THE ARF AND SATO LINK CONCORDANCE INVARIANTS

RACHEL STURM BEISS

Abstract. The Kervaire-Arf invariant is a $\mathbb{Z}/2$ valued concordance invariant of knots and proper links. The $\beta$ invariant (or Sato’s invariant) is a $\mathbb{Z}$ valued concordance invariant of two component links of linking number zero discovered by J. Levine and studied by Sato, Cochran, and Daniel Ruberman. Cochran has found a sequence of invariants $\{\beta_i\}$ associated with a two component link of linking number zero where each $\beta_i$ is a $\mathbb{Z}$ valued concordance invariant and $\beta_0 = \beta$. In this paper we demonstrate a formula for the Arf invariant of a two component link $L = X \cup Y$ of linking number zero in terms of the $\beta$ invariant of the link:

$$\text{arf}(X \cup Y) = \text{arf}(X) + \text{arf}(Y) + \beta(X \cup Y) \pmod{2}.$$ 

This leads to the result that the Arf invariant of a link of linking number zero is independent of the orientation of the link’s components. We then establish a formula for $|\beta|$ in terms of the link’s Alexander polynomial $\Delta(x, y) = (x - 1)(y - 1)f(x, y)$:

$$|\beta(L)| = |f(1, 1)|.$$ 

Finally we find a relationship between the $\beta_i$ invariants and linking numbers of lifts of $X$ and $Y$ in a $\mathbb{Z}/2$ cover of the compliment of $X \cup Y$.

1. Introduction

The Kervaire-Arf invariant $[KM, R]$ is a $\mathbb{Z}/2$ valued concordance invariant of knots and proper links. The $\beta$ invariant (or Sato’s invariant) is a $\mathbb{Z}$ valued concordance invariant of two component links of linking number zero discovered by Levine (unpublished) and studied by Sato [S], Cochran [C], and Daniel Ruberman. Cochran [C] has found a sequence of invariants $\{\beta_i\}$ associated with a two component link of linking number zero where each $\beta_i$ is a $\mathbb{Z}$ valued concordance invariant and $\beta_0 = \beta$. 23 Theorem 1 demonstrates a formula for the Arf invariant of a two component link $L = X \cup Y$ of linking number zero in terms of the $\beta$ invariant of the link:

Theorem 1. $\text{arf}(X \cup Y) = \text{arf}(X) + \text{arf}(Y) + \beta(X \cup Y) \pmod{2}.$

This leads to the result that the Arf invariant of a link of linking number zero is independent of the orientation of the link’s components. In Theorem 2 we establish a formula for $|\beta|$ in terms of the link’s Alexander polynomial $\Delta(x, y) = (x - 1)(y - 1)f(x, y)$.

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Theorem 2. \(|\beta(L)| = |f(1, 1)|\).

Finally, in Theorem 3 we find a relationship between the invariants and linking numbers of lifts of \(X\) and \(Y\) in a \(Z/2\) cover of the compliment of \(X \cup Y\):

Theorem 3. \(-\frac{1}{2} \text{lk}(Y^0, Y^1) = \sum_{j=0}^{N-1} 4^j \beta_j(L) - 2(4)^{N-1} \text{lk}(c_N^0, c_N^1)\).

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We will begin by recalling a few definitions.

Let \(V\) be a vector space of dimension \(2n\) over \(Z/2\). Let \(\phi: V \times V \to Z/2\) be a nonsingular symmetric bilinear form such that \(\phi(a, a) = 0\) for every \(a\) in \(V\). The map \(q: V \to Z/2\) is a quadratic form with respect to \(\phi\) if and only if

\[q(a + b) = q(a) + q(b) + \phi(a, b)\] for every \(a, b\) in \(V\).

Definition 1.1. The set \(\{a_1, b_1, \ldots, a_n, b_n\}\) is a symplectic basis of \(V\) with respect to \(\phi\) if it is a basis of \(V\) and if \(\phi(a_i, b_j) = \delta_{ij}\) and \(\phi(a_i, a_j) = \phi(b_i, b_j) = 0\).

