THE ARF AND SATO LINK CONCORDANCE INVARIENTS

RACHEL STURM BEISS

Abstract. The Kervaire-Arf invariant is a $\mathbb{Z}/2$ valued concordance invariant of knots and proper links. The $\beta$ invariant (or Sato's invariant) is a $\mathbb{Z}$ valued concordance invariant of two component links of linking number zero discovered by J. Levine and studied by Sato, Cochran, and Daniel Ruberman. Cochran has found a sequence of invariants $\{\beta_i\}$ associated with a two component link of linking number zero where each $\beta_i$ is a $\mathbb{Z}$ valued concordance invariant and $\beta_0 = \beta$. In this paper we demonstrate a formula for the Arf invariant of a two component link $L = X \cup Y$ of linking number zero in terms of the $\beta$ invariant of the link:

$$\text{arf}(X \cup Y) = \text{arf}(X) + \text{arf}(Y) + \beta(X \cup Y) \pmod{2}.$$ 

This leads to the result that the Arf invariant of a link of linking number zero is independent of the orientation of the link's components. We then establish a formula for $|\beta|$ in terms of the link's Alexander polynomial $\Delta(x, y) = (x - 1)(y - 1)f(x, y)$:

$$|\beta(L)| = |f(1, 1)|.$$ 

Finally we find a relationship between the $\beta_i$ invariants and linking numbers of lifts of $X$ and $Y$ in a $\mathbb{Z}/2$ cover of the compliment of $X \cup Y$.

1. Introduction

The Kervaire-Arf invariant [KM, R] is a $\mathbb{Z}/2$ valued concordance invariant of knots and proper links. The $\beta$ invariant (or Sato's invariant) is a $\mathbb{Z}$ valued concordance invariant of two component links of linking number zero discovered by Levine (unpublished) and studied by Sato [S], Cochran [C], and Daniel Ruberman. Cochran [C] has found a sequence of invariants $\{\beta_i\}$ associated with a two component link of linking number zero where each $\beta_i$ is a $\mathbb{Z}$ valued concordance invariant and $\beta_0 = \beta$. Theorem 1 demonstrates a formula for the Arf invariant of a two component link $L = X \cup Y$ of linking number zero in terms of the $\beta$ invariant of the link:

**Theorem 1.** $\text{arf}(X \cup Y) = \text{arf}(X) + \text{arf}(Y) + \beta(X \cup Y) \pmod{2}$.

This leads to the result that the Arf invariant of a link of linking number zero is independent of the orientation of the link's components. In Theorem 2 we establish a formula for $|\beta|$ in terms of the link's Alexander polynomial $\Delta(x, y) = (x - 1)(y - 1)f(x, y)$.
Theorem 2. \(|\beta(L)| = |f(1, 1)|.\)

Finally, in Theorem 3 we find a relationship between the invariants and linking numbers of lifts of \(X\) and \(Y\) in a \(Z/2\) cover of the compliment of \(X \cup Y:\)

Theorem 3. 

\[-\frac{1}{2} \text{lk}(Y^0, Y^1) = \sum_{j=0}^{N-1} 4^j \beta_j(L) - 2(4)^{N-1} \text{lk}(c_0^N, c_1^N).\]

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We will begin by recalling a few definitions.

Let \(V\) be a vector space of dimension \(2n\) over \(Z/2.\) Let \(\varphi: V \times V \rightarrow Z/2\) be a nonsingular symmetric bilinear form such that \(\varphi(a, a) = 0\) for every \(a\) in \(V.\) The map \(q: V \rightarrow Z/2\) is a quadratic form with respect to \(\varphi\) if and only if

\[q(a + b) = q(a) + q(b) + \varphi(a, b)\quad \text{for every } a, b \text{ in } V.\]

Definition 1.1. The set \(\{a_1, b_1, \ldots, a_n, b_n\}\) is a symplectic basis of \(V\) with respect to \(\varphi\) if it is a basis of \(V\) and if \(\varphi(a_i, b_j) = \delta_{ij}\) and \(\varphi(a_i, a_j) \varphi(b_i, b_j) = 0.\)

Definition 1.2 (Arf invariant of a quadratic form). Let \(\{a_1, b_1, \ldots, a_n, b_n\}\) be a symplectic basis of \(V\) with respect to \(\varphi.\) Then

\[\text{arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i)\]

is the Arf invariant of \(q.\)

The above definition is independent of the choice of symplectic basis (see Arf [A]).

