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INDECOMPOSABLE COHEN-MACAULAY MODULES AND THEIR MULTIPLICITIES

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ABSTRACT. The main aim of this paper is to find a large class of rings for which there are indecomposable maximal Cohen-Macaulay modules of arbitrary high multiplicity (or rank in the case of domains).

1. INTRODUCTION

Let $(A, \mathfrak{m})$ be a (commutative) henselian Cohen-Macaulay local ring and let $\mathrm{CM}(A)$ be the category of maximal Cohen-Macaulay A-modules (shortly MCM A-modules), i.e. of finitely generated modules M with $\operatorname{depth} M = \dim A$. For $s \in \mathbf{N}$ let $n_A(s)$ be the cardinal of isomorphism classes of indecomposable modules M from $\mathrm{CM}(A)$ whose multiplicity $e_A(M) = e(\mathfrak{m}, M) = s$. Take $n_A = \sum_{s \in \mathbf{N}} n_A(s)$.

(1.1) **First Brauer-Thrall type conjecture.** *If $n_A = \infty$, then $n_A(s) \neq 0$ for infinitely many s.*

When $\dim A = 0$ then $e_A(M) = \operatorname{length}_A(M)$ and (1.1) holds by A. V. Roiter's theorem [R, Au_1] or [P, (7.7)]. Using the Auslander-Reiten theory for MCM modules (see [Au_3, P, AR_1, Ya or Y, Appendix]) Y. Yoshino succeeded in solving positively (1.1) for reduced analytic algebras A over a perfect valued field k which are isolated singularities. Our Theorem (5.4) gives in particular the following

(1.2) **Theorem.** *Let (A, m) be a reduced excellent henselian local CM-ring, $k := A/\mathfrak{m}$, $p := \operatorname{char} k$. Suppose that*

(i) $[k : k^p] < \infty$ *if* $p > 0$,
(ii) *A is an isolated singularity,*
(iii) *A/pA is an isolated singularity.*

Then (1.1) *holds.*

Note that (iii) follows from (ii) when A contains a field (i.e. the equal characteristic case). When A is a domain, $e_A(M) = e(A) \cdot \operatorname{rank}(M)$ by [M_2, (14.8)]

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so in the hypothesis of Theorem (1.2) there are indecomposable MCM-modules of arbitrary high rank if $n_A = \infty$. The proof follows [Y] entirely, our contribution being mainly to extend his Lemmas (2.10) and (2.12) in the following form (see (4.8)):

(1.3) **Theorem.** *Let* $(A, \mathfrak{m})$ *be a reduced excellent henselian local CM-ring,* $k := A/\mathfrak{m}$, $p := \operatorname{char} k$ *and* $I_s(A)$ *the ideal defining the singular locus of* A, *i.e.* $I_s(A) = \bigcap_{q \notin \operatorname{Reg} A} q$. *Suppose that*

(i) $[k : k^p] < \infty$ *if* $p > 0$,
(ii) *if* $pA \neq 0$ *then* A_q/pA_q *is regular for every* $q \in \operatorname{Reg} A$ *containing* pA,
(iii) $I_s(A) \subseteq \mathfrak{m}$, *i.e.* A *is not regular.*

Then there exists a positive integer r *such that*

(1) *an MCM* A*-module* M *is indecomposable iff* $M/I_s(A)^r M$ *is indecomposable,*
(2) *two indecomposable MCM* A*-modules* M, N *are isomorphic iff* $M/I_s(A)^r M$ *and* $N/I_s(A)^r N$ *are isomorphic.*

In particular this theorem gives large classes of isolated singularities for which there exist Dieterich [D] reduction ideals.

In the hypothesis of (1.3) we get $n_A \leq n_{\hat{A}}$ (see (4.10)) where $\hat{A}$ is the completion of A. In particular we can improve the result from [K] and [BGS] for excellent henselian local rings (see (4.11)). Though (4.11) can also be obtained using the property of Artin approximation of excellent henselian local rings (see [Po, (1.3)] as we indicate in (5.6), we choose here an easier method (see §§3–4) which is entirely self-contained and proves to be more powerful for these questions. Our §2 contains just preliminaries arranged more or less after [Y] which we include here for completeness.

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2. The singular locus of an excellent local ring

Let A be an excellent ring. Then $\operatorname{Reg} A = \{q \in \operatorname{Spec} A | A_q \text{ is regular}\}$ is an open set and $I_s(A) = \bigcap_{q \notin \operatorname{Reg} A} q$ defines the singular locus of A, i.e. $V(I_s(A)) = \operatorname{Spec} A \backslash \operatorname{Reg} A$.

(2.1) **Lemma.** *Let* $u: A \to B$ *be a flat morphism of excellent rings. Then* $I_s(B) \subset \sqrt{u(I_s(A))B}$.

Proof. If $q \in \operatorname{Reg} B$ then $q \cap A \in \operatorname{Reg} A$ by [M_1, (21.D)]. Thus a prime ideal from B containing $u(I_s(A))$ must also contain $I_s(B)$. □

(2.2) It will also be useful to express $I_s(A)$ as the radical of a certain ideal of A whose elements can be precisely described. This is already well known for rings A which are essentially of finite type over a perfect field k because in that case the Jacobian criterion for smoothness [M_1, (29.C)] applies and we have

$I_s(A) = H_{A/k}$. In general, given a finite presentation A-algebra $B = A[X]/\mathfrak{a}$, $X = (X_1, \dots, X_n)$, the nonsmooth locus of B over A is defined by the ideal

$$H_{B/A} = \sqrt{\sum_f \Delta_f((f) : \mathfrak{a})B},$$

where the sum is taken over all systems f of r-polynomials from $\mathfrak{a}$, and Δ_f is the ideal generated by all $r \times r$-minors of $\partial f / \partial Y$, $r = 1, \dots, n$, being variable (see [Po, (2.1)]). Using [Y, §2] we will present such a description of $I_s(A)$ when A is a Noetherian complete local ring having some additional properties.

(2.3) Till the end of this section $(R, \mathfrak{m})$ *is a reduced Noetherian complete local ring* with a *perfect residue field* k. Then either R contains k or R is an algebra over a Cohen ring of residue field k, i.e. a complete DVR (T, t) which is an unramified extension of $\mathbf{Z}_{(p)}$, $p := \operatorname{char} k > 0$, $t := p \cdot 1 \in T$. When R contains k we put $T := k$ and $t = 0$ in order to unify both situations.

Let $\mathscr{R}(T, R)$ be the set of all prime ideals $q \subseteq R$ for which $T \to R_q$ is a regular morphism. Clearly $\mathscr{R}(T, R) \subseteq \operatorname{Reg} R$ because T is regular and regular morphisms preserve this property [M_1, (33.B)]. When R contains k the other inclusion also holds, k being perfect. When R is in the unequal characteristic case $(pR \neq 0)$ then we suppose that

$$(*) \qquad R_q/pR_q \textit{ is regular for every } q \in \operatorname{Reg} R.$$

Thus in both situations we have $\mathscr{R}(T, R) = \operatorname{Reg} R$.

(2.4) Let $x = (x_1, \dots, x_n)$ be a system of elements from R such that (t, x) forms a system of parameters in R. Then the canonical map $T[[X]] \to R$, $X = (X_1, \dots, X_n) \to x$, is finite by Cohen Structure Theorems.

