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BASIC DUAL HOMOTOPY INVARIANTS OF RIEMANNIAN FOLIATIONS

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ABSTRACT. In this paper, we use the Sullivan minimal model construction to produce invariants for Riemannian foliations. Existence and vanishing results are proved for these invariants.

1. INTRODUCTION

In this paper, we prove existence and vanishing results for dual homotopy invariants of Riemannian foliations. We shall view a foliation as a decomposition of a manifold into submanifolds, called leaves, of the same dimension. Those differential forms, which are orthogonal to and invariant along the leaves, are called basic; they form a subdifferential graded algebra of the de Rham algebra. We observe that for Riemannian foliations, the truncated characteristic homomorphism factors into the algebra of basic forms. Using the Sullivan minimal model construction [S], we define basic dual homotopy invariants and explain how they are related to the dual homotopy invariants of Hurder [Hu]. It follows that our vanishing result provides criterion for vanishing of Hurder's dual homotopy invariants in the case of Riemannian foliations.

We point out that the idea of defining dual homotopy invariants for other complexes, besides the de Rham complex of the manifold as done by Hurder, has already been pursued by Lehmann [L]. However, the extensive structure theory of Riemannian foliations due to Molino [M] allows us to draw explicit conclusions concerning basic dual homotopy invariants. Corollaries (4.4) and (4.7) are of this nature.

This paper is organized as follows: in §2, some basic facts from Sullivan's theory of minimal models are recalled. In §3, we follow Hurder's construction of dual homotopy invariants and define basic dual homotopy invariants. In §4, our main results are stated and proved. Several corollaries also appear.

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All manifolds are assumed connected, and geometric structures are supposed to be C^∞.

2. Minimal models

In this section, we briefly summarize the results on minimal models, which will be required in the following. First, we recall the following notions. Some references are [S, GM, Ha1].

Definition (2.1). A differential graded algebra $(DGA)(A, d)$ is called minimal if

(i) it is freely generated as an algebra by a graded vector space with a well-ordered indexing, which respects the grading. If V denotes the set of generators, then

$$V = \{v_{(i)},\ i \in I = \text{some well-ordered index set}\}$$

with $\deg v_{(i)} > \deg v_{(j)} \Rightarrow i > j$; and

$A = \bigwedge V =$ the exterior algebra generated by the odd degree generators $\otimes$ the polynomial algebra generated by the even degree generators;

(ii) the differential d satisfies

$$dv_{(i)} \in \bigwedge\{v_{(<i)}\}.$$

Definition (2.2). Two DGA maps ϕ, $\psi\colon A \to B$ are said to be algebraic homotopic if there exists a DGA map

$$H\colon A \to B \otimes \bigwedge\langle t, dt\rangle$$

such that $H|_{t=0, dt=0} = \phi$, $H|_{t=1, dt=0} = \psi$; where $\bigwedge\langle t, dt\rangle$ denotes the DGA generated by t where $\deg t = 0$, and dt where $\deg dt = 1$, with differential d satisfying $d(t) = dt$, $d(dt) = 0$.

Theorem (2.3). *Let (A, d) be a connected DGA, i..e, $H^0(A, d) =$ the ground field. Then there exists a minimal DGA $\mathscr{M}$ and a DGA map $\phi\colon \mathscr{M} \xrightarrow{\sim} A$, such that ϕ induces an isomorphism in cohomology. Furthermore, $\mathscr{M}$ is unique up to algebraic homotopy. $(\mathscr{M}, \phi)$, sometimes denoted just by $\mathscr{M}$, is called the minimal model of A. The indecomposable elements of $\mathscr{M}$, $\mathscr{M}^+/(\mathscr{M}^+)^2$, which is isomorphic to a set of generators of $\mathscr{M}$, is well defined and is denoted by $\pi^*(A)$, the dual homotopy of A. If $\mathscr{M}$ is finite dimensional in each degree, the desuspension of the dual of $\pi^*(A)$, $s^{-1}\pi_*(A)$, forms a graded Lie algebra, called the desuspended homotopy of A.*

