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VARIETIES OF GROUP REPRESENTATIONS AND CASSON'S INVARIANT FOR RATIONAL HOMOLOGY 3-SPHERES

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ABSTRACT. Andrew Casson's $\mathbf{Z}$-valued invariant for $\mathbf{Z}$-homology 3-spheres is shown to extend to a $\mathbf{Q}$-valued invariant for $\mathbf{Q}$-homology 3-spheres which is additive with respect to connected sums. We analyze conditions under which the set of abelian $SL_2(\mathbf{C})$ and $SU(2)$ representations of a finitely generated group is isolated. A formula for the dimension of the Zariski tangent space to an abelian $SL_2(\mathbf{C})$ or $SU(2)$ representation is obtained. We also derive a sum theorem for Casson's invariant with respect to toroidal splittings of a $\mathbf{Z}$-homology 3-sphere.

Andrew Casson's lectures at MSRI in the spring of 1985 introduced an integer valued invariant of oriented integral homology 3-spheres. This invariant, constructed by means of representation spaces, yields interesting new results in low dimensional topology. In this paper we examine the extent to which Casson's procedure for defining his invariant can be used to obtain a rational valued invariant for oriented rational homology 3-spheres.

Let π be a finitely generated group and G a Lie group. It is well known that the set $R(\pi, G)$ of all homomorphisms of π into G can be given the structure of an analytic set in a natural manner. If G is an algebraic group, $R(\pi, G)$ becomes an algebraic set. The closed subspace of $R(\pi, G)$ consisting of representations $\pi \to G$ with abelian image will be denoted by $A(\pi, G)$. Let $R^n(\pi, G)$ be the union of those components of $R(\pi, G)$ which do not meet $A(\pi, G)$. When G is understood from the context, it will be dropped from the notation.

If R is a commutative ring, an R-homology 3-sphere is a closed, orientable (over $\mathbf{Z}$) 3-manifold with homology isomorphic to $H_*(S^3; R)$. Let $H(R)$ be the set of oriented homeomorphism types of oriented R-homology 3-spheres. For $M \in H(\mathbf{Z})$ Casson defined an integer valued invariant $\lambda(M)$. We briefly recall his definition (see [AM] for a comprehensive exposition of Casson's MSRI lectures). Let $M = W_1 \cup_F W_2$ be a Heegard decomposition of M, where $F = \partial W_i$ is of genus g and let F^* be F punctured once. The diagram of

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inclusions:

$$F^* \to F \begin{smallmatrix} \nearrow & W_1 & \searrow \\ & & \\ \searrow & W_2 & \nearrow \end{smallmatrix} M$$

induces a corresponding diagram of spaces of SU(2) representations:

$$R(\pi_1 M) \begin{smallmatrix} \nearrow & R(\pi_1 W_1) & \searrow \\ & & \\ \searrow & R(\pi_1 W_2) & \nearrow \end{smallmatrix} R(\pi_1 F) \to R(\pi_1 F^*).$$

Let

$$\begin{aligned} Q_i &= \text{image of } R(\pi_1 W_i) \text{ in } R^* = R(\pi_1 F^*), \\ R &= \text{image of } R(\pi_1 F) \text{ in } R^*, \\ A_i &= \text{image of } A(\pi_1 W_i) \text{ in } R^*, \\ A &= \text{image of } A(\pi_1 F) \text{ in } R^*. \end{aligned}$$

$R - A$ is an open manifold on which SU(2) /center acts freely by conjugation. Let

$$\begin{aligned} \widehat{R} &= R - A \text{ modulo action by conjugation}, \\ \widehat{Q}_i &= Q_i - A_i \text{ modulo action by conjugation.} \end{aligned}$$

The $\widehat{Q}_i$ embed properly in $\widehat{R}$ and their intersection is compact. The orientation of M can be used to determine an orientation of Q_i, R^*, $\widehat{Q}_i$, and $\widehat{R}$. Let $\langle Q_1, Q_2\rangle_{R^*}$ be the homological intersection number of the compact manifolds Q_1 and Q_2 in R^*. Casson proves that an algebraic intersection number $\langle \widehat{Q}_1, \widehat{Q}_2\rangle_{\widehat{R}}$ can be defined. His invariant is given by

Definition.

$$\lambda(M) = (-1)^g \langle \widehat{Q}_1, \widehat{Q}_2\rangle_{\widehat{R}} / 2\langle Q_1, Q_2\rangle_{R^*}.$$

Casson proves that this number is an integer and is independent of the Heegard decomposition of M. A key point in Casson's theory is that for $M \in H(\mathbf{Z})$, the trivial representation is an isolated point in $R(\pi_1 M, \mathrm{SU}(2))$; it is this fact that allows him to conclude that $\widehat{Q}_1 \cap \widehat{Q}_2$ is compact and thus making it possible to define the intersection number $\langle \widehat{Q}_1, \widehat{Q}_2\rangle_{\widehat{R}}$.

Let

$$\begin{aligned} \widehat{R}(\pi_1 M) &= \text{image of } R(\pi_1 M) - A(\pi_1 M) \text{ in } R^* \text{ modulo action by conjugation,} \\ \widehat{R}^n(\pi_1 M) &= \text{image of } R^n(\pi_1 M) \text{ in } R^* \text{ modulo action by conjugation.} \end{aligned}$$

$\widehat{R}(\pi_1 M)$ may fail to be compact; however, $\widehat{R}^n(\pi_1 M)$ is compact and is in fact the union of the compact components of $\widehat{R}(\pi_1 M)$. Our starting point is the observation that Casson's procedure for defining the intersection number $\langle \widehat{Q}_1, \widehat{Q}_2\rangle_{\widehat{R}}$ remains valid for an $M \in H(\mathbf{Q})$ *provided* one restricts attention to the compact components of $\widehat{Q}_1 \cap \widehat{Q}_2 = \widehat{R}(\pi_1 M)$. Using this interpretation of

$\langle \widehat{Q}_1, \widehat{Q}_2 \rangle_{\widehat{R}}$, we can now define $\lambda(M)$ for any $M \in H(\mathbf{Q})$ as in the $\mathbf{Z}$-homology sphere case. A routine homology computation reveals that $\langle Q_1, Q_2 \rangle_{R^*} = \pm$ order of $H_1(M)$. Thus in general, $\lambda(M)$ will be a rational number. Casson's arguments immediately extend to show that $\lambda(M)$ remains independent of the Heegard decomposition and that the property $\lambda(-M) = -\lambda(M)$ is retained.

