AN ASYMPTOTIC FORMULA FOR HYPO-ANALYTIC PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. An asymptotic expansion formula for hypo-analytic pseudodifferential operators is proved and applications are given.

INTRODUCTION

In [2] we introduced hypo-analytic pseudodifferential operators that are naturally associated with the hypo-analytic structures of [1]. In this paper we establish an asymptotic formula for these operators. Such an expansion is essential in several applications. It allows us to define, in a natural way, the symbol of a hypo-analytic pseudodifferential operator, as well as the symbols of the adjoint, transpose and composition of operators. The paper is organized as follows. In Chapter I we discuss and develop the asymptotic formula. Chapter II applies this formula to two results.

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1. ASYMPTOTIC EXPANSION

1. Hypo-analytic structures. We will deal with structures which are a special case of the hypo-analytic structures introduced by Baouendi, Chang and Treves in [1]. We shall summarize the relevant concepts here. Let \( \Omega \) be a \( C^\infty \) manifold of dimension \( m \). A hypo-analytic structure of maximal dimension on \( \Omega \) is the data of an open covering \( (U_\alpha) \) of \( \Omega \) and for each index \( \alpha \), of \( m \) \( C^\infty \) functions \( Z_1, \ldots, Z_m \) satisfying the following two conditions:

1. \( dZ_1, \ldots, dZ_m \) are linearly independent at each point of \( U_\alpha \);
2. if \( U_\alpha \cap U_\beta \neq \emptyset \), there are open neighborhoods \( \mathcal{O}_\alpha \) of \( Z_\alpha(U_\alpha \cap U_\beta) \) and \( \mathcal{O}_\beta \) of \( Z_\beta(U_\alpha \cap U_\beta) \) and a holomorphic map \( F_\beta^{\alpha} \) of \( \mathcal{O}_\alpha \) onto \( \mathcal{O}_\beta \) such that

\[
Z_\beta = F_\beta^{\alpha} \circ Z_\alpha \quad \text{on} \quad U_\alpha \cap U_\beta .
\]

We will use the notation \( Z_\alpha = (Z_\alpha^1, \ldots, Z_\alpha^m) : U_\alpha \to C^m \). A distribution \( h \) defined in an open neighborhood of a point \( p_0 \) of \( \Omega \) is hypo-analytic at \( p_0 \)
if there is a chart \((U_\alpha, Z_\alpha)\) of the above type whose domain contains \(p_0\) and a holomorphic function \(\hat{h}\) defined on an open neighborhood of \(Z_\alpha(p_0)\) in \(C^m\) such that \(h = \hat{h} \circ Z_\alpha\) in a neighborhood of \(p_0\). By a hypo-analytic local chart we mean an \(m+1\)-tuple \((U, Z^1, \ldots, Z^m)\) [abbreviated \((U, Z)\)] consisting of an open subset \(U\) of \(\Omega\) and of \(m\) hypo-analytic functions whose differentials are linearly independent at every point of \(U\).

We will now reason in a hypo-analytic local chart \((U, Z)\) of \(\Omega\). Assume that the open set \(U\) has been contracted sufficiently so that the mapping \(Z = (Z^1, \ldots, Z^m) : U \to C^m\) is a diffeomorphism of \(U\) onto \(Z(U)\) and that \(U\) is the domain of local coordinates \(x_j\) \((1 \leq j \leq m)\) all vanishing at a "central point" which will be denoted by \(0\). We will suppose \(Z(0) = 0\) and denote by \(Z_x\) the Jacobian matrix of the \(Z^j\) with respect to the \(x^k\). Substitution of \(Z_x(0)^{-1}Z(x)\) for \(Z(x)\) will allow us to assume that \(Z_x(0) = \) the identity matrix. Therefore the real part of the \(Z^j\) \((j = 1, \ldots, m)\) can serve as coordinates and in these new coordinates

\[
Z^j = x^j + \sqrt{-1}\phi^j(x), \quad j = 1, \ldots, m,
\]

where \(\phi = (\phi^1, \ldots, \phi^m)\) is real valued with 0 differential at the origin.

Moreover, the functions \(Z^j\) are selected so that all the derivatives of order two of the \(\phi^j\) vanish at the origin. Indeed if this is not already so it suffices to replace each \(Z^j\) by

\[
Z^j - \frac{\sqrt{-1}}{2} \sum_k \sum_l \frac{\partial^2 \phi^j}{\partial x^k \partial x^l}(0)Z^k Z^l.
\]

We will use \(\hat{Z}_x\) to denote the transpose of the inverse of the matrix \(Z_x\). Since the first and second derivatives of all the \(\phi^j\) are zero at the origin, after contracting \(U\) if necessary, we can find a number \(c\), \(0 < c < 1\) such that for all \(x, y\) in \(U\) and for all \(\xi\) in \(R_m\)

\[
|\Re \hat{Z}_x(x)\xi| \leq c|\Im \hat{Z}_x(x)\xi| \quad \text{and} \quad
(1.1) \quad \Re \{\sqrt{-1}\hat{Z}_x(x)\xi \cdot (Z(x) - Z(y)) - \langle \hat{Z}_x(x)\xi, (Z(x) - Z(y))^2 \rangle\}
\]

\[
\leq -c|\xi||Z(x) - Z(y)|^2,
\]

where \(\langle \xi \rangle = (\xi^1 + \cdots + \xi^m)^{\frac{1}{2}}\) for \(|\Im \xi| < |\Re \xi|\).

2. Hypo-analytic pseudodifferential operators. We will continue to work in the chart \((U, Z)\) of §1. Our aim now is to briefly describe the hypo-analytic pseudodifferential operators.

Definition 2.1. Let \(d\) be a real number. We denote by \(\tilde{S}^d(U, U)\) the space of holomorphic functions \(\tilde{a}(z, w, \theta)\) in a product set \(\mathcal{C} \times \mathcal{C} \times \mathcal{C}\) with \(\mathcal{C}\) an open neighborhood of \(Z(U)\), and \(\mathcal{C}\) an open cone in \(C_m \setminus \{0\}\) containing \(R_m \setminus \{0\}\) which have the following property:
Given any compact subset $K$ of $\mathcal{C}$ and any closed cone $\mathcal{C}' \subset \mathcal{C}$ whose interior contains $R_m \setminus \{0\}$, there is a constant $r > 0$ such that for all $z, w$ in $K$ and all $\theta$ in $\mathcal{C}'$, we have

$$|\hat{a}(z, w, \theta)| \leq r(1 + |\theta|)^d.$$ 

**Definition 2.2.** We say that a $C^\infty$ function $a(x, y, \theta)$ in $U \times U \times R_m$ is a hypo-analytic amplitude of degree $d$ and we write $a \in \mathcal{S}^d(U, U)$ if there is $\hat{a} \in \hat{S}^d(U, U)$ such that $a(x, y, \theta) = \hat{a}(Z(x), Z(y), \theta)$, for all $x$ in $U$, $y$ in $U$, $0 \neq \theta \in R_m$.

Let $a(x, y, \theta) = \hat{a}(Z(x), Z(y), \theta)$ be a hypo-analytic amplitude of degree $d \in R$ in $U \times U$. For any $\epsilon > 0$ and $u \in C^0_c(U)$ we define the linear operator

$$A^\epsilon u(x) = \left(\frac{1}{4\pi^3}\right)^q \int_{U} \int_{R_m} \exp(\sqrt{-1} \xi \cdot (Z(x) - Z(y) - \epsilon|\xi|^2)) \cdot a(x, y, \xi)u(y) dZ(y) d\xi$$

We contract $U$ sufficiently so that for every $x, y \in U$ and $\xi \in R_m$ the point $\tilde{Z}(x, y, \xi) = Z(x) + \sqrt{-1}((\hat{Z}(x, y, \xi))(Z(x) - Z(y)))$ will remain in the cone in which $a(x, y, \cdot)$ is defined. We observe that each $A^\epsilon u$ is a hypo-analytic function. The results of [2] may be consolidated into:

**Theorem 2.1.** When $\epsilon \to 0$, $A^\epsilon$ converges to a continuous linear operator $A$: $E'(U) \to \mathcal{D}(U)$ which maps $C^\infty(U)$ into $C^\infty(U)$ continuously. If $u$ is hypo-analytic at $0$ then $Au$ is hypo-analytic at $0$.