Theorem (2.4). *Let A and B be connected DGA's and let $\phi: A \to B$ be a DGA map. Then ϕ induces functorially a map $\mathscr{M}\phi$ between the minimal models of A and B:*

$$\begin{array}{ccc} \mathscr{M}A & \xrightarrow{\mathscr{M}\phi} & \mathscr{M}B \\ \sim\big\downarrow & & \big\downarrow\sim \\ A & \xrightarrow[\phi]{} & B \end{array}$$

such that the above diagram commutes up to algebraic homotopy. $\mathscr{M}\phi$ is uniquely defined up to algebraic homotopy by the algebraic homotopy commutativity of this diagram, and depends only on the algebraic homotopy class of ϕ. Furthermore, on the level of indecomposable elements, the induced (functorial) map

$$\phi^{\sharp}: \pi^*(A) \to \pi^*(B)$$

is well defined, and the corresponding desuspended dual

$$s^{-1}\phi_{\sharp}: s^{-1}\pi_*(B) \to s^{-1}\pi_*(A)$$

is a graded Lie algebra homomorphism.

3. Basic dual homotopy invariants

Let $\mathscr{F}$ be a Riemannian (or $SO(q)$) foliation of codimension q and dimension p on a manifold M. Denote by $Q(\mathscr{F})$ the principal $SO(q)$-normal bundle of $\mathscr{F}$, and by ω an adapted connection in $Q(\mathscr{F})$ [KT].

The usual Chern-Weil construction yields a map

$$h(\omega): I(SO(q)) \to \Omega_{\mathrm{DR}}(M)$$

where $I(SO(q))$ denotes the algebra of invariant polynomials on $so(q)$ and $\Omega_{\mathrm{DR}}(M)$ is the de Rham algebra of smooth differential forms on M.

Remark (3.1). We follow the convention that $I(SO(q))$ vanishes in odd degrees.

We recall the notions of basic connections and basic differential forms.

Definition (3.2). Let $(M, \mathscr{F})$ be a G-foliated manifold and ω an adapted connection in the normal bundle $Q(\mathscr{F})$. Then ω is said to be basic if

$$L_X\omega = 0$$

for every partially horizontal vector field X on $Q(\mathscr{F})$ (i.e., a global section of the Bott connection), where L_X denotes the Lie derivative along X.

Remark (3.3). Pasternak [P] showed that the normal bundle of every Riemannian foliation has a basic connection.

Definition (3.4). A smooth differential form φ on a foliated manifold $(M, \mathscr{F})$ is called basic, if

$$i(X)\varphi = L_X\varphi = 0$$

for every vector field X tangent to the leaves of $\mathscr{F}$. Here, i denotes interior product.

Remark (3.5). The basic differential forms constitute a sub-DGA of the de Rham algebra, denoted by $\Omega_B(\mathscr{F})$.

Theorem (3.6). *Let $(M, \mathscr{F})$ be a Riemannian-foliated manifold of codimension q, and ω a basic connection in the normal bundle $Q(\mathscr{F})$. Then the Chern-Weil construction induces a map*

$$h_{l,B}\colon I(SO(q))_l \to \Omega_B(\mathscr{F})$$

where $I(SO(q))_l$ denotes the truncated invariant polynomial algebra:

$$I(SO(q))_l = I(SO(q))/\{\textit{elements of degree} > l\}$$

for $l \geq q+1$. The map $h_{l,B}$ is independent of the choice of ω up to algebraic homotopy.

Proof. First, denote by Ω the curvature of ω:

$$\Omega = d\omega + \tfrac{1}{2}[\omega, \omega].$$

Then Ω is a basic form: for X a partially horizontal vector field,

$$\begin{aligned} i(X)\Omega &= i(X)\,d\omega + \tfrac{1}{2}i(X)[\omega, \omega] \\ &= i(X)\,d\omega, \quad \text{since } i(X)\omega = 0 \text{ by definition}, \\ &= i(X)\,d\omega + d\,i(X)\omega = L_X\omega = 0. \\ L_X\Omega &= L_X\{d\omega + \tfrac{1}{2}[\omega, \omega]\} \\ &= dL_X\omega + [L_X\omega, \omega] = 0. \end{aligned}$$