The above definition raises the question: When is $\widehat{R}(\pi_1 M) = \widehat{R}^n(\pi_1 M)$? This occurs precisely when $\pi = \pi_1 M$ satisfies

Property A. *$A(\pi, \mathrm{SU}(2))$ is a union of components of $R(\pi, \mathrm{SU}(2))$.*

We will be interested in conditions on M which imply that Property A holds for $\pi_1 M$. This motivates the next definition and result.

Definition. A finitely generated group π is *cyclically finite* (CF) if each normal subgroup with finite cyclic quotient, other than the ones of maximal even index (on which there is no condition), has finite abelianization.

Theorem A. *Let π be a finitely generated group with finite abelianization. Then π is CF if and only if $A(\pi, \mathrm{SL}_2(\mathbf{C}))$ is a union of components of $R(\pi, \mathrm{SL}_2(\mathbf{C}))$.*

Theorem A is a consequence of the more general result, Theorem 1.1, which is the computation of the dimension of the Zariski tangent space to an abelian representation $\rho: \pi \to \mathrm{SL}_2(\mathbf{C})$ where π is an arbitrary finitely generated group and ρ has finite image. Since $\mathrm{SU}(2)$ is included as a Lie subgroup of $\mathrm{SL}_2(\mathbf{C})$ Theorem A implies the following:

Theorem B. *Let $M \in H(\mathbf{Q})$ be such that $\pi_1 M$ is CF. Then $\pi_1 M$ satisfies Property* A.

Many 3-manifold groups are CF; for example, any nonzero Dehn surgery on a knot in a $\mathbf{Z}$-homology 3-sphere whose Alexander polynomial has no roots of unity as zeros result in a manifold whose fundamental group is CF.

Another condition that implies that Property A holds for $\pi_1 M$, $M \in H(\mathbf{Q})$ can be deduced from Bass' $\mathrm{SL}_2(\mathbf{C})$ subgroup theorem. A closed 3-manifold will be called *not sufficiently large*, abbreviated NSL, if it is irreducible and contains no orientable incompressible surface of positive genus.

Theorem C. *Suppose $M \in H(\mathbf{Q})$ is NSL. Then $\pi_1 M$ satisfies Property* A.

We show by example that the hypotheses of these two theorems are independent; furthermore, we produce an irreducible, atoroidal $M \in H(\mathbf{Q})$ which does not satisfy the hypothesis of either.

In general the free product of two groups with Property A will not have Property A; however, we prove

Theorem. *Suppose $M \in H(\mathbf{Q})$ is a connected sum $M = M_1 \# M_2$. Then $\pi_1 M$ has Property* A *if and only if both $\pi_1 M_1$ and $\pi_1 M_2$ have Property* A *and at least one of $H_1(M_i)$ $i = 1, 2$ is a $\mathbf{Z}/2$ vector space.*

Suppose $M \in H(\mathbf{Q})$ decomposes as a connected sum $M = M_1 \# M_2$. When $M \in H(\mathbf{Z})$, results of Casson imply that $\lambda(M) = \lambda(M_1) + \lambda(M_2)$. Our generalized invariant is also additive with respect to connected sums.

Theorem D. *Let* $M_1, M_2 \in H(\mathbf{Q})$. *Then* $\lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2)$.

Now consider an $M \in H(\mathbf{Z})$, which splits along an embedded torus, i.e., $M = M_1 \cup M_2$ where $\partial M_1 = \partial M_2 \cong S^1 \times S^1$. Each ∂M_j can be canonically identified with $\partial(S^1 \times D^2)$ and thus there is a well-defined "closure" $\overline{M}_j = M_j \cup (S^1 \times D^2) \in H(\mathbf{Z})$.

Theorem E. $\lambda(M) = \lambda(\overline{M}_1) + \lambda(\overline{M}_2)$.

A more general formula for arbitrary (noncanonical) "closures" follows easily from Theorem E and Casson's Dehn surgery formula. Theorem E has been independently discovered by Akbulut and McCarthy (private communication) and by Fukuhara and Maruyama [FM]. Suppose $M \in H(\mathbf{Q})$ is prime. Then by results of Jaco and Shalen [JS] and Johannson [Jo], M has a canonical torus decomposition $M = \bigcup_i M_i$ and we define

$$\bar{\bar{\lambda}}(M) = \sum_i \lambda(\overline{M}_i).$$

Theorem E implies that for $M \in H(\mathbf{Z})$, $\bar{\bar{\lambda}}(M) = \lambda(M)$.

This study suggests a number of interesting problems and open questions, some of which have been included in the subsequent sections.

1

Throughout this section all representation spaces should be understood to consist of $\mathrm{SL}_2(\mathbf{C})$ representations.

Let π be a finitely generated group and suppose $\rho \in A(\pi)$ has finite image. This image is necessarily finite cyclic, say image $(\rho) \cong C_r$, $r \geq 1$. As the adjoint representation $\mathrm{Ad} : \mathrm{SL}_2(\mathbf{C}) \to \mathrm{Aut}(\mathrm{sl}_2(\mathbf{C}))$ has kernel $\{\pm I\}$, $\mathrm{Ad} \circ \rho$ has image C_n where

$$n = \begin{cases} r & \text{if } r \text{ is odd}, \\ r/2 & \text{otherwise}. \end{cases}$$

For each $d | n$, let $\Psi_d : C_n \to C_d$ be the surjection $\Psi_d(x) = x^{n/d}$ (the group law in C_n will be written multiplicatively). Define the homomorphism χ_d to be the composite:

$$\begin{tikzcd} \pi \arrow[r, "\mathrm{Ad} \circ \rho"] \arrow[dr, "\chi_d"'] & C_n \arrow[d, "\Psi_d"] \\ & C_d \end{tikzcd}$$

Set $\pi_d = \text{kernel}(\chi_d)$ and let $b_1(\pi_d)$ denote the rank of the abelianization of π_d.

