THE DIFFEOTOPY GROUP  
OF THE TWISTED 2-SPHERE BUNDLE OVER THE CIRCLE  

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Abstract. The diffeotopy group of the nontrivial 2-sphere bundle over the circle is shown to be isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The first generator is induced by a reflection across the base circle, while a second generator comes from rotating the 2-sphere fiber as one travels around the base circle. The technique employed also shows that homotopic diffeomorphisms are diffeotopic.

1. Introduction  
Gluck proved in [G] that the diffeotopy group $\mathcal{H}$ of $S^1 \times S^2$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In fact, for a large class of 3-dimensional manifolds, Rubinstein, Laudenbach, Waldhausen and others computed the diffeotopy groups. The methods exploited there do not seem to work for the twisted $S^2$-bundle over $S^1$, $S^1 \times S^2$. Therefore, we use different methods in conjunction with Gluck's arguments to compute the diffeotopy group $S^1 \times S^2$. In this paper, we shall prove the following:

Theorem. The diffeotopy group $\mathcal{H}$ of the twisted 2-sphere bundle over $S^1$, $S^1 \times S^2$, is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Corollary. Each diffeomorphism homotopic to the identity is diffeotopic to the identity.

2. Notation and proof of the theorem  
We adopt the following notation. Let $I$ be the unit interval $[0, 1]$ and $S^1$ the unit circle in the plane, i.e., the set of all complex numbers whose absolute value is 1. We will use $\exp 2\pi i \theta$ as a point of $S^1$, where $\theta$ is a real number and $i$ is $\sqrt{-1}$. Let $S^2$ be the unit sphere in the 3-dimensional Euclidean space. We will write $\nu$ as a point of $S^2$. In $S^1 \times S^2$, $\sim$ means every $(\exp 2\pi i \theta, \nu)$ in $S^1 \times S^2$ is identified with $(-\exp 2\pi i \theta, -\nu)$. $D^2$ will be the unit disk in the
plane which is the set of all complex numbers whose absolute value is less than or equal to 1.

To compute $\mathcal{G}$, we need the following crucial lemma.

(2.1) **Lemma.** Let $\text{Map}^1(S^2, S^2)$ be the set of all degree one continuous maps from $S^2$ to $S^2$. We assume that the topology is induced from the compact open topology.

Define a $\mathbb{Z}_2$-action on $\text{Map}^1(S^2, S^2)$ by

$$\mathbb{Z}_2 \times \text{Map}^1(S^2, S^2) \to \text{Map}^1(S^2, S^2)$$

$$\lambda \to A \circ \lambda \circ A$$

where $A$ is the antipodal mapping of $S^2$. Then the fundamental group of the quotient space is $\mathbb{Z}_2$.

**Proof.** Put $E = \{[\alpha] \mid \alpha : I \to \text{Map}^1(S^2, S^2), \alpha(0) = \text{the identity map}\}$, where $[\alpha] = [\beta]$ means that $\alpha(1) = \beta(1)$ and $\alpha \ast \overline{\beta}$ is homotopic to a constant path. More precisely, $\alpha \ast \overline{\beta}$ is the composition of $\alpha$ and $\beta$, i.e.,

$$\alpha \ast \overline{\beta}(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq 1/2, \\ \overline{\beta}(2t - 1) = \beta(2 - 2t), & 1/2 \leq t \leq 1. \end{cases}$$

Define $\pi : E \to \text{Map}^1(S^2, S^2)$ by $\pi([\alpha]) = \alpha(1)$. The space $E$ is simply connected, since $\text{Map}^1(S^2, S^2)$ is path-connected, and has a universal covering space (cf. [M, p. 394]).