Definition 1.2 (Arf invariant of a quadratic form). Let \(\{a_1, b_1, \ldots, a_n, b_n\}\) be a symplectic basis of \(V\) with respect to \(\phi\). Then

\[\text{arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i)\]

is the Arf invariant of \(q\).

The above definition is independent of the choice of symplectic basis (see Arf [A]).

Let \(K\) be a knot in \(S^3\) and \(M\) an oriented Seifert surface spanning \(K\). Then \(H_1(M; Z/2)\) is a vector field over \(Z/2\) of dimension \(2q\) where \(q\) = genus of \(M\). We have a symmetric nonsingular bilinear form

\[\text{int}_2: H_1(M; Z/2) \times H_1(M; Z/2) \to Z/2\]

defined by \(\text{int}_2(a, b) = \text{mod} 2\) intersection number of cycles \(a\) and \(b\) which represent \(a\) and \(b\). Since \(M\) is orientable, \(\text{int}_2(a, a) = 0\) for every \(a\) in \(H_1(M; Z/2)\).

From now on we will assume we have fixed an orientation of \(S^3\). Since \(M\) and \(S^3\) are now oriented we can distinguish between the positive and negative normal directions to \(M\).

Given \(a\), an element of \(H_1(M)\), let \(a\) be a curve in \(M\) which represents \(a\) and \(a^-\), \(a^+\) the push offs of \(a\) in the positive and negative normal directions to \(M\). Define \(\delta_2: H_1(M; Z/2) \to Z/2\) by \(\delta_2(a) = \text{lk}(a, a^+) \mod 2\) the modulo
two linking number between \( a \) and \( a^+ \). We have

\[
\delta_2(a + b) = \delta_2(a) + \delta_2(b) + \text{int}_2(a, b)
\]

[KM], so \( \delta_2 \) is a quadratic form associated with \( \text{int}_2 \).

**Definition 1.3 (Arf invariant of a knot).** The Arf invariant of a knot \( K \) is equal to the Arf invariant of the quadratic form \( \delta_2 \).

Let \( L = X \cup Y \) be a two component link with even linking number in \( S^3 \) whose components \( X \) and \( Y \) are oriented. Let \( M \) be a connected Seifert surface for \( L \). Then \( H_1(M; \mathbb{Z}/2) \) is a vector space over \( \mathbb{Z}/2 \) of dimension \( 2g + 1 \) where \( g \) = genus of \( M \). The modulo two intersection form is nonsingular on the quotient space of \( H_1(M; \mathbb{Z}/2) \) given by \( H_1(M; \mathbb{Z}/2)/J \), where \( J \) is a one-dimensional subspace generated by one boundary component of \( M \).

**Definition 1.4 (Arf invariant of a link).** The Arf invariant of link \( L \) is equal to the Arf invariant of the quadratic form \( \delta_2 \) restricted to the quotient space \( H_1(M; \mathbb{Z}/2)/J \) of \( H_1(M; \mathbb{Z}/2) \).

The above definition is well defined for links of even linking number \([R]\).

If \( L = X \cup Y \) has linking number zero then there exist Seifert surfaces \( M_x, M_y \) for \( X \) and \( Y \) such that \( X \cap M_y = Y \cap M_x = \emptyset \). Therefore \( M_x \cap M_y = C_1 \cup C_2 \cup \cdots \cup C_n \) where \( C_j \) is a circle embedded in \( M_x \) and \( M_y \). Orient \( C_j \) so that a positive unit tangent to \( C_j \) together with the positive unit normals to \( M_x \) and \( M_y \) give the chosen orientation of \( S^3 \).

The intersection \( M_x \cap M_y \) has a natural framing of its normal bundle given by the normal 1-fields \((\vec{v}, \vec{w})\). By the Thom-Pontryagin construction \((M_x \cap M_y, \vec{v}, \vec{w})\) corresponds to an element of \( \Pi_3(S^2) \).

**Definition 1.5 (the \( \beta \) or Sato invariant).** \( \beta(L) \) is the element of \( \Pi_3(S^2) \) given by \((M_x \cap M_y, \vec{v}, \vec{w})\).