Consider a field k. If V is a k-vector space and $\alpha: \pi \to \text{End}_k(V)$ a representation, we will use the notation V_α for the $k[\pi]$-module V. The jth cohomology of π over k with coefficients in V_α will be denoted $H^j(\pi; V_\alpha)$ or simply $H^j(\pi; \alpha)$ when V is clear from the context. Define

$$b_j(\pi; V_\alpha) = b_j(\pi; \alpha) = \dim_k H^j(\pi; V_\alpha).$$

The first result of this section is the calculation of $b_1(\pi; \text{Ad}\circ\rho)$ and of the dimension of T_ρ, the Zariski tangent space of $R(\pi)$ at ρ (see §1A of [Mu]) in terms of the numbers $b_1(\pi_d)$.

Let $\mu, \varphi: \mathbf{Z}_+ \to \mathbf{Z}$ be the Möbius and Euler functions [HW, Chapter XVI].

(1.1) **Theorem.**

(i) $b_1(\pi; \text{Ad}\circ\rho) = b_1(\pi) + \frac{2}{\varphi(n)} \sum_{d|n} \mu(n/d) b_1(\pi_d)$.

(ii) *If* $Z(\rho)$ *denotes the centralizer of* $\rho(\pi)$ *in* $\text{SL}_2(\mathbf{C})$, *then* $\dim_{\mathbf{C}} T_\rho = 3 - \dim_{\mathbf{C}} Z(\rho) + b_1(\pi) + \frac{2}{\varphi(n)} \sum_{d|n} \mu(n/d) b_1(\pi_d)$.

Proof. We prove (i) first.

It is easy to verify that $\text{Ad}\circ\rho$ splits as the sum of three 1-dimensional representations, two with image C_n plus the trivial representation. Let $\beta: \pi \to S^1$ be one of the former. For each $d|n$ and $j \in \mathbf{Z}$ define $\beta_d^j: \pi \to S^1$ by $\beta_d^j(x) = \beta(x)^{nj/d}$. Then

$$\text{(I.1)} \qquad \text{(i)} \quad \text{sl}_2(\mathbf{C})_{\text{Ad}\circ\rho} \cong \mathbf{C}_{\beta_1^1} \oplus \mathbf{C}_{\beta_n^1} \oplus \mathbf{C}_{\beta_n^{n-1}},$$

$$\text{(I.2)} \qquad \text{(ii)} \quad \mathbf{C}[C_d]_{\chi_d} \cong \bigoplus_{j=1}^{d} \mathbf{C}_{\beta_d^j},$$

$$\text{(I.3)} \qquad \text{(ii)} \quad \text{for } e|d,\ \beta_d^d = \beta_{d/e}^1.$$

According to the isomorphism (I.1),

$$\text{(I.4)} \qquad b_1(\pi; \text{Ad}\circ\rho) = b_1(\pi; \beta_1^1) + b_1(\pi; \beta_n^1) + b_1(\pi; \beta_n^{n-1}).$$

To compute the right-hand side of this equation we note that by Shapiro's Lemma [Br, p. 73]

$$b_1(\pi_d) = b_1(\pi; \mathbf{C}[\pi/\pi_d]) = b_i(\pi; \mathbf{C}[C_d]_{\chi_d}) = \sum_{j=1}^{d} b_1(\pi; \beta_d^j)$$

by (I.2). But if j and k have the same order in $\mathbf{Z}/d$, then $b_1(\pi; \beta_d^j) = b_1(\pi; \beta_d^k)$. This is true because for any $i \in \mathbf{Z}$, $b_1(\pi; \mathbf{C}_{\beta_d^i}) = b_1(\pi; \mathbf{Q}[\zeta]_{\beta_d^j})$ where ζ is a primitive dth root of unity and the fact that there is a Galois automorphism of $\mathbf{Q}[\zeta]/\mathbf{Q}$ which induces a π-module automorphism $\mathbf{Q}[\zeta]_{\beta_d^j} \cong \mathbf{Q}[\zeta]_{\beta_d^k}$. Thus we may write

$$\begin{aligned} b_1(\pi_d) &= \sum_{e|d} \varphi(e) b_1(\pi\,;\, \beta_d^{d|e}) \\ &= \sum_{e|d} \varphi(e) b_1(\pi\,;\, \beta_e^1) \end{aligned}$$

by (I.3). Now this equation holds for each $d|n$ so by the Möbius inversion formula [HW, Chapter XVI],

$$b_1(\pi\,;\, \beta_d^1) = \frac{1}{\varphi(d)} \sum_{e|d} \mu(d/e) b_1(\pi_e).$$

In particular $b_1(\pi\,;\, \beta_1^1) = b_1(\pi)$ and

$$b_1(\pi\,;\, \beta_n^{n-1}) = b_1(\pi\,;\, \beta_n^1) = \frac{1}{\varphi(n)} \sum_{d|n} \mu(n/d) b_1(\pi_d).$$

Plugging these values into equation (I.4) completes the proof of (i).

To deduce (ii), we use the identity

$$\dim_{\mathbf{C}} T_\rho = 3 - \dim_{\mathbf{C}} Z(\rho) + \dim_{\mathbf{C}} H^1(\pi\,;\, \mathrm{Ad}\circ\rho)$$

(compare with §2 of [Go]) and apply (i). □

(1.2) *Remarks.* (i) Similar techniques to those used in the preceding proof can be used to calculate $b_1(\pi\,;\, \mathrm{Ad}\circ\rho)$, and therefore $\dim_{\mathbf{C}} T_\rho$, at any representation $\rho \in R(\pi)$ having finite image.

(ii) The conclusions of Theorem 1.1 hold when $\mathrm{SL}_2(\mathbf{C})$ is replaced by $\mathrm{SU}(2)$ and dimensions are taken over $\mathbf{R}$; the calculation is formally identical.

(1.3) **Definition.** An element $\rho \in R(\pi)$ is called *rigid* if it has a neighborhood in $R(\pi)$ consisting entirely of conjugates of ρ.