Let \(K\) be a knot in \(S^3\) and \(M\) an oriented Seifert surface spanning \(K.\) Then \(H_1(M; Z/2)\) is a vector field over \(Z/2\) of dimension \(2q\) where \(q = \text{genus}\) of \(M.\) We have a symmetric nonsingular bilinear form

\[\text{int}_2: H_1(M; Z/2) \times H_1(M; Z/2) \rightarrow Z/2\]

defined by \(\text{int}_2(a, b) = \text{the mod } 2 \text{ intersection number of cycles } a, b \text{ which represent } a \text{ and } b.\) Since \(M\) is orientable, \(\text{int}_2(a, a) = 0\) for every \(a\) in \(H_1(M; Z/2).\)

From now on we will assume we have fixed an orientation of \(S^3.\) Since \(M\) and \(S^3\) are now oriented we can distinguish between the positive and negative normal directions to \(M.\)

Given \(a,\) an element of \(H_1(M),\) let \(a\) be a curve in \(M\) which represents \(a\) and \(a^+, a^-\) the push offs of \(a\) in the positive and negative normal directions to \(M.\) Define \(\delta_2: H_1(M; Z/2) \rightarrow Z/2\) by \(\delta_2(a) = \text{lk}(a, a^+) \text{ mod } 2\) the modulo
two linking number between $a$ and $a^+$. We have
\[ \delta_2(a + b) = \delta_2(a) + \delta_2(b) + \text{int}_2(a, b) \]
[KM], so $\delta_2$ is a quadratic form associated with $\text{int}_2$.

**Definition 1.3** (Arf invariant of a knot). The Arf invariant of a knot $K$ is equal to the Arf invariant of the quadratic form $\delta_2$.

Let $L = X \cup Y$ be a two component link with even linking number in $S^3$ whose components $X$ and $Y$ are oriented. Let $M$ be a connected Seifert surface for $L$. Then $H_1(M; \mathbb{Z}/2)$ is a vector space over $\mathbb{Z}/2$ of dimension $2g + 1$ where $g = \text{genus of } M$. The modulo two intersection form is nonsingular on the quotient space of $H_1(M; \mathbb{Z}/2)$ given by $H_1(M; \mathbb{Z}/2)/J$, where $J$ is a one-dimensional subspace generated by one boundary component of $M$.

**Definition 1.4** (Arf invariant of a link). The Arf invariant of link $L$ is equal to the Arf invariant of the quadratic form $\delta_2$ restricted to the quotient space $H_1(M; \mathbb{Z}/2)/J$ of $H_1(M; \mathbb{Z}/2)$.

The above definition is well defined for links of even linking number $[R]$.

If $L = X \cup Y$ has linking number zero then there exist Seifert surfaces $M_x, M_y$ for $X$ and $Y$ such that $X \cap M_y = Y \cap M_x = \emptyset$. Therefore $M_x \cap M_y = C_1 \cup C_2 \cup \cdots \cup C_n$ where $C_j$ is a circle embedded in $M_x$ and $M_y$. Orient $C_j$ so that a positive unit tangent to $C_j$ together with the positive unit normals to $M_x$ and $M_y$ give the chosen orientation of $S^3$.

The intersection $M_x \cap M_y$ has a natural framing of its normal bundle given by the normal 1-fields $(\vec{v}, \vec{w})$. By the Thom-Pontryagin construction $(M_x \cap M_y, \vec{v}, \vec{w})$ corresponds to an element of $\Pi_3(S^2)$.

