a$$

and that $R = I_S(A) = I_S(B)$. Note that $M_bM_a = A \cap B$. Hence by [2, Corollary 1.4], M_b/M_bM_a is semisimple as a left R -module. Since $B(A/M_bM_a) = 0$ and $A((M_a \cap M_b)/M_bM_a) = 0$, A/M_bM_a is a homogeneous component, so fully invariant in M_b/M_bM_a as left R -modules. Since $M_b \not\rightsquigarrow M_a$, $M_aM_b = M_a \cap M_b \supset M_bM_a$. Suppose that M_b is a generative right ideal of T . Then note that $M_b^2 = M_bTM_b = M_bT = M_b$. Hence

$$M_bM_a \supset M_bM_aM_bT \supset M_bM_bM_aT = M_bM_aT \supset M_bM_a,$$

so that $M_bM_a = M_bM_aT$. Therefore for any $t \in T$, $f_t: x + M_bM_a \mapsto xt + M_bM_a$ is a well-defined mapping in $\text{End}_R(M_b/M_bM_a)$. Hence

$$(At + M_bM_a)/M_bM_a = f_t(A/M_bM_a) \subset A/M_bM_a$$

and thus A is a right ideal of T . Moreover by [2, Lemma 1.1(b)], $TA = TSA = TM_b = T$. Therefore A is a generative right ideal of T . In a similar way, we can show that if M_a is a generative left ideal of T then so is B . Consequently S and R are tame subidealizers in T . It is now easy to complete the proof by [3, Theorems 2.2 and 2.10].

When S is a noetherian prime ring, the theorem is rephrased as follows.

Corollary 1.2. *Let S be a noetherian prime ring and M_a, M_b maximal ideals of S with $M_a \rightsquigarrow M_b$ and $M_b \not\rightsquigarrow M_a$, and let R be an idealizer w.r.t. $M_a \rightsquigarrow M_b$. If M_b is idempotent and left S -projective (or M_a is idempotent and right S -projective), then*

$$\text{gl.dim } S - 1 \leq \text{gl.dim } R \leq \text{gl.dim } S + 1.$$

Proof. Recall that a noetherian prime ring is an order in a simple artinian ring. Put $T = O_r(M_b)$. Suppose that M_b is idempotent and left S -projective. Then $T = (S: M_b)_r$ and $(S: M_b)_r M_b = T$, so that M_b is a generative right ideal of T (and vice versa (cf. [3, Proposition 1.1])). Thus T satisfies the hypotheses of the theorem. This completes the proof.

Definition. We call a link between maximal ideals *projective* if it satisfies the hypotheses of the theorem.

Remarks. (1) In [8], Tarsy conjectured that if $\Lambda \supset \Gamma$ are successive D -orders of finite global dimension then their global dimensions differ by at most one. A counterexample to this conjecture is given in [4]. However, if Λ and Γ are basic tiled D -orders and if the associated link is projective then this conjecture follows from the theorem.

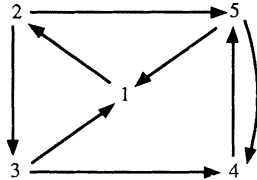
(2) The hypothesis ‘ $M_b \not\rightsquigarrow M_a$ ’ is necessary (cf. [2, Proposition 5.6]).

(3) In the theorem, $\text{gl.dim } S - 1$ and $\text{gl.dim } S + 1$ are the best bounds of $\text{gl.dim } R$.