(2.5) **Lemma** (Scheja-Storch [S, (4.2)]). *There exists x as above such that* $\operatorname{ht}(H_{R/T[[x]]}) \geq 1$, *i.e. for every minimal prime ideal $q \subseteq R$ the fraction field extension* $\operatorname{Fr}(T[[x]]) \hookrightarrow R_q$ *is (finite) separable.*

Proof. When $T \neq k$ there is nothing to show because $\operatorname{char} T = 0$. Suppose $T = k$. Let $q_1, \dots, q_s$ be the minimal prime ideals of R and take an arbitrary system of parameters $y = (y_1, \dots, y_n)$ of R. If the field extensions $\alpha_i : k((y)) \to R_{q_i}$, $1 \leq i \leq s$, are all separable then $\operatorname{ht}(H_{R/k[[y]]}) \geq 1$ by the Jacobian criterion for smoothness [M_1, (29.C)]. Suppose that the $(\alpha_i)_{1 \leq i \leq e}$ are not separable for a certain e, $1 \leq e \leq s$. Then $p > 0$ and for every i, $1 \leq i \leq e$, there exists an element $z_i \in R_{q_i} \backslash k((y))$ such that $z_i^p \in k((y))$. Since α_i is finite we have

$$k((y)) \otimes_{k[[y]]} R \cong \prod_{i=1}^{s} R_{q_i}.$$

Thus we can find one $z \in R$ and $w \in k[[y]]$, $w \neq 0$, such that z/w corresponds to $(z_1, \dots, z_e, y_n, \dots, y_n)$ by the above isomorphism. Then

$h := z^p \in k[[y]]$ and $z \notin k[[y]]$. Adding a constant to z we can suppose that $h \in (y)k[[y]]$. If $h \in k[[y^p]]$ then $h \in k^p[[y^p]]$ (k is perfect) and so $z \in k[[y]]$ which is not possible.

Suppose that $h \notin k[[y_1, \dots, y_{n-1}, y_n^p]]$. After a coordinate transformation we can suppose also that h is regular in y_n. Applying the Weierstrass Preparation Theorem for $U - h$ in $k[[y, U]]$ we find a distinguished polynomial

$$(1) \qquad P = y_n^r + \sum_{i=1}^{r} a_i y_n^{r-i}, \qquad a_i \in k[[y_1, \dots, y_{n-1}, U]], \ a_i(0) = 0,$$

and an invertible formal power series $g \in k[[y, U]]$ such that

$$(2) \qquad U - h = Pg.$$

Substituting $U = h$ in P we get

$$(3) \qquad y_n^r + \sum_{i=1}^{r} a_i(y_1, \dots, y_{n-1}, h) y_n^{r-i} = 0$$

because $g(U = h) \neq 0$ since $g(0) \neq 0$ and $h(0) = 0$. Applying $\partial/\partial y_n$ in (2) we obtain

$$(\partial P/\partial y_n)g + P(\partial g/\partial y_n) = -\partial h/\partial y_n \neq 0$$

and substituting $U = h$ we get $(\partial P/\partial y_n)(U = h) \neq 0$. Thus (3) defines a separable equation for y_n over $k[[y_1, \dots, y_{n-1}, z^p]]$. In particular y_n is separable over $S := k[[y_1, \dots, y_{n-1}, z]]$. Denote $y' = (y_1, \dots, y_{n-1}, z)$. We have

$$[R_{q_i} : k((y'))]_{ins} = [R_{q_i} : k((y))]_{ins} - p$$

for every $i = 1, \dots, e$, where $[\ ]_{ins}$ denotes the inseparable degree. Repeating this procedure inductively we finally find a system of parameters x in R such that $k((x)) \hookrightarrow R_{q_i}$ is separable for every i. □

(2.6) *Remark.* If k is not perfect then the above lemma does not hold. If $a \in k \backslash k^p$ then $A = k[[X, Y]]/(X^p + aY^p)$ (after [Y, (2.7)] is a counterexample.

From now on we suppose that R is a CM-ring. Then the canonical map $T[[X]] \to R$, $X \to x$ is flat (so free) by [M_1, (36.B)].

(2.7) **Lemma.** *Let $q \in \operatorname{Reg} R$. Then there exists a system of elements x in R such that*

(i) *(t, x) is a system of parameters in R,*
(ii) *$H_{R/T[[x]]} \not\subset q$.*

Proof. If $t = 0$ then we choose a system of elements y in R which forms a regular system of parameters in R_q. If $t \in q$ then by condition (2.3)(*) we get R_q/tR_q regular. Thus there exists y such that (t, y) forms a regular system of parameters in R_q. By Lemma (2.5) there exists a system of elements z in R which forms a system of parameters $\bar{z}$ in $R/\mathfrak{a}$, $\mathfrak{a} = \sqrt{(t, y)}$ such that the map $(T/T \cap \mathfrak{a})[[\bar{z}]] \to R/\mathfrak{a}$ is generically smooth. Since q is a minimal prime

ideal containing $\mathfrak{a}$ we get $(T/T \cap q)[[\bar{z}]] \to R/q$ separable and so the map $T[[y, z]] \to R_q$ is etale. Thus $x = (y, z)$ works.

Suppose now $t \notin q$. Then as above we can choose y in R which forms a regular system of parameters in R_q. Take a system of elements z in R such that (t, z) forms modulo q a system of parameters in R/q. Then (t, y, z) forms a system of parameters in R and $T[[y, z]] \to R_q$ is etale (char $R/q = 0$). Thus $x = (y, z)$ works.

(2.8) **Corollary.** $I_s(R) = \sqrt{\sum_x H_{R/T[[x]]}}$, *where the sum is taken over all systems of elements* x *such that* (t, x) *forms a system of parameters of* R.

Proof. If $q \in \operatorname{Spec} R$ does not contain $H_{R/T[[x]]}$ for a certain system x then the map $T[[x]] \to R_q$ is etale and so R_q is regular because $T[[x]]$ is so. Conversely if $q \in \operatorname{Reg} R$ then by Lemma (2.7) there exists x such that $q \not\supset H_{R/T[[x]]}$. □

(2.9) Let $S \subseteq R$ be a regular local subring such that R is a finitely generated free S-module, $R^e = R \otimes_S R$ is the *enveloping algebra* of R over S and $\mu: R^e \to R$ is the multiplication map. Denote $I := \operatorname{Ker} \mu$. The ideal $\mathcal{N}_S^R = \mu(\operatorname{Ann}_{R^e} I)$ is called the *Noether different* of R over S.

(2.10) **Lemma.** $\mathcal{N}_S^R \cdot \Omega_{R/S} = 0$ *and* $H_{R/S} = \sqrt{\mathcal{N}_S^R}$.