**Definition 1.5** (the $\beta$ or Sato invariant). $\beta(L)$ is the element of $\Pi_3(S^2)$ given by $(M_x \cap M_y, \vec{v}, \vec{w})$.

**Alternate definition to 1.5a** (the $\beta$ or Sato invariant).
\[ \beta(L) = 2 \sum_{i<j} \text{lk}(C_i, C_j) + \sum_{j=1}^n \text{lk}(C_j, C_j^+) \]
where $\text{lk}(a, b)$ is the linking number between $a$ and $b$ with respect to the chosen orientation of $S^3$.

Definitions 1.5 and 1.5a are obviously the same if $M_x \cap M_y$ is connected. In Lemma 2.1 we show that if $M_x \cap M_y$ has two or more components then $M_x$ and $M_y$ can be altered so that the altered surfaces intersect in a connected manifold. Figure 1.1 illustrates how the definitions are the same when $M_x \cap M_y = c_1 \cup c_2$ has two components and $M_x, M_y$ are altered so that the new intersection $c$ is connected.
In this section we establish a relationship between the $\beta$ invariant and the Arf invariant of a link of linking number zero.

**Theorem 1.** If $L = X \cup Y$ is an oriented link of linking number zero then $\text{arf}(L) = \text{arf}(X) + \text{arf}(Y) + \beta(L) \pmod{2}$.

First we will find two oriented Seifert surfaces $M_X, M_Y$ for $X, Y$ respectively such that $M_X \cap M_Y$ consists of only one circle. We cut $M_X$ and $M_Y$ along this circle and reglue to obtain an oriented Seifert surface $M$ for $L$. Surface $M$ will be used to compute $\text{arf}(L)$.

**Lemma 2.1.** Let $L = X \cup Y$ be a link of linking number zero. There exist orientable Seifert surfaces $M_X$ and $M_Y$ spanning $X$ and $Y$ such that $M_X \cap M_Y = \emptyset$ or one circle. Furthermore this circle is not null homologous in $H_1(M_X)$ of $H_1(M_Y)$.

**Proof.** Let $N_X, N_Y$ be oriented Seifert surfaces for $X$ and $Y$ such that $N_X \cap N_Y = \{c_1, \ldots, c_n\}$, where each $c_j$ is an embedded circle. Change circle inter-
sections $c_1$ and $c_2$ to band intersections by the following method of Cooper [Co]. Let $\alpha_1, \alpha_2$ be paths from $\partial N_Y$ to points on $c_1$ and $c_2$ respectively. Push $N_Y$ in along $\alpha_1$ and $\alpha_2$ so $c_1$ and $c_2$ become band intersections (see Figure 2.1). Let $p_1$ and $p_2$ be two points on the resulting band intersections.

Let $\gamma$ be an arc in $N_X$ from $p_1$ to $p_2$ and oriented from $p_1$ to $p_2$ such that the intersection number of $\gamma$ with $N_Y$ at $p_1$ is $+1$ and the intersection number of $\gamma$ with $N_X$ at $p_2$ is $-1$ (see Figure 2.2(i)). Now attach a handle to $N_Y$ whose core coincides with $\gamma$ (see Figure 2.2(ii)). Our two band intersections have been transformed into two different band intersections. If we attach two handles to $N_X$ whose cores coincide with the two arcs in $\partial N_Y = Y$ which were pushed in along paths $\alpha_1$ and $\alpha_2$ then the band intersections will become one circle intersection (see Figure 2.2(iii)).

Let us call our new surfaces $N'_X$ and $N'_Y$. These new surfaces intersect in $N'_X \cap N'_Y = \{a, c_3, \ldots, c_n\}$, where $a$ was formed from $c_1$ and $c_2$. If we repeat the above process we can find oriented surfaces $M_X$ and $M_Y$ such that $M'_X \cap M'_Y = \text{one circle } c$.