Example 1.3. Let Λ be the basic tiled D -order

$$\begin{pmatrix} D & \pi D & \pi D & \pi D & \pi D \\ D & D & \pi D & \pi D & \pi D \\ D & D & D & \pi D & \pi D \\ \pi D & \pi D & \pi D & D & \pi D \\ D & D & \pi D & D & D \end{pmatrix}.$$

Then $\text{gl.dim } \Lambda = 3$ and the link graph of Λ is given by



Let M_i be the set of all matrices in Λ whose (i, i) -entries are in πD ($i = 1, \dots, 5$). Then M_1, \dots, M_5 are the maximal ideals of Λ . Note that M_1 and M_4 are left Λ -projective. Let α_1, α_2 , and α_3 be the links $M_3 \rightsquigarrow M_4, M_3 \rightsquigarrow M_1$, and $M_5 \rightsquigarrow M_1$, respectively, and let Γ_i be the idealizer w.r.t. α_i ($i = 1, 2, 3$). Then $\text{gl.dim } \Gamma_1 = 2$, $\text{gl.dim } \Gamma_2 = 3$, and $\text{gl.dim } \Gamma_3 = 4$.

2. TILED ORDERS OF FINITE GLOBAL DIMENSION

Let D be a local Dedekind domain with a unique maximal ideal πD and the quotient ring K . The main object of this section is to prove

Theorem 2.1. *Suppose that $2 \leq n \leq 5$. Let Λ be a basic tiled D -order in $(K)_n$ and $\text{gl.dim } \Lambda < \infty$. Then there exists a chain of basic tiled D -orders $\Lambda = \Lambda_0 \subset \Lambda_1 \subset \dots \subset \Lambda_t$ such that Λ_t is hereditary and that Λ_{i-1} is the idealizer w.r.t. a projective link in Λ_i where $i = 1, \dots, t$. If $n \geq 6$ then there exists a tiled D -order of finite global dimension in $(K)_n$ which does not have the above property.*

We now recall some definitoins and notation of tiled D -orders. Let $n \geq 2$ and let $\Lambda = (\pi^{\lambda_{ij}} D)$ be a basic tiled D -order in $(K)_n$, where λ_{ij} 's are nonnegative integers, $\lambda_{ii} = 0$, $\lambda_{ij} + \lambda_{jk} \geq \lambda_{ik}$, and $\lambda_{ij} + \lambda_{ji} > 0$ (if $i \neq j$) for any $1 \leq i, j, k \leq n$. Put $m_{kij} = 1$ (if $i = j = k$) and λ_{ij} (otherwise) for any $1 \leq i, j, k \leq n$ and $M_k = (\pi^{m_{kij}} D)$ for any $1 \leq k \leq n$. Then M_1, \dots, M_n are the maximal ideals of Λ and $J = M_1 \cap \dots \cap M_n$ is the Jacobson radical of Λ . Let e_i be the matrix in $(K)_n$ whose (i, i) -entry is 1 and the others are 0. $\mathcal{Q}(\Lambda) = (\mathcal{Q}(\Lambda)_0, \mathcal{Q}(\Lambda)_1, v)$ is the valued quiver of Λ provided that $\mathcal{Q}(\Lambda)_0 = \{1, \dots, n\}$ is the set of vertices, $\mathcal{Q}(\Lambda)_1$ is the set of arrows defined by $i \rightarrow j \in \mathcal{Q}(\Lambda)_1$ if $e_j(J/J^2)e_i \neq 0$ and that v is the mapping from $\mathcal{Q}(\Lambda)_1$ to nonnegative integers such that for any $\alpha: i \rightarrow j \in \mathcal{Q}(\Lambda)_1$, $v(\alpha) = \lambda_{ji}$ (if $i \neq j$) and 1 (if $i = j$). Let