André Weil [W, §3] has shown that $\rho \in R(\pi)$ is rigid as long as

$$H^1(\pi\,;\, \mathrm{sl}_2(\mathbf{C})_{\mathrm{Ad}\circ\rho}) \cong 0;$$

that is, as long as $b_1(\pi\,;\, \mathrm{Ad}\circ\rho) = 0$.

Proof of Theorem A. First we assume that π is CF.

Let $\rho \in A(\pi)$ be arbitrary. As $b_1(\pi) = 0$, ρ has a finite image and so we may calculate $b_1(\pi\,;\, \mathrm{Ad}\circ\rho)$ by Theorem (1.1)(i). Since π is CF, this identity shows $b_1(\pi\,;\, \mathrm{Ad}\circ\rho) = 0$. By [W], ρ is rigid and therefore has an $R(\pi)$-neighborhood lying entirely in $A(\pi)$. It follows that $A(\pi)$ is both open and closed in $R(\pi)$ and hence is a union of components of $R(\pi)$.

Now assume that $A(\pi)$ is a union of components of $R(\pi)$. Since $R(\pi)$ is locally connected, $A(\pi)$ is open in $R(\pi)$; furthermore, $A(\pi)$, being an algebraic set, is a *finite* union of components of $R(\pi)$ [Mi, Appendix A]. The hypothesis that $b_1(\pi) = 0$ implies that $A(\pi)$ is a finite union of $\mathrm{SL}_2(\mathbf{C})$-orbits, each closed (and therefore open) in $R(\pi)$. It follows that the component of $R(\pi)$ containing a particular $\rho \in R(\pi)$ is homeomorphic to the manifold $\mathrm{SL}_2(\mathbf{C})/Z(\rho)$. As in the remark on p. 13 of [Mi], we have

$$\dim_{\mathbf{C}} T_\rho = \dim_{\mathbf{C}} \mathrm{SL}_2(\mathbf{C})/Z(\rho) = 3 - \dim_{\mathbf{C}} Z(\rho)$$

(it is at this point our argument breaks down if $\mathrm{SL}_2(\mathbf{C})$ is replaced by $\mathrm{SU}(2)$). Comparing this last equation with Theorem (1.1)(ii) shows that for each $\rho \in A(\pi)$ with associated $n \geq 1$ and $\pi_d \subseteq \pi$, $d|n$, defined as above,

$$\sum_{d|n} \mu(n/d) b_1(\pi_d) = 0. \tag{I.5}$$

To prove π is CF we must show $b_1(\pi') = 0$ for $\pi' \triangleleft \pi$ such that $\pi/\pi' \cong C_n$ $(n \geq 1)$ and for which there is a subgroup π'' of π' with $\pi'' \triangleleft \pi$ and $\pi/\pi'' \cong C_r$, where

$$r = \begin{cases} n & \text{if } n \text{ is odd,} \\ 2n & \text{otherwise.} \end{cases}$$

We do this by inducting on n.

The case $n = 1$ is handled by our hypothesis that $b_1(\pi) = 0$. Assume $n > 1$ and that the result is known for all $d < n$. Given π' as above, define a homomorphism $\rho \in A(\pi)$ as the composition

$$\pi \to \pi/\pi'' \to \mathrm{SL}_2(\mathbf{C}).$$

According to (I.5),

$$0 = \sum_{d|n} \mu(n/d) b_1(\pi_d) = b_1(\pi_n)$$

by the inductive hypothesis. But $\pi_n = \pi'$, so $b_1(\pi') = 0$ and the induction is complete.

This finishes the proof of Theorem A. □

The following lemma provides a general criterion for recognizing 3-manifolds whose groups are CF.

(1.4) **Lemma.** *Suppose K is a smooth knot in some $M \in H(\mathbf{Z})$ and let $K(r/s)$ denote the result of an (r/s) Dehn surgery of M along K. Then for $r \neq 0$, $\pi_1(K(r/s))$ is CF if and only if $\Delta_K(t)$, the Alexander polynomial of K, has no nth root of unity as a zero. Here $n = r/2$ or r depending on whether r is even or odd.*

Proof. Since $H_1(K(r/s)) \cong C_r$, the cyclic covers of $K(r/s)$ correspond to the set of positive divisors of r, each such d producing a d-fold cyclic cover $N_d \to K(r/s)$. Let M_d be the d-fold branched cyclic cover M, branched along K. It may be shown

$$H_1(N_d) \cong H_1(M_d) \oplus C_{r/d}.$$

As $|H_1(M_d)| = \pm \prod_{j=1}^d \Delta_K(e^{2\pi i j/d})$ [BZ, Theorem 8.21], the lemma follows. □

According to Theorem B, $\pi_1(K(r/s))$ has Property A for those knots K and nonzero integers r satisfying the hypotheses of the preceding lemma. For instance, if $K \subseteq S^3$ is the figure-eight knot, $\pi_1(K(r/s))$ has Property A for all $r \neq 0$. If K is the trefoil knot, $\pi_1(K(r/s))$ has Property A for all r/s except those with $r \equiv 0 \pmod 6$, $r \neq \pm 6$. Finally, it can be shown using Lemma (1.4) that $\pi_1(M)$ has Property A for any $M \in H(\mathbf{Q})$ whose first homology group is cyclic of prime power order.

2

In this section we prove Theorem C and provide examples illustrating its relationship to Theorem B.

Theorem C. *Suppose $M \in H(\mathbf{Q})$ is NSL. Then $\pi_1(M)$ satisfies Property A.*

Proof. Since M is not sufficiently large, $\pi = \pi_1(M)$ cannot be written as a nontrivial free product with amalgamation [Sh, Proposition 4]. According to Bass [Ba, Corollary 3], there are only finitely many $\mathrm{GL}_2(\mathbf{C})$-orbits in $R(\pi, \mathrm{GL}_2(\mathbf{C}))$. Hence, by [Wo, Lemma 4.7.1], $R(\pi, \mathrm{SU}(2))$ is a finite union of $\mathrm{SU}(2)$-orbits. Each such orbit is both open and closed in $R(\pi, \mathrm{SU}(2))$ and so $A(\pi, \mathrm{SU}(2))$ is a union of components of this space. Thus π satisfies Property A. □

The core of this proof is the $\mathrm{GL}_2(\mathbf{C})$ subgroup theorem of Bass. For the convenience of the reader we provide an elementary and brief account of how this result implies $R(\pi, \mathrm{SU}(2))$ consists of only finitely many $\mathrm{SU}(2)$ orbits.