We define two commuting $\mathbb{Z}_2$-actions on $E$. The first is given by

$$(\varphi, [\alpha]) \to [A \circ \alpha \circ A]$$

where $\varphi$ denotes the nontrivial element of $\mathbb{Z}_2$ in the first action. Next, we are going to use the fact, proved in [Hu], that $\Pi_1 \text{Map}^1(S^2, S^2) = \mathbb{Z}_2$. Then, given any path $\alpha$ starting at the identity, we can find a path $\gamma$, also, starting at the identity map with $\alpha(1) = \gamma(1)$ and $\alpha \ast \overline{\gamma}$ not homotopic to a constant path. We define the second involution

$$(\varphi, [\alpha]) \to [\gamma]$$

where $\varphi$ is the nontrivial element of $\mathbb{Z}_2$. This involution describes the generator of the group of covering transformations on $E$. This enables us to conclude that

$$\Pi_1(\text{Map}^1(S^2, S^2)/\mathbb{Z}_2) = \mathbb{Z}_2,$$

e i.e. $\Pi_1(E/\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \mathbb{Z}_2$,

because of M. Armstrong's result in [A]:

Let $G$ be a discontinuous group of homeomorphism of a simply connected, locally path connected, Hausdorff space $X$. Then the fundamental group of the quotient $X/G$ is $G/N$ where $N$ is the subgroup of $G$ generated by those elements which have fixed points.

Therefore, we have only to show that there exists no $[\alpha]$ in $E$ such that $\varphi \varphi [\alpha] = [\alpha]$ since $\varphi$ has no fixed point and $\varphi$ has a fixed point.
Suppose there exists such \([\alpha]\). Then we get
\[ x_{\gamma}[\alpha] = x[\gamma] = [A \circ \gamma \circ A] = [\alpha]. \]
Since \(\alpha(1) = A \circ \gamma(1) \circ A = \gamma(1), \) \(\alpha(1)\) is in the set \(\mathcal{F}\) of fixed points of the \(\mathbb{Z}_2\)-action on \(\text{Map}^1(S^2, S^2)\). Since \([\gamma] = [A \circ \alpha \circ A], (A \circ \alpha \circ A) \ast \bar{\gamma} = 0,\) and \(\alpha \ast \bar{\gamma}\) is not homotopic to a constant path. Since \((A \circ \alpha \circ A) \ast \bar{\gamma} \ast \gamma \ast \bar{\alpha}\) is not homotopic to a constant path, \((A \circ \alpha \circ A) \ast \bar{\alpha}\) is not homotopic to a constant path. This will lead to a contradiction.

We claim that there is a path \(\beta\) in \(\text{Map}^1(S^2, S^2)\) going from identity to \(\alpha(1)\) which is fixed under the involution on \(\text{Map}^1(S^2, S^2)\). We use the fact that the set of self-homotopy equivalences of \(RP^2\) is path-connected (see [GK]). This set corresponds exactly to the set of maps in \(\text{Map}^1(S^2, S^2)\) that are fixed by the given action \(\lambda \rightarrow A \circ \lambda \circ A\). So there is a map \(\beta: I \rightarrow \text{Map}^1(S^2, S^2)\) with \(\beta(0) = \text{identity and } \beta(1) = \alpha(1), \) such that \(\beta(t)\) lies in the fixed points set \(\mathcal{F}\). Observe that \((A \circ \alpha \circ A) \ast \bar{\beta} \ast \beta \ast \bar{\alpha}\) is not homotopic to a constant path.

Let \(\delta = \beta \ast \bar{\alpha}\). Then \(A \circ \delta \circ A = (A \circ \beta \circ A) \ast (A \circ \alpha \circ A) = \beta \ast (A \circ \alpha \circ A).\) Since \(\lambda \rightarrow A \circ \lambda \circ A\) is an involution and \(\Pi_1(\text{Map}^1(S^2, S^2)) = \mathbb{Z}_2, A \circ \delta \circ A\) is homotopic to \(\delta\). Also, each element is its own inverse and \(\beta \ast (A \circ \alpha \circ A)\) is homotopic to \((A \circ \alpha \circ A) \ast \bar{\beta}\. Thus \((A \circ \alpha \circ A) \ast \bar{\beta} \ast \beta \ast \bar{\alpha}\) is homotopic to \(\delta \ast \delta\), which is trivial. This is a contradiction. We have proved the lemma. \(\square\)