The first part of the theorem is proved by first deforming the path of $\xi$-integration from $R_m$ to the image of $R_m$ under the map

$$\xi \to \xi(\xi) = \tilde{Z}(x)(\xi) + \sqrt{-1}(\hat{Z}(x, y, \xi))(Z(x) - Z(y)).$$

The second inequality in (1.1) will then force the exponential term in (2.3) to be bounded. The integral can then be treated as an oscillatory integral.

Following [2] we will call $A$ a hypo-analytic pseudodifferential operator. When $Z(x) = x$ this specializes to the usual analytic pseudodifferential operator.

3. Formal hypo-analytic amplitudes. In this section $(U, Z)$ will be as in §2. Our aim is to establish an asymptotic expansion formula for hypo-analytic amplitudes.

Fix a neighborhood $\mathcal{O}$ of $Z(U)$ in $C^m$, a cone $\mathcal{C}$ in $C^m \setminus \{0\}$ and let $R_0(z, w)$ be a positive continuous function on $\mathcal{O} \times \mathcal{O}$. For each $j = 0, 1, 2, \ldots$ let $k_j(z, w, \theta)$ be a holomorphic function in the set

$$\{ (z, w, \theta) \in \mathcal{O} \times \mathcal{O} \times \mathcal{C}; |\theta| > R_0(z, w) \sup(j, 1) \}.$$ 

Set $k_j(x, y, \theta) = \hat{k}_j(Z(x), Z(y), \theta)$.
Definition 3.1. We will say that the series \[ \sum_{j=0}^{\infty} k_j(x, y, \theta) \] defines a formal hypo-analytic amplitude of degree \( d \) if there exists a continuous function \( c_0(z, w) > 0 \) on \( \mathcal{O} \times \mathcal{O} \) such that for all \((z, w)\) in \( \mathcal{O} \times \mathcal{O} \) and all \( \theta \) in \( \mathcal{E} \), \(|\theta| > R_0(z, w)\sup(j, 1)\),

\[ |\hat{k}_j(z, w, \theta)| \leq c_0(z, w)^{j+1} j!|\theta|^{d-j}. \]

We now show how to construct a true hypo-analytic amplitude from the formal one given above. We will work in a compact set \( K \subseteq \mathcal{U} \) and a relatively compact neighborhood \( \mathcal{O}_K \) of \( Z(K) \) in \( \mathcal{O} \). This enables us to replace the functions \( c_0(z, w) \) and \( R_0(z, w) \) of the above definition by constants \( C_0 \) and \( R_0 \). We will also assume that the cone \( \mathcal{E} \) has been shrunk to satisfy: for some \( \delta > 0 \), whenever \( \theta = \xi + \sqrt{-1}\eta \in \mathcal{E} \), then \( \delta|\theta| \leq |\xi| \). Let \( R > \max(R_0, C_0) \).

We will use a sequence of smooth cutoff functions \( \phi_j(\xi) \) having the following properties:

\[ 0 \leq \phi_j(\xi) \text{ for all } \xi, \quad \text{and} \quad \phi_j(\xi) = 0 \text{ in } |\xi| < 2R \sup(j, 1), \]

\[ \phi_j(\xi) = 1 \text{ if } |\xi| > 3R \sup(j, 1); \]

\[ \phi_j(\xi) = 0 \text{ if } |\xi| < 2; \]

See [8] for the construction of such cutoffs. Define

\[ \hat{k}(z, w, \theta) = \sum_{j=0}^{\infty} \phi_j(\xi)^j k_j(z, w, \theta) \]

for \((z, w) \in \mathcal{O}_K \times \mathcal{O}_K\) and \( \theta = \xi + \sqrt{-1}\eta \in \mathcal{E} \). \( \hat{k} \) is a \( C^\infty \) function of \((z, w, \theta)\) holomorphic in \((z, w)\). \( \hat{k} \) satisfies the following estimates:

\[ |\hat{k}(z, w, \theta)| \leq \sum_{0 \leq j < d} |\hat{k}_j(z, w, \theta)| + \sum_{j \geq d} \phi_j(\xi)^j |k_j(z, w, \theta)|; \]

\[ \leq \sum_{0 \leq j < d} |\hat{k}_j(z, w, \theta)| + \sum_{j \geq d} \phi_j(\xi)c_0^{j+1} j!|\theta|^{d-j}; \]

\[ \leq \sum_{0 \leq j < d} |\hat{k}_j(z, w, \theta)| + \sum_{j \geq d} \phi_j(\xi)c_0^{j+1} j!|\xi|^{d-j}. \]

Since for \( j \geq d \) the \( j \)th term lives on the set \( \{\xi : |\xi| \geq 2Rj\} \), the latter

\[ \leq \sum_{0 \leq j < d} |\hat{k}_j(z, w, \theta)| + |\xi|^d \sum_{j \geq d} c_0^{j+1} j! \left( \frac{1}{2Rj} \right)^j \]

\[ \leq \sum_{0 \leq j < d} |\hat{k}_j(z, w, \theta)| + \text{constant } |\xi|^d \]

\[ \leq \text{constant } |\theta|^d. \]
\[ \partial_{\theta} \hat{k}(z, w, \theta) \leq \sum_{j=0}^{\infty} |\partial_{\theta} \phi_j(\xi) \hat{k}_j(z, w, \theta)| \]
\[ \leq \left( \sum_{j=0}^{\infty} |\partial_{\theta} \phi_j(\xi)| c_0^{j+1} \frac{j!}{|\xi|^j} \right) \]
\[ \leq \delta^d |\xi|^d \left( \sum_{j=0}^{\infty} |\partial_{\theta} \phi_j(\xi)| c_0^{j+1} \frac{j!}{|\xi|^j} \right) \]

We now use the fact that \( \partial_{\theta} \phi_j(\xi) \) lives in the set \{\xi : 2Rj \leq |\xi| \leq 3Rj\};
\[ \leq \text{constant} |\xi|^d \left( \sum_{j=0}^{\infty} c_0^{j+1} \frac{j!}{2Rj} \right) \]
Since \( j! / j! \leq e^{-j} \), the latter \( \leq \text{constant} |\xi|^d \sum_{j=0}^{\infty} (c_0/j^2) e^{-j} \).
Recalling that \( 2Rj \leq |\xi| \leq 3Rj \), we get
\[ \leq \text{constant} e^{-\frac{|\xi|^2}{4R}} \]
\[ \leq \text{constant} e^{-\frac{|\xi|^2}{4R} |\theta|} \]

Thus for \( (z, w, \theta) \in \mathcal{O}_K \times \mathcal{O}_K \times \mathcal{C} \), we have: \( |\hat{k}(z, w, \theta)| \leq \text{const.} |\theta|^d \)
and \( |\partial_{\theta} \hat{k}(z, w, \theta)| \leq \text{const.} e^{-\frac{|\xi|^2}{4R} |\theta|} \).

We may assume that the shape of \( \mathcal{C} \) has been modified to allow us to solve the Cauchy-Riemann equations in \( \mathcal{C} \) (see [5]) \( \partial_{\theta} \hat{k}_1 = \partial_{\theta} \hat{k} \) in such a way that the solution \( \hat{k}_1 \) is holomorphic with respect to \( (z, w) \) in \( \mathcal{O}_K \times \mathcal{O}_K \) and the following estimate holds on sets of the kind \( K_1 \times K_2 \times \mathcal{C}(K_1, K_2 \subset \in \mathcal{O}_K) \) and \( \mathcal{C}_1 \) a cone whose closure is contained in \( \mathcal{C} \):
\[ |\hat{k}_1(z, w, \theta)| \leq \text{const.} e^{-\frac{|\xi|^2}{4R} |\theta|} \]

Define then \( h = \hat{k} - \hat{k}_1 \). We now have, in \( \mathcal{O}_K \times \mathcal{O}_K \times \mathcal{C}_1 \), \( \partial_{\theta} \hat{h} = 0 \) and \( \hat{k} - \hat{h} \) decays exponentially as \( |\theta| \to \infty \) (uniformly, provided \( (z, w, \theta) \) stays in sets like \( K_1 \times K_2 \times \mathcal{C}_1 \) as above).