**Alternate definition to 1.5a (the \( \beta \) or Sato invariant).**

\[
\beta(L) = 2 \sum_{i<j} \text{lk}(C_i, C_j) + \sum_{j=1}^n \text{lk}(C_j, C_j^+)
\]

where \( \text{lk}(a, b) \) is the linking number between \( a \) and \( b \) with respect to the chosen orientation of \( S^3 \).

Definitions 1.5 and 1.5a are obviously the same if \( M_x \cap M_y \) is connected. In Lemma 2.1 we show that if \( M_x \cap M_y \) has two or more components then \( M_x \) and \( M_y \) can be altered so that the altered surfaces intersect in a connected manifold. Figure 1.1 illustrates how the definitions are the same when \( M_x \cap M_y = c_1 \cup c_2 \) has two components and \( M_x, M_y \) are altered so that the new intersection \( c \) is connected.
2. A RELATIONSHIP BETWEEN THE $\beta$ AND ARF INVARIANTS

In this section we establish a relationship between the $\beta$ invariant and the Arf invariant of a link of linking number zero.

**Theorem 1.** If $L = X \cup Y$ is an oriented link of linking number zero then $\text{arf}(L) = \text{arf}(X) + \text{arf}(Y) + \beta(L) \pmod{2}$.

First we will find two oriented Seifert surfaces $M_X$, $M_Y$ for $X$, $Y$ respectively such that $M_X \cap M_Y$ consists of only one circle. We cut $M_X$ and $M_Y$ along this circle and reglue to obtain an oriented Seifert surface $M$ for $L$. Surface $M$ will be used to compute $\text{arf}(L)$.

**Lemma 2.1.** Let $L = X \cup Y$ be a link of linking number zero. There exist orientable Seifert surfaces $M_X$ and $M_Y$ spanning $X$ and $Y$ such that $M_X \cap M_Y = \emptyset$ or one circle. Furthermore this circle is not null homologous in $H_1(M_X)$ of $H_1(M_Y)$.

**Proof.** Let $N_X$, $N_Y$ be oriented Seifert surfaces for $X$ and $Y$ such that $N_X \cap N_Y = \{c_1, \ldots, c_n\}$, where each $c_j$ is an embedded circle. Change circle inter-
sections $c_1$ and $c_2$ to band intersections by the following method of Cooper [Co]. Let $\alpha_1, \alpha_2$ be paths from $\partial N_Y$ to points on $c_1$ and $c_2$ respectively. Push $N_Y$ in along $\alpha_1$ and $\alpha_2$ so $c_1$ and $c_2$ become band intersections (see Figure 2.1). Let $p_1$ and $p_2$ be two points on the resulting band intersections. Let $\gamma$ be an arc in $N_X$ from $p_1$ to $p_2$ and oriented from $p_1$ to $p_2$ such that the intersection number of $\gamma$ with $N_Y$ at $p_1$ is $+1$ and the intersection number of $\gamma$ with $N_X$ at $p_2$ is $-1$ (see Figure 2.2(i)). Now attach a handle to $N_Y$ whose core coincides with $\gamma$ (see Figure 2.2(ii)). Our two band intersections have been transformed into two different band intersections. If we attach two handles to $N_X$ whose cores coincide with the two arcs in $\partial N_Y = Y$ which were pushed in along paths $\alpha_1$ and $\alpha_2$ then the band intersections will become one circle intersection (see Figure 2.2(iii)).

Let us call our new surfaces $N'_X$ and $N'_Y$. These new surfaces intersect in $N'_X \cap N'_Y = \{a, c_3, \ldots, c_n\}$, where $a$ was formed from $c_1$ and $c_2$. If we repeat the above process we can find oriented surfaces $M'_X$ and $M'_Y$ such that $M'_X \cap M'_Y = \text{one circle } c$.