(2.2) **Corollary.** Let \(g\) be the self-diffeomorphism of \(S^1 \times S^2\) defined by
\[
[\exp 2\pi i \theta, v] \rightarrow \left[ \exp 2\pi i \theta, \begin{pmatrix} \cos 4\pi \theta & \sin 4\pi \theta & 0 \\ -\sin 4\pi \theta & \cos 4\pi \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (v) \right].
\]
Then \(g\) cannot be extended to a map from \((D^2 \times S^2)/\simeq\) to itself, where \(\simeq\) means every point \((r \exp 2\pi i \theta, v)\) in \(D^2 \times S^2\) is identified with \((-r \exp 2\pi i \theta, -v), 0 \leq r \leq 1.\)

**Proof.** Suppose there exists an extension \(k\) of \(g.\) Since \(D^2 \times S^2\) is the universal covering space of \((D^2 \times S^2)/\simeq\), we can lift \(k\) to \(\hat{k}\) on \(D^2 \times S^2\). Let us examine the value of the second coordinate under the mapping \(\hat{k}\), i.e., consider the following commutative diagram
\[
\begin{array}{ccc}
(r \exp 2\pi i \theta, v) & \xrightarrow{(-, K(r \exp 2\pi i \theta, v))} & (r \exp 2\pi i \theta, v) \\
\downarrow & & \downarrow \\
(D^2 \times S^2)/\simeq & \xrightarrow{k} & (D^2 \times S^2)/\simeq
\end{array}
\]
Without loss of generality, since $\text{Map}^1(S^2, S^2)$ is path-connected, we may assume that $K$ is a map from $D^2 \times S^2$ to $S^2$ such that
\[
K(\exp 2\pi i \theta, v) = \begin{pmatrix} \cos 4\pi \theta & \sin 4\pi \theta & 0 \\ -\sin 4\pi \theta & \cos 4\pi \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (v),
\]
and
\[
-K(-r \exp 2\pi i \theta, -v) = K(r \exp 2\pi i \theta, v).
\]
Define $\tilde{K}$ from $D^2$ to $\text{Map}^1(S^2, S^2)$ by
\[
\exp 2\pi i \theta \mapsto K(r \exp 2\pi i \theta, v).
\]
From the diagram above, we get $\tilde{K}(r \exp 2\pi i \theta) = A \circ \tilde{K}(r \exp 2\pi i \theta) \circ A$. Consider $p \circ \tilde{K}$. Then $p \circ \tilde{K}(r \exp 2\pi i \theta) = p \circ \tilde{K}(r \exp 2\pi i \theta)$ where $p$ is the projection from $\text{Map}^1(S^2, S^2)$ to $\text{Map}^1(S^2, S^2)/\mathbb{Z}_2$.
Define $\tilde{K}$ from $D^2$ to $\text{Map}^1(S^2, S^2)/\mathbb{Z}_2$ by
\[
\tilde{K}(\exp 2\pi i \theta) = p \circ \tilde{K}(r \exp 2\pi i \theta).
\]
This is well defined and
\[
\tilde{K}(\exp 2\pi i \theta) = \left( \begin{pmatrix} \cos 2\pi \theta & \sin 2\pi \theta & 0 \\ -\sin 2\pi \theta & \cos 2\pi \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).
\]
in $\text{Map}^1(S^2, S^2)/\mathbb{Z}_2$ where $\langle \rangle$ means the image under the projection $p$.
\[
\tilde{K}(0) = \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).
\]
This implies $\tilde{K}$ is a homotopy between following two maps
\[
\exp 2\pi i \theta \mapsto \left( \begin{pmatrix} \cos 2\pi \theta & \sin 2\pi \theta & 0 \\ -\sin 2\pi \theta & \cos 2\pi \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)
\]
and
\[
\exp 2\pi i \theta \mapsto \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).
\]
Since $p_* : \Pi_1(\text{Map}^1(S^2, S^2) \rightarrow \Pi_1(\text{Map}^1(S^2, S^2)/\mathbb{Z}_2) = \mathbb{Z}_2$ is onto, and the map $\tilde{T}$
\[
\exp 2\pi i \theta \mapsto \left( \begin{pmatrix} \cos 2\pi \theta & \sin 2\pi \theta & 0 \\ -\sin 2\pi \theta & \cos 2\pi \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)
\]
is the nontrivial loop in \(\text{Map}(S^2, S^2)\) (which we shall show later),