This implies that the Chern-Weil construction factors through the basic forms:

$$h(\omega)\colon I(SO(q)) \overset{h_B(\omega)}{\to} \Omega_B(\mathscr{F}) \subset \Omega_{\mathrm{DR}}(M).$$

Now, let ω_0, ω_1 be two basic connections and let $l \geq q+1$. Clearly, since $\Omega_B(\mathscr{F})$ vanishes in degrees exceeding q, $h(\omega_0)$, and $h(\omega_1)$ induce maps on the truncated invariant polynomial algebra:

$$h_{l,B}(\omega_0), h_{l,B}(\omega_1)\colon I(SO(q))_l \to \Omega_B(\mathscr{F}).$$

We will exhibit an algebraic homotopy

$$H\colon I(SO(q))_l \to \Omega_B(\mathscr{F}) \otimes \textstyle\bigwedge\langle t, dt\rangle,$$

with $H|_{t=0,dt=0} = h_{l,B}(\omega_0)$ and $H|_{t=1,dt=0} = h_{l,B}(\omega_1)$. Define

$$\begin{aligned} \Omega_t &= d\omega_0 \otimes t + d\omega_1 \otimes (1-t) - (\omega_0 - \omega_1) \otimes dt + [\omega_0, \omega_1] \otimes t(1-t) \\ &\quad + \tfrac{1}{2}[\omega_0, \omega_0] \otimes t^2 + \tfrac{1}{2}[\omega_1, \omega_1] \otimes (1-t)^2 \\ &\in \Omega^2(Q(\mathscr{F}), so(q)) \otimes \textstyle\bigwedge\langle t, dt\rangle. \end{aligned}$$

Observe that $d\omega_0$ $d\omega_1$, $\omega_0 - \omega_1$, $[\omega_0, \omega_1]$, $[\omega_0, \omega_0]$, $[\omega_1, \omega_1]$ are all annihilated by $i(X)$ and L_X for any partially horizontal vector field X on $Q(\mathscr{F})$. We define H as follows: for $\varphi \in I^{2r}(SO(q))$,

$$H: \varphi \mapsto \varphi(\Omega_t, \dots, \Omega_t) \quad (r \text{ times of } \Omega_t).$$

Then by the above observation and the $SO(q)$-invariance of φ, we see that

$$\begin{aligned} H(\varphi) &\in \Omega_B^{2r}(\mathscr{F}) \otimes \mathbf{R}[t] \oplus \Omega_B^{2r-1}(\mathscr{F}) \otimes \mathbf{R}[t] \otimes dt \\ &\subset \Omega_B(\mathscr{F}) \otimes \bigwedge\langle t, dt\rangle. \end{aligned}$$

Note that for $\varphi \in I^{2r}(SO(q))$, $2r > q+1$, $H(\varphi) \in \Omega_B^{>q}(\mathscr{F}) \otimes \bigwedge\langle t, dt\rangle = 0$. Thus,

$$H: I(SO(q))_l \to \Omega_B(\mathscr{F}) \otimes \bigwedge\langle t, dt\rangle$$

is well defined for $l \geq q+1$. Obviously, H satisfies the restriction requirements at $t = dt = 0$, and $t = 1$, $dt = 0$. □

Theorem (3.7). *In the same setting as the above theorem,* $h_{l,B}$ *induces*

$$\mathscr{M}h_{l,B}: \mathscr{M}I(SO(q))_l \to \mathscr{M}\Omega_B(\mathscr{F}),$$

which is well defined up to algebraic homotopy when $l \geq q+1$; *and*

$$h^\sharp_{l,B}: \pi^*(I(SO(q))_l) \to \pi^*(\Omega_B(\mathscr{F})),$$

which is well defined when $l \geq q+1$.

Proof. Follows from Theorem (2.5). □

Remark (3.8). $\mathscr{M}h_{l,B}$ and $h^\sharp_{l,B}$ will be denoted simply by $\mathscr{M}h$ and $h^\sharp$, respectively, if no confusion arises.