Suppose $M_X \cap M_Y = c$ and $c$ is null homologous in $H_1(M_X)$. Then $c$ bounds a surface $N$ in $M_Y$. Look at a neighborhood of $N$ in $S^3$ parametrized by $N \times [1, -1]$, where $N \times 0$ coincides with $N$. Replace $M_X$ with $M_X - N \cup \partial N \times [1, 0] \cup N \times 1$. Now round off the corner at $N \times 0$ and we will no longer have an intersection along $c$.  \[\square\]
Proof of Theorem 1. By Lemma 2.1 we can assume we have oriented Seifert surfaces $M_X$ and $M_Y$ for $X$, $Y$ respectively such that $M_X \cap M_Y = \emptyset$ or $M_X \cap M_Y = c$, where $c$ is an embedded circle and $c$ not homologous to zero in $H_1(M_X)$ or $H_1(M_Y)$.

If $M_X \cap M_Y = \emptyset$ then $L$ is a boundary link. From the methods of Robertello [R] it easily follows that $\text{arf}(L) = \text{arf}(X) + \text{arf}(Y)$. Since the $\beta$ invariant for a boundary link is zero the formula holds in this case.

Assume that $M_X \cap M_Y = c$. We can find symplectic bases:

$$\{c, d_1, e_1, d_2, e_2, \ldots, d_g, e_g\}, \quad \{c, d'_1, e_1, f_1, \ldots, e_h, f_h\}$$

for $H_1(M_X; \mathbb{Z}/2)$ and $H_1(M_Y; \mathbb{Z}/2)$ respectively, where $g = \text{genus } M_X$ and $h = \text{genus } M_Y$, and $c$ represents $c$.

We now construct $M$, an oriented Seifert surface for $L$, from $M_X$ and $M_Y$. Cut $M_X$ and $M_Y$ along $c$. Attach $M_X - c$ to $M_Y - c$ so that the positive sides of $M_X - c$ and $M_Y - c$ form the positive side of $M$. $M$ is orientable. Let $H_1(M; \mathbb{Z}/2) = I \oplus J$, where $J = \text{null space of form } \text{int}_2$ and $I$ is the subspace with symplectic basis:

$$\{c, d, e_2, \ldots, e_g, d_g, e_2, f_1, \ldots, e_h, f_h\}$$

where $d = d_1 \cup d'_1$ represents $d$. The self-linking numbers of the basis element of $I$ have not changed with the exception of $d$: $\delta_2(d) = \delta_2(d_1) + \delta_2(d'_1) + 1$. 

Figure 2.2
The extra 1 appears because $d_1$ and $d'_1$ intersect at a point. Figure 2.3 is a local picture of $M_x \cup M_y$ near $d_1$ and $d'_1$ and shows how the extra linking number appears.

Use the above symplectic basis of $I$ to calculate $\text{arf}(L)$:

$$\text{arf}(L) \equiv \delta_2(c)\delta_2(d) + \sum_{j=2}^{g} \delta_2(c_j)\delta_2(d_j) + \sum_{j=2}^{h} \delta_2(e_j)\delta_2(f_j)$$

$$\equiv \delta_2(c)(\delta_2(d_1) + \delta_2(d'_1) + 1)$$

$$+ \sum_{j=2}^{g} \delta_2(c_j)\delta_2(d_j) + \sum_{j=2}^{h} \delta_2(e_j)\delta_2(f_j)$$

$$\equiv \delta_2(c) + \text{arf}(X) + \text{arf}(Y) \mod 2$$
and \( \delta_2(c) \equiv \beta(L) \pmod{2} \) so,

\[
\text{arf}(L) \equiv \text{arf}(X) + \text{arf}(Y) + \beta(L) \pmod{2}.
\]

\( \blacksquare \)

**Corollary 1.** The Arf invariant of a link of linking number zero is independent of the orientation of the components of the link.