Proof. The first equality is trivial because $\Omega_{R/S} = I/I^2$. Let $q \subseteq R$ be a prime ideal. If $q \not\supset \mathcal{N}_S^R$ then $\Omega_{R_q/S} = \Omega_{R/S} \otimes_R R_q = 0$ as above. Since $S \subseteq R$ is finite free we get $S \to R_q$ etale, i.e. $q \not\supset H_{R/S}$. Conversely if $q \not\supset H_{R/S}$ then $S \to R_q$ is etale and so $\Omega_{R/S} \otimes_R R_q = 0$. Thus $I_Q = I_Q^2$ for a certain prime ideal $Q \subseteq R^e$, $Q \supseteq I$ such that $\mu(Q) = q$. By Nakayama's Lemma we get $I_Q = 0$ and so $Q \not\supset \operatorname{Ann}_{R^e} I$. Thus $\mathcal{N}_S^R \not\subset q$.

(2.11) We end this section by listing some facts from Hochschild cohomology, which can be found in [P, Chapter 11]. Let $B \subseteq A$ be an extension of rings. The nth Hochschild cohomology functors $H_B^n(A, -)$, $n \geq 0$, are defined on the category of A-bimodules with values in the category of B-modules and have the following properties:

(i) $H_B^0(A, M) = M^{(A)} := \{x \in M | ax = xa$ for every $a \in A\}$ for all A-bimodules M.

(ii) If M, N are two A-modules then $\operatorname{Hom}_B(M, N)$ is an A-bimodule [the left (resp. right) action of A on $\operatorname{Hom}_B(M, N)$ is given as the one induced from the action on N (resp. M)] and $H_B^0(A, \operatorname{Hom}_B(M, N)) = \operatorname{Hom}_A(M, N)$.

(iii) $H_B^1(A, M)$ is a factor A-module of $\operatorname{Der}_B(A, M) = \operatorname{Hom}_A(\Omega_{A/B}, M)$.

(iv) If A is a projective module over B and

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of A-bimodules then there exist some B-morphisms $\partial^{(n)}: H_B^n(A, M'') \to H_B^{n+1}(A, M')$, $n \geq 0$, such that the following sequence is

exact:

$$0 \to H_B^0(A, M') \to H_B^0(A, M) \to H_B^0(A, M'') \to H_B^1(A, M')$$
$$\to \cdots \to H_B^n(A, M') \to H_B^n(A, M) \to H_B^n(A, M'') \to H_B^{n+1}(A, M') \to \cdots .$$

(2.12) **Lemma.** *Let $S \subseteq R$ be as in (2.9) and M an R-bimodule. Then $\mathcal{N}_S^R \cdot H_S^1(R, M) = 0$.*

Proof. By Lemma (2.10) we have $\mathcal{N}_S^R \Omega_{R/S} = 0$ and so $\mathcal{N}_S^R \cdot \mathrm{Hom}_R(\Omega_{R/S}, M) = 0$. Now apply (2.11)(iii). □

3. CM-APPROXIMATION

(3.1) Let R be a CM local ring and $a \subset R$ a proper ideal. The couple (A, a) is a CM-*approximation* if there exists a function $\nu: \mathbf{N} \to \mathbf{N}$ (called CM-*function*) such that for every $s \in \mathbf{N}$, every two MCM R-modules M, N and every linear R-map $\varphi: M \to N/a^{\nu(s)}N$ there exists a linear R-map $\psi: M \to N$ such that $(A/a^s) \otimes_A \varphi \cong (A/a^s) \otimes_A \psi$, in other words the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N/a^{\nu(s)}N \\ {\scriptstyle\psi}\downarrow & & \downarrow \\ N & \longrightarrow & N/a^s N \end{array}$$

But given M, N then there exists ν such that for every linear R-map $\varphi: M \to N/a^{\nu(s)}N$ there exists a linear R-map $\psi: M \to N$ such that $(A/a^s) \otimes_A \varphi \cong (A/a^s) \otimes_A \psi$ (this follows from a linear form of the strong approximation theorem; see e.g. [Po, (1.5)] which holds in fact in every Noetherian local ring). Thus the above definition asks for a unique function for all MCM R-modules M, N.

As we shall see in the next section the CM-approximation plays an important role in the proof of (1.3). The aim of this section is to give sufficient conditions when $(R, I_s(R))$ is a CM-approximation.

(3.2) **Lemma.** *Let $S \subset R$ be an extension of Noetherian rings such that R is a finitely generated projective module over S, x an element from $\mathcal{N}_S^R$ and M, N two finitely generated R-modules such that M is projective over S. Let $e \in \mathbf{N}$ be a positive integer such that $\mathrm{Ann}_N x^e := \{z \in N | x^e z = 0\} = \mathrm{Ann}_N x^{e+1}$ and $s \in \mathbf{N}$. Then for every linear R-map $\varphi: M \to N/x^{e+s+1}N$ there exists a linear R-map $\psi: M \to N$ which makes commutative the following diagram:*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N/x^{e+s+1}N \\ {\scriptstyle\psi}\downarrow & & \downarrow \\ N & \longrightarrow & N/x^{e+s}N \end{array}$$

Proof. Let $N' = \operatorname{Ann}_N x^e$. We have the following commutative diagram:

$$(1)\qquad \begin{tikzcd} 0 \arrow[r] & N/N' \arrow[r, "x^{e+s+1}"] \arrow[d, "x"'] & N/N' \arrow[r] \arrow[d, "\wr"] & N/N' + x^{e+s+1}N \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N/N' \arrow[r, "x^{e+s}"] & N/N' \arrow[r] & N/N' + x^{e+s}N \arrow[r] & 0 \end{tikzcd}$$

in which the bases are exact. Indeed if $x^{e+s}z \in N'$ for a certain $z \in N$ then $x^{2e+s}z = 0$ and so $z \in \operatorname{Ann}_N x^{2e+s} = N'$. Applying the functor $\operatorname{Hom}_S(M, -)$ to (1) we get the following commutative diagram:

$$(2)\qquad \begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_S(M, N/N') \arrow[r] \arrow[d, "x"'] & \operatorname{Hom}_S(M, N/N') \arrow[r] \arrow[d, equal] & \operatorname{Hom}_S(M, N/N' + x^{e+s+1}N) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{Hom}_S(M, N/N') \arrow[r] & \operatorname{Hom}_S(M, N/N') \arrow[r] & \operatorname{Hom}_S(M, N/N' + x^{e+s}N) \arrow[r] & 0 \end{tikzcd}$$

where the bases are exact because M is projective over S. Clearly these bases are also exact sequences of R-bimodules and applying the Hochschild cohomology functors we get the following commutative diagram (see (2.11)(ii)):

$$(3)\qquad \begin{tikzcd} \operatorname{Hom}_R(M, N/N') \arrow[r] \arrow[d, equal] & \operatorname{Hom}_R(M, N/N' + x^{e+s+1}N) \arrow[r] \arrow[d] & H^1_S(R, \operatorname{Hom}_S(M, N/N')) \arrow[d, "x"] \\ \operatorname{Hom}_R(M, N/N') \arrow[r] & \operatorname{Hom}_R(M, N/N' + x^{e+s}N) \arrow[r] & H^1_S(R, \operatorname{Hom}_S(M, N/N')) \end{tikzcd}$$

in which the bases are exact (see (2.11)(iv)). Since the last vertical map is zero by Lemma (2.12) we get a linear R-map $\alpha\colon M \to N/N'$ such that the following diagram is commutative:

$$(4)\qquad \begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, dashed, "\alpha"'] & N/x^{e+s+1}N \arrow[r] & N/N' + x^{e+s+1}N \arrow[d] \\ N/N' \arrow[rr] & & N/N' + x^{e+s}N \end{tikzcd}$$

Note that in the diagram

$$(5)\qquad \begin{tikzcd} M \arrow[rr] \arrow[d, equal] & & N/x^{e+s+1}N \arrow[d] \\ M \arrow[r, dashed, "\psi"] \arrow[d, equal] & N/N' \cap x^{e+s}N \arrow[r] \arrow[d] & N/x^{e+s}N \arrow[d] \\ M \arrow[r] & N/N' \arrow[r] & N/N' + x^{e+s}N \end{tikzcd}$$

the small square is cartesian and so there exists ψ which makes (5) commutative.