\[
\exp 2\pi i \theta \rightarrow \begin{pmatrix}
\cos 2\pi \theta & \sin 2\pi \theta & 0 \\
-\sin 2\pi \theta & \cos 2\pi \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

represents the nontrivial loop, we have a contradiction. It remains to show \(\tilde{T}\) is nontrivial. Consider the nontrivial \(S^2\)-fiber bundle over \(S^2\) with structure group \(SO(3)\), i.e., the space is given as follows:

\[
(D^2 \times S^2) \cup_T (D^2 \times S^2)
\]

= gluing two copies of \(D^2 \times S^2\) along the boundary by \(T\).

where \(T\) is a map from \(S^1 \times S^2\) to itself given by

\[
(\exp 2\pi i \theta, v) \rightarrow (\exp 2\pi i \theta, \tilde{T}(\exp 2\pi i \theta)(v)).
\]

Suppose \(\tilde{T}\) is homotopic to a constant path. Then we can get a homotopy equivalence from \((D^2 \times S^2) \cup_{id} (D^2 \times S^2)(= S^2 \times S^2)\) to \((D^2 \times S^2) \cup_T (D^2 \times S^2),\)

and we have a contradiction (see [S]). This completes the proof. \(\square\)

**Remark.** If we use the results in [KKR], we can give a shorter proof of the corollary above. More precisely, if \(g\) can be extended, then we may construct two spaces which must be homotopy equivalent, but by the homotopy invariant in [KKR], the two spaces can not be homotopy equivalent. So we have a contradiction.

*(2.3) Theorem.* The diffeotopy group \(\mathscr{G}\) of \(S^1 \times S^2\) is \(Z_2 \oplus Z_2\).

**Proof.** Our argument is divided into 4 steps.

1. We construct a map \(\varphi\) from \(\mathscr{G}\) to \(\mathscr{H}/Z_2\) (= \(Z_2 \oplus Z_2\)).
2. We show that the image of \(\varphi\) is \(Z_2\).
3. \(\text{Ker} \varphi\) is \(Z_2\).
4. \(Z_2 \rightarrow \mathscr{G} \rightarrow Z_2\) is split.

Recall that \(\mathscr{H} = \text{Diff}(S^1 \times S^2)/\sim\) where \(\sim\) is the normal subgroup consisting of those diffeomorphisms which are diffeotopic to identity and \(\mathscr{G} = \text{Diff}(S^1 \times S^2)/\sim\), where \(\sim\) means the same as above.

According to Gluck,

\[
\mathscr{H} = Z_2 \oplus Z_2 \oplus Z_2 = \langle T \rangle \oplus \langle f \rangle \oplus \langle h \rangle,
\]

\(S^1 \times S^2 \rightarrow S^1 \times S^2,\)

\(T: (\exp 2\pi i \theta, v) \rightarrow \left( \exp 2\pi i \theta, \begin{pmatrix} \cos 2\pi \theta & \sin 2\pi \theta & 0 \\ -\sin 2\pi \theta & \cos 2\pi \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}(v) \right),\)

\(f: (\exp 2\pi i \theta, v) \rightarrow (\exp(-2\pi i \theta), v),\)