This decay together with Theorem 2.1 of §2 imply that if for \( u \in \mathcal{E}'(U), U \)
sufficiently small, we define
\[ \text{op} \hat{k}^\varepsilon u(x) = \left( \frac{1}{4\pi^3} \right)^{\frac{3}{2}} \int_U \int_{\mathbb{R}^3} e^{-\frac{1}{4\varepsilon}|Z(x) - Z(y)|^2} \cdot \hat{k}(Z(x), Z(y), \xi) u(y) dZ(y) d\xi \]
then as \( \varepsilon \to 0^+ \), \( \text{op} \hat{k}^\varepsilon \) will converge to an operator \( \text{op} \hat{k} \) having the properties in Theorem 2.1, §2. Moreover, for any \( u \in \mathcal{E}'(U) \), \( \text{op} \hat{k} u - \text{op} \hat{h} u \) is a hypo-analytic function. We will therefore replace \( \hat{k} \) by the hypo-analytic amplitude.
\( \hat{h} \) and think of \( \hat{h} \) as being the true amplitude constructed from the formal one given by \( \sum_{j=0}^{\infty} k_j(x, y, \theta) \).

4. Asymptotic expansion. Let \( k(x, y, \theta) \) be a hypo-analytic amplitude of degree \( d \) say \( k(x, y, \theta) = \hat{k}(Z(x), Z(y), \theta) \) where \( \hat{k} \) is holomorphic in \( \mathcal{O} \times \mathcal{O} \times \mathbb{C} \), \( \mathcal{O} \) and \( \mathbb{C} \) are as in §1. For each \( j = 1, \ldots, m \), let \( N_j \) denote the vector field \( N_j Z^k = -\sqrt{-1} \delta_j \).

If \( K \subset U \) is any compact subset, by Cauchy’s inequality we have \( c > 0 \) such that:

\[
\left| \frac{1}{\alpha!} \partial_\xi^\alpha N_j^\alpha k(x, x, \xi) \right| \leq c |\alpha|^{\alpha} \alpha! (1 + |\xi|)^{d-|\alpha|}
\]
for \( x \in K, \xi \in \mathbb{R}^m \).

Thus if we define

\[
k_j(x, \xi) = \sum_{|\alpha| = j} \frac{1}{\alpha!} \partial_\xi^\alpha N_j^\alpha k(x, x, \xi)
\]
then \( \sum_{j=0}^{\infty} k_j(x, \xi) \) can be thought of as a formal hypo-analytic symbol. Let \( (\phi_j)_j \) be the cutoff functions of the previous section. If \( U' \) is any relatively compact subset of \( U \), we can form a true symbol by setting

\[
k(x, \xi) = \sum_{j=0}^{\infty} k_j(x, \xi) \phi_j(\xi)
\]
We then have two operators \( \text{op} k(x, y, \xi) \) and \( \text{op} \hat{k}(x, \xi) : \mathcal{E}'(U') \rightarrow D'(U') \)
where for \( u \in \mathcal{E}'(U') \),

\[
\text{op} \hat{k} u(x) = \lim_{\varepsilon \to 0^+} \left( \frac{1}{4\pi^3} \right)^{\frac{m}{2}} \int \int e^{-\varepsilon (|Z(x) - Z(y)|^2 + |\xi - \xi'|^2)} k(x, \xi) u(y) dZ(y) d\xi
\]
and

\[
\text{op} k u(x) = \lim_{\varepsilon \to 0^+} \left( \frac{1}{4\pi^3} \right)^{\frac{m}{2}} \int \int e^{-\varepsilon (|Z(x) - Z(y)|^2 + |\xi - \xi'|^2)} k(x, y, \xi) u(y) dZ(y) d\xi
\]

The next theorem proves that if \( U' \) is small enough, modulo a hypo-analytic regularizing operator, \( \text{op} k = \text{op} \hat{k} \).

**Theorem 4.1.** If the neighborhood \( U' \) is sufficiently small, \( \text{op} k \equiv \text{op} \hat{k} \) in the sense that for any \( u \in \mathcal{E}'(U') \), \( \text{op} k u - \text{op} \hat{k} \) is a hypo-analytic function.

**Proof.** Assume \( U' \) is an open ball centered at 0, its size to be determined later. We first take \( u \in \mathcal{E}^0(U') \). The theorem will be proved by first establishing:

(i) \( \text{op} k - \text{op} \hat{k} \) is in \( C^\infty(U') \), and

(ii) There exists \( c > 0 \) such that for all \( \alpha \in \mathbb{Z}^+ \),

\[
|M_\alpha^{\alpha} (\text{op} k - \text{op} \hat{k}) u(x)| \leq c |\alpha|^{\alpha} |\alpha|! \quad \text{where} \quad M_j = \sqrt{-1} N_j
\]
for each \( j = 1, \ldots, m \).
Taylor expansion in $U'$ gives

$$k(x, y, \xi) = \sum_{|\alpha| \leq N} \frac{(Z(y) - Z(x))^{\alpha}}{\alpha!} \frac{\partial^{|\alpha|} k(x, x, \xi)}{\partial x^{|\alpha|}} + \sum_{|\alpha| = N+1} (Z(y) - Z(x))^{\alpha} k_\alpha(x, y, \xi)$$

where $k_\alpha(x, y, \xi) = (N + 1) \int_0^1 M_y^\alpha k(x, x + t(y - x), \xi)(1 - t)^N dt$.

For each $N = 1, 2, \ldots$ we define the amplitudes

$$k_N(x, y, \xi) = \phi_{N+1}(\xi) k(x, y, \xi), \quad \tilde{k}_N(x, y, \xi) = \sum_{j \leq N} \phi_j(\xi) k_j(x, \xi),$$

$$r_N(x, \xi) = \sum_{j \leq N} (\phi_{N+1}(\xi) - \phi_j(\xi)) k_j(x, \xi),$$

$$s_N(x, y, \xi) = \left( \sum_{|\alpha| = N+1} \frac{1}{\alpha!} D_\xi^\alpha k_\alpha(x, y, \xi) \right) \phi_{N+1}(\xi), \quad \text{and}$$

$$t_N(x, y, \xi) = \sum_{|\alpha| \leq N+1} \frac{1}{\alpha!} \{D_\xi^\alpha \phi_{N+1}(\xi) k_\alpha(x, y, \xi) - \phi_{N+1}(\xi) D_\xi^\alpha k_\alpha \}.$$

Let $K_N$, $\tilde{K}_N$, $R_N$, $S_N$ and $T_N$ denote the respective operators that are defined in the same fashion as $op_k$. We have

$$(op_k - op\tilde{k})u = (K_N - op_k)u + (op_k - K_N)u + R_Nu + S_Nu + T_Nu.$$

Our estimates will show that given any positive integer $l$, there exists a positive integer $N$ such that each term on the right-hand side of the above equation is in $C^l$—thus establishing that $(op_k - op\tilde{k})u \in C^\infty(U')$.

(A) **Estimate of $M^\alpha (op_k - K_N)u$**. Since the $\xi$-support of

$$(1 - \phi_{N+1}(\xi))k(x, y, \xi)$$

is compact, $(op_k - K_N)u$ is hypo-analytic and therefore in particular, $C^\infty$.