It remains to show that $N' \cap x^{e+s}N = 0$. Indeed let $y \in N' \cap x^{e+s}N$ and $z \in N$ with $y = x^{e+s}z$. Then $0 = x^e y = x^{2e+s}z$ and so $z \in N'$, i.e. $y = x^{e+s}z = 0$. $\square$

(3.3) **Lemma.** *Let $B \hookrightarrow A$ be a finite flat extension of Noetherian rings, $\mathfrak{a} \subset A$ an ideal and $x \in H_{A/B}$ an element. Then there exists a positive integer r such that for every finitely generated A-module N which is free over B*

$$(\mathfrak{a}N : x^r)_N = (\mathfrak{a}N : x^{r+1})_N$$

holds, where $(\mathfrak{a}N : x^r)_N = \{z \in N | x^r z \in \mathfrak{a}N\}$.

Proof. Step 1. *Reduction to the case* $(\mathfrak{a} : x) = \mathfrak{a}$. Since A is Noetherian we have $\mathfrak{a}' := (\mathfrak{a} : x^n) = (\mathfrak{a} : x^{n+1})$ for a certain positive integer n. If $xy \in \mathfrak{a}'$ for a certain $y \in A$ then $x^{n+1}y \in \mathfrak{a}$ and so $y \in \mathfrak{a}'$, i.e. $(\mathfrak{a}' : x) = \mathfrak{a}'$.

Suppose that $r' \in \mathbf{N}$ satisfies our lemma for x and $\mathfrak{a}'$. Then $r = n + r'$ works. Indeed, let N be as in our lemma. If $x^s z \in \mathfrak{a}N \subseteq \mathfrak{a}'N$ for some $s \in \mathbf{N}$ and $z \in N$ then $x^{r'} z \in \mathfrak{a}'N$ because $(\mathfrak{a}'N : x^{r'})_N = (\mathfrak{a}'N : x^{r'+1})_N$. Thus $x^r z \in x^n \mathfrak{a}'N \subseteq \mathfrak{a}N$.

Remark. $\mathrm{Ass}_A(A/\mathfrak{a}') = \{q \in \mathrm{Ass}_A(A/\mathfrak{a}) | x \notin q\}$.

Let $\mathfrak{a} = \bigcap_{i=1}^e Q_i$ be an irredundant primary decomposition of $\mathfrak{a}$, $q_i := \sqrt{Q_i}$, $q_i' := q_i \cap B$, $Q_i' := Q_i \cap B$, $\mathfrak{b} := \mathfrak{a} \cap B = \bigcap_{i=1}^e Q_i'$ and $k_i' \subseteq k_i$ the residue field extension of $B_{q_i'} \subset A_{q_i}$.

Step 2. *Case when* $k_i' = k_i$, $1 \le i \le e$. By Step 1 we may suppose that $(\mathfrak{a} : x) = \mathfrak{a}$. Fix an i, $1 \le i \le e$. Clearly $x \notin q_i$ because x is a nonzero divisor of $A/\mathfrak{a}$. Then the map $B_{q_i'} \to A_{q_i}$ is etale and so $q_i A_{q_i} = q_i' A_{q_i}$. Since $k_i' = k_i$ the extension $B_{q_i'} \subset A_{q_i}$ is dense. In particular we have

$$B_{q_i'}/Q_i' B_{q_i'} \cong A_{q_i}/Q_i' A_{q_i}$$

and it follows that $Q_i' A_{q_i} = Q_i A_{q_i}$.

We show that $r = 0$ satisfies this case. Let N be as in our lemma, and $z \in N$ such that $xz \in \mathfrak{a}N$. Then $z \in Q_i N_{q_i} = Q_i' N_{q_i}$. Thus there exists an element $y_i \in A \backslash q_i$ such that $y_i z \in Q_i' N$. Since $B/q_i' \to A/q_i$ is finite we get $(y_i A) \cap (B \backslash q_i') \neq \varnothing$. Thus changing y_i by one of its multiples we may suppose that $y_i \in B \backslash q_i'$, i.e. $z \in Q_i' N_{q_i'}$. Since N is free over B we have

$$\mathfrak{b}N = \bigcap_{j=1}^e Q_j' N$$

and $Q_j' N$ is exactly the q_j'-primary submodule of N associated to $\mathfrak{b}N$. Then $N \cap Q_j' N_{q_j'} = Q_j' N$ and so

$$z \in N \cap \left(\bigcap_{j=1}^e Q_j' N_{q_j'} \right) = \mathfrak{b}N \subseteq \mathfrak{a}N.$$

Step 3. *Case when there exists a faithfully flat B-algebra C such that for every prime ideal q associated to $C \otimes_B \mathfrak{a}$ in $D = C \otimes_B A$ the residue field extension of*

$C_{q\cap C} \hookrightarrow D_q$ *is trivial.* We apply Step 2 to the case $C \subseteq D$, $\mathfrak{a}D$, $x' = 1\otimes x \in D$. Clearly $x' \in D\otimes_A H_{A/B} \subseteq H_{D/C}$. Then there exists r such that for every finitely generated D-module N' which is free over C it follows that

$$(\mathfrak{a}N' : x'^r)_{N'} = (\mathfrak{a}N' : x'^{r+1})_{N'}.$$

Let N be a finitely generated A-module which is free over B and take $N'' = D\otimes_A N$. Then N'' is free over C and so we get in particular

$$(\mathfrak{a}N'' : x'^r)_{N''} = (\mathfrak{a}N'' : x'^{r+1})_{N''}.$$

But $(\mathfrak{a}N'' : x'^r)_{N''} = D\otimes_A(\mathfrak{a}N : x^r)_N$. Indeed, $(\mathfrak{a}N : x^r)_N$ is exactly the kernel of the composed map $f\colon N \xrightarrow{x^r} N \to N/\mathfrak{a}N$ and by flatness $\operatorname{Ker}(D\otimes_A f) \cong D\otimes_A \operatorname{Ker} f$. Thus the inclusion $u\colon (\mathfrak{a}N : x^r)_N \to (\mathfrak{a}N : x^{r+1})_N$ goes by base change in an equality. Since D is a faithfully flat A-algebra we get u surjective too.

Step 4. *General case—reduction to Step* 3. We need the following

(3.4) **Lemma.** *Let $S \subseteq R$ be a finite flat extension of Noetherian rings and denote*

$$d_{R/S} = \max_{q'\in \operatorname{Spec} S} \sum_{\substack{q\in \operatorname{Spec} R\\ q\cap S = q'}} ([k(q) : k(q')] - 1),$$

where $k(q)$ denotes the residue field of R_q. Then $d_{R/S} < \infty$ and $d_{R\otimes_S R/R} \le d_{R/S}$ if $d_{R/S} > 0$, where the structural map $R \to R\otimes_S R$ is given by $y \to y\otimes 1$.