**Proof.** We must show that \( \beta(L) \pmod{2} \) is independent of the link’s orientation.

Definition 1.5 for \( \beta(L) \) shows

\[
\beta(L) = 2 \sum_{i<j} \text{lk}(c_i, c_j) L + \sum_{j=1}^{2p} \text{lk}(c_j, c_j^+) \pmod{2}.
\]

Each term \( \text{lk}(c_j, c_j^+) \) is independent of the orientation of the components of the link. \( \blacksquare \)

3. A FORMULA FOR \( |\beta(L)| \) IN TERMS OF THE ALEXANDER POLYNOMIAL OF \( L \)

In this section we establish a relationship between the \( \beta \) invariant of a link and its Alexander polynomial \( \Delta(x, y) \). If \( L = X \cup Y \) is a link of linking number zero then it has an Alexander polynomial of the form

\[
\Delta(x, y) = (x - 1)(y - 1)f(x, y).
\]

**Theorem 2.** \( |\beta(L)| = |f(1, 1)| \).

We compute \( \Delta(x, y) \) using an algorithm by Cooper [Co] which is summarized in the following paragraph:

Let \( L = X \cup Y \) be an oriented link in \( S^3 \) and \( M_X, M_Y \) Seifert surfaces with orientations inherited from those of \( X \) and \( Y \) such that \( M_X \cap M_Y \) consists of only clasp intersections (see Figure 3.1(i)). Cooper defines two bilinear forms

\[
u, v : H_1(M_X \cup M_Y) \times H_1(M_X \cup M_Y) \rightarrow \mathbb{Z}
\]

by \( u(a, b) = \text{lk}(a, b^{-+}) \) and \( v(a, b) = \text{lk}(a, b^{-+}) \), where \( a, b \) are curves in \( M_X \) and \( M_Y \) respectively which represent \( a \) and \( b \). The notation \( \text{lk}(a, b^{-+}) \) stands for the linking number between \( a \) and \( b^{-+} \) where \( b^{-+} \) is the push off of \( b \) in the negative normal direction to \( M_X \) and negative normal direction to \( M_Y \). Similarly \( \text{lk}(a, b^{-+}) \) is the linking number between \( a \) and the push off of \( b \) in the negative normal direction to \( M_X \) and positive normal direction to \( M_Y \). When \( a \) or \( b \) passes from \( M_X \) to \( M_Y \) it must contain the clasp intersection. Near a clasp the push off of a curve will lie in one of the four quadrants formed by the clasp intersection as suggested in Figure 3.1(ii). Cooper observed that

\( |f(1, 1)| \) is equal to \( |a_1| \), where \( a_1 \) is a Conway polynomial coefficient. In [C] the Conway polynomial coefficients are found to be related to Milnor’s \( \mu \)-invariants and to certain linking numbers.

\( ^1 \)
(i) A clasp intersection

\[ M_X \]

\[ M_Y \]

(ii)

\[ b^{+-} \quad b^{++} \]

\[ b^{--} \quad b^{-+} \]

\[ + \quad - \]

\[ M_Y \]

\[ M_X \]

(iii) Changing a circle intersection to two clasps

\[ \text{Figure 3.1} \]

\[ \Delta(x, y) = (x - 1)^{-2g}(y - 1)^{-2h} \det(xyA + A^T - xB - yB^T), \]

where \( g \) = genus of \( M_X \) and \( h \) = genus of \( M_Y \).

Proof of Theorem 2. By Lemma 2.1 we can assume that we have oriented Seifert surfaces \( N'_X \) and \( N'_Y \) for \( X \) and \( Y \) such that \( N'_X \cap N'_Y \) is a single circle. The circle intersection can be changed as in [Co] to two clasp intersections (see Figure 3.1(iii)). Let \( N_X, N_Y \) be the resulting Seifert surfaces for \( X \) and \( Y \) such that \( N_X \cap N_Y \) is two clasps. If \( N'_X \cap N'_Y = c \) then the invariant of \( L \),
\( \beta(L) \), is by definition equal to \( \text{lk}(c, c^{-\ominus}) = \text{lk}(c, c^{-+}) \). Let us choose a basis of \( H_1(N_X \cup N_Y) \) consisting of a basis for \( H_1(N_X) \) union a basis for \( H_1(N_Y) \) union \( \zeta \) where \( c \) runs through the two clasps as above and represents \( \zeta \). Notice that if curve \( b \) lies in \( M_X \) then \( u(a, b) = v(a, b) = \text{lk}(a, b^{-\ominus}) \) where \( b^{-\ominus} \) is the push off of \( b \) in the negative normal direction to \( M_X \). If \( A_X = A \) restricted to \( H_1(N_X) \) and \( B_X = B \) restricted to \( H_1(N_X) \) then \( A_X = B_X \). Similarly if \( A_Y = A \) restricted to \( H_1(N_Y) \) and \( B_Y = B \) restricted to \( H_1(N_Y) \) then \( B_Y = A_Y^T \). Let \( n = \beta(L) = \text{lk}(c, c^{-\ominus}) = \text{lk}(c, c^{-+}) \). Then with respect to the above basis for \( H_1(N_X \cup N_Y) \), \( A \) and \( B \) have the following form:

\[
A = \begin{bmatrix} A_X & q_{ij} \\ q_{ji} & A_Y \end{bmatrix}, \quad B = \begin{bmatrix} A_X & q_{ij} \\ A_Y^T & q_{ji} \end{bmatrix}
\]

where \( q_{ij} = q_{ji} \). \( A \) has matrices \( A_X \), \( A_Y \) and \( (n) \) down the diagonal. Away from these submatrices \( A \) has \( ij \)th entry designated by \( q_{ij} \). The \( q_{ij} \) entries represent linking numbers between nonintersecting curves in \( N_X \) and \( N_Y \), thus \( q_{ij} = q_{ji} \). \( B \) has matrices \( A_X \), \( A_Y^T \) and \( (n) \) down the diagonal. The \( q_{ij} \) entries of \( B \) are equal to those of \( A \).

By Cooper’s algorithm

\[
\Delta(x, y) = (x - 1)^{-2g}(y - 1)^{-2h} \det_{xy} \begin{bmatrix} A_X & q_{ij} \\ q_{ji} & A_Y \end{bmatrix} \]
\[
= (x - 1)^{-2g}(y - 1)^{-2h} \cdot \det \begin{bmatrix} A_X^T & q_{ji} \\ q_{ij} & A_Y \end{bmatrix} - x \begin{bmatrix} A_X & q_{ij} \\ q_{ji} & A_Y^T \end{bmatrix} - y \begin{bmatrix} A_X^T & q_{ij} \\ q_{ji} & A_Y \end{bmatrix}
\]
\[
= (x - 1)^{-2g}(y - 1)^{-2h} \cdot \det \begin{bmatrix} (y - 1)(xA_X - A_X^T) \\ (x - 1)(y - 1)q_{ji} \\ (x - 1)(y - 1)q_{ji} \\ n \\ n \end{bmatrix}
\]

Now factor out \( (x - 1) \) from the first \( 2g \) rows and \( (y - 1) \) from the next \( 2h \) rows:

\[
= \det \begin{bmatrix} xA_X - A_X^T & \cdots & (x - 1)q \\ (y - 1)q & \cdots & yA_Y - A_Y^T & (y - 1)q \\ (x - 1)(y - 1)q & \cdots & (x - 1)(y - 1)n \end{bmatrix}
\]

Factor our \( (x - 1)(y - 1) \) from the last row:

\[
= (x - 1)(y - 1) \det \begin{bmatrix} xA_X - A_X^T & (x - 1)q \\ (y - 1)q & \cdots & yA_Y - A_Y^T & (y - 1)q \\ q & \cdots & n \end{bmatrix}
\]
\[
= (x - 1)(y - 1)f(x, y).
\]
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\[
\begin{bmatrix}
A_X - A_X^T & 0 \\
A_Y - A_Y^T & \\
q & \cdots & q & n
\end{bmatrix}
\]

\[
f(1, 1) = \det(A_X - A_X^T) \det(A_Y - A_Y^T) n = \pm n = \pm \beta(L)
\]
since \(A_X - A_X^T\) and \(A_Y - A_Y^T\) are intersection forms on \(H_1(N_X)\) and \(H_1(N_Y)\) and therefore have determinant \(\pm 1\).