Suppose $|Z(x) - Z(y)| \leq A$ for all $x, y$ in $U'$.

$$|\langle op_k - K_N \rangle u(x)\rangle| = \left(\frac{1}{4\pi^3}\right)^{\frac{d}{2}} \left| \int_{\mathbb{R}^d} \int_{|\xi| \leq 3R(N+1)} e^{-\frac{1}{4}(Z(x) - Z(y))^2} \cdot k(x, y, \xi)(1 - \phi_{N}(\xi)) dZ(y) d\xi \right|$$

$$\leq \text{const.} \int_{|\xi| \leq 3R(N+1)} e^{A|\xi|} (1 + |\xi|)^d d\xi$$

(the constant is independent of $N$)

$$\leq \text{const.} \left( e^{3RA} \right)^{N+1} (N+1)^{d+m}$$

$$\leq c_1^{N+1}$$

for some $c_1 > 0$ independent of $N$. 
Moreover, since each \((\text{op} k - K_N)u\) is hypo-analytic in a common domain, for example some neighborhood of the compact set \(U'\), we can find a constant \(c_1 > 0\) independent of \(N\) such that for all \(\alpha \in \mathbb{Z}_m^+\),

\[
|M^\alpha(\text{op} k - K_N)u(x)| \leq c_1^{(|\alpha|+1)}c_1^{N+1}\alpha!
\]

(B) **Estimate of** \(M^\alpha(S_Nu)\). Write

\[
s_N(x, y, \xi) = \phi_{N+1}(\xi) \sum_{|\alpha| = N+1} D^\alpha_{\xi} \phi_{N}(x, y, \xi) = \phi_{N+1}(\xi)\hat{s}_N(x, y, \xi).
\]

For \(|\alpha| = N + 1\), we have

\[
\left|\frac{D^\alpha_{\xi} \phi_{N}(x, y, \xi)}{\alpha!}\right| \leq c_2^{(|\alpha|+1)}(1 + |\xi|)^{d-N-1}.
\]

It follows that \(|\hat{s}_N(x, y, \xi)| \leq c_2^{N+1}N!(1 + |\xi|)^{d-N-1}\) for some \(c_2 > 0\).

Let

\[
I^e_N(x) = \left(\frac{1}{4\pi^3}\right)^{\frac{q}{2}} \int \int e^{\sqrt{-1}(Z(x)-Z(y))\cdot\xi-\varepsilon|\xi|^2} \phi_{N+1}(\xi)\hat{s}_N(x, y, \xi)u(y)\,dZ(y)d\xi.
\]

We note that \(s_Nu(x) = \lim_{\varepsilon \to 0^+} I^e_N(x)\).

We will deform the path of \(\xi\)-integration from \(R_m\) to the image of \(R_m\) under the map

\[
\xi \to \theta(\xi) = \phi_{2N}(\xi)\zeta(\xi) + (1 - \phi_{2N}(\xi))\xi
\]

where

\[
\zeta(\xi) = \tilde{Z}_x(x)\xi + \sqrt{-1}(\tilde{Z}_x(x)\xi)(Z(x) - Z(y)).
\]

The deformation is allowed since it takes place in a region where \(\phi_{N+1}(\xi)\) is analytic.

We have

\[
|\theta(\xi)| = \left\{
\begin{array}{ll}
\xi, & \text{for } |\xi| \leq 4RN, \\
\zeta(\xi), & \text{for } |\xi| \geq 6RN.
\end{array}
\right.
\]

\[
|M^\alpha(I^e_N(x))| \leq \left(\frac{1}{4\pi^3}\right)^{\frac{q}{2}} \sum_{\beta < \alpha} \left(\frac{\alpha}{\beta}\right) \int \int |\xi|^{\alpha - \beta} e^{\sqrt{-1}(Z(x)-Z(y))\cdot\xi-\varepsilon|\xi|^2}
\cdot \phi_{N+1}(\xi) M^\beta \hat{s}_N(x, y, \xi)u(y)\,dZ(y)d\xi|
\]

We use the above contour and pass to the limit to get:

\[
|(M^\alpha s_Nu)(x)| \leq \left|\left(\frac{1}{4\pi^3}\right)^{\frac{q}{2}} \sum_{\beta \leq \alpha} \left(\frac{\alpha}{\beta}\right) \int_{2R(N+1) \leq |\xi| \leq 6RN} \int (\theta(\xi))^{\alpha - \beta}
\cdot e^{\sqrt{-1}(Z(x)-Z(y))\cdot\theta(\xi)} \phi_{N+1}(\xi) M^\beta \hat{s}_N(x, y, \xi)u(y)\,d\theta dZ(y)
\right|
\]

\[
+ \int_{|\xi| \geq 6RN} \int (\zeta(\xi))^{\alpha - \beta} e^{\sqrt{-1}(Z(x)-Z(y))\cdot\zeta(\xi)} \phi_{N+1}(\xi)
\cdot M^\beta \hat{s}_N(x, y, \zeta(\xi))u(y)\,dZ(y)d\xi
\]
We recall that the exponential in the second integral is bounded (§1, (1.1)). By hypo-analyticity we get $c_3 > 0$ such that
\[
\forall \beta, |M^\beta s_N(x, y, \xi)| \leq c_3 |\beta| + 1 \beta! c_2^{N+1} N!(1 + |\xi|)^{d-N-1}.
\]

These observations imply that
\[
|\alpha| \leq \alpha \omega_2 W' J_{2R(N+1)} < \alpha \omega_2 J_{2R(N+1)}
\]
\[
\int_{6RN \leq |\xi|} |\xi|^{\alpha + \beta + 2} c_3 \beta! N!(1 + |\xi|)^{d-N-1} d\xi
\]
for some $c_3 \geq \max(c_3, c_2)$. Hence, after modifying $c_3$ if necessary, we get
\[
|M^\alpha s_N u(x)| \leq \alpha! \left( \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!} \int_{2R(N+1) \leq |\xi|} (1 + |\xi|)^{|\alpha - \beta| + d - N - 1} N! d\xi \right) c_3^N
\]
\[
\leq \alpha! c_3^N \left( \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!} \left( \frac{1}{1 + 2RN} \right)^{\alpha - \beta - d - m + 1} N! \right)
\]
\[
\leq \alpha! c_3^N \left( \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!} \left( 1 + 2RN \right)^{|\alpha - \beta| + d + m - 1} \right) \left( \frac{1}{2R} \right)^N N!
\]
\[
\leq \alpha! c_3^N \left( \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!} \left( |\alpha - \beta| + d + m - 1 \right) e^{1+2RN} \right) \left( \frac{1}{2Re} \right)^N Ne
\]
\[
\leq \alpha! c_3^N \left( \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!} \left( |\alpha - \beta| + d + m - 1 \right) \right) \left( \frac{e^{2R}}{2Re} \right)^N Ne^2.
\]

Using the inequality: $(k + l)! \leq 2^{k+l} k! l!$ for any positive integers $k$ and $l$, the latter is dominated by
\[
\alpha! c_3^N \left( \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!} |\alpha - \beta|! \right) 2^{|\alpha| + d + m - 1} \left( \frac{e^{2R}}{2Re} \right)^N Ne^2.
\]

For $|\alpha| \leq N$, we can find another constant which we will still call $c_3$ such that the above quantity $\leq \alpha! c_3^N$. 
(C) **Estimate of \( M^\alpha (\text{op} \hat{k} - \hat{K}_N)u \).** Let

\[
J^\alpha u(x) = \left( \frac{1}{4\pi^3} \right)^q \int \int e^{ \sqrt{-1}(Z(x) - Z(y)) \cdot \xi - e|\xi|^2 } 
\cdot \left( \sum_{j > N} \phi_j(\xi) k_j(x, \xi) \right) u(y) dZ(y) d\xi .
\]

For each \( j > N \), we will use the contour

\[
\theta_j(\xi) = \phi_{2j}(\xi) \xi(\xi) + (1 - \phi_{2j}(\xi)) \xi = \begin{cases} 
\xi, & \text{when } |\xi| \leq 4R_j, \\
\xi(\xi), & \text{when } |\xi| \geq 6R_j.
\end{cases}
\]

In the quantity

\[
M^\alpha(J^\alpha u)(x) = \left( \frac{1}{4\pi^3} \right)^q \sum_{j > N} \left[ \sum_{\beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) \int \int \theta_j(\xi) e^{ \sqrt{-1}(Z(x) - Z(y)) \cdot \theta_j(\xi) } 
\cdot \phi_j(\xi) M^\beta k_j(x, \xi) u(y) dZ d\xi \right]
\]

we use the contours \( \theta_j \) in each term of the sum and take limits to get

\[
M^\alpha(\text{op} \hat{k} - \hat{K}_N)u(x) = \sum_{j > N} (I^1_j(x) + I^2_j(x))
\]

where

\[
I^1_j(x) = \left( \frac{1}{4\pi^3} \right)^q \sum_{\beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) \int_{2R_j \leq |\xi| \leq 6R_j} \int \theta_j(\xi) e^{ \sqrt{-1}(Z(x) - Z(y)) \cdot \theta_j(\xi) } 
\cdot \phi_j(\xi) M^\beta k_j(x, \theta_j(\xi)) u(y) dZ d\theta_j
\]

while \( I^2_j(x) \) is a similar expression except that the integration in \( \xi \) is carried out over the region \( \{ \xi : |\xi| \geq 6R_j \} \).