$$p: x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_m} x_m$$

be a path in $\mathcal{Q}(\Lambda)$. Then we put $v(p) = \sum_{i=1}^m v(\alpha_i)$ and we call p a $v(p)$ -path. $\mathcal{P}(\Lambda)$ denotes the set of all λ_{ij} -paths from j to i in $\mathcal{Q}(\Lambda)$ where $i, j \in \mathcal{Q}(\Lambda)_0$ and $i \neq j$. An order is associated with $\mathcal{P}(\Lambda)$, i.e., for $p_1, p_2 \in \mathcal{P}(\Lambda)$, $p_1 \leq p_2$ if p_1 is a subpath of p_2 . For $i \in \mathcal{Q}(\Lambda)_0$, i^+ (resp. i^-) denotes the set $\{x \in \mathcal{Q}(\Lambda)_0 \mid i \rightarrow x \in \mathcal{Q}(\Lambda)_1\}$ (resp. $\{y \in \mathcal{Q}(\Lambda)_0 \mid y \rightarrow i \in \mathcal{Q}(\Lambda)_1\}$). Note that $M_i \rightsquigarrow M_j$ if and only if $i \rightarrow j \in \mathcal{Q}(\Lambda)_1$. (See [9] and [2, §4].)

Let $I = (\pi^{a_{ij}} D)$ and $H = (\pi^{b_{ij}} D)$ be ideals of Λ , and put

$$x_{ij} = \text{Max}\{b_{kj} - a_{ki} \mid 1 \leq k \leq n\}$$

and

$$y_{ij} = \text{Max}\{b_{ik} - a_{jk} \mid 1 \leq k \leq n\}$$

for any $1 \leq i, j \leq n$. Then it holds that $(H:I)_r = \{q \in (K)_n \mid Iq \subset H\} = (\pi^{x_{ij}} D)$ and $(H:I)_l = \{q \in (K)_n \mid qI \subset H\} = (\pi^{y_{ij}} D)$.

Lemma 2.2. *Let M_b be a maximal ideal of Λ and put $\Gamma = O_r(M_b)$. Then $\Lambda = \mathbf{I}_\Gamma(M_b)$.*

Proof. Put $\Gamma = (\pi^{\gamma_{ij}} D)$ and $O_l(M_b) = (\pi^{y_{ij}} D)$. Since $\Lambda \subset O_r(M_b) \cap O_l(M_b)$, $\gamma_{ij} \leq \lambda_{ij}$ and $y_{ij} \leq \lambda_{ij}$ for any $1 \leq i, j \leq n$. It is easily checked that $\gamma_{ij} \geq \lambda_{ij}$ (if $i \neq b$) and $y_{ij} \geq \lambda_{ij}$ (if $j \neq b$). This completes the proof.

Lemma 2.3. *Let M_b be a maximal ideal of Λ and projective as a left Λ -module. Put $\Gamma = O_r(M_b) = (\pi^{\gamma_{ij}} D)$. Then there exists $l \in \mathcal{Q}(\Lambda)_0$ such that $b^+ = \{l\}$ and $l \neq b$. Moreover,*

- (1) *for any $1 \leq i \leq n$, $m_{bib} = m_{bil} + m_{blb}$, in particular $\lambda_{bl} + \lambda_{lb} = 1$.*
- (2) *$\gamma_{ij} = \lambda_{ij} - \lambda_{lb}$ (if $i = b$) and λ_{ij} (otherwise).*

Proof. Since M_b is left Λ -projective, $b^+ = \{l\}$ for some $l \in \mathcal{Q}(\Lambda)_0$ with $l \neq b$.

(1) Let $i \neq b$. Then any λ_{ib} -path from b to i contains the arrow $b \rightarrow l$. Hence $\lambda_{ib} = \lambda_{il} + \lambda_{lb}$ and so $m_{bib} = m_{bil} + m_{blb}$. Let $i = b$. Then for any $k \neq b, l$, $\lambda_{bk} + \lambda_{kb} = \lambda_{bk} + \lambda_{kl} + \lambda_{lb} \geq \lambda_{bl} + \lambda_{lb}$. Since $b \rightarrow b \notin \mathcal{Q}(\Lambda)_1$,

$$1 = m_{bbb} = \text{Min}\{m_{bbk} + m_{bkb} \mid 1 \leq k \leq n\} = \lambda_{bl} + \lambda_{lb}.$$

This completes the proof of (1).

(2) This follows from (1) and the proof of Lemma 2.2.

A right ideal of a ring is *semimaximal* if it is an intersection of finitely many maximal right ideals.