\(h: (\exp 2\pi i \theta, v) \rightarrow (\exp 2\pi i \theta, -v).\)
(i) First step. Define a $\mathbb{Z}_2$-action on $\mathcal{H}$ as follows:

$$
\mathbb{Z}_2 \times \mathcal{H} \to \mathcal{H}
$$

$$
\langle \mathcal{L} \rangle \to \langle \phi \circ \mathcal{L} \rangle
$$

where $\phi: S^1 \times S^2 \to S^1 \times S^2$ is a diffeomorphism defined by

$$(\exp 2\pi i \theta, v) \to (-\exp 2\pi i \theta, -v).$$

We want to construct a map $\rho$ from $\mathcal{G}$ to $\mathcal{H}/\langle \phi \rangle$. To do it, we have to show that, given any diffeomorphism $w$ of $S^1 \times S^2$, we have a lift defined up to a covering transformation, i.e.,

$$
\begin{array}{ccc}
S^1 \times S^2 & \xrightarrow{\overline{w}} & S^1 \times S^2 \\
p \downarrow & & \downarrow p \\
S^1 \tilde{\times} S^2 & \xrightarrow{w} & S^1 \tilde{\times} S^2
\end{array}
$$

where $p: S^1 \times S^2 \to S^1 \tilde{\times} S^2$ is the natural projection.

To show the existence of $\overline{w}$, consider $(w \circ p)_* \Pi_1(S^1 \times S^2)$. $S^1 \times S^2$ is a double covering of $S^1 \tilde{\times} S^2$, so $p_* \Pi_1(S^1 \times S^2)$ is $2\mathbb{Z} \subset \mathbb{Z} = \Pi_1(S^1 \tilde{\times} S^2)$. Hence, any automorphism of $\Pi_1(S^1 \tilde{\times} S^2)$ preserves $p_* \Pi_1(S^1 \times S^2)$. By the lifting lemma, there exist $\overline{w}$ from $S^1 \times S^2$ to itself such that $p \circ \overline{w} = w \circ p$. Note that $p \circ \phi \circ \overline{w} = w \circ p$, since $\phi$ is a regular covering transformation. Furthermore, if $w_1$ is isotopic to $w_2$ by an isotopy $H$ on $S^1 \tilde{\times} S^2$, then, as in the above argument, we have $\overline{H}: I \times S^1 \times S^2 \to S^1 \times S^2$ such that $p \circ \overline{H} = H \circ (\text{Id} \times p)$. Now we can define a map $\phi$ from $\mathcal{G}$ to $\mathcal{H}/\mathbb{Z}_2$ by $\langle \eta \rangle \to \langle \overline{\eta} \rangle$.

Since $f \circ \phi = \phi \circ f$, $f$ induces a self-diffeomorphism on $S^1 \tilde{\times} S^2$. Thus $\text{Im} \phi \supset \mathbb{Z}_2$. Note that since $h$ is isotopic to $\phi$, $\langle h \rangle$ is trivial in $\mathcal{H}/\mathbb{Z}_2$.

(ii) Second step. We know that $\text{Im} \phi \supset \mathbb{Z}_2$, from First step. To demonstrate $\text{Im} \phi = \mathbb{Z}_2$, we shall show that there exist no $T'$ in $\mathcal{H}$ which is isotopic to $T$ and commutes with $\phi$.