Assuming that \( |\alpha| \leq N - d - m \), we have

\[
|I^1_j(x)| \leq \text{const} . \sum_{\beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) \int_{2R_j \leq |\xi| \leq 6R_j} (1 + |\xi|)^{d-j+|\alpha|-|\beta|} \left( e^{6R} \right)^j c_0^j |\beta| + 1 \beta! d\xi
\]

\[
\leq \text{const} . \left( c_0 e^{6R} \right)^j \alpha! \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!} \int_{2R_j \leq |\xi| \leq 6R_j} (1 + |\xi|)^{d-j+|\alpha|-|\beta|} |j|! \left( e^{6R} \right)^j c_0^N
\]

\[
\leq \text{const} . \left( c_0 e^{6R} \right)^j \alpha! \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!} \int_{0 \leq \rho \leq 6R_j} \rho^{d-j+|\alpha|-|\beta|+m-1} |j|! d\rho \left( e^{6R} \right)^j c_0^N
\]

(We have used the fact that \( d - j + |\alpha| \leq 0 \).)

\[
\leq \text{const} . \alpha! \left( \frac{c_0 e^{6R}}{6R} \right)^j (6R)^{d+m+N} c_0^N \left( \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!} \right) .
\]
Therefore, for some \( \tilde{c}_4 > 0 \) independent of \( j \) and \( N \),
\[
|I_1^j(x)| \leq \alpha! c_4^{N+1} \left( \frac{c_0 e^{6RA}}{6R} \right)^j
\]
Similarly, after modifying the constant \( \tilde{c}_4 \) if necessary,
\[
|I_2^j(x)| \leq \text{const.} \alpha! \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!} \left( \frac{1}{1 + 6Rj} \right)^{j-d-m-|\alpha| + |\beta|} c_0^{|\beta| + j + 1} j!
\]
\[
\leq \alpha! c_4^{N+1} \left( \frac{c_0}{6R} \right)^j
\]
We recall that \( c_0 \leq R \). At this point we choose \( U' \) so small that if \( A = \sup_{x, y \in U'} |Z(x) - Z(y)| \), then \( c_0 e^{6RA} < 6R \).
We then get a constant \( c_4 > 0 \) such that: \( |M^\alpha (\text{op} \tilde{K} - \tilde{K}_N) u(x)| \leq \alpha! c_4^{N+1} \) for \( |\alpha| \leq N - d - m \).

(D) Estimate of \( M^\alpha(R_N u) \).
\[
R_N u(x) = \left( \frac{1}{4\pi^2} \right)^{\frac{j}{2}} \int \int e^{\sqrt{-1}(Z(x) - Z(y)) \cdot \xi} u(y)
\]
\[
\cdot \left( \sum_{j \leq N} (\phi_{N+1}(\xi) - \phi_j(\xi)) k_j(x, \xi) \right) dZ(y) d\xi
\]
is hypo-analytic since each \( \phi_{N+1} - \phi_j \) is supported in \( 2Rj \leq |\xi| \leq 3R(N + 1) \).

We estimate
\[
\left| \sum_{j \leq N} (\phi_{N+1}(\xi) - \phi_j(\xi)) k_j(x, \xi) \right| \leq \left( \sum_{j \leq N} c_0^{j+1} j! |\xi|^{-j} \right)|\xi|^d
\]
\[
\leq \left( \sum_{j \leq N} c_0^{j+1} j! \left( \frac{1}{2Rj} \right)^j \right)|\xi|^d \quad \text{(since } 2Rj \leq |\xi|\text{)}
\]
\[
\leq \left( \sum_{j \leq N} \left( \frac{c_0}{2Re} \right)^j \right) c_0 e|\xi|^d \quad \text{since } \frac{j!}{j!} \leq je^{-j+1}.
\]
It follows that
\[
|R_N u(x)| \leq \text{constant} \int_{|\xi| \leq 3R(N + 1)} |\xi|^d d\xi \leq \text{const.} 3R(N + 1)^{d+m}
\]
which in turn implies that there is a constant \( \tilde{c}_5 > 0 \) such that \( |R_N u(x)| \leq \tilde{c}_5^{N+1} \). Moreover, by hypo-analyticity, we get \( c_5 > 0 \) satisfying \( |M^\alpha R_N u(x)| \leq \alpha! c_5^{N+1} \) for all \( |\alpha| \leq N \).

(E) Estimate of \( M^\alpha(T_N u) \).
\[
T_N u(x) = \lim_{\varepsilon \to 0} \left( \frac{1}{4\pi^3} \right)^{\frac{j}{2}} \int \int e^{\sqrt{-1}(Z(x) - Z(y)) \cdot \xi - \varepsilon|\xi|^2} t_N(x, y, \xi) u(y) dZ(y) d\xi
\]
where
\[ t_N(x, y, \xi) = \sum_{|\alpha| \leq N+1} \frac{1}{\alpha!} \{(D_{\xi}^\alpha (\phi_{N+1}(\xi)k_\alpha(x, y, \xi)) - \phi_{N+1}(\xi)D_{\xi}^\alpha k_\alpha(x, y, \xi)\}. \]

We can therefore take the limit under the integral sign and write
\[ T_N u(x) = \sum_{|\alpha| \leq N+1} A_\alpha(x), \]
where for each \( \alpha, |\alpha| \leq N + 1, \)
\[ A_\alpha(x) = \left(\frac{1}{4\pi^3}\right)^{|\alpha|} \sum_{0 \neq \beta \leq N} \int_{2R(N+1) \leq |\xi| \leq 3R(N+1)} \int e^{\sqrt{-1}Z((x) - Z(y))\xi} \frac{1}{\beta!} \]
\[ \cdot (D_{\xi}^\beta \phi_{N+1}(\xi)) \frac{D_{\xi}^\alpha - \beta k_\alpha(x, y, \xi)}{(\alpha - \beta)!} u(y) dZ(y) d\xi. \]

Therefore
\[ |A_\alpha(x)| \leq \text{const.} \alpha ! c_0^{|\alpha|+1} (e^{3R(N+1)} \sum_{0 \neq \beta \leq |\alpha|} \frac{1}{\beta!} \left[ \frac{[3R(N+1)]^{d+m+1}}{[2R(N+1)]^{|\alpha| - |\beta|}} \right]^{|\beta|} \left( \frac{c_0}{R} \right)^{|\beta|} \]
\[ \leq \text{const.} \frac{\alpha !}{[2R(N+1)]^{|\alpha|}} c_0^{|\alpha|+1} \left( \sum_{0 \leq \beta \leq |\alpha|} \frac{[2(N+1)c_0]^{|\beta|}}{\beta!} \right). \]

Since \( |\alpha| \leq N \) and \( R \) may be taken to be larger than 1, we know that the factor \( \frac{\alpha !}{[2R(N+1)]^{|\alpha|}} \leq 1. \) Therefore, we conclude that there is a constant \( c_6 \geq 0 \) for which \( |M''(T_N u)| \leq c_6^{N+1} N! \) whenever \( |\alpha| \leq N. \)

From (a)–(e) we conclude that there is a positive number \( c \) such that
\[ |M''(\text{op } k - \text{op } \hat{k})u(x)| \leq c^{N+1} N! \]
for all \( \alpha, |\alpha| \leq N - m - d. \)

If we take \( |\alpha| = N - m - d \), we can get a constant \( \hat{c} \geq c \) satisfying:
\[ \forall \alpha, |M''(\text{op } k - \text{op } \hat{k})u(x)| \leq \hat{c}^{|\alpha|+1} \alpha! \text{ for every } x \in U'. \]