Suppose $M'_X \cap M'_Y = c$ and $c$ is null homologous in $H_1(M'_X)$. Then $c$ bounds a surface $N$ in $M'_Y$. Look at a neighborhood of $N$ in $S^3$ parametrized by $N \times [1, -1]$, where $N \times 0$ coincides with $N$. Replace $M'_X$ with $M'_X - N \cup \partial N \times [1, 0] \cup N \times 1$. Now round off the corner at $N \times 0$ and we will no longer have an intersection along $c$. \hfill \Box

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Figure 2.2

**Proof of Theorem 1.** By Lemma 2.1 we can assume we have oriented Seifert surfaces \( M_x \) and \( M_y \) for \( X, Y \) respectively such that \( M_x \cap M_y = \emptyset \) or \( M_x \cap M_y = c \), where \( c \) is an embedded circle and \( c \) not homologous to zero in \( H_1(M_x) \) or \( H_1(M_y) \).

If \( M_x \cap M_y = \emptyset \) then \( L \) is a boundary link. From the methods of Robertello [R] it easily follows that \( \text{arf}(L) = \text{arf}(X) + \text{arf}(Y) \). Since the \( \beta \) invariant for a boundary link is zero the formula holds in this case.

Assume that \( M_x \cap M_y = c \). We can find symplectic bases:

\[
\{c, d_1, e_2, d_2, \ldots, e_g, d_g\}, \quad \{c, d'_1, e_2, d_2, \ldots, e_h, f_h\}
\]

for \( H_1(M_x; \mathbb{Z}/2) \) and \( H_1(M_y; \mathbb{Z}/2) \) respectively, where \( g = \text{genus } M_x \) and \( h = \text{genus } M_y \), and \( c \) represents \( c \).

We now construct \( M \), an oriented Seifert surface for \( L \), from \( M_x \) and \( M_y \). Cut \( M_x \) and \( M_y \) along \( c \). Attach \( M_x - c \) to \( M_y - c \) so that the positive sides of \( M_x - c \) and \( M_y - c \) form the positive side of \( M \). \( M \) is orientable. Let \( H_1(M; \mathbb{Z}/2) = I \oplus J \), where \( J = \text{null space of form } \text{int}_2 \) and \( I \) is the subspace with symplectic basis:

\[
\{c, d, e_2, d_2, \ldots, e_g, d_g, e_2, d_2, \ldots, e_h, f_h\}
\]

where \( d = d_1 \cup d'_1 \) represents \( d \). The self-linking numbers of the basis element of \( I \) have not changed with the exception of \( d \):

\[
\delta_2(d) = \delta_2(d_1) + \delta_2(d'_1) + 1.
\]
The extra 1 appears because $d_1$ and $d_1'$ intersects at a point. Figure 2.3 is a local picture of $M_x \cup M_y$ near $d_1$ and $d_1'$ and shows how the extra linking number appears.

Use the above symplectic basis of $I$ to calculate $\text{arf}(L)$:

$$\text{arf}(L) \equiv \delta_2(c) \delta_2(d) + \sum_{j=2}^{g} \delta_2(c_j) \delta_2(d_j) + \sum_{j=2}^{h} \delta_2(e_j) \delta_2(f_j)$$

$$\equiv \delta_2(c)(\delta_2(d_1) + \delta_2(d_1') + 1)$$

$$+ \sum_{j=2}^{g} \delta_2(c_j) \delta_2(d_j) + \sum_{j=2}^{h} \delta_2(e_j) \delta_2(f_j)$$

$$\equiv \delta_3(c) + \text{arf}(X) + \text{arf}(Y) \mod 2$$
and \( \delta_2(c) \equiv \beta(L) \pmod{2} \) so,

\[
\text{arf}(L) \equiv \text{arf}(X) + \text{arf}(Y) + \beta(L) \pmod{2}.
\]

\[\square\]

**Corollary 1.** The Arf invariant of a link of linking number zero is independent of the orientation of the components of the link.