The set $\overline{R}(\pi) = R(\pi, \mathrm{SU}(2))/$ conjugation is a compact, semialgebraic set [B] and as such admits a finite triangulation [Hi]. As noted above, the hypotheses on M imply that $b_1(\pi) = 0$ and π is not a nontrivial free product with amalgamation. According to [Ba], each element of $R(\pi, \mathrm{SU}(2))$ is conjugate in $\mathrm{GL}_2(\mathbf{C})$ to a homomorphism ρ whose image lies in one of

$$\text{(i)} \qquad \left\{ \begin{bmatrix} u & b \\ 0 & \overline{u} \end{bmatrix} \middle| u \text{ is a root of unity} \right\},$$

or

(ii) $\mathrm{SL}_2(A)$ for some ring of algebraic integers A.

In the first case ρ has abelian image and thus, up to conjugation, is one of only finitely many possibilities. In the second case, the fact that the algebraic closure of $\mathbf{Q}$ is countable implies that there are only countably many possibilities for ρ up to conjugation. Two $\mathrm{SU}(2)$ representations are conjugate over $\mathrm{GL}_2(\mathbf{C})$ if and only if they are conjugate over $\mathrm{SU}(2)$ [Wo, Lemma 4.7.1] and thus the compact polyhedron $\overline{R}(\pi)$ is countable. It is therefore finite and hence $R(\pi, \mathrm{SU}(2))$ consists of finitely many $\mathrm{SU}(2)$ orbits.

(2.1) **Examples.** Consider the figure-eight knot $K \subseteq S^3$. According to the remarks at the end of §1, $K(4)$, the $+4$ Dehn surgery of S^3 along K, has a group which is CF. On the other hand, Thurston [T, §4.11] shows that $K(4)$ is Haken. Thus for $M \in H(\mathbf{Q})$, $\pi_1(M)$ being CF does not imply M is NSL.

Next let K be the trefoil knot and consider $K(12)$. By [Mo], this is a Seifert fiber space with three singular fibers and orbit S^2. By Theorem VI.15 of [Ja], $K(12)$ is NSL. But as noted in §1, $\pi_1(K(12))$ is not CF. Hence for $M \in H(\mathbf{Q})$, M being NSL does not imply $\pi_1(M)$ is CF.

Finally we show that the conditions $\pi_1(M)$ being CF or M being NSL do not exhaust $H(\mathbf{Q})$. Using the notation of [Oe], we take K to be the star knot $K(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}) \subseteq S^3$. This knot has Alexander polynomial $\Delta_K(t) = (t^2 - t + 1)(2 - 5t + 2t^2)$ and so, by Lemma (1.4), $\pi_1(K(12))$ is not CF. Further, by Corollary (4b) of [Oe], $K(12)$ is Haken and therefore is not NSL. This is the desired example.

(2.2) *Remark.* Let K be as in the last example. By considering the manifold $K(12/s)$, $s \gg 0$, we can show there is a manifold $M \in H(\mathbf{Q})$, such that M is not NSL, $\pi_1(M)$ is not CF, and M is atoroidal (compare §3).

In the proof of Theorem C we noted that if M is NSL, then $R(\pi_1(M), \mathrm{SU}(2))$ is the union of a finite number of $\mathrm{SU}(2)$ orbits. Casson's invariant for $M \in H(\mathbf{Z})$ gives an algebraic count of one-half the number of the orbits corresponding to irreducible representations. It is natural then to ask whether or not this is an exact count when M is NSL. Consider the following example. Let K be the (p, q) torus knot. When $r/s \neq 0, pq$, $K(r/s)$ is NSL (see [Mo and Ja, Theorem VI.15]).

If $m, n \in \mathbf{Z}$ are chosen to satisfy $pm - qn = 1$, then $\pi_1(S^3 \backslash K)$ has a presentation $\langle x, y \mid x^q = y^p \rangle$ where K has meridian $\mu = x^m y^{-n}$ and preferred longitude $\lambda = x^q (x^m y^{-n})^{-pq}$. Then

$$\pi_1(K(r/s)) \cong \langle x, y \mid x^q = y^p, \mu^r \lambda^s = e \rangle.$$

It follows that the set of irreducible $\mathrm{SU}(2)$-representations of $\pi_1(K(r/s))$ corresponds bijectively to

$$\{(a, b) \mid [a, b] \neq e, a^q = b^p = \varepsilon e, \varepsilon = \pm 1, (a^m b^{-n})^{r-spq} = \varepsilon^s e\} \subseteq \mathrm{SU}(2) \times \mathrm{SU}(2).$$

Let $S^1_+ = \{u \mid \mathrm{Im}(u) > 0\} \subseteq S^1$ and let $D = \{(u, v) \mid u^q = v^p = \varepsilon \in \{\pm 1\}\} \subseteq S^1_+ \times S^1_+$. It can be shown that the set of conjugacy classes of irreducible $\mathrm{SU}(2)$ representations of $\pi_1(K(r/s))$, $\widehat{R}(\pi_1(K(r/s)))$, is in 1-1 correspondence with

$$\bigcup_{(u, v) \in D} \{\lambda \in S^1_+ \mid \lambda^{r-spq} = \varepsilon^s, \varepsilon = \pm 1, \ \mathrm{Re}(\lambda) \text{ lies between } \mathrm{Re}(u^m v^{-n}) \text{ and } \mathrm{Re}(u^m v^n)\}.$$

When $p = 2$, r is odd, and $s > 0$, there are

$$\left| \frac{(q^2 - 1)}{4} s - \sum_{k=1}^{(q-1)/2} \left(\left[\frac{(q + 2k)r}{4q} \right] - \left[\frac{(q - 2k)r}{4q} \right] \right) \right|$$

such classes ("[ ]" denotes the greatest integer less than or equal to function). Taking $r = 1$ gives $K(1/s) \in H(\mathbf{Z})$ with $|\widehat{R}(\pi, (K(1/s)))| = (q^2 - 1)s/4$. It follows from Casson's Dehn surgery formula that this quantity agrees with $\lambda(K(1/s))$.