Lemma 2.4. *Let M_b be left Λ -projective with $b^+ = \{l\}$ and $\Gamma = O_r(M_b)$. Then the following statements are equivalent.*

- (1) $l^- = \{b\}$.
- (2) Γ is left Λ -projective.
- (3) M_b is a (semi)maximal right ideal of Γ .

Proof. (1) \Rightarrow (2). Let $j \neq b, l$. Then by the hypothesis, $\lambda_{lj} = \lambda_{lb} + \lambda_{bj}$. Hence $\lambda_{lj} - \lambda_{lb} = \lambda_{bj}$. Hence by Lemma 2.3(2), $\Gamma e_j = \Lambda e_j$. Moreover, by Lemma 2.3, $\Gamma e_b = \Lambda e_b$ and $\Gamma e_l \cong \Lambda e_b$, so that Γ is left Λ -projective.

(2) \Rightarrow (1). Since ${}_{\Lambda}\Gamma$ is finitely generated and projective, $\Gamma/M_b \cong \Lambda/M_b \otimes_{\Lambda} \Gamma$ is simple as a right Γ -module. By Lemma 2.2, $\Lambda = \mathbf{I}_{\Gamma}(M_b)$. Thus it follows from [7, Theorem 1.3] that $0 \rightarrow \Lambda/M_b \rightarrow \Gamma/M_b \rightarrow \Gamma/\Lambda \rightarrow 0$ is a nonsplit exact sequence of right Λ -modules and $(\Gamma/\Lambda)_{\Lambda}$ is simple. Since $b^+ = \{l\}$, $M_l = \text{ann}_{\Lambda}(\Gamma/\Lambda) = (\Lambda:\Gamma)_r$. Since ${}_{\Lambda}\Gamma$ is projective, $M_l\Gamma = O_r(\Gamma) = \Gamma$. Hence M_l is right Λ -projective, so that $l^- = \{b\}$.

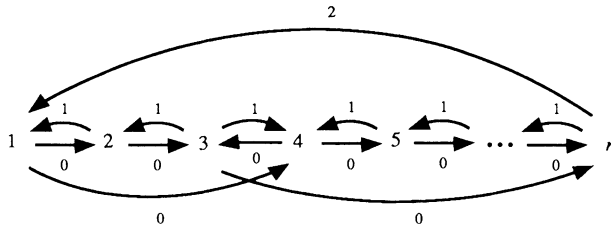
(2) \Leftrightarrow (3). This follows from [3, Proposition 1.3] and Lemma 2.2.

Let n_0 be the maximum number among n 's satisfying the following condition:

(J) If Λ is a basic tiled D -order in $(K)_n$ and $\text{gl.dim } \Lambda < \infty$ then there exists $i \in \mathcal{Q}(\Lambda)_0$ such that $|i^+| = 1$ or $|i^-| = 1$.

It follows from [5, Theorem 4.2; 4, Lemma 5] that $n_0 \geq 4$. The following example shows that $n_0 \leq 5$, which is also a counterexample to Tarsy's conjecture.

Example 2.5. Let Λ be the basic tiled D -order whose valued quiver is given by



here $n \geq 6$. Then $\text{gl.dim } \Lambda = n$ and (J) does not hold.

Remarks. (1) This example is something like Tarsy's [8, Theorem 11].

(2) It is shown in [5] that for a fixed n , there are only finitely many isomorphism classes of tiled D -orders of finite global dimension in $(K)_n$. Hence there is the maximum finite global dimension of a tiled D -order in $(K)_n$. Although, at present, n is the maximum among known examples, we do not have enough reason to conjecture that n is the maximum.