**Proof.** We must show that \( \beta(L) \pmod{2} \) is independent of the link's orientation.

Definition 1.5 for \( \beta(L) \) shows

\[
\beta(L) = 2 \sum_{i<j} \text{lk}(c_i, c_j) L + \sum_{j=1}^{2p} \text{lk}(c_j^+, c_j^-) 
\]

\[
\equiv \sum_{j=1}^{2p} \text{lk}(c_j^+, c_j^-) \pmod{2}.
\]

Each term \( \text{lk}(c_j^+, c_j^-) \) is independent of the orientation of the components of the link. \[\square\]

3. A FORMULA FOR \( |\beta(L)| \) IN TERMS OF THE ALEXANDER POLYNOMIAL OF \( L \)

In this section we establish a relationship between the \( \beta \) invariant of a link and its Alexander polynomial \( \Delta(x, y) \). If \( L = X \cup Y \) is a link of linking number zero then it has an Alexander polynomial of the form

\[
\Delta(x, y) = (x - 1)(y - 1)f(x, y).
\]

**Theorem 2.** \( |\beta(L)| = |f(1, 1)|.\)

We compute \( \Delta(x, y) \) using an algorithm by Cooper [Co] which is summarized in the following paragraph:

Let \( L = X \cup Y \) be an oriented link in \( S^3 \) and \( M_X, M_Y \) Seifert surfaces with orientations inherited from those of \( X \) and \( Y \) such that \( M_X \cap M_Y \) consists of only clasp intersections (see Figure 3.1(i)). Cooper defines two bilinear forms

\[
u, v : H_1(M_X \cup M_Y) \times H_1(M_X \cup M_Y) \to \mathbb{Z}
\]

by \( u(a, b) = \text{lk}(a, b^-) \) and \( v(a, b) = \text{lk}(a, b^+) \), where \( a, b \) are curves in \( M_X \) and \( M_Y \) respectively which represent \( a \) and \( b \). The notation \( \text{lk}(a, b^-) \) stands for the linking number between \( a \) and \( b^- \) where \( b^- \) is the push off of \( b \) in the negative normal direction to \( M_X \) and negative normal direction to \( M_Y \). Similarly \( \text{lk}(a, b^+) \) is the linking number between \( a \) and the push off of \( b \) in the negative normal direction to \( M_X \) and positive normal direction to \( M_Y \). When \( a \) or \( b \) passes from \( M_X \) to \( M_Y \) it must contain the clasp intersection. Near a clasp the push off of a curve will lie in one of the four quadrants formed by the clasp intersection as suggested in Figure 3.1(ii). Cooper observed that

\[\text{The quantity } |f(1, 1)| \text{ is equal to } |a_i|, \text{ where } a_i \text{ is a Conway polynomial coefficient. In [C] the Conway polynomial coefficients are found to be related to Milnor's } \mu\text{-invariants and to certain linking numbers.}\]
(i) A clasp intersection

\[ M_X \quad \text{---} \quad M_Y \]

(ii) Changing a circle intersection to two clasps

\[ \begin{array}{c}
  b^+ - \\
  - + \\
  + - \\
  b^- + \\
\end{array} \quad \begin{array}{c}
  b^+ - \\
  - + \\
  + - \\
  b^- + \\
\end{array} \]

(iii) Changing a circle intersection to two clasps

\[ H_1(M_X \cup M_Y) \cong H_1(M_X) \oplus H_1(M_Y) \oplus \tilde{H}_0(M_X \cap M_Y). \]

Loops circling through the clasps form a basis for the component of \( H_1(M_X \cup M_Y) \) isomorphic to \( \tilde{H}_0(M_X \cap M_Y) \). Let \( A, B \) be matrices representing \( u \) and \( v \) respectively. Cooper shows that

\[ \Delta(x, y) = (x - 1)^{-2g} (y - 1)^{-2h} \det(xyA + A^T - xB - yB^T), \]

where \( g = \) genus of \( M_X \) and \( h = \) genus of \( M_Y \).