Applying by recurrence the above lemma we get finally a finite flat B-algebra C of the form $A\otimes_B A\otimes_B \cdots \otimes_B A$ such that $d_{C\otimes_B A/C} = 0$, i.e. $k(q\cap C) = k(q)$ for all $q \in \operatorname{Spec}(C\otimes_B A)$. Since a finite flat extension is faithfully flat we are ready. □

Proof of Lemma (3.4). Let $q' \in \operatorname{Spec} S$. Then

$$d_{R/S,q'} = \sum_{\substack{q\in \operatorname{Spec} R\\ q\cap S = q'}} ([k(q) : k(q')] - 1) < \operatorname{rank}_{k(q')} k(q')\otimes_S R,$$

the last number being bounded by the minimal number of generators of R over S. It is enough to show that

$$d_{R\otimes_S R/R,q} < d_{R/S,q'}$$

for every $q \in \operatorname{Spec} R$ lying over q' and such that $d_{R/S,q'} > 0$. So by base change we reduce the question to the case when $S = k(q') =: k$. Then R is Artinian. Let $(k_i)_{1\le i\le e}$ be its residue fields. It is enough to show that

$$d_{k_1\otimes_k k_i/k_1} \le d_{k_i/k_1}, \quad 1 \le i \le e, \qquad d_{k_1\otimes_k k_1/k_1} < d_{k_1/k} \quad \text{if } k \ne k_1.$$

First inequality is clear because

$$1 + d_{k_1\otimes_k k_i/k_1} \le \operatorname{rank}_{k_1} k_1\otimes_k k_i = \operatorname{rank}_k k_i = d_{k_i/k} + 1.$$

The equality holds only when $k_1 \otimes_k k_i$ is a field. But $k_1 \otimes_k k_1$ is not a field so the second inequality holds too. □

(3.5) **Lemma.** *Let $S \subseteq R$ be an extension of Noetherian rings such that R is a finitely generated projective module over S, x an element from $\mathcal{N}_S^R$ and $\mathfrak{a} \subseteq R$ an ideal. Then there exists an increasing function $\nu: \mathbf{N} \to \mathbf{N}$ such that for every $s \in \mathbf{N}$, for two finitely generated R-modules M, N which are free over S and for every linear R-map $\varphi: M \to N/(\mathfrak{a}, x^{\nu(s)})N$ there exists a linear R-map $\psi: M \to N/\mathfrak{a}N$ which makes commutative the following diagram:*

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, dashed, "\psi"'] & N/(\mathfrak{a}, x_\nu^{(s)})N \arrow[d] \\ N/\mathfrak{a}N \arrow[r] & N/(\mathfrak{a}, x^s)N \end{tikzcd}$$

Proof. Let r be the integer given by Lemma (3.3) for x and $\mathfrak{a}$. Define ν by $\nu(s) = 1 + \max\{r, s\}$. Then given M, N, s, φ as in our lemma we find the wanted ψ by applying Lemma (3.2) for x, $\overline{N} = N/\mathfrak{a}N$ and $e = r$. □

(3.6) **Lemma.** *Let $x = (x_1, \dots, x_n)$ be a system of elements from a Noetherian ring R such that for every i, $1 \le i \le n$, there exists a Noetherian subring S_i of R such that*

(i) *R is finite free over S_i,*
(ii) *$x_i \in \mathcal{N}_{S_i}^R$.*

Then there exists an increasing function $\nu: \mathbf{N} \to \mathbf{N}$ such that for every $s \in \mathbf{N}$, for two finitely generated R-modules M, N which are free over all $(S_i)_{1 \le i \le n}$ and for every linear R-map $\varphi: M \to N/x^{\nu(s)}N$ there exists a linear R-map $\psi: M \to N$ which makes commutative the following diagram:

$$(*) \qquad \begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, dashed, "\psi"'] & N/x^{\nu(s)}N \arrow[d] \\ N \arrow[r] & N/x^s N \end{tikzcd}$$

Proof. Denote $\mathfrak{b}_i = (x_1, \dots, x_i)$, $i = 1, \dots, n$. Apply induction on n. If $n = 1$ then apply Lemma (3.5) for x_1 and $\mathfrak{a} = 0$. Suppose now that a function ν' is given which works for $\mathfrak{b}_{n-1}$. Let $s \in \mathbf{N}$ and ν_s'' be the function given by Lemma (3.5) for x_n and $\mathfrak{a} := \mathfrak{b}_{n-1}^{\nu'(s)}$. Define $\nu: \mathbf{N} \to \mathbf{N}$ by $\nu(s) = \nu'(s) + \nu_s''(s)$. Let M, N be two finitely generated R-modules which are free over all $(S_i)_{1 \le i \le n}$ and $\varphi: M \to N/b_n^{\nu(s)}N$ a linear R-map. Then there exists a linear R-

map $\alpha\colon M \to \overline{N} = N/\mathfrak{b}_{n-1}^{\nu'(s)}N$ which makes commutative the following diagram:

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, dashed, "\alpha"'] & N/b_n^{\nu(s)}N \arrow[d] \\
\overline{N} \arrow[r] & \overline{N}/x_n^{\nu_s''(s)}\overline{N} \cong N/(\mathfrak{b}_{n-1}^{\nu'(s)}, x_n^{\nu_s''(s)})N \arrow[d] \\
 & \overline{N}/x_n^{s}\overline{N} \cong N/(\mathfrak{b}_{n-1}^{\nu'(s)}, x_n^{s})N
\end{tikzcd}$$

Thus there exists a linear R-map $\psi\colon M \to N$ which makes commutative the following diagram:

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d, dashed, "\psi"'] & \overline{N} \cong N/b_{n-1}^{\nu'(s)}N \arrow[r] \arrow[d] & N/(\mathfrak{b}_{n-1}^{\nu'(s)}, x_n^{s})N \arrow[d] \\
N \arrow[r] & N/\mathfrak{b}_{n-1}^{s}N \arrow[r] & N/\mathfrak{b}_n^{s}N
\end{tikzcd}$$

Clearly ψ also makes $(*)$ commutative. □

(3.7) **Proposition.** *Let $(R, \mathfrak{m})$ be a reduced complete local CM-ring with a perfect residue field k, $p := \operatorname{char} k$ and $I_s(R)$ the ideal defining the singular locus of R. Suppose that for every $q \in \operatorname{Reg} R$ containing pR the ring R_q/pR_q is regular and $I_s(R) \subseteq m$. Then $(R, I_s(R))$ is a CM-approximation.*

Proof. Let $T \subseteq R$ be the Cohen ring of the residue field k (see (2.3)). By Lemma (2.10) and Corollary (2.8) we have

$$I_s(R) = \sqrt{\sum_x \mathcal{N}^R_{T[[x]]}},$$

where the sum is taken over all systems of elements x such that (t, x) forms a system of parameters of R. Then we can find a system of elements $y = (y_1, \dots, y_r)$ in $I_s(R)$ such that

(1) $I_s(R) = \sqrt{yR}$,

(2) for every $i = 1, \dots, r$ there exists a system of elements $x^{(i)}$ of R such that $(t, x^{(i)}))$ forms a system of parameters of R and $y_i \in \mathcal{N}^R_{T[[x^{(i)}]]}$.