Definition (3.9). The image of

$$h^\sharp: \pi^*(I(SO(q))_l) \to \pi^*(\Omega_b(\mathscr{F})),$$

for $l \geq q+1$ are called basic dual homotopy invariants of $\mathscr{F}$.

Remark (3.10). In the following, we describe briefly the structure of $\pi^*(I(SO(q))_l)$ following Hurder [Hu] and Hurder and Kamber [HK]. First, recall that

$$I(SO(q)) = \begin{cases} \mathbf{R}[p_1, \dots, p_\alpha] & \text{for } q \text{ odd}, \\ \mathbf{R}[p_1, \dots, p_\alpha, e_q] & \text{for } q \text{ even}, \end{cases}$$

where

(i) $\alpha =$ the largest integer $\leq (q-1)/2$,
(ii) $\deg p_j = 4j$,
(iii) $\deg e_q = q$.

Denote the minimal model of $I(SO(q))_l$ by $\Phi: \mathscr{M}I(SO(q))_l \to I(SO(q))_l$, and denote by x_i (and x_e) the generators of $\mathscr{M}I(SO(q))_l$ such that

$$\Phi(x_i) = p_i \quad (\Phi(x_e) = e_q).$$

Denote by Z_l the DGA with zero differential and zero multiplication consisting of elements of the form

$$P_J \otimes y_I = p_1^{j_1} \cdots p_r^{j_r}(e_q^{j_e}) \otimes y_{i_1} \cdots y_{i_s}(y_e),$$

with $1 \leq i_1 \leq \cdots \leq i_s \leq r$ such that

(i) $\deg p_J \leq l$,
(ii) $\deg p_{i_1} p_J > l$,
(iii) $j_k = 0$, if $\deg p_k < \deg p_{i_1}$.

Z_l is sometimes called a Vey basis for the cohomology of the truncated Weil algebra $A(SO(q))_l$ [KT].

Since Z_l has zero differential and zero multiplication, it is formal and coformal [MN]. Consequently, the desuspended homotopy of Z_l forms a free Lie algebra generated by the desuspended dual of Z_l, denoted by $\mathscr{L}(s^{-1}Z_l^*)$. Then

$$s^{-1}\pi_*(I(SO(q))_l) \cong \langle s^{-1}x_1^*, \ldots, s^{-1}x_r^*, (s^{-1}x_e^*)\rangle \oplus \mathscr{L}(s^{-1}Z_l^*)$$

as vector spaces. Thus, as a (graded) Lie algebra, $s^{-1}\pi_*(I(SO(q))_l)$ is the extension of a free Lie algebra by an abelian one. The structure of this extension is described in [HK].

Remark (3.11). Hurder [Hu] defined dual homotopy invariants for a G-foliation $\mathscr{F}$ of codimension q on a manifold $\mathscr{M}$:

$$h_l^\sharp \colon \pi^*(I(G)_l) \to \pi^*(M),$$

where $l \geq q$ if $Q(\mathscr{F})$ has a basic connection and $l \geq 2q$ otherwise. In the case of a Riemannian foliation, they are related to the basic dual homotopy invariants as follows: for $l \geq q+1$,

$$\begin{tikzcd} \pi^*(I(SO(q))_l) \arrow[r, "h^\sharp"] \arrow[rr, bend right, "h_l^\sharp"'] & \pi^*(\Omega_B(\mathscr{F})) \arrow[r, "i^\sharp"] & \pi^*(\mathscr{M}) \end{tikzcd}$$

is commutative where $\iota^\#$ is induced by the inclusion $\iota\colon \Omega_B(\mathscr{F}) \subset \Omega_{\mathrm{DR}}(\mathscr{M})$. ($\pi^*(M) = \pi^*(\Omega_{\mathrm{DR}}(M))$.)