By using integration by parts we also reach the same conclusion for \( u \in C_c^1(U'). \) Indeed all we need is a representation of the form \( u = \sum_{|\alpha| \leq N} M'' u_\alpha \)
where each \( u_\alpha \in C_c^0(U') \) which is always possible. We have thus shown that \( (\text{op } k - \text{op } \hat{k})u \) is in \( C_{\alpha}^\infty(U') \) and that there is \( c > 0 \) such that for all \( \alpha \in Z^+, \)
\[ |M''(\text{op } k - \text{op } \hat{k})u(x)| \leq c^{|\alpha|+1} \alpha!. \]

By Theorem 3.1 of [1] it follows that \( \text{op } k u - \text{op } \hat{k} u \) is a hypo-analytic function.
II. Applications

1. Parametrix for an elliptic operator. As an application of Theorem 4.1 we consider here the construction of a parametrix for an elliptic hypo-analytic differential operator. We will begin by composing a hypo-analytic differential operator $A$ with a hypo-analytic pseudodifferential operator $B$. In [3] we introduced hypo-analytic differential operators. In the local chart $(U, Z)$, the operator $A$ is given by $A = \sum_{|\alpha| \leq n} a_{\alpha}(x) N^{\alpha}$ where each $a_{\alpha}(x)$ is a hypo-analytic function and $N_j = -\sqrt{-1} M_j$ for $j = 1, \ldots, m$.

Theorem 4.1 of the previous chapter allows us to represent the operator $B$ by a symbol $b(x, \theta)$. From §2, Theorem 2.1 we know that both $B \circ A$ and $A \circ B$ are continuous linear maps from $\mathcal{E}'(U)$ to $\mathcal{D}'(U)$. We first assume that the operator $A = a(x) N^{\beta}$ for some hypo-analytic function $a(x)$ and some index $\beta$. Then $B(Au)(x)$ is by definition the limit as $\varepsilon \to +0$ of

$$B^\varepsilon (Au)(x) = \left( \frac{1}{4\pi^3} \right)^{\frac{n}{2}} \int \int e^{\sqrt{-1}(Z(x) - Z(y)) \cdot \xi - \varepsilon |\xi|^2} b(x, \xi) a(y) N^{\beta} u(y) dZ(y) d\xi.$$ 

On the other hand, $\lim_{\varepsilon \to 0^+} B^\varepsilon (Au)(x) = C \circ (N^{\beta} u)(x)$ where $C$ is a hypo-analytic pseudodifferential operator with amplitude given by $b(x, \xi) a(y)$. Therefore, Theorem 4.1 tells us that $C$ can be represented by the symbol $c(x, \xi) = \sum_{\alpha} \frac{\partial^\alpha b(a(x) \xi^{\beta})}{\alpha!}$. It follows that modulo a hypo-analytic function, we can write

$$B(Au)(x) = \lim_{\varepsilon \to 0^+} \left( \frac{1}{4\pi^3} \right)^{\frac{n}{2}} \int \int e^{\sqrt{-1}(Z(x) - Z(y)) \cdot \xi - \varepsilon |\xi|^2} c(x, \xi) u(y) dZ(y) d\xi.$$ 

The latter says that a symbol of $B \circ A$ is given by

$$\xi^{\beta} c(x, \xi) = \sum_{\alpha} \frac{\partial^\alpha b(a(x) \xi^{\beta})}{\alpha!}.$$ 

On the other hand, applying the operator $A$ to

$$B^\varepsilon u(x) = \left( \frac{1}{4\pi^3} \right)^{\frac{n}{2}} \int \int e^{\sqrt{-1}(Z(x) - Z(y)) \cdot \xi - \varepsilon |\xi|^2} b(x, \xi) u(y) dZ(y) d\xi$$

gives

$$A(B^\varepsilon u(x)) = \left( \frac{1}{4\pi^3} \right)^{\frac{n}{2}} \int \int e^{\sqrt{-1}(Z(x) - Z(y)) \cdot \xi - \varepsilon |\xi|^2} \left( \sum_{\gamma \leq \beta} \xi^{\beta - \gamma} a(x) N^{\beta} b(x, \xi) \right) u(y) dZ d\xi$$

$$= \left( \frac{1}{4\pi^3} \right)^{\frac{n}{2}} \int \int e^{\sqrt{-1}(Z(x) - (Z(y)) \cdot \xi - \varepsilon |\xi|^2} \left( \sum_{\alpha} \frac{\partial^\alpha (a(x) \xi^{\beta}) N^{\alpha} b(x, \xi)}{\alpha!} \right) u(y) dZ d\xi.$$
This means that $A \circ B$ has a symbol given by
\[
\sum \frac{\partial^\alpha (a(x)\xi^\beta) N^\alpha b(x, \xi)}{\alpha!}
\]
By linearity, we will have the same formulas for the symbol of $B \circ A$ and $A \circ B$ when $A$ is also given by $A = \sum_{|\alpha| \leq n} a_\alpha(x) N^\alpha$.
We have thus shown that if either $A$ or $B$ is hypo-analytic differential operator, the composition $A \circ B$ is hypo-analytic pseudodifferential operator with symbol
\[
\sum \frac{\partial^\alpha a(x, \xi) N^\alpha b(x, \xi)}{\alpha!}
\]
**Definition 1.1.** Let $P = \sum_{|\alpha| \leq k} a_\alpha(Z(x)) M^\alpha$ where the $a_\alpha(z)$ are holomorphic in a neighborhood of $Z(U)$ in $C^m$. We say a point $(x, \xi) \in T^* U \setminus \{0\}$ is in the characteristic set of $P$ if the point $(Z(x), Z(x) \xi)$ is in the characteristic set of $P_Z = \sum_{|\alpha| \leq k} a_\alpha(z) (\frac{\partial}{\partial z})^\alpha$.
**Notation.** $\text{Char } P = \text{ the characteristic set of } P$ as given by Definition 1.1.

**Definition 1.2.** A hypo-analytic differential operator $P$ is said to be elliptic at a point $x \in \Omega$ if for every $(x, \xi) \in T^* \Omega \setminus \{0\}$, $(x, \xi) \notin \text{Char } P$.

Now suppose $P = \sum_{|\alpha| \leq k} a_\alpha(Z(x)) M^\alpha$ is a hypo-analytic differential operator that is elliptic at our central point $0 \in U$. Since $Z(0) = 0$ and $dZ(0) = \Id$, we can find a neighborhood $\mathcal{V}$ of $0$ in $C^m$, a cone $\mathcal{C}$ in $C_m$ containing $R_m \setminus \{0\}$ and constants $c, R > 0$ such that: when $z \in \mathcal{V}$ and $\xi \in \mathcal{C}$, $|\xi| \geq R$ we have $|\sum_{|\alpha| \leq k} a_\alpha(z) \xi^\alpha| \geq c|\xi|^k$.

We now have all the ingredients we need to state

**Theorem 1.1.** Let $A$ be hypo-analytic differential operator in $\Omega$ that is elliptic of order $d$. Given any relatively compact open subset $\Omega$ of $\Omega$, there is a hypo-analytic pseudodifferential operator $B$ in $\hat{\Omega}$ of order $-d$ such that $AB - I$ and $BA - I$ are hypo-analytic regularizing in $\hat{\Omega}$.

The proof of this theorem is a simple adaptation of that of the corresponding theorem for analytic pseudodifferential operators as given by Treves [8]. Therefore we omit it.

2. Propagation of hypo-analyticity. In [3] it was shown that hypo-analytic singularities for solutions propagate along the bicharacteristics of hypo-analytic differential operators. Here we extend this result to what may be called classical hypo-analytic pseudodifferential operator. This result may also be viewed as an extension of a theorem of Hanges [4].

We will work in the hypo-analytic local chart $(U, Z)$ of Chapter I. Let $P$ be a classical hypo-analytic pseudodifferential operator with principal symbol $p$. Let $t \rightarrow (x(t), \xi(t)) = \gamma(t)$ be a curve in $T^* U \setminus \{0\}$ and set $\hat{\gamma}(t) = (\hat{x}(t), \hat{\xi}(t)) = (Z(x(t)), \hat{Z}_x(x(t)) \xi(t))$. 