4. The \(\beta_j\) invariants

If \(L = X \cup Y\) is a link of linking number zero then Cochran has defined a series of integral concordance invariants \(\beta_j\), where \(\beta_0 = \beta\). To define \(\beta_j\) we must first define the notion of derived links of a link.

Definition 4.1 (Cochran [C]). If \(L = X \cup Y\) is a link of linking number zero and \(M_X, M_Y\) are Seifert surfaces spanning \(X\) and \(Y\) such that \(M_X \cap M_Y = c_1\) where \(c_1\) is an embedded circle, then \(L = X \cup c_1\) is a first derived link of \(L\). Link \(L_j = X \cup c_j\) is a \(j\)th derived link of \(L\) if it is a first derived link of \(L_{j-1} = X \cup c_{j-1}\). (Note: Each derived link has the same first component.)

Definition 4.2 (Cochran [C]). The \(\beta_j\) invariant of \(L\), \(j > 0\), is equal to the \(\beta\) invariant of a \(j\)th derived link of \(L\) and \(\beta_0 = \beta\).

Theorem 3. Let \(L = X \cup Y\) be a link of linking number zero where \(X\) is the unknot. Let \(\Sigma\) be the \(\mathbb{Z}/2\) cover of \(S^3\) branched over \(X\) and \(Y^0, Y^1\) the two lifts of \(Y\) in \(\Sigma\). Then

\[
\begin{align*}
\frac{1}{2} \text{lk}(Y^0, Y^1) &= \sum_{j=0}^{N-1} 4^j \beta_j(L) - 2 \circ 4^{N-1} \text{lk}(c_N^0, c_N^1), \\
\end{align*}
\]

where \(L_N = X \cup c_N\) is an \(N\)th derived link of \(L\), and \(c_N^0, c_N^1\) are the two lifts of \(c_N\) in \(\Sigma\).

Corollary 3. \(\frac{1}{2} \text{lk}(Y^0, Y^1) \equiv \beta(L) \mod 2\).

Proof. This is just line (1) modulo two.

Proof of Theorem 5. Let \(M_X, M_Y\) be oriented Seifert surfaces for \(X\) and \(Y\) such that \(M_X \cap M_Y = c_1\) where \(c_1\) is an embedded circle. In \(\Sigma\), \(M_Y\) (cut along \(c_1\)) lifts to \(M_Y^0\) and \(M_Y^1\); \(c_1\) lifts to \(c_1^0\) and \(c_1^1\). Now,

\[
\partial M_Y^0 = c_1^0 - c_1^1 + Y^0 \quad \text{and} \quad \partial M_Y^1 = c_1^1 - c_1^0 + Y^1.
\]

Curve \(c_1\) has a neighborhood in \(M_Y\) parametrized by \(c_1 \times [-1, 1]\). We will choose \(\gamma\) in \(M_Y\) to be equal to \(c_1 \times -1\) or \(c_1 \times 1\). The two lifts of \(\gamma\) will be contained in collar neighborhoods of \(\partial M_Y^0\) and \(\partial M_Y^1\). Choose \(\gamma\) so that \(\gamma^0\), the lift of \(\gamma\) in \(M_Y^0\), is contained in a neighborhood of \(c_1^0\) in \(M_Y^0\), then \(\gamma^1\), the lift of \(\gamma\) in \(M_Y\) will be contained in a neighborhood of \(c_1^1\) in \(M_Y^1\).
We would like to compute $\text{lk}(Y^0, Y^1)$. First note that since $X$ is the unknot $\Sigma$ is homeomorphic to $S^3$ so the linking number is well defined. In $\Sigma - Y^1$, $Y^0$ is homologous to $c^1_0 - \gamma^0$ and $Y^1$ is homologous to $c^0_1 - \gamma^1$ (see Figure 4.1). So $\text{lk}(Y^0, Y^1) = \text{lk}((\gamma^0 - c^1_0), (Y^1))$. In $\Sigma - (\gamma^0 - c^1_0)$, $Y^1$ is homologous to $\gamma^1 - c^0_1$, so