Remark (3.12). Hurder [Hu] proved that if a G-foliation $\mathscr{F}$ has a trivial normal bundle, the following diagram is commutative:

$$\begin{tikzcd} \pi^*(I(G)_l) \arrow[r, "h_l^\sharp"] & \pi^*(M) \\ Z_l \cong H^*(A(G)_l) \arrow[u, "j"] \arrow[r, "\Delta_*"'] & H^*_{\mathrm{DR}}(M), \arrow[u, "\mathscr{H}^*"'] \end{tikzcd}$$

where l is appropriately chosen according to the last remark, $\mathscr{H}^*$ denotes the dual Hurewicz map, and Δ_* is the characteristic homomorphism, whose images being the secondary characteristic invariants of $\mathscr{F}$ [KT].

Remark (3.13). For a Riemannian foliation $\mathscr{F}$ of codimension q on a manifold M, consider the diagram

$$\begin{tikzcd}
\pi^*(I(SO(q))_{q+1}) \arrow[r, "h^\sharp_{q+1,B}"] \arrow[d] & \pi^*(\Omega_B(\mathscr{F})) \arrow[r, "\imath^\sharp"] & \pi^*(M) \\
\pi^*(I(SO(q))_q) \arrow[urr, "h^\sharp_q"'] & &
\end{tikzcd}$$

It can easily be seen that

$$\operatorname{im}(\imath^\sharp \circ h^\sharp_{q+1,B}) \subseteq \operatorname{im} h^\sharp_q.$$

Furthermore, using the identification $Z_l \subseteq \pi^*(I(SO(q))_l)$, it is easily seen that

$$h^\sharp_q(Z_q)/\imath^\sharp \circ h^\sharp_{q+1,B}(Z_{q+1}) \cong h^\sharp_q(\mathscr{V}),$$

where $\mathscr{V} = \{p_J \otimes y_I \mid \deg p_i p_J = q+1\} \subseteq Z_q$. $\mathscr{V}$ is nonempty only when $q \equiv 3 \bmod 4$. In the case that the normal bundle of $\mathscr{F}$ is trivial, $\mathscr{V}$ consists exactly of the universal secondary invariants, which are variable [LP, Hu]. We remark further that if we choose a specific basic connection ω in the normal bundle of $\mathscr{F}$ and consider the map

$$h(\omega)^\sharp_{q,B} : \pi^*(I(SO(q))_q) \to \pi^*(\Omega_B(\mathscr{F})),$$

then $\mathscr{V}$ shows up in those universal basic dual homotopy invariants, which depend on the choice of ω.

4. Existence and vanishing of basic dual homotopy invariants

Theorem (4.1). *Let $\mathscr{F}$ be a codimension q Riemannian foliation on a manifold M, and let ω be a basic connection in the normal bundle of $\mathscr{F}$. Consider the map*

$$h(\omega)^\sharp = h(\omega)^\sharp_{q,B} : \pi^*(I(SO(q))_q) \to \pi^*(\Omega_B(\mathscr{F}))$$

induced by the Chern-Weil construction. Suppose there are two linearly independent elements α, $\beta \in \pi^*(I(SO(q))_q)$, $\deg\alpha$, $\deg\beta \le q$, whose images under $h(\omega)^\sharp$ are nontrivial and such that $\deg\alpha + \deg\beta > q$. Then the image of $h(\omega)^\sharp$ contains the suspended dual of a free (graded) Lie algebra with two generators. In particular, $s^{-1}\pi_*(\Omega_B(\mathscr{F}))$ contains a free Lie algebra with two generators.

Proof. We first show that $s^{-1}\alpha^*$ and $s^{-1}\beta^*$ generate a free Lie algebra in $s^{-1}\pi_*(I(SO(q))_q)$. Let $\langle\alpha, \beta\rangle$ denote the free DGA generated by α and β with zero multiplication and zero differential. Consider the inclusion

$$\langle\alpha, \beta\rangle \hookrightarrow I(SO(q))_q$$

and its minimal model [Ha1]:

$$\begin{tikzcd}
\langle\alpha, \beta\rangle \arrow[r] \arrow[dr] & (\langle\alpha, \beta\rangle \otimes \mathscr{M}, d) \arrow[r] \arrow[d] & (\mathscr{M}, d_{\mathscr{M}}) \\
& I(SO(q))_q. &
\end{tikzcd}$$