Suppose that there exists such $T'$. That means the following:

$$
S^1 \times S^2 \times I \to S^1 \times S^2,
$$

$$
\{(\exp 2\pi i \theta, v), t\} \to (G_t(\exp 2\pi i \theta, v), F_t(\exp 2\pi i \theta, v))
$$

where $t = 0$, $G_0 = \exp 2\pi i \theta$, and

$$
F_0(\exp 2\pi i \theta, v) = \begin{pmatrix}
\cos 2\pi \theta & \sin 2\pi \theta & 0 \\
-\sin 2\pi \theta & \cos 2\pi \theta & 0 \\
0 & 0 & 1
\end{pmatrix}(v).
$$
Note that $F_0$ is the second coordinate of $T$. For $t = 1$, we have

\[
\begin{align*}
\begin{array}{ccc}
(\exp 2\pi i \theta, v) & \longrightarrow & (G_1(\exp 2\pi i \theta, F_1(\exp 2\pi i \theta, v)) \\
\cap & & \cap \\
S^1 \times S^2 & \longrightarrow & S^1 \times S^2 \\
\phi & \downarrow & \phi \\
S^1 \times S^2 & \longrightarrow & S^1 \times S^2 \\
\psi & & \psi \\
\end{array}
\end{align*}
\]

\[
(-\exp 2\pi i \theta, -v) \longrightarrow (G_1(-\exp 2\pi i \theta, -v), F_1(-\exp 2\pi i \theta, -v)).
\]

From the commutativity of the above diagram (since $T' \circ \phi = \phi \circ T'$)

\[
F_1(\exp 2\pi i \theta, -v) = -F_1(\exp 2\pi i \theta, v)
\]

Define $F'_i$ from $S^1$ to $\text{Map}^1(S^1, S^2)$ by

\[
\exp 2\pi i \theta \rightarrow F_i(\exp 2\pi i \theta, \_).
\]

Note that, since $\Pi_1(\text{Map}^1(S^2, S^2)) = Z_2$ is abelian, we need not worry about choosing a base point.

Recall the $Z_2$-action on $\text{Map}^1(S^2, S^2)$

\[
Z_2 \times \text{Map}^1(S^2, S^2) \rightarrow \text{Map}^1(S^2, S^2)
\]

\[
\lambda \rightarrow A \circ \lambda \circ A
\]

For each $t$,

\[
[F'_i] \in \Pi_1(\text{Map}^1(S^2, S^2)) \quad \text{and} \quad [p \circ F'_i] \in \Pi_1(\text{Map}^1(S^2, S^2)/Z_2),
\]

where $[\_]$ means the equivalence class of loops. Then $[p \circ F'_i]$ is trivial, since $p \circ F'_1(-\exp 2\pi i \theta) = p \circ F'_1(\exp 2\pi i \theta)$. Therefore $[p \circ F'_0]$ is trivial, since $p \circ F'_0$ is homotopic to $p \circ F'_0$.

By (2.1), $\Pi_1(\text{Map}^1(S^2, S^2)/Z_2) = Z_2$. Since $\Pi_1(\text{Map}^1(S^2, S^2)) = Z_2$ and the nontrivial element is represented by $F'_0: S^1 \rightarrow \text{Map}^1(S^2, S^2)$

\[
\exp 2\pi i \theta \rightarrow \begin{pmatrix}
\cos 2\pi \theta & \sin 2\pi \theta & 0 \\
-\sin 2\pi \theta & \cos 2\pi \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

then, the fact that the nontrivial element in $\Pi_1(\text{Map}^1(S^2, S^2)/Z_2)$ is lifted as the nontrivial element in $\Pi_1(\text{Map}^1(S^2, S^2))$ implies a contradiction. Thus, we can conclude that $\text{Im} \phi = Z_2$. 
(iii) Third step. Suppose \((q)\) in \(\ker \varphi\). Then \(q\) is isotopic to the identity or \(\phi \circ q\) is isotopic to the identity of \(S^1 \times S^2\). By a straightforward argument of Gluck (see [G, pp. 315–316] and cf. [T]), we deform \(q\) so that the restriction to \(\{1, -1\} \times S^2\) is the identity. Since \(\overline{q}\) (or \(\phi \circ q\)) is isotopic to the identity, \(q\) can be considered as a diffeomorphism \(\overline{q}\) from \(I \times S^2\) to \(I \times S^2\) such that the restriction to \(\{1, -1\} \times S^2\) of \(\overline{q}\) is the identity.