**Proof of Theorem 2.** By Lemma 2.1 we can assume that we have oriented Seifert surfaces \( N'_X \) and \( N'_Y \) for \( X \) and \( Y \) such that \( N'_X \cap N'_Y \) is a single circle. The circle intersection can be changed as in [Co] to two clasp intersections (see Figure 3.1(iii)). Let \( N_X, N_Y \) be the resulting Seifert surfaces for \( X \) and \( Y \) such that \( N_X \cap N_Y \) is two clasps. If \( N'_X \cap N'_Y = c \) then the invariant of \( L \),
Let us choose a basis of \( H_1(N_X \cup N_Y) \) consisting of a basis for \( H_1(N_X) \) union a basis for \( H_1(N_Y) \) union \( \xi \) where \( c \) runs through the two clasps as above and represents \( \xi \). Notice that if curve \( b \) lies in \( M_X \) then \( u(a, b) = v(a, b) = \text{lk}(a, b^\circ) \) where \( b^\circ \) is the push off of \( b \) in the negative normal direction to \( M_X \). If \( A_X = A \) restricted to \( H_1(N_X) \) and \( B_X = B \) restricted to \( H_1(N_X) \) then \( A_X = B_X \). Similarly if \( A_Y = A \) restricted to \( H_1(N_Y) \) and \( B_Y = B \) restricted to \( H_1(N_Y) \) then \( B_Y = A_Y^T \). Let \( n = \beta(L) = \text{lk}(c, c^{-\circ}) = \text{lk}(c, c^{++}) \). Then with respect to the above basis for \( H_1(N_X \cup N_Y) \), \( A \) and \( B \) have the following form:

\[
A = \begin{bmatrix}
A_X & q_{ij} \\
q_{ji} & A_Y \\
\end{bmatrix}, \quad B = \begin{bmatrix}
A_X & q_{ij} \\
q_{ji} & A_Y \\
\end{bmatrix}
\]

where \( q_{ij} = q_{ji} \). \( A \) has matrices \( A_X, A_Y \) and \( n \) down the diagonal. Away from these submatrices \( A \) has \( ij \)th entry designated by \( q_{ij} \). The \( q_{ij} \) entries represent linking numbers between nonintersecting curves in \( N_X \) and \( N_Y \), thus \( q_{ij} = q_{ji} \). \( B \) has matrices \( A_X, A_Y^T \) and \( n \) down the diagonal. The \( q_{ij} \) entries of \( B \) are equal to those of \( A \).

By Cooper’s algorithm

\[
\Delta(x, y) = (x - 1)^{-2g}(y - 1)^{-2h} \operatorname{det}_{xy} \begin{bmatrix}
A_X & q_{ij} \\
q_{ji} & n \\
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
A_X^T & q_{ji} \\
q_{ij} & A_Y \\
\end{bmatrix} - x \begin{bmatrix}
A_X & q_{ij} \\
q_{ji} & n \\
\end{bmatrix} - y \begin{bmatrix}
A_Y^T & q_{ij} \\
q_{ji} & n \\
\end{bmatrix}
\]

\[
= (x - 1)^{-2g}(y - 1)^{-2h} \cdot \det \begin{bmatrix}
(x - 1)(yA_X - A_X^T) & (x - 1)(y - 1)q_{ij} \\
(x - 1)(y - 1)q_{ji} & (x - 1)(y - 1)n \\
\end{bmatrix}.
\]

Now factor out \( (x - 1) \) from the first \( 2g \) rows and \( (y - 1) \) from the next \( 2h \) rows:

\[
= \text{det} \begin{bmatrix}
xA_X - A_X^T & \cdots & (x - 1)q \\
(y - 1)q \cdots yA_Y - A_Y^T & (y - 1)q \\
(x - 1)(y - 1)q \cdots (x - 1)(y - 1)n \\
\end{bmatrix}.
\]

Factor our \( (x - 1)(y - 1) \) from the last row:

\[
= (x - 1)(y - 1)\text{det} \begin{bmatrix}
xA_X - A_X^T & (x - 1)q \\
(y - 1)q \cdots yA_Y - A_Y^T & (y - 1)q \\
q & n \\
\end{bmatrix}
\]

\[
= (x - 1)(y - 1)f(x, y).
\]
\[ f(1, 1) = \det \begin{bmatrix} A_X - A_X^T & 0 \\ A_Y - A_Y^T & \vdots \\ q & \cdots & q & n \end{bmatrix} = \det(A_X - A_X^T) \det(A_Y - A_Y^T)n = \pm n = \pm \beta(L) \]

since \( A_X - A_X^T \) and \( A_Y - A_Y^T \) are intersection forms on \( H_1(N_X) \) and \( H_1(N_Y) \) and therefore have determinant \( \pm 1 \).