Clearly, $(\langle\alpha, \beta\rangle \otimes \mathscr{M}, d)$ is the tensor product of the two DGA's $(\langle\alpha, \beta\rangle, 0)$ and $(\mathscr{M}, d_{\mathscr{M}})$. Now, denote the minimal model of $\langle\alpha, \beta\rangle$ by $\mathscr{M}' \xrightarrow{\varphi} \langle\alpha, \beta\rangle$, and consider the DGA map

$$\varphi \otimes 1 : \mathscr{M}' \otimes \mathscr{M} \to \langle\alpha, \beta\rangle \otimes \mathscr{M}.$$

Clearly, $\varphi \otimes 1$ induces an isomorphism in cohomology, and therefore is a minimal model of $\langle\alpha, \beta\rangle \otimes \mathscr{M}$. So, by the uniqueness of minimal models, it is a minimal model of $I(SO(q))q$. Therefore,

$$\pi^*(I(SO(q))_q) = \pi^*(\mathscr{M}') \oplus \pi^*(\mathscr{M}).$$

Now, $\langle\alpha, \beta\rangle$ has zero multiplication and differential; thus, it has the real homotopy of a wedge of spheres, i.e., $s^{-1}\pi_*(\mathscr{M}')$ is a free Lie algebra generated by $s^{-1}\alpha^*$ and $s^{-1}\beta^*$ [MN, HK]. Hence $s^{-1}\pi_*(I(SO(q))_q)$ contains a free Lie algebra generated by $s^{-1}\alpha^*$ and $s^{-1}\beta^*$. To finish the proof, we note that the desuspended dual of $h^\sharp$:

$$s^{-1}h_\sharp : s^{-1}\pi_*(\Omega_B(\mathscr{F})) \to s^{-1}\pi_*(I(SO(q))_q)$$

is a (graded) Lie algebra map. According to the hypothesis, the image of $s^{-1}h_\sharp$ contains a free Lie algebra with 2 generators. So the preimage must contain a free Lie algebra with 2 generators. □

Theorem (4.2). *Let $\mathscr{F}$ be a codimension q Riemannian foliation on a manifold M. Consider the map*

$$h^*_{q+1,B} : H^*(I(SO(q))_{q+1}) \cong I(SO(q))_{q+1} \to H^*_B(\mathscr{F})$$

induced by the Chern-Weil construction. Suppose $h^*_{q+1,B}$ is the zero map; i.e., there are no nontrivial primary characteristic classes in the cohomology of the basic forms, then

$$h^\sharp_{q+1,B} : \pi^*(I(SO(q))_{q+1}) \to \pi^*(\Omega_B(\mathscr{F}))$$

is the zero map as well; i.e., there are no nontrivial basic dual homotopy invariants; in particular,

$$h^\sharp_{q+1} : \pi^*(I(SO(q))_{q+1} \to \pi^*(M)$$

is the zero map.

Proof. Let ω be a basic connection in the normal bundle of $\mathscr{F}$. Since $h^*_{q+1,B} = 0$, we have

$$p_i \xrightarrow{h(\omega)} h(\omega)(p_i) = d\alpha_i \in \Omega_B(\mathscr{F}), \quad \text{for some } \alpha_i \in \Omega_B(\mathscr{F}).$$

This allows us to extend $h(\omega)_{q+1,B}$ to the Weil algebra

$$\tilde{h}(\omega) : A(SO(q))_{q+1} = I(SO(q))_{q+1} \otimes \Lambda P_{q+1} \to \Omega_B(\mathscr{F})$$

by defining

$$\tilde{h}(\omega)(p_i) = h(\omega)(p_i),$$
$$\tilde{h}(\omega)(y_i) = \alpha_i, \quad \text{where } d\alpha_i = h(\omega)(p_i).$$