Since R is CM the inclusion $S_i := T[[x^{(i)}]] \subset R$ is finite flat (so free). Let $\nu'\colon \mathbf{N} \to \mathbf{N}$ be the function given by Lemma (3.6) for y. If M, N are two MCM R-modules then $(t, x^{(i)})$ is a regular M or N-sequence for all i. Thus M and N are finitely generated flat over S_i, $1 \le i \le r$, and so free (see [M_1, (20.C)]).

Now let u be a positive integer such that $I_s(R)^u \subset yR$ and note that ν given by $\nu(s) = u\nu'(s)$ works. □

4. CM-REDUCTION IDEALS

The aim of this section is to extend Proposition (3.7) to noncomplete rings and to apply CM-approximation in proving (1.3).

(4.1) **Lemma.** *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring and* $\mathfrak{a} \subset A$ *an ideal. The following statements are equivalent*:

(i) $(A, \mathfrak{a})$ *is a CM-approximation,*
(ii) $(A, \sqrt{\mathfrak{a}})$ *is a CM-approximation.*

Proof. Let u be a positive integer such that $(\sqrt{\mathfrak{a}})^u \subset \mathfrak{a}$. If (i) holds and $\nu: \mathbf{N} \to \mathbf{N}$ is the associated CM-function then as in the proof of Proposition (3.7) the function $\bar{\nu}$ given by $\bar{\nu}(s) = u\nu(s)$ works for $(A, \sqrt{\mathfrak{a}})$. If (ii) holds and $\bar{\nu}$ is the associated CM-function then the function ν given by $\nu(s) = \bar{\nu}(su)$ works. Indeed, let M, N be two MCM A-modules, $s \in \mathbf{N}$ and $\varphi: M \to N/\mathfrak{a}^{\nu(s)}N$ a linear A-map. Then there exists a linear map $\psi: M \to N$ such that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, dashed, "\psi"'] & N/\mathfrak{a}^{\bar{\nu}(su)}N \arrow[r] & N/(\sqrt{\mathfrak{a}})^{\bar{\nu}(su)}N \arrow[r] \arrow[d] & N/\mathfrak{a}^{s}N \arrow[d, equal] \\
N \arrow[rr] & & N/(\sqrt{\mathfrak{a}})^{su}N \arrow[r] & N/\mathfrak{a}^{s}N
\end{tikzcd}$$

(4.2) **Lemma.** *Let* $A \to B$ *be a flat local morphism of CM-local rings and* $\mathfrak{a} \subset A$ *an ideal. If* $(B, \mathfrak{a}B)$ *is a CM-approximation then* $(A, \mathfrak{a})$ *is too.*

Proof. We claim that the CM-function ν associated to $(B, \mathfrak{a}B)$ works also for $(A, \mathfrak{a})$. Indeed, let M, N be two MCM A-modules, $s \in \mathbf{N}$ and $\varphi: M \to N/\mathfrak{a}^{\nu(s)}N$ a linear A-map. Then $\overline{M} = B \otimes_A M$, $\overline{N} = B \otimes_A N$ are MCM B-modules since by flatness

$$\operatorname{depth} \overline{M} = \operatorname{depth}_A M + \operatorname{depth}(B/\mathfrak{m}B) = \operatorname{depth} A + \operatorname{depth}(B/\mathfrak{m}B) = \operatorname{depth} B$$

where $\mathfrak{m}$ denotes the maximal ideal of A (see e.g. [M_2, (23.3)]). Thus there exists a linear B-map $\bar{\psi}: \overline{M} \to \overline{N}$ such that the following diagram commutes:

$$\begin{tikzcd}
\overline{M} \arrow[r, "B\otimes_A\varphi"] \arrow[d, dashed, "\bar{\psi}"'] & \overline{N}/\mathfrak{a}^{\nu(s)}\overline{N} \arrow[d] \\
\overline{N} \arrow[r] & \overline{N}/a^{s}\overline{N}
\end{tikzcd}$$

Since M, N are finitely generated modules, the existence of $\bar{\psi}: \overline{M} \to \overline{N}$ such that the above diagram commutes means in other words that a certain linear system of equations L over A has a solution in B. Indeed, let $M \cong A^n/(z_1, \ldots, z_e)$, $z_i = (z_{ij})_{1 \le j \le n}$, $N = A^{n'}/(z'_1, \ldots, z'_{e'})$, $z'_\lambda = (z'_{\lambda\mu})_{1 \le \mu \le n'}$, $a = (a_1, \ldots, a_v)$ a system of generators of a^s and φ is given by the matrix

$(w_{j\mu})_{1\le j\le n,\ 1\le\mu\le n'}$. Then L has the following form:

$$\sum_{j=1}^{n} z_{i\eta}X_{\eta\mu} = \sum_{\lambda=1}^{e'} Y_{i\lambda}z'_{\lambda\mu}, \qquad 1 \le i \le e,\ 1 \le \mu \le n',$$

$$X_{j\mu} - w_{j\mu} = \sum_{\alpha=1}^{v} a_\alpha U_{\alpha j\mu} + \sum_{\lambda=1}^{e'} Y'_{j\lambda}z'_{\lambda\mu}, \qquad 1 \le j \le n.$$

Clearly $\bar{\psi}$ gives a solution of L in B. By faithfully flatness L also has a solution $(x_{j\mu}, y_{i\lambda}, y'_{j\lambda}, u_{\alpha j\mu})$ in A and the matrix $(x_{j\mu})$ defines a map $\psi\colon M \to N$ such that a diagram as above commutes. □

(4.3) **Proposition.** *Let $(A, \mathfrak{m})$ be an excellent local CM-ring, $p := \operatorname{char}(A/\mathfrak{m})$, and $I_s(A)$ the ideal defining the singular locus of A. Suppose that*

(i) *for every $q \in \operatorname{Reg} A$ containing pA the ring A_q/pA_q is regular,*
(ii) *there exists a flat, reduced noetherian complete local A-algebra $(B, \mathfrak{n})$ such that*
(ii$_1$) *$(B, \mathfrak{n})$ is CM and its residue field K is perfect,*
(ii$_2$) *for every $q \in \operatorname{Reg} A$ the map $A_q \to A_q \otimes_A B$ is regular,*
(iii) *$I_s(A) \subseteq \mathfrak{m}$.*

Then $(A, I_s(A))$ is a CM-approximation.

Proof. Let $q' \in \operatorname{Spec} B$ and $q := q' \cap A$. If $q \in \operatorname{Reg} A$ then $A_q \to B_{q'}$ is regular by (ii$_2$) and so $q' \in \operatorname{Reg} B$. Thus if $q' \not\supset I_s(A)$ then $q' \not\supset I_s(B)$, i.e. $I_s(B) \supset I_s(A)B$. Moreover $I_s(B) = \sqrt{I_s(A)B}$ by Lemma (2.1).