Definition 2.1. The curve \( \gamma(t) \) is said to be a bicharacteristic for \( P \) if the equations
\[
\frac{d\tilde{x}}{dt} = \frac{\partial p}{\partial \xi}(\tilde{x}(t), \tilde{\xi}(x)), \quad \frac{d\tilde{\xi}}{dt} = -\frac{\partial p}{\partial z}(\tilde{x}(t), \tilde{\xi}(t))
\]
hold.

We can now state the theorem of this section.

Theorem 2.1. Assume \( p(0, \xi_0) = 0 \) and \( P \) is of principal type at \( (0, \xi_0) \). Suppose \( \gamma = \{(x(t), \xi(t))\} \) is a bicharacteristic for \( P \) through \( (x(0), \xi(0)) = (0, \xi_0) \) and that \( Pu \) is hypo-analytic on \( \gamma \). Then either \( u \) is hypo-analytic at every point of \( \gamma \) or \( u \) is not hypo-analytic at any point of \( \gamma \).

The proof will use a version of the FBI transform as developed by Sjöstrand in [7]. We will therefore first discuss Sjöstrand's FBI transformations adapted to our situation here.

Let \( H \) be a totally real submanifold of \( C^m \) of maximal dimension with defining functions \( h_1, \ldots, h_m \).

Define
\[
\Lambda_H = \left\{ (x, \frac{2}{i} \partial h(x)) : h \in C^\infty(C^m, R), h \equiv 0 \text{ on } H \right\}.
\]

Note that if \( x_0 \in H \), then \( (x_0, \xi_0) \in \Lambda_H \) iff \( \exists \) real numbers \( t_1, \ldots, t_m \) such that
\[
\xi_0 = \frac{2}{i} \sum_{j=1}^m t_j \partial h_j(x_0).
\]

Fix a point \( (y_0, \eta_0) \in \Lambda_H \). Let \( \varphi \) be a holomorphic function defined near \( (x_0, y_0) \)
\begin{align*}
(2.1) \quad \frac{\partial \varphi}{\partial y}(x_0, y_0) &= -\eta_0, \\
(2.2) \quad \det \frac{\partial^2 \varphi}{\partial x \partial y}(x_0, y_0) &\neq 0, \\
(2.3) \quad \Re \varphi_{y y}(x_0, y_0) \mid_{T_{y_0}H \times T_{y_0}H} &> 0.
\end{align*}

Here \( \Re \varphi \) is considered as a function on \( C^n \times H \), defined locally.

Set
\[
\varphi_1(x, y) = -\Re \varphi(x, y).
\]

Condition (2.1) implies that \( H \ni y \mapsto \varphi_1(x_0, y) \) has a critical point at \( y_0 \) since \( \frac{\partial \varphi}{\partial y}(x_0, y_0) = \frac{\partial \varphi}{\partial y}(x_0, y_0) = -\eta_0 \) and that therefore \( d_y \varphi_1(x_0, y_0) = d_h(y_0) \) for some \( h \) vanishing on \( H \). This together with condition (2.3) and the implicit function theorem give us neighborhoods \( N(x_0) \) of \( x_0 \) in \( C^m \), \( N(y_0) \) of \( y_0 \) in \( H \) and a unique \( C^\infty \) function \( y = y(x) : N(x_0) \to N(y_0) \) such that \( y(x) \) is the unique critical point for \( H \ni y \mapsto \varphi_1(x, y) \), \( x \in N(x_0) \). We next note that for \( x \in N(x_0) \), \( (y(x), -\frac{2}{i} \frac{\partial \varphi}{\partial y}(x, y(x))) \in \Lambda_H \). Indeed, this follows from the fact that \( H \ni y \mapsto \varphi_1(x, y) \) has a critical point at \( y(x) \) and that \( h_1, \ldots, h_m \) are the defining functions for \( H \).
For \( x \in N(x_0) \), let \( \eta(x) = -\frac{-2}{i} \frac{\partial \varphi}{\partial y}(x, y(x)) \). Then
\[
(y(x), \eta(x)) = \left( y(x), -\frac{-2}{i} \frac{\partial \varphi}{\partial y}(x, y(x)) \right) \in \Lambda_H.
\]
Moreover, for \( x \in N(x_0) \), \( y(x) \) is the unique point in \( N(y_0) \) such that
\[
\frac{-\partial \varphi}{\partial y}(x, y(x)) = -\frac{-2}{i} \frac{\partial \varphi}{\partial y}(x, y(x)) \in (\Lambda_H)_{y(x)}.
\]
This is due to the uniqueness of the critical point.

Let \( \Phi(x) = \varphi_1(x, y(x)) \). Let \( a(x, y, \lambda) \) be a classical analytic symbol defined near \( (x_0, y_0) \) and elliptic at this point. For \( \Psi \) a real-valued function defined on an open set \( W \) in \( C^m \), we define the space \( H^\text{loc}_\Psi(W) = \{ v : W \times \mathbb{R} \to \mathbb{C} : v(z, \lambda) \) is holomorphic in \( z \) and for any \( K \subset W \) and \( \varepsilon > 0 \) \( \exists \delta \exists |v(z, \lambda)| \leq c e^{\delta (|\Psi(z)|+\varepsilon)} \) for all \( z \in K, \lambda \geq 1 \} \).

Let \( u \in D'(N(y_0)) \), and for \( z \in N(x_0) \) set
\[
Tu(z, \lambda) = \int_{H} e^{i\lambda \Phi(z, y)} a(z, y, \lambda) \chi(y) u(y) dy
\]
where \( \chi \in C^\infty(\mathbb{N}(y_0)) \), \( \chi \equiv 1 \) near \( y_0 \).

Here we are assuming that the neighborhoods \( N(y_0) \) and \( N(x_0) \) have been contracted so that the symbol \( a \) and the phase function \( \varphi \) are defined. It is easily checked that
\[
T : D'(N(y_0)) \to H^\text{loc}_\Phi(N(x_0)).
\]

In the sequel, \( WF_{h}u \) denotes the hypo-analytic wave front set of Baouendi-Chang-Treves [1]. Our proof of Theorem 2.1 will use the following proposition of Sjöstrand [7].

**Proposition 2.1.** Let \( z_1 \in N(y_0) \). Then \( (y(z_1), \eta(z_1)) \notin WF_{h}u \) iff \( Tu \in H^\text{loc}_{\Phi-\xi_0}(W) \) for some \( \xi_0 > 0 \) and some neighborhood \( W \) of \( z_1 \).

**Proof of Theorem 2.1.** In order to obtain a suitable phase function, we will need the following two lemmas from [6]. For notational convenience we will use \( y_0 \) for \( 0 \in Z(U) = H \).

**Lemma 2.1.** Set \( z_0 = (y_0' - i\xi_0', 0) \in C^{n-1} \times C \). There exists a holomorphic function \( \varphi \) defined near \( (z_0, y_0) \) which solves
\[
\frac{\partial \varphi}{\partial z_n}(z, y) = p \left( y, \frac{-\partial \varphi}{\partial y}(z, y) \right)
\]
and satisfies (2.1)–(2.3) with \( \eta_0 = \xi_0 \).

We remark that the lemma is proved by using the Cauchy-Kovalevska theorem, which guarantees the existence of a holomorphic \( \varphi \) that solves the initial value problem
\[
\frac{\partial \varphi}{\partial z_n} = p \left( y, \frac{-\partial \varphi}{\partial y} \right)
\]
and
\[ \varphi(z, 0, y) = \frac{i}{2} \sum_{j=1}^{n-1} (z_j - y_j)^2 - (\xi_0)_n y_n + iC(y_n - (y_0)_n)^2 \]
where \( RC \) is chosen sufficiently large. In the sequel, the neighborhoods \( N(z_0), N(y_0) \) and the function \( \Phi \) are related to the \( \varphi \) of Lemma 2.1 as before.

**Lemma 2.2.** There is an elliptic analytic symbol \( a(z, y, \lambda) \) such that the FBI transformation \( T \) with phase \( \varphi \) and symbol \( a \) satisfies \( D_{z_n} T = TP \) in \( H_{\Phi_{-e}}^{loc}(W) \).