(2.3) *Question.* Let $M \in H(\mathbf{Q})$ be NSL and $\pi = \pi_1(M)$. What is the relationship between $\lambda(M)$ and $|R(\pi)|$?

In an earlier version of this paper it was erroneously asserted that $\lambda(K(1/s)) \neq \frac{1}{2}|\widehat{R}(\pi_1(K(1/s)))|$. We thank Eric Klassen for bringing to our attention an error in our application of Casson's Dehn surgery formula. For further interesting calculations concerning Casson's invariant, see the forthcoming thesis of E. Klassen (Cornell University, 1987).

3

A closed 3-manifold M has a canonical prime and torus decomposition. In this section we investigate the additivity of $\lambda(M)$ with respect to these decompositions.

The following theorem shows that Property A is usually *not* preserved under free products.

(3.1) **Theorem.** *Suppose π is a finitely generated free product $\pi = \pi_1 * \pi_2$ with finite abelianization. Then π has Property* A *if and only if both π_1 and π_2 have Property* A *and at least one of $H_1(\pi_1)$ and $H_1(\pi_2)$ is a $\mathbf{Z}/2$ vector space.*

Proof. In what follows, all representation spaces will consist of SU(2) representations.

Assume that π satisfies Property A and let $f_i \in A(\pi_i)$, $i = 1, 2$, be arbitrary. There is an $x \in \mathrm{SU}(2)$ such that $f = (f_1, xf_2x^{-1}) \in R(\pi_1) \times R(\pi_2) = R(\pi)$ is abelian. If C, C_1, and C_2 denote the components of f, f_1, and f_2 in their respective spaces then $C = C_1 \times C_2$.

Now by assumption $C \subseteq A(\pi)$ and thus $C_i \subseteq A(\pi_i)$, $i = 1, 2$. Since f_1 and f_2 were arbitrary, both π_1 and π_2 have Property A. Furthermore, if neither $H_1(\pi_1)$ nor $H_1(\pi_2)$ is a $\mathbf{Z}/2$ vector space, we may choose noncentral $f_i \in A(\pi_i)$, $i = 1, 2$ and a $y \in \mathrm{SU}(2)$ so that $(f_1, yf_2y^{-1}) \in C_1 \times C_2 = C$ is not abelian, contradicting our assumption that π has Property A. Thus $H_1(\pi_i)$ is a $\mathbf{Z}/2$ vector space for some i. The proof of the converse is similar. □

Each $M \in H(\mathbf{Q})$ admits a prime decomposition $M = \#_{i=1}^n M_i$, unique up to the ordering of the factors, where each $M_i \in H(\mathbf{Q})$ and is irreducible. This suggest the question: Is λ additive with respect to connected sums? If $M \in H(\mathbf{Z})$ Casson has answered the question in the affirmative (see [AM]). It is also true in general; we prove

Theorem D. *Let M_1 and M_2 be* **Q***-homology* 3*-spheres. Then $\lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2)$.*

Proof. Choose genus g_i Heegard splittings $M_i = W_{i1} \cup_{F_i} W_{i2}$ and let $M = W_1 \cup_F W_2$ be the genus $g = g_1 + g_2$ Heegard splitting of $M = M_1 \# M_2$ given by

$$W_j = W_{1j} \natural W_{2j}, \qquad j = 1, 2,$$
$$F = F_1 \# F_2.$$

We adopt the following notation and conventions for the purposes of this proof.

All representation spaces will consist of $\mathrm{SU}(2)$ representations. Let

$$\begin{aligned}
R(M_i) &= R(\pi_1(M_i)), \qquad i = 1, 2, \\
R(M) &= R(\pi_1(M)) \cong R(M_1) \times R(M_2), \\
Q_{ij} &= R(\pi_1(W_{ij})), \qquad 1 \le i, j \le 2, \\
Q_j &= R(\pi_1(W_j)) \cong Q_{1j} \times Q_{2j}, \qquad j = 1, 2, \\
R_i &= R(\pi_1(F_i)), \quad R_i^* = R(\pi_1(F_i^*)), \qquad i = 1, 2, \\
R &= R(\pi_1(F)), \quad R^* = R(\pi_1(F^*)).
\end{aligned}$$

There are smooth maps

$$\partial_i : R_i^* \to \mathrm{SU}(2), \quad \partial_i : \rho \to \rho(\partial F_i^*).$$

Evidently, $R_i = \partial_i^{-1}(e)$, $i = 1, 2$.

A superscript "I" appended to a space of representations will indicate the subset of all irreducible representations in the space.

If X is an $\mathrm{SU}(2)$-space, $\widehat{X}$ will denote the associated orbit space. We warn the reader that *this notation differs from that used in the introduction.*

Choose a neighborhood R_i^N of $R^n(M_i)$ in R_i^I such that $R(M_i) \cap R_i^N = R^n(M_i)$, $i = 1, 2$, and set $Q_{ij}^N = Q_{ij} \cap R_i^N$. Clearly $Q_{i1}^N \cap Q_{i2}^N = R^n(M_i)$ and

(III.1) $$\langle \widehat{Q}_{i1}^N, \widehat{Q}_{i2}^N \rangle_{\widehat{R}_i^N} = 2(-1)^{g_i} \langle Q_{i1}, Q_{i2} \rangle_{R_i^*} \lambda(M_i).$$

Now

(III.2) $$2(-1)^g \langle Q_1, Q_2 \rangle_{R^*} \lambda(M) = \sum_{C \subseteq R^n(M)} i(\widehat{C}),$$

where the sum ranges over the components C of $R^n(M)$ and $i(\widehat{C})$ is the intersection of $\widehat{Q}_1^I$ and $\widehat{Q}_2^I$ near $\widehat{C}$.