Then, we have the following diagram:

$$\begin{tikzcd}
I(SO(q))_{q+1} \arrow[r, hook] \arrow[rr, bend left, "h(\omega)"] & A(SO(q))_{q+1} \arrow[r, "\tilde{h}(\omega)"] & \Omega_B(\mathcal{F}) \\
\mathcal{M}I(SO(q))_{q+1} \arrow[r] \arrow[u, "\sim \psi"] & \mathcal{M}A(SO(q))_{q+1} \arrow[r, "\mathcal{M}\tilde{h}(\omega)"] \arrow[u, "\sim \theta"] & \mathcal{M}\Omega_B(\mathcal{F}), \arrow[u, "\sim \phi"]
\end{tikzcd}$$

which commutes up to algebraic homotopy. By Theorem (2.4), $\mathcal{M}\tilde{h}(\omega)$ is determined (up to algebraic homotopy) by the algebraic homotopy commutativity of the square on the right. Since $A(SO(q))_{q+1}$ is q-connected, it is easily seen that $\mathcal{M}A(SO(q))_{q+1}$ vanishes in degrees smaller than $q+1$. Thus, for degree reasons, $\tilde{h}(\omega)\circ\theta = 0$, and $\mathcal{M}\tilde{h}(\omega)$ is determined by $\phi\circ\mathcal{M}\tilde{h}(\omega)\sim 0$. Hence, $\mathcal{M}\tilde{h}(\omega) = 0$ up to algebraic homotopy. On the dual homotopy level, $h^\sharp$ is the composition of

$$\pi^*(I(SO(q))_{q+1}) \to \pi^*(A(SO(q))_{q+1}) \xrightarrow{0} \pi^*(\Omega_B(\mathcal{F})).$$

Thus, we have $h^\sharp_{q+1,B} = 0$, as claimed. □

By the same method, the following theorem can be proved.

Theorem (4.3). *Let $\mathcal{F}$ be a G-foliation on a manifold M with trivial normal bundle. Denote by ω an adapted connection in the normal bundle of $\mathcal{F}$ and by $\Delta(\omega)$ the characteristic homomorphism* [KT]

$$\Delta(\omega)\colon A(G)_l \to \Omega_{\mathrm{DR}}(M).$$

Suppose $\dim M \le 2l-2$ *and* $\Delta_* = \Delta(\omega)_*\colon H^*(A(G)_l) \to H^*_{\mathrm{DR}}(M)$ *is the zero map*; *i.e., there are no nontrivial secondary characteristic invariants. Then,*

$$h^\sharp = h^\sharp_l\colon \pi^*(I(G)_l) \to \pi^*(M)$$

is also the zero map; *i.e., there are no nontrivial dual homotopy invariants.*

Proof. We recall that $A(G)_l$ has a subalgebra Z_l consisting of cocycles which has zero multiplication, and that the inclusion $Z_l \subset A(G)_l$ induces an isomorphism in cohomology [KT]. Denumerate the elements of Z_l by $\{u_i\colon i = 1, 2, 3, \ldots\}$ such that $\deg u_i \le \deg u_j$, if $i \le j$. We recall that $\deg u_1 \ge l$. Now, since $\Delta_* = 0$, we have

$$\Delta(u_i) = d\rho_i, \quad \text{for some } \rho_i \in \Omega_{\mathrm{DR}}(M).$$

Denote by $B(G)_l$ the DGA $A(G)_l \otimes \Lambda\langle v_1, v_2, \ldots\rangle$ with differential δ satisfying $\delta v_i = u_i$. Then, Δ extends to a map on $B(G)_l$:

$$\widetilde{\Delta}\colon B(G)_l \to \Omega_{\mathrm{DR}}(M),$$

with

$$\widetilde{\Delta}(v_i) = \rho_i, \quad \text{where } d\rho_i = \Delta(u_i).$$

Consider the following diagram, which commutes up to algebraic homotopy:

$$\begin{tikzcd}
I(G)_l \arrow[r] \arrow[rrr, bend left=40, "h(\omega)"] & A(G)_l \arrow[r] \arrow[rr, bend left=30, "\Delta(\omega)"] & B(G)_l \arrow[r, "\widetilde{\Delta}(\omega)"] & \Omega_{\mathrm{DR}}(M) \\
\mathscr{M}I(G)_l \arrow[r] \arrow[u, "\sim"', "\psi"] & \mathscr{M}A(G)_l \arrow[r] \arrow[u, "\theta", "\sim"'] & \mathscr{M}B(G)_l \arrow[r, "\mathscr{M}\widetilde{\Delta}(\omega)"] \arrow[u, "\sim"', "\zeta"] & \mathscr{M}_M . \arrow[u, "\sim"', "\phi"]
\end{tikzcd}$$