4. The \( \beta_j \) Invariants

If \( L = X \cup Y \) is a link of linking number zero then Cochran has defined a series of integral concordance invariants \( \beta_j \), where \( \beta_0 = \beta \). To define \( \beta_j \) we must first define the notion of derived links of a link.

**Definition 4.1** (Cochran [C]). If \( L = X \cup Y \) is a link of linking number zero and \( M_X, M_Y \) are Seifert surfaces spanning \( X \) and \( Y \) such that \( M_X \cap M_Y = c_1 \) where \( c_1 \) is an embedded circle, then \( L = X \cup c_1 \) is a first derived link of \( L \). Link \( L_j = X \cup c_j \) is a \( j \)th derived link of \( L \) if it is a first derived link of \( L_{j-1} = X \cup c_{j-1} \). (Note: Each derived link has the same first component.)

**Definition 4.2** (Cochran [C]). The \( \beta_j \) invariant of \( L, j > 0 \), is equal to the \( \beta \) invariant of a \( j \)th derived link of \( L \) and \( \beta_0 = \beta \).

**Theorem 3.** Let \( L = X \cup Y \) be a link of linking number zero where \( X \) is the unknot. Let \( \Sigma \) be the \( \mathbb{Z}/2 \) cover of \( S^3 \) branched over \( X \) and \( Y^0, Y^1 \) the two lifts of \( Y \) in \( \Sigma \). Then

\[
\frac{1}{2} \text{lk}(Y^0, Y^1) = \sum_{j=0}^{N-1} 4^j \beta_j(L) - 2 \circ 4^{N-1} \text{lk}(c_N^0, c_N^1),
\]

where \( L_N = X \cup c_N \) is an \( N \)th derived link of \( L \), and \( c_N^0, c_N^1 \) are the two lifts of \( c_N \) in \( \Sigma \).

**Corollary 3.** \( \frac{1}{2} \text{lk}(Y^0, Y^1) \equiv \beta(L) \mod 2 \).

**Proof.** This is just line (1) modulo two. \( \square \)

**Proof of Theorem 5.** Let \( M_X, M_Y \) be oriented Seifert surfaces for \( X \) and \( Y \) such that \( M_X \cap M_Y = c_1 \) where \( c_1 \) is an embedded circle. In \( \Sigma \), \( M_Y \) (cut along \( c_1 \)) lifts to \( M_Y^0 \) and \( M_Y^1 \); \( c_1 \) lifts to \( c_1^0 \) and \( c_1^1 \). Now,

\[
\partial M_Y^0 = c_1^0 - c_1^1 + Y^0 \quad \text{and} \quad \partial M_Y^1 = c_1^1 - c_1^0 + Y^1.
\]

Curve \( c_1 \) has a neighborhood in \( M_Y \) parametrized by \( c_1 \times [-1, 1] \). We will choose \( \gamma \) in \( M_Y \) to be equal to \( c_1 \times -1 \) or \( c_1 \times 1 \). The two lifts of \( \gamma \) will be contained in collar neighborhoods of \( \partial M_Y^0 \) and \( \partial M_Y^1 \). Choose \( \gamma \) so that \( \gamma^0 \), the lift of \( \gamma \) in \( M_Y^0 \), is contained in a neighborhood of \( c_1^0 \) in \( M_Y^0 \), then \( \gamma^1 \), the lift of \( \gamma \) in \( M_Y \) will be contained in a neighborhood of \( c_1^1 \) in \( M_Y^1 \).
We would like to compute $\text{lk}(Y^0, Y^1)$. First note that since $X$ is the unknot $\Sigma$ is homeomorphic to $S^3$ so the linking number is well defined. In $\Sigma - Y^1$, $Y^0$ is homologous to $c_1^0 - \gamma^0$ and $Y^1$ is homologous to $c_1^0 - \gamma^1$ (see Figure 4.1). So $\text{lk}(Y^0, Y^1) = \text{lk}(\gamma^0 - c_1^1, Y^1)$. In $\Sigma - (\gamma^0 - c_1^1)$, $Y^1$ is homologous to $\gamma^1 - c_1^0$, so