For many components $C \subseteq R^n(M)$, $i(\widehat{C}) = 0$. Indeed R_1^I has a trivial normal bundle in $(R_1^*)^I$, thus

$$\begin{aligned}
&\langle \widehat{Q_{11}^N \times Q_{21}^N}, \widehat{Q_{12}^N \times Q_{22}^N} \rangle \\
&\quad = \pm \langle Q_{11}^N, Q_{12}^N \rangle \langle \widehat{Q_{21}^N}, \widehat{Q_{22}^N} \rangle \quad \text{(compare Lemma (3.2) below)} \\
&\quad = 0.
\end{aligned}$$

Thus, $i(\widehat{C}) = 0$ for each $C \subseteq R_1^I \times R_2^I \subseteq R^I$. From equation (III.2) we then see that

$$\text{(III.3)}\qquad 2(-1)^g\langle Q_1, Q_2\rangle_{R^*}\lambda(M) = \langle Q_{11}\widehat{\times} Q_{21}^N, Q_{12}\widehat{\times} Q_{22}^N\rangle + \langle Q_{11}^N\widehat{\times} Q_{21}, Q_{12}^N\widehat{\times} Q_{22}\rangle.$$

To compute the right-hand side of this equation, choose a small open 3-ball neighborhood B of the identity in $\mathrm{SU}(2)$. An $\mathrm{SU}(2)$-invariant metric on R_2^* can be used to construct an equivariant tubular neighborhood of $R_2^I = (\partial_2|(R_2^*)^I)^{-1}(e)$ of the form $T: R_2^I \times B \to (R_2^*)^I$ where $\partial_2(T(g, x)) = x^{-1}$. Define

$$\begin{array}{rcl} \Psi: \partial_1^{-1}(B)\times R_2^I & \to & R^I, \\ (f, g) & \to & (f, T(g, \partial_1(f))). \end{array}$$

Ψ is an equivariant injection extending the inclusion $R_1 \times R_2^I \to R^I$ whose image is a neighborhood of $R_1 \times R^n(M_2)$ in R^I.

Consider the following commutative diagram, each of whose columns is a smooth fiber bundle and each of whose horizontal arrows is an inclusion.

$$\begin{array}{ccccccccc} Q_{11}, & Q_{12} & \to & R_1 & \to & \partial_1^{-1}(B) & \xrightarrow{\text{cod.0}} & R_1^* \\ \downarrow & \downarrow & & \downarrow & & \downarrow & & \\ Q_{11}\widehat{\times} Q_{21}^N, & Q_{12}\widehat{\times} Q_{22}^N & \to & R_1\widehat{\times} R_2^N & \to & \widehat{\partial_1^{-1}(B)\times R_2^N} & \xrightarrow[\text{cod.0}]{\widehat{\Psi}} & \widehat{R}^I \\ \downarrow & \downarrow & & \downarrow & & \downarrow & & \\ \widehat{Q_{21}^N}, & \widehat{Q_{22}^N} & \to & \widehat{R}_2^N & \to & \widehat{R}_2^N & & \end{array}$$

It follows from the diagram and Lemma (3.2) below that

$$\begin{aligned} \langle Q_{11}\widehat{\times} Q_{21}^N, Q_{12}\widehat{\times} Q_{22}^N\rangle &= (-1)^{g_1(g_2-1)}\langle Q_{11}Q_{12}\rangle_{R_1^*}\langle \widehat{Q}_{21}^N, \widehat{Q}_{22}^N\rangle_{\widehat{R}_2^N} \\ &= (-1)^{g_1(g_2-1)}2(-1)^{g_2}\langle Q_{11}, Q_{12}\rangle_{R_1^*}\langle Q_{21}, Q_{22}\rangle_{R_2^*}\lambda(M_2) \\ &\hspace{20em}\text{by (III.1)} \\ &= 2(-1)^{g_1+g_2}\langle Q_1, Q_2\rangle\lambda(M_2). \end{aligned}$$

Similarly

$$\langle Q_{11}^N\widehat{\times} Q_{21}, Q_{12}^N\widehat{\times} Q_{22}\rangle = 2(-1)^{g_1+g_2}\langle Q_1, Q_2\rangle\lambda(M_1).$$

Substituting these calculations into equation (III.3) shows $\lambda(M) = \lambda(M_1) + \lambda(M_2)$. $\square$

(3.2) **Lemma.** *Consider a diagram of smooth, compatibly oriented fiber bundles*

$$\begin{array}{ccccc} F_1, F_2 & \longrightarrow & E_1, E_2 & \longrightarrow & B_1, B_2 \\ \cap & & \cap & & \cap \\ F & \longrightarrow & E & \longrightarrow & B \end{array}$$

where each inclusion is proper and $F_1^{n_1}$, $F_2^{n_2}$ and $B_1^{m_1}$, $B_2^{m_2}$ have complementary dimensions in F and B. Then if $E_1 \cap E_2$ is compact.

$$\langle E_1, E_2\rangle_E = (-1)^{n_1 m_2}\langle B_1, B_2\rangle_B\langle F_1, F_2\rangle_F.$$

Proof. First assume that F is closed.

Any isotopy of B with support in a chart of the bundle $E \to B$ lifts to a fiber preserving isotopy of E with support lying over this chart. By hypothesis, $B_1 \cap B_2$ is compact and so after a finite number of such moves we may construct a compactly supported isotopy of B which at $t = 1$ has made B_1 transverse to B_2 and which lifts to a compactly supported, fiber-preserving isotopy of E. We are thus reduced to the case where B_1 and B_2 intersect transversely in one point and the bundle $E \to B$ is trivial.

Now E_1 and E_2 intersect only within the fiber F. An isotopy of F making F_1 transverse to F_2 extends to a compactly supported isotopy of E, making E_1 transverse to E_2.

Note that now $E_1 \cap E_2 = F_1 \cap F_2$. Taking care of the signs involved we obtain

$$\langle E_1, E_2\rangle_E = (-1)^{n_1 m_2}\langle B_1, B_2\rangle_B \langle F_1, F_2\rangle_F.$$

Finally when F is not closed, we may use the same argument with the proviso that when we lift an isotopy of B supported in a relatively compact open chart of $E \to B$, we must taper it to the identity outside a given compact neighborhood of $E_1 \cap E_2$. □

Now suppose $M \in H(\mathbf{Q})$ is prime. According to Jaco and Shalen [JS] and Johansson [Jo], there is a canonical decomposition $M = \bigcup_{i=1}^{n} M_i$ where M_i contains no essential, nonperipheral tori and ∂M_i is a union of tori.