$$\text{lk}(\gamma^0 - c^1_0, Y^1) = \text{lk}(\gamma^0 - c^1_0, \gamma^1 - c^0_1) = \text{lk}(\gamma^0, \gamma^1) - \text{lk}(\gamma^0, c^0_1) - \text{lk}(c^1_0, \gamma^1) + \text{lk}(c^1_0, c^0_1) = 2\text{lk}(c^0_1, c^1_0) - 2\text{lk}(\gamma^0, c^0_1)$$

since $\text{lk}(\gamma^0, \gamma^1) = \text{lk}(c^0_1, c^1_0)$ and $\text{lk}(\gamma^0, c^0_1) = \text{lk}(c^1_0, \gamma^1)$.

We now use a general lifting formula of linking numbers: Let $p: \tilde{W} \to W$ be a covering space projection, and $G$ the group of covering transformations of $\tilde{W}$. Assume linking numbers are well defined in $W$ and $W$. Let $a, b$ be two closed curves in $W$. Then $\text{lk}_W(a, b) = \sum_{g \in G} \text{lk}_{\tilde{W}}(a, b)$. We have

$$\beta(L) = \text{lk}_{\tilde{W}}(c^0_1, c^1_0) = \text{lk}_{\tilde{W}}(c^0_1, \gamma) = \text{lk}(c^0_1, \gamma) + \text{lk}(c^0_1, \gamma^1) = \text{lk}(c^0_1, \gamma^0) + \text{lk}(c^0_1, c^1_0).$$

So

$$\text{lk}(Y^0, Y^1) = 2 \text{lk}(c^0_1, c^1_0) - 2 \text{lk}(\gamma^0, c^0_1) = 2 \text{lk}(c^0_1, c^1_0) - 2 \text{lk}(\gamma^0, c^0_1) - 2 \text{lk}(c^0_1, c^1_0) + 2 \text{lk}(c^0_1, c^1_0) = 4 \text{lk}(c^0_1, c^1_0) - 2\text{lk}(\gamma^0, c^0_1) + \text{lk}(c^0_1, c^1_0)] = 4 \text{lk}(c^0_1, c^1_0) - 2\beta(L).$$

and

$$(2) \quad \text{lk}(Y^0, Y^1) = -2\beta(L) + 4 \text{lk}(c^0_1, c^1_0).$$
The above equation is a recursive one. Consider the link $L_1 = X \cup c_1$, a first derived link of $L$. By (2)

$$\text{lk}(c_1^0, c_1^1) = -2\beta_1(L) + 4 \text{lk}(c_2^0, c_2^1)$$

where $L_2 = X \cup c_2$ is a second derived link of $L$. Thus,

$$\text{lk}(y^0, y^1) = -2\beta(L) + 4[ -2\beta_1(L) + 4 \text{lk}(c_2^0, c_2^1)] = -2\beta(L) - 2 \cdot 4\beta_1(L) + 4^2 \text{lk}(c_1^0, c_1^2).$$

In general $\text{lk}(c_j^0, c_j^1) = -2\beta_j(L) + 4 \text{lk}(c_{j+1}^0, c_{j+1}^1)$, so we have

$$\text{lk}(y^0, y^1) = -2\beta_0(L) - 2 \cdot 4\beta_1(L) - 2 \cdot 4^2\beta_2(L) - 2 \cdot 4^3\beta_3(L) - \ldots + 4^N \text{lk}(c_N^0, c_N^1)$$

or

$$-\frac{1}{2} \text{lk}(y^0, y^1) = \sum_{j=0}^{N-1} \beta_j(L) - 2 \cdot 4^{N-1} \text{lk}(c_N^0, c_N^1). \quad \square$$

References


Department of Decision Sciences, Concordia University, 1455 De Maisonneuve Boulevard West, Montreal, Quebec H3G 1M8, Canada