By [5, Theorem 2.3], we can determine whether n_0 is 4 or 5 at least in principle. Making use of the idealizers w.r.t. links [2, Proposition 4.6], we can shorten the computation and it is verified that when $n = 5$, (J) holds. Thus we obtain

Proposition 2.6. *The maximum number satisfying (J) is 5.*

Lemma 2.7. *Let Λ be a basic tiled D -order in $(K)_n$ and M_b a maximal ideal of Λ . Then the following statements are equivalent.*

- (1) M_b is left Λ -reflexive.
- (2) $O_r(M_b) \not\cong \Lambda$.
- (3) There exists $a \in \mathcal{Q}(\Lambda)_0$ such that $a \neq b$ and any λ_{ba} -path from a to b is maximal in $\mathcal{P}(\Lambda)$.

Proof. Since M_b is a maximal ideal, M_b is left Λ -reflexive if and only if $(\Lambda: (\Lambda: M_b)_r)_l \subsetneq \Lambda$ if and only if $(\Lambda: M_b)_r \supsetneq \Lambda$. As in the proof of Lemma 2.2, it is shown that $O_r(M_b) = (\Lambda: M_b)_r$. Hence (1) is equivalent to (2).

Put $O_r(M_b) = (\pi^{\gamma_{ij}}D)$. If $i \neq b$ then $\gamma_{ij} = \lambda_{ij}$ holds. Hence $O_r(M_b) \supsetneq \Lambda$ if and only if there is $a \in \mathcal{Q}(\Lambda)_0$ such that $a \neq b$ and $\lambda_{ba} > \text{Max}\{m_{bka} - m_{bkb} \mid 1 \leq k \leq n\}$ if and only if there is $a \in \mathcal{Q}(\Lambda)_0$ such that $a \neq b$ and $\lambda_{ba} + \lambda_{kb} > \lambda_{ka}$ for any $k \neq a, b$. Retaking a if necessary, this is equivalent to (3). This completes the proof.

Although the next proposition deviates from the proof of the theorem, we obtain it as an application of Lemma 2.7.

Proposition 2.8. *Let Λ be a basic tiled D -order in $(K)_n$ and $\text{gl.dim } \Lambda = 2$. Then there exist $a, b \in \mathcal{Q}(\Lambda)_0$ such that $|a^-| = 1$ and $|b^+| = 1$. Moreover, there exists a chain of tiled D -orders $\Lambda = \Gamma_0 \subset \Gamma_1 \subset \dots \subset \Gamma_m$ such that Γ_m is a maximal order, Γ_{i-1} is obtained as an idealizer in Γ_i ($1 \leq i \leq m$) and that for some $1 \leq l \leq m$, $\text{gl.dim } \Gamma_i = 1$ (if $l \leq i \leq m$) and 2 (if $0 \leq i < l$).*

Proof. Since $\mathcal{P}(\Lambda)$ is a finite set, there exist $a, b \in \mathcal{Q}(\Lambda)_0$ such that any λ_{ba} -path from a to b in $\mathcal{Q}(\Lambda)$ is maximal in $\mathcal{P}(\Lambda)$. Hence by Lemma 2.7, M_a (resp. M_b) is right (resp. left) Λ -reflexive. Since $\text{gl.dim } \Lambda = 2$, it follows from [1, Proposition 5.2] that M_a (resp. M_b) is right (resp. left) Λ -projective, i.e., $|a^-| = 1$ and $|b^+| = 1$. Put $\Gamma_1 = O_r(M_b)$. Then Γ_1 is finitely generated and projective as a right Λ -module. Hence $\text{gl.dim } \Gamma_1 \leq \text{gl.dim } \Lambda$. By Lemma 2.2, $\Lambda = \mathbf{I}_{\Gamma_1}(M_b)$. Repeating a similar argument, we obtain a desired chain.

We are now in a position to prove the theorem.