If q' contains pA then A_q/pA_q is regular (see (i)). Since $A_q/pA_q \to B_{q'}/pB_{q'}$ is regular by base change we get $B_{q'}/pB_{q'}$ regular too. Applying Proposition (3.7) to $(B, \mathfrak{n})$ we note that $(B, I_s(B))$ is a CM-approximation. By Lemma (4.1) $(B, I_s(A)B)$ is a CM-approximation and so $(A, I_s(A))$ is too (see Lemma (4.2)). □

(4.4) **Theorem.** *Let $(A, \mathfrak{m})$ be a reduced excellent local CM-ring, $k := A/\mathfrak{m}$, $p := \operatorname{char} k$ and $I_s(A)$ the ideal defining the singular locus of A. Suppose that*

(i) *$[k : k^p] < \infty$ if $p > 0$,*
(ii) *for every $q \in \operatorname{Reg} A$ containing pA the ring A_q/pA_q is regular,*
(iii) *$I_s(A) \subseteq \mathfrak{m}$.*

Then $(A, I_s(A))$ is a CM-approximation.

Proof. If k is perfect then apply Proposition (4.3) for $B = \hat{A}$ the completion of $(A, \mathfrak{m})$ (the map $A \to \hat{A}$ is regular because A is excellent and $\hat{A}$ is reduced because A is so).

If k is not perfect let $K = k^{1/p^\infty}$ and let P be its prime subfield. Then from the exact sequence

$$\Gamma_{K/P} = 0 \to \Gamma_{K/k} \to \Omega_{k/P} \otimes_k K$$

we get $\operatorname{rank}_K \Gamma_{K/k} \leq \operatorname{rank}_k \Omega_{k/P} = \operatorname{rank}_k \Omega_{k/k^p} < \infty$, where $\Gamma_{K/k}$ denotes the imperfection module [M_1, (39.B)].

Using [EGA, (22.2.6)] or [NP, Corollary (3.6)] there exists a formally smooth Noetherian complete local A-algebra $(B, \mathfrak{n})$ such that

(1) $B/\mathfrak{n} \cong K$,

(2) $\dim B = \dim A + \operatorname{rank}_K \Gamma_{K/k}$.

Then the structural morphism $A \to B$ is regular by André-Radu's Theorem (see [An, BR_1, BR_2]) because A is excellent. Moreover B is a reduced CM-ring by [M_1, (33.B)]. Now apply Proposition (4.3).

(4.5) **Lemma.** *Let (A, m) be a Noetherian henselian local ring and $\mathfrak{a} \subset A$ an ideal. Suppose that $(A, \mathfrak{a})$ is a CM-approximation. Let $\nu: \mathbf{N} \to \mathbf{N}$ be its CM-function and $r = \nu(1)$. Then an MCM A-module M is indecomposable iff $M/\mathfrak{a}^r M$ is indecomposable over $A/\mathfrak{a}^r$.*

Proof (inspired by [Y, (2.10)]). Clearly $M/\mathfrak{a}^r M$ is decomposable if M is so (use Nakayama's Lemma). If M is indecomposable then $\operatorname{End}_A(M)$ is a local A-algebra, A being henselian. Let f be an idempotent from $\operatorname{End}_A(M/\mathfrak{a}^r M)$. Then there exists a linear A-map $g: M \to M$ such that $\bar{g} := (A/\mathfrak{a}) \otimes g = (A/\mathfrak{a}) \otimes f$ (ν is a CM-function). Clearly $\bar{g}$ is an idempotent. Since $\operatorname{End}_A(M)$ is local the sub-A-algebra

$$\{(A/\mathfrak{a}) \otimes h | h \in \operatorname{End}_A(M)\} \subset \operatorname{End}_A(M/\mathfrak{a}M)$$

is local too. Thus $\bar{g} = 0$ or $\bar{g} = 1$. Then $\mathfrak{a} \cdot (M/a^r M)$ contains either $\operatorname{Im} f$ or $\operatorname{Im}(1 - f)$. Since f is idempotent we get either $\operatorname{Im} f = \operatorname{Im} f^r = 0$ or $\operatorname{Im}(1 - f) = \operatorname{Im}(1 - f)^r = 0$. Thus $f = 0$ or $f = 1$. □

(4.6) **Lemma.** *Conserving the hypothesis and the notations from* (4.5), *let M, N be two MCM A-modules such that M (resp. N) is indecomposable and $h: M \to N$ is a linear A-map. Suppose that $(A/\mathfrak{a}^r) \otimes_A h$ has a retraction (resp. section). Then h has a retraction (resp. section).*

Proof. Since $(A, \mathfrak{a})$ is a CM-approximation there exists a linear A-map $g: N \to M$ such that $(A/\mathfrak{a}) \otimes g$ is a retraction (resp. section) of $(A/\mathfrak{a}) \otimes h$. Then $\operatorname{Im}(1 - gh) \subseteq \mathfrak{a}M$ (resp. $\operatorname{Im}(1 - hg) \subseteq \mathfrak{a}N$). Since $\operatorname{End}_A(M)$ (resp. $\operatorname{End}_A(N)$) is a local ring we get $gh = 1 - (1 - gh)$ (resp. hg) bijective. Thus h has a retraction $(gh)^{-1} g$ (resp. a section).

(4.7) Let $\mathfrak{b}$ be an ideal in a Noetherian local ring $(B, \mathfrak{n})$. Then $\mathfrak{b}$ is a CM-*reduction ideal* if the following statements hold:

(i) An MCM B-module M is indecomposable iff $M/\mathfrak{b}M$ is indecomposable over $B/\mathfrak{b}$.

(ii) Two indecomposable MCM B-modules M, N are isomorphic iff $M/\mathfrak{b}M$ and $N/\mathfrak{b}N$ are isomorphic over $B/\mathfrak{b}$.

Note that our CM-reduction ideal is not necessarily $\mathfrak{n}$-primary as in [D]. If $\mathfrak{b}$ is a CM-reduction ideal of B then $\mathfrak{b}^s$ is also one for every $s \in \mathbf{N}$.

(4.8) **Theorem.** *Let* $(A, \mathfrak{m})$ *be a reduced excellent henselian local CM-ring,* $k := A/\mathfrak{m}$, $p := \operatorname{char} k$ *and* $I_s(A)$ *the ideal defining the singular locus of* A. *Suppose that*

(i) $[k : k^p] < \infty$ *if* $p > 0$,
(ii) *for every* $q \in \operatorname{Reg} A$ *containing* pA *the ring* A_q/pA_q *is regular,*
(iii) $I_s(A) \subseteq \mathfrak{m}$.

Then there exists a positive integer r *such that* $I_s(A)^r$ *is a CM-reduction ideal of* A.

The proof follows from Lemmas (4.5) and (4.6).

Let n_A be the cardinal of the *isomorphism classes of indecomposable MCM A-modules.*

(4.9) **Corollary.** *Conserving the notations and hypothesis of Theorem* (4.8) *let* B *be the completion of* A *with respect to* $I_s(A)$. *Then*

(i) *An MCM* A*-module* M *is indecomposable iff* $B \otimes_A M$ *is an indecomposable MCM* B*-module*

(ii) *Two indecomposable MCM* B*-modules* M, N *are isomorphic iff* $B \otimes_A M$, $B \otimes_A N$ *are isomorphic over* B.

In particular $n_A \leq n_B$.