By Gluck (cf. [Ha]), \(\overline{q}\) is isotopic to the identity or to \(\overline{d}\) while fixing the \(\{1, -1\} \times S^2\), where \(\overline{d}: I \times S^2 \to I \times S^2\)

\[
(t, v) \to \left( t, \begin{pmatrix}
\cos 2\pi t & \sin 2\pi t & 0 \\
-\sin 2\pi t & \cos 2\pi t & 0 \\
0 & 0 & 1
\end{pmatrix} \right) (v).
\]

We claim that \(\overline{d}\) is \(\overline{g}\), where \(g\) is the self-diffeomorphism of \(S^1 \times S^2\) in (2.2). Obviously, \(g\) is the identity on \(\{1, -1\} \times S^2 \subset S^1 \times S^2\). Restrict \(g\) to \(S^1 \times S^2 - \{1, -1\} \times S^2\), and under the following identification,

\[
(\theta, v) \quad \longleftrightarrow \quad [(\exp 2\pi i \theta, v)]
\]

we get \(g': (0, 1/2) \times S^2 \to (0, 1/2) \times S^2\)

\[
(\theta, v) \to \left( \exp 2\pi i \theta, \begin{pmatrix}
\cos 4\pi \theta & \sin 4\pi \theta & 0 \\
-\sin 4\pi \theta & \cos 4\pi \theta & 0 \\
0 & 0 & 1
\end{pmatrix} \right) (v)
\]

Identify \((0, 1/2) \times S^2\) with \((0, 1) \times S^2\) by \((\theta, v) \to (2\theta, v)\).

Then, \(g'': (0, 1) \times S^2 \to (0, 1) \times S^2\)

\[
(2\theta, v) \to \left( 2\theta, \begin{pmatrix}
\cos 4\pi \theta & \sin 4\pi \theta & 0 \\
-\sin 4\pi \theta & \cos 4\pi \theta & 0 \\
0 & 0 & 1
\end{pmatrix} \right) (v), \quad 0 < \theta < 1/2.
\]
If we replace $2\theta$ by $\theta$, we get
\[
g'' = I : (0, 1) \times S^2 \to (0, 1) \times S^2
\]
\[
(\theta, v) \mapsto \begin{pmatrix}
\cos 2\pi \theta & \sin 2\pi \theta & 0 \\
-\sin 2\pi \theta & \cos 2\pi \theta & 0 \\
0 & 0 & 1
\end{pmatrix} (v).
\]
This proves the claim, i.e., $\bar{g} = \bar{d}$. Hence, $q$ is isotopic to the identity or $g$; $g$ is not isotopic to the identity, otherwise $g$ could be extended to $(D^2 \times S^2)/\sim$ (see (2.2)). To show $\text{Ker} \varphi = \mathbb{Z}_2$, it remains to show that $(g)^2 = \text{id}$, i.e., $g^2$ is isotopic to the identity. An isotopy is constructed as follows:
\[
S^1 \simeq S^2 \times I \to S^1 \simeq S^2
\]
\[
\{[(\exp 2\pi i \theta, v)], t\} \to [\exp 2\pi i \theta, H(\exp 4\pi i \theta, t(v))]
\]
where $H : S^1 \times I \to SO(3)$ is a homotopy between the maps
\[
\exp 2\pi i \theta \mapsto \begin{pmatrix}
\cos 4\pi \theta & \sin 4\pi \theta & 0 \\
-\sin 4\pi \theta & \cos 4\pi \theta & 0 \\
0 & 0 & 1
\end{pmatrix},
\exp 2\pi i \theta \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
(iv) Fourth step. The splitting follows from the fact that $(f)$ has order 2. Now we have completed the proof. $\square$

From (2.2) and (2.3), we get the following result.

**Corollary.** Any self-diffeomorphism of $S^1 \simeq S^2$ homotopic to the identity is diffeotopic to the identity.

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