Proof of Theorem 2.1. Let Λ be a basic tiled D -order in $(K)_n$ with $a \rightarrow b \in \mathcal{Q}(\Lambda)_1$ and $a \neq b$, and let Γ be the idealizer w.r.t. $a \rightarrow b$ in Λ . Then it follows from [2, Theorem 4.9] that the new arrows in $\mathcal{Q}(\Gamma)_1$ are in the form of $a \rightarrow y$ or $x \rightarrow b$ where $x \in a^-$ and $y \in b^+$. Hence if $a \rightarrow b$ is a projective link then $|b^+| = 1$ or $|a^-| = 1$ in $\mathcal{Q}(\Gamma)$. Therefore any basic tiled D -order obtained by iterating the idealizers w.r.t. projective links from a hereditary order satisfies (J). Hence for $n \geq 6$, the tiled D -order of Example 2.5 is not obtained in such a way.

Suppose that $2 \leq n \leq 5$. Let Λ be a basic tiled D -order with $\text{gl.dim } \Lambda < \infty$. It follows from Proposition 2.6 that there exists $b \in \mathcal{Q}(\Lambda)_0$ such that $|b^-| = 1$ or $|b^+| = 1$. Let $b^+ = \{l\}$. Put

$$u = \begin{pmatrix} \pi^{\lambda_{l1}} & & 0 \\ & \ddots & \\ 0 & & \pi^{\lambda_{ln}} \end{pmatrix}$$

and $\Lambda' = u\Lambda u^{-1} = (\pi^{\lambda'_{ij}}D)$. Then $\lambda'_{ij} = 0$ for any $1 \leq j \leq n$ and $\lambda'_{ij} \geq 0$ for any $1 \leq i, j \leq n$ (cf. [5, Lemma 11]). Note that $\lambda'_{ij} + \lambda'_{ji} = \lambda_{ij} + \lambda_{ji}$. Hence

$\lambda'_{bl} = \lambda'_{bl} + \lambda'_{lb} = \lambda_{bl} + \lambda_{lb} = 1$ by Lemma 2.3(1). Hence for any $1 \leq j \leq n$, $0 = \lambda'_{lj} \leq \lambda'_{lb} + \lambda'_{bj} = \lambda'_{bj} \leq \lambda'_{bl} + \lambda'_{lj} = 1$, so that $\lambda'_{bj} = 0$ or 1 .

Suppose that $l^- \neq \{b\}$. Then for some a with $a \neq b, l$, $\lambda'_{ba} = 1$. Since $b \rightarrow l \in \mathcal{Q}(\Lambda')_1$, $0 = \lambda'_{lb} < \lambda'_{la} + \lambda'_{ab} = \lambda'_{ab}$. Hence $\lambda'_{ba} + \lambda'_{ab} \geq 2$. Since M_b is left Λ -projective, it follows from Lemma 2.7 that retaking a if necessary, any λ_{ba} -path a to b is maximal in $\mathcal{P}(\Lambda')$.

Put $\Lambda'_1 = (\pi^{\lambda'_{ij}} D)$ where $\lambda'_{1ij} = \lambda'_{ij} - 1$ (if $(i, j) = (b, a)$) and λ'_{ij} (otherwise). Then it follows from [2, Lemmas 4.3 and 4.5] that Λ'_1 is a basic tiled D -order and Λ' is the idealizer w.r.t. $a \rightarrow b$ in Λ'_1 . Put $\Gamma' = O_r(M'_b)$. Then $M'_{1b} = \Lambda'_1 M'_b$ is a maximal ideal of Λ'_1 by [2, Lemma 1.1], $\Gamma' = O_r(M'_{1b})$ and $\Gamma' M'_{1b} = \Gamma' M'_b = \Gamma'$. Therefore M'_{1b} is left Λ'_1 -projective, so that $b^+ = \{l\}$ in $\mathcal{Q}(\Lambda'_1)$. If $b \rightarrow a \in \mathcal{Q}(\Lambda'_1)_1$ then $a = l$, so $\lambda'_{1ab} = \lambda'_{lb} = 0$ and so $\lambda'_{1ab} + \lambda'_{1ba} = 0$, which is a contradiction. Thus $a \rightarrow b$ is a projective link in Λ'_1 . It follows from Theorem 1.1 that $\text{gl.dim } \Lambda'_1 < \infty$.