$$\text{lk}(\gamma^0 - c_1^1, Y^1) = \text{lk}(\gamma^0 - c_1^1, \gamma^1 - c_1^1)$$

$$= \text{lk}(\gamma^0, \gamma^1) - \text{lk}(\gamma^0, c_1^0) - \text{lk}(c_1^1, \gamma^1) + \text{lk}(c_1^1, c_1^0)$$

$$= 2 \text{lk}(c_1^0, c_1^1) - 2 \text{lk}(\gamma^0, c_1^0)$$

since $\text{lk}(\gamma^0, \gamma^1) = \text{lk}(c_1^0, c_1^1)$ and $\text{lk}(\gamma^0, c_1^0) = \text{lk}(c_1^1, \gamma^1)$.

We now use a general lifting formula of linking numbers: Let $p: \tilde{W} \rightarrow W$ be a covering space projection, and $G$ the group of covering transformations of $\tilde{W}$. Assume linking numbers are well defined in $W$ and $\tilde{W}$. Let $a, b$ be two closed curves in $W$. Then $\text{lk}_W(a, b) = \sum_{\tau \in G} \text{lk}_{\tilde{W}}(\tau a, b)$. We have

$$\beta(L) = \text{lk}_{\tilde{W}}(c_1, c_1) = \text{lk}_G(c_1, \gamma)$$

$$= \text{lk}(c_1^0, \gamma^0) + \text{lk}(c_1^0, \gamma^1) = \text{lk}(c_1^0, \gamma^0) + \text{lk}(c_1^0, c_1^1).$$

So

$$\text{lk}(Y^0, Y^1) = 2 \text{lk}(c_1^0, c_1^1) - 2 \text{lk}(\gamma^0, c_1^0)$$

$$= 2 \text{lk}(c_1^0, c_1^1) - 2 \text{lk}(\gamma^0, c_1^0) - 2 \text{lk}(c_1^0, c_1^1) + 2 \text{lk}(c_1^0, c_1^1)$$

$$= 4 \text{lk}(c_1^0, c_1^1) - 2[\text{lk}(\gamma^0, c_1^0) + \text{lk}(c_1^0, c_1^1)]$$

$$= 4 \text{lk}(c_1^0, c_1^1) - 2 \beta(L).$$

and

(2)  \hspace{1cm} \text{lk}(Y^0, Y^1) = -2 \beta(L) + 4 \text{lk}(c_1^0, c_1^1).
The above equation is a recursive one. Consider the link \( L_1 = X \cup c_1 \) a first derived link of \( L \). By (2)
\[
\text{lk}(c_1^0, c_1^1) = -2\beta_1(L) + 4 \text{lk}(c_2^0, c_2^1)
\]
where \( L_2 = X \cup c_2 \) is a second derived link of \( L \). Thus,
\[
\text{lk}(Y^0, Y^1) = -2\beta(L) + 4[-2\beta_1(L) + 4 \text{lk}(c_2^0, c_2^1)]
\]
\[
= -2\beta(L) - 2 \cdot 4\beta_1(L) + 4^2 \text{lk}(c_1^0, c_1^1).
\]
In general \( \text{lk}(c_j^0, c_j^1) = -2\beta_j(L) + 4 \text{lk}(c_{j+1}^0, c_{j+1}^1) \), so we have
\[
\text{lk}(Y^0, Y^1) = -2\beta_0(L) - 2 \cdot 4\beta_1(L) - 2 \cdot 4^2\beta_2(L) - 2 \cdot 4^3\beta_3(L) - \cdots + 4^N \text{lk}(c_N^0, c_N^1)
\]
or
\[
-\frac{1}{2} \text{lk}(Y^0, Y^1) = \sum_{j=0}^{N-1} \beta_j(L) - 2 \cdot 4^{N-1} \text{lk}(c_N^0, c_N^1). \quad \square
\]

REFERENCES