That is, if \( Y \subseteq Z(U) = H \) is a small neighborhood of \( y_0 \), then for \( z \) in \( W \subseteq C^m \) a small neighborhood of \( z_0 = (y'_0 - i\xi'_0, 0) \) and \( u \in \mathcal{E}'(Y) \) we have
\[ D_{z_n} Tu - TPu \in H_{\Phi_{-e}}^{loc}(W) \]
for some \( e > 0 \).

The symbol \( a(z, y, \lambda) \) is constructed by solving the transport equations at each degree of homogeneity.

We recall now that
\[ \gamma(t) = (\hat{x}(t), \hat{\xi}(t)) \]
and
\[ \gamma(0) = (y_0, \xi_0) = (Z(x(0)), \dot{Z}_x(x(0))\xi_0). \]

Write \( y_0 = (y'_0, (y_n)_n) \) and \( \xi_0 = (\xi'_0, (\xi_n)_n) \).

We will use the equations
\[ \frac{\partial \varphi}{\partial z}(z, y) = p \left( y, -\frac{\partial \varphi}{\partial y}(z, y) \right), \]
and
\[ \frac{\partial \varphi}{\partial y}(z_0, y_0) = -\xi_0 \]
to prove that
\[ \hat{\xi}(t) = -\frac{\partial \varphi}{\partial y}(y'_0 - i\xi'_0, t, \hat{x}(t)) \]
We recall that
\[ \frac{d}{dt} = \frac{\partial}{\partial \xi} \quad \text{and} \quad \frac{d}{dt} = -\frac{\partial}{\partial x}. \]

Hence
\[ \frac{d}{dt} \left[ \frac{\partial \varphi}{\partial y}(y'_0 - i\xi'_0, t, \hat{x}(t)) \right] = \varphi_{y\zeta}(y'_0 - i\xi'_0, t, \hat{x}(t)) + \varphi_{yy}(y'_0 - i\xi'_0, t, \hat{x}(t)) \frac{d\hat{x}}{dt} \]
\[ \frac{d}{dt} \left[ \frac{\partial \varphi}{\partial y}(y'_0 - i\xi'_0, t, \hat{x}(t)) + \varphi_{yy}(y'_0 - i\xi'_0, t, \hat{x}(t)) \frac{\partial}{\partial \xi}(\hat{x}(t), \hat{\xi}(t)) \right] \]
Now (2.4) implies that
\[ \varphi_{yz}(z, y) = \frac{\partial p}{\partial y} \left( y, -\frac{\partial \varphi}{\partial y} \right) - \frac{\partial p}{\partial \zeta} \left( y, -\frac{\partial \varphi}{\partial y} \right) \varphi_{yy}(z, y). \]
It follows that
\[
\frac{d}{dt} \left[ -\frac{\partial \phi}{\partial y} (y_0' - i\xi_0', t, \hat{x}(t)) \right]
\]
\begin{equation}
(2.5)
= -\frac{\partial p}{\partial y} \left( \hat{x}(t), -\frac{\partial \phi}{\partial y} (y_0' - \xi_0', t, \hat{x}(t)) \right).
\end{equation}

\[
+ \frac{\partial p}{\partial \zeta} \left( \hat{x}(t), -\frac{\partial \phi}{\partial y} (y_0' - i\xi_0', t, \hat{x}(t)) \right) \phi_{yy} (y_0' - i\xi_0', t, \hat{x}(t))
\]
\[
- \phi_{yy} (y_0' - i\xi_0', t, \hat{x}(t)) \frac{\partial p}{\partial \zeta} (\hat{x}(t), \hat{\xi}(t)).
\]

But \( \hat{\xi}(t) \) also satisfies (2.5) since
\[
\frac{d\hat{\xi}}{dt} = -\frac{\partial p}{\partial y} (\hat{x}(t), \hat{\xi}(t)) + \frac{\partial p}{\partial \zeta} (\hat{x}(t), \hat{\xi}(t)) \phi_{yy} (y_0' - i\xi_0', t, \hat{x}(t))
\]
\[
- \phi_{yy} (y_0' - i\xi_0', t, \hat{x}(t)) \frac{\partial p}{\partial \zeta} (\hat{x}(t), \hat{\xi}(t))
\]
\[
= -\frac{\partial p}{\partial y} (\hat{x}(t), \hat{\xi}(t)).
\]

Moreover, by 2.4, \( -\frac{\partial p}{\partial y} (y_0' - i\xi_0', 0, y_0) = \xi_0 = \hat{\xi}(0) \).

We conclude that
\begin{equation}
(2.6)
\hat{\xi}(t) = -\frac{\partial \phi}{\partial y} (y_0' - i\xi_0', t, \hat{x}(t)).
\end{equation}

For \( t \in [0, 1] \), let
\[
z(t) = z_0 + (0', t) = (y_0' - i\xi_0', t) \in C^{n-1} \times \mathbb{R}.
\]

We now recall that for \( z \) near \( z_0 \), \( y(z) \) is the unique point in \( N(y_0) \subseteq H \) such that
\[
y(z_0) = y_0 \quad \text{and} \quad \frac{\partial \phi}{\partial y} (z, y(z)) \in (\Lambda_H)_{y(z)}.
\]

But by (2.6), \( \hat{\xi}(t) = -\frac{\partial \phi}{\partial y} (z(t), \hat{x}(t)) \) and since the forms \( \frac{1}{i} \partial h_1, \ldots, \frac{1}{i} \partial h_n \) are real on \( H = Z(U) \) and span all of \( T^*H \), we know that
\[
\hat{\xi}(t) = Z_{x}(x(t))\xi(t) \in (\Lambda_H)_{x(t)}.
\]

It therefore follows that
\[
y(z(t)) = \hat{x}(t).
\]

In our previous notation,
\[
\eta(z(t)) = -\frac{\partial \phi}{\partial y} (z(t), y(z)) = -\frac{\partial \phi}{\partial y} (z(t), \hat{x}(t)) = \hat{\xi}(t).
\]

Thus
\begin{equation}
(2.7)
(\hat{x}(t), \hat{\xi}(t)) = (y(z(t)), \eta(z(t))).
\end{equation}

Since \( WF_{ha}(Pu) \cap \gamma = \emptyset \) and \( \gamma \) is compact, (2.7) and Proposition (2.1) tell us that
\[
T(Pu) \subseteq H_{\Phi-\epsilon_0}(N)
\]
for some $\varepsilon_0 > 0$ and a neighborhood $N$ of $\{z(t) = 0 \leq t \leq 1\}$ in $C^m$. If $W$ is chosen as in Lemma 2.2, then

$$D_{z_n} Tu \in H^\text{loc}_{\Phi - \varepsilon_0}(N \cap W).$$

This may require a modification of $\varepsilon_0$.

Now $z(0) = z_0 \in N \cap W$. Therefore, $\exists t_1 > 0$ such that $N \cap W$ is a neighborhood of $\{z(t) : 0 \leq t \leq t_1\}$. It is crucial to note that the size of $t_1$ is independent of the distribution $u$.

If now $K$ is a compact neighborhood of $\{z(t) : 0 < t < t_1\}$, then $\exists c > 0$ such that

$$|D_{z_n} Tu(z, \lambda)| \leq ce^{\lambda(\Phi(z) - \frac{t_0}{2})} \quad \forall z \in K \text{ and } \lambda \geq 1. \quad (2.8)$$

If $(y_0, \xi_0) = (y(z(0)), \eta(z(0))) \notin WF_{ha} u$, we know that, after modifying $c$ and $\varepsilon_0$,

$$|Tu(z, \lambda)| \leq ce^{\lambda(\Phi(z) - \frac{t_0}{2})} \quad \forall \lambda \geq 1 \text{ and } \forall z \text{ near } z_0. \quad (2.9)$$

From (2.7), (2.8) and (2.9), it follows that

$$WF_{ha}(u) \cap \{(y(t), \xi(t)) : 0 \leq t \leq t_1\} = \emptyset.$$

**REFERENCES**


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