After at most $n-2$ times repetition of the above procedure, we obtain a chain of basic tiled D -orders of finite global dimension $\Lambda' = \Lambda'_0 \subset \Lambda'_1 \subset \dots \subset \Lambda'_r$ such that Λ'_{i-1} is the idealizer w.r.t. a projective link $a_i \rightarrow b$ in Λ'_i ($1 \leq i \leq r$) and $\lambda'_{rbj} = 0$ for any $j \neq l$ where $\Lambda'_r = (\pi^{\lambda'_{ij}} D)$. Hence $l^- = \{b\}$ in $\mathcal{Q}(\Lambda'_r)$. When $|b^-| = 1$, consider a similar argument as above. If Λ'_r is not hereditary, repeat the above argument. It is now easy to complete the proof by

Lemma 2.9. *Suppose that $2 \leq n \leq 5$. Let Λ be a basic tiled D -order in $(K)_n$ and $\text{gl.dim } \Lambda < \infty$. Then the following statements are equivalent.*

- (1) *For any $1 \leq i, j \leq n$ with $i \neq j$, $i^+ = \{j\}$ if and only if $j^- = \{i\}$.*
- (2) *Λ is a hereditary order.*

Proof. It is clear that (2) implies (1). We prove (1) \Rightarrow (2) by induction on n . If $n = 2$ then $\text{gl.dim } \Lambda < \infty$ if and only if Λ is hereditary, so that the assertion holds. Let $n \geq 3$. Then by Proposition 2.6 and the assumption, there exist $1 \leq b, l \leq n$ such that $b \neq l$, $b^+ = \{l\}$ and $l^- = \{b\}$. Put $\Gamma = O_r(M_b)$ and $\Gamma' = e\Gamma e$ where $e = \sum_{i \neq l} e_i$. Then Γ' satisfies (1) and by the induction hypothesis Γ' is hereditary. Since Γ is Morita equivalent to Γ' , Γ is also hereditary. By Lemma 2.2, $\Lambda = \mathbf{I}_\Gamma(M_b)$ and by Lemma 2.4, M_b is a semimaximal right ideal of Γ . Therefore it follows from [7, Theorem 4.3] that Λ is a hereditary order. This completes the proof.

Corollary 2.10. *Suppose that $2 \leq n \leq 5$. Then the set of basic tiled D -orders of finite global dimension in $(K)_n$ coincides with the set of basic tiled D -orders in $(K)_n$ obtained by finite iterations of the idealizers w.r.t. projective links from a hereditary order.*

Proof. This follows from Theorems 1.1 and 2.1.

In connection with the corollary and [2, Proposition 4.6], it might be a natural question whether any basic tiled D -order of finite global dimension appears in a

chain of successive tiled D -orders of finite global dimension from a hereditary order. The following example shows that this is not true.

Example 2.11. Let $\Lambda = (\pi^{\lambda_{ij}} D)$ be the basic tiled D -order

$$\begin{pmatrix} D & D & D & D & D & D \\ \pi D & D & \pi D & \pi D & D & D \\ \pi D & \pi D & D & D & D & D \\ \pi^2 D & \pi D & \pi^2 D & D & \pi D & D \\ \pi^2 D & \pi^2 D & \pi D & \pi D & D & D \\ \pi^2 D & \pi^2 D & \pi^2 D & \pi D & \pi D & D \end{pmatrix}.$$

Then $\text{gl.dim } \Lambda = 5$. Let $(a_1, b_1) = (6, 1)$, $(a_2, b_2) = (4, 2)$, $(a_3, b_3) = (3, 4)$, and $(a_4, b_4) = (2, 5)$. For $1 \leq k \leq 4$ and $1 \leq i, j \leq 6$, put $\gamma_{kij} = \lambda_{ij} - 1$ (if $(j, i) = (a_k, b_k)$) and λ_{ij} (otherwise) and put $\Gamma_k = (\pi^{\gamma_{kij}} D)$. Then $\Gamma_1, \dots, \Gamma_4$ are the successive tiled D -orders containing Λ and $\text{gl.dim } \Gamma_k = \infty$ for all $k = 1, \dots, 4$.