Fix M_i and set $\partial M_i = \bigcup_{j=1}^{m} T_j$, each T_j a torus. Now each T_j separates M into two $\mathbf{Q}$-homology circles, precisely one of which, E_j say, misses $\mathring{M}_i$. Let γ_i be the unique essential simple closed curve on T_j which bounds a 2-chain in E_j rationally. We define the *closure* of M_i in M to be the union

$$\overline{M}_i = M_i \cup \left(\bigcup_{j=1}^{m} S^1 \times D^2\right),$$

where the solid tori $S^1 \times D^2$ are attached to the boundary components of M_i in any way which identifies each γ_j to some $\{*\} \times \partial D^2$. $\overline{M}_i$ is a well-defined element of $H(\mathbf{Q})$.

Set

$$\overline{\overline{\lambda}}(M) = \sum_{i=1}^{n} \lambda(\overline{M}_i).$$

(3.3) *Question.* Does $\overline{\overline{\lambda}}(M) = \lambda(M)$?

Our next theorem answers this question in the affirmative when $M \in H(\mathbf{Z})$. (See Theorem E of the introduction.)

(3.4) **Theorem.** *Suppose $M = M_1 \cup_T M_2 \in H(\mathbf{Z})$ where T is a torus. Then the closures $\overline{M}_1$ and $\overline{M}_2$ are elements of $H(\mathbf{Z})$ and $\lambda(M) = \lambda(\overline{M}_1) + \lambda(\overline{M}_2)$.*

(3.5) *Remarks.* (i) Although we defined the closure operation only for the components of the torus decomposition of a prime $M \in H(\mathbf{Q})$, we make a similar definition for any decomposition of an $M \in H(\mathbf{Q})$ into components whose boundaries consist of tori.

(ii) Theorem (3.4) and an inductive argument shows that $\overline{\overline{\lambda}}(M) = \lambda(M)$ when $M \in H(\mathbf{Z})$.

Proof of Theorem (3.4). Denote by K_i the core of $S^1 \times D^2$ in $\overline{M}_i$, $i = 1, 2$, and let $K_i(1)$ be the element of $H(\mathbf{Z})$ resulting from the $+1$ surgery of $\overline{M}_i$ along K_i. If K_i' is the core of the surgery solid torus in $K_i(1)$, then K_i' has exterior equal to M_i and furthermore, $K_i'(-1) = \overline{M}_i$. Hence

$$\begin{aligned} M = M_1 \cup M_2 &= M(K_1', K_2'; -1, 0, 1, 1) \\ &= (K_1' \# K_2')(-1) \qquad\qquad [\text{G1}, \S 3.1] \end{aligned}$$

by the remarks following Lemma 7.1 of [G2]. Then Casson's Dehn surgery formula [AM] implies

$$\begin{aligned} \lambda(M) &= \lambda((K_1' \# K_2')(-1)) \\ &= \lambda(K_1(1) \# K_2(1)) - \tfrac{1}{2}{}'\Delta^{(2)}_{K_1' \# K_2'}(1) \\ &= (\lambda(K_1(1)) - \tfrac{1}{2}\Delta^{(2)}_{K_1'}(1)) + (\lambda(K_2(1)) - \tfrac{1}{2}\Delta^{(2)}_{K_2'}(1)) \\ &= \lambda(K_1'(-1)) + \lambda(K_2'(-1)) \\ &= \lambda(\overline{M}_1) + \lambda(\overline{M}_2). \quad \square \end{aligned}$$

(3.6) *Question.* Is there a splitting formula for surfaces of higher genus?

Consider again a prime manifold $M \in H(\mathbf{Q})$ with torus decomposition $M = \bigcup_{i=1}^n M_i$. If $\overline{M}_i$ is prime and atoroidal, then $\overline{M}_i$ is either NSL, in which case Theorem C implies $\pi_1(\overline{M}_i)$ has Property A, or $\overline{M}_i$ is Haken and therefore hyperbolic by Thurston's hyperbolization theorem [T].

(3.7) *Question.* If $M \in H(\mathbf{Q})$ is Haken and hyperbolic, does $\pi_1(M)$ satisfy Property A?

(3.8) *Remark.* A closed, prime, atoroidal $M \in H(\mathbf{Q})$ which is Seifert fibered is NSL [Ja, Chapter VI].

If Thurston's geometrization conjecture [T] is true, then each M_i is either Seifert fibered or hyperbolic. However $\overline{M}_i$ may fail to be prime or atoroidal. Thus it would be convenient to define a Casson type invariant for an M^3, which is the interior of a compact N^3 whose boundary is a union of tori and such that the inclusion $\partial N \to N$ induces a surjection $H_1(\partial N; \mathbf{Q}) \to H_1(N; \mathbf{Q})$.

(3.9) **Problem.** Define such an invariant.

We close this section with some general observations concerning the relationship between Casson type invariants and the μ-invariant.

Casson proved (see [AM]) that for $M \in H(\mathbf{Z})$, $\lambda(M) \equiv \mu(M) \pmod 2$. For $M \in H(\mathbf{Z}/2)$ with $\pi_1(M)$ satisfying Property A, this identity will not hold in general. For instance, if M is a lens space $\lambda(M)$ is defined and is 0, since $\pi_1(M)$ is abelian. On the other hand, $\mu(M)$ can take on any even value (mod 16). It would be of interest to determine the connection, if any, between these two quantities. Walter Neumann [N] has defined an integer valued invariant for $\mathbf{Z}/2$-homology 3-spheres of plumbed type, which reduces (mod 16) to the μ-invariant.

(3.10) *Question.* What is the relationship between Neumann's invariant and Casson type invariants?

Finally, we point out that in general, the μ-invariant is defined for framed 3-manifolds. This suggests that the appropriate definition of $\lambda(M)$ should depend on a framing of M.

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