Notice that $\mathscr{M}B(G)_l$ vanishes in degrees less than $2l-1$. On the other hand, under our assumption, $\Omega_{\mathrm{DR}}(M)$ vanishes in degrees larger than $2l-2$. Thus, $\widetilde{\Delta}(\omega)\circ\zeta = 0$ and $\mathscr{M}\widetilde{\Delta}(\omega) = 0$ up to algebraic homotopy. Hence,

$$h^\sharp \colon \pi^*(I(G)_l) \to \pi^*(A(G)_l) \to \pi^*(B(G)_l) \xrightarrow{0} \pi^*(M)$$

is the zero map as claimed. □

Corollary (4.4). *Let $\mathscr{F}$ be a transversally parallelizable foliation of codimension q. Then $\mathscr{F}$ has no nontrivial dual homotopy invariants of degree exceeding q.*

Proof. $\mathscr{F}$ being transversally parallelizable implies that $\mathscr{F}$ has a flat basic connection. Thus, the Chern-Weil map vanishes and Theorem (4.2) applies. □

Remark (4.5). Examples of transversally parallelizable foliations are Lie foliations [F].

Remark (4.6). In contrast, we point out that Hurder [Hu] has proved the existence of nontrivial dual homotopy invariants for many foliations with trivial normal bundle.

The following corollaries can easily be proved. We refer the readers to [Pa] for details.

Corollary (4.7). (i) *Let $\mathscr{F}$ be a G-foliation of codimension q on a rationally elliptic manifold M* [Ha2]. *Then the dual homotopy invariants of $\mathscr{F}$ of degree exceeding l have rank at most* 2.

(ii) *Let $\mathscr{F}$ be a G-foliation of codimension q on a manifold M such that the normal bundle is equipped with a basic connection. Suppose $\Omega_B(\mathscr{F})$ is rationally elliptic. Then the dual homotopy invariants of degree exceeding q have rank at most* 1, *with the nontrivial invariants* (*if any*) *of odd degree.*

Proof. We recall that the desuspended dual of $\pi^{>l}(I(G)_l)$ forms a free Lie algebra [HK]. Suppose two elements of $\pi^{>l}(I(G)_l)$, which correspond to linearly independent generators in the desuspended dual, give rise to nontrivial dual homotopy invariants; then the argument of Theorem (4.1) shows that the desuspended dual of these invariants generates a free Lie algebra in $s^{-1}\pi_*(M)$, which will have elements of arbitrarily high degree, contradicting the rational ellipticity of M. The second part follows easily from the fact that for a rationally elliptic DGA A with $H^i(A) = 0$ for $i > N$, $\pi^j(A) = 0$, if $j > N$ and j even, or $j > 2N-1$ and j odd [Ha2]. □

Remark (4.8). Blumenthal [B] proved that for a homogeneous G/K foliation $\mathscr{F}$ on a compact manifold, where G is a compact, connected Lie group and G/K is simply connected, there is a map

$$\varphi: \Omega_B(\mathscr{F}) \to \Omega_{\mathrm{DR}}(G/K),$$

which induces an isomorphism in cohomology. Thus, $\Omega_B(\mathscr{F})$ is rationally elliptic.

Corollary (4.9). *Let $\mathscr{F}$ be a codimension q Riemannian foliation. Suppose $\Omega_B(\mathscr{F})$ is coformal* [MN]; *then the only way in which basic dual homotopy invariants of degree exceeding q arise is the one described in Theorem* (4.1).

Proof. This result follows from the fact that coformality is equivalent to the differential of the minimal model being quadratic [MN]. □

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