3. APPENDIX

In this appendix we shall give a list of the representatives of isomorphism classes of tiled D -orders of finite global dimension in $(K)_n$ where $n = 4, 5$. This list is obtained by computation using Theorem 2.1. When $n \leq 3$, see [8].

Let $\Lambda = (\pi^{\lambda_{ij}} D)$ be a tiled D -order. Then we put $d = d(\Lambda) = \sum_{i,j} \lambda_{ij}$. In the following matrices Λ , 2^i stands for $\pi^i D$ ($0 \leq i \leq 4$).

$d = 13$

$$\Lambda = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 1 & 1 \\ 4 & 2 & 2 & 1 & 1 \\ 4 & 2 & 2 & 2 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{ccccc} 1 & & & & 5 \\ \circ \uparrow & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \end{array}$$

$\text{gl.dim } \Lambda = 2$

$d = 13$

$$\Lambda = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 \\ 4 & 2 & 2 & 1 & 1 \\ 4 & 4 & 2 & 2 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{ccccc} 1 & & & & 5 \\ \circ \uparrow & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \end{array}$$

$\text{gl.dim } \Lambda = 2$

$d = 13$

$$\Lambda = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 \\ 4 & 2 & 2 & 1 & 2 \\ 4 & 2 & 2 & 2 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{ccccc} 1 & & & & 5 \\ \circ \uparrow & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \end{array}$$

$\text{gl.dim } \Lambda = 2$

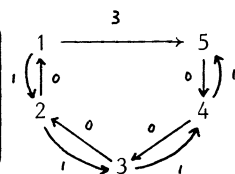
$d = 14$

$$\Lambda = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 1 & 1 \\ 4 & 2 & 2 & 1 & 1 \\ 4 & 4 & 2 & 2 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{ccccc} 1 & & & & 5 \\ \circ \uparrow & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \\ \begin{array}{ccccc} & & & & 5 \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \\ & & & & \circ \end{array} \end{array}$$

$\text{gl.dim } \Lambda = 3$

$d = 19$

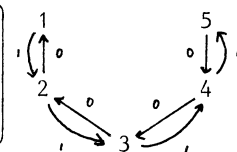
$$\Lambda = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 & 1 \\ 8 & 8 & 4 & 2 & 1 \end{pmatrix}$$



$\text{gl.dim } \Lambda = 3$

$d = 20$

$$\Lambda = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \end{pmatrix}$$



$\text{gl.dim } \Lambda = 2$

REFERENCES

1. H. Bass, *Finitistic dimension and a homological generalization of semiprimary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488.
2. H. Fujita, *An idealizer with respect to a link between maximal ideals*, Comm. Algebra **16** (1988), 1053–1082.
3. K. R. Goodearl, *Subrings of idealizer rings*, J. Algebra **33** (1975), 405–429.
4. V. A. Jategaonkar, *Global dimension of triangular orders over a discrete valuation ring*, Proc. Amer. Math. Soc. **38** (1973), 8–14.
5. ———, *Global dimension of tiled orders over a discrete valuation ring*, Trans. Amer. Math. Soc. **196** (1974), 313–330.
6. E. Kirkman and J. Kuzmanovich, *Global dimension of a class of orders*, preprint, Wake Forest Univ., 1988.
7. J. C. Robson, *Idealizers and hereditary noetherian prime rings*, J. Algebra **22** (1972), 45–81.
8. R. Tarsy, *Global dimension of orders*, Trans. Amer. Math. Soc. **151** (1970), 335–340.
9. A. Wiedemann and K. W. Roggenkamp, *Path orders of global dimension two*, J. Algebra **80** (1983), 113–133.

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