Proof. (i) By Theorem (4.8) there exists $r \in \mathbf{N}$ such that $I_s(A)^r$ is a CM-reduction ideal of A. Let M be an indecomposable MCM A-module. Then $B \otimes_A M$ is an MCM B-module by flatness and $\overline{M} := M/I_s(A)^r M$ is indecomposable over $\overline{A} := A/I_s(A)^r$. Since $\overline{A} \cong B/I_s(A)^r B$ it follows that $(B \otimes_A \overline{A}) \otimes_A M$ is indecomposable over $B \otimes_A \overline{A}$ and so $B \otimes_A M$ is indecomposable too.

Conversely if $B \otimes_A M$ is an indecomposable MCM B-module and x is a system of parameters in A then x is a $(B \otimes_A M)$-regular sequence. Since $A \to B$ is faithfully flat it follows that x is an M-regular sequence, i.e. M is an MCM A-module. Clearly M must be indecomposable because $B \otimes_A M$ is so.

(ii) If $B \otimes_A M \cong B \otimes_A N$ then $B \otimes_A \overline{M} \cong B \otimes_A \overline{N}$ and so $\overline{A} \otimes_A M \cong \overline{A} \otimes_A N$. Thus $M \cong N$ because $I_s(A)^r$ is a CM-reduction ideal of A. □

(4.10) **Corollary.** *Let* $(A, \mathfrak{m})$ *be a reduced excellent henselian local CM-ring,* $k := A/\mathfrak{m}$ *and* $p := \operatorname{char} k$. *Suppose that*

(i) A *is an isolated singularity, i.e.* $\mathfrak{m} = I_s(A)$,
(ii) $[k : k^p] < \infty$ *if* $p > 0$,
(iii) *for every* $q \in \operatorname{Reg} A$ *containing* pA *the ring* A_q/pA_q *is regular.*

Then $n_A \leq n_{\widehat{A}}$, *where* $\widehat{A}$ *is the completion of* $(A, \mathfrak{m})$.

Remark. When k is perfect and $pA = 0$ then Corollary (4.10) is an easy consequence of [Y, (2.10), (2.12); Po, (1.3)] (see our (5.6)).

(4.11) **Corollary.** *Let* $(A, \mathfrak{m})$ *be a reduced excellent henselian Gorenstein local ring. Suppose that*

(i) A *is an isolated singularity of equal characteristic,*
(ii) $k := A/\mathfrak{m}$ *is algebraically closed and* $\operatorname{char} k \neq 2$,
(iii) *the completion* $\widehat{A}$ *is a simple hypersurface singularity.*

Then A *is of finite CM-type, i.e.* A *has just a finite set of isomorphic classes of indecomposable MCM* A*-modules.*

The proof follows by [GK, K, BGS] and our (4.10).

5. Bounded multiplicity CM-type

(5.1) This section is devoted to an extension of (1.2). First we list some definitions and facts from the Auslander-Reiten theory for the MCM modules (see [Au_3, AR_1, P, Ya; Y Appendix]).

Let $(A, \mathfrak{m})$ be a henselian local CM-ring and M, N two indecomposable MCM A-modules. A linear A-map $f\colon M \to N$ is *irreducible* if f is not an isomorphism and given any factorization $f = gh$ in the category $\mathrm{CM}(A)$, g has a section or h has a retraction. The AR-*quiver of* A is a directed graph which has as vertices the isomorphic classes of indecomposable MCM modules over A and there is an arrow from the isomorphic class of M to that of N provided there is an irreducible linear map from M to N. A *chain of irreducible maps* from M to N is a sequence of irreducible linear maps:

$$M_0 \xrightarrow{f_1} M_1 \to \cdots \xrightarrow{f_n} M_n$$

with all M_i indecomposable MCM A-modules; n is called the *length of the chain.* If A is an isolated singularity then the AR-graph of A is *locally finite,* i.e. each vertex may be incident to only a finite number of other vertices (see [Au_2, AR_2, Y, (A.18)])

The following two lemmas are just variants of [Y, Lemmas (3.1), (3.2)] or [D, §1].

(5.2) **Lemma** (Harada-Sai lemma for MCM-modules). *Let* n *be a positive integer,* M_i, $0 \leq i \leq 2^n$, *some indecomposable MCM* A*-modules and* $f_i\colon M_{i-1} \to M_i$, $1 \leq i \leq 2^n$, *some nonisomorphic linear* A*-maps. Suppose that*

(i) $\mathfrak{m}^r$ *is a CM-reduction ideal of* A *for a certain* $r \in \mathbf{N}$,
(ii) $\operatorname{length}(M_i/\mathfrak{m}^r M_i) \leq n$, $0 \leq i \leq n$.

Then $(A/\mathfrak{m}^r) \otimes (f_{2^n} \circ \cdots \circ f_1) = 0$.

The proof follows easily from [HS, Lemma 12] and our Lemma (4.6).

(5.3) **Lemma.** *Let* n *be a positive integer,* M, N *two indecomposable MCM* A*-modules and* $\varphi\colon M \to N$ *a linear* A*-map. Suppose that*

(1) $\mathfrak{m}^r$ *is a CM-reduction ideal of* A *for a certain* $r \in \mathbf{N}$,
(2) $(A/\mathfrak{m}^r) \otimes \varphi \neq 0$,

(3) *there is no chain of irreducible maps from* M *to* N *of length* $< n$ *which is nontrivial modulo* $\mathfrak{m}^r$.

Then

(i) *There exist a chain of irreducible maps*

$$M = M_0 \xrightarrow{f_1} M_1 \to \cdots \xrightarrow{f_n} M_n$$

and a linear A*-map* $g\colon M_n \to N$ *such that* $(A/\mathfrak{m}^r) \otimes (g \circ f_n \circ \cdots \circ f_1) \neq 0$.

(ii) *There exist a chain of irreducible maps*

$$N_n \xrightarrow{g_n} N_{n-1} \to \cdots \xrightarrow{g_1} N_0 = N$$

and a linear A*-map* $f\colon M \to N_n$ *such that* $A/\mathfrak{m}^r \otimes (g_1 \circ \cdots \circ g_n \circ f) \neq 0$.

The proof follows as in [Y, (3.2)].

(5.4) **Theorem.** *Let* $(A, \mathfrak{m})$ *be a reduced excellent henselian local CM-ring,* $k := A/\mathfrak{m}$, $p := \operatorname{char} k$, Γ *the AR-quiver of* A *and* Γ^0 *a connected component of* Γ. *Suppose that*

(i) A *is an isolated singularity,*

(ii) Γ^0 *is of bounded multiplicity type, i.e. there exists* $n \in \mathbf{N}$ *such that all indecomposable MCM modules* M *whose isomorphic classes are vertices in* Γ^0 *hold,* $e(M) \leq n$.

(iii) $[k : k^p] < \infty$ *if* $p > 0$,

(iv) *for every* $q \in \operatorname{Reg} A$ *containing* pA *the ring* A_q/pA_q *is regular.*

Then $\Gamma = \Gamma^0$ *and* Γ *is a finite graph. In particular* A *is of finite CM-type.*

Proof (inspired from [Y, (3.3)]). By (i) we have $I_s(A) = \mathfrak{m}$ and it follows that $\mathfrak{m}^r$ is a CM-reduction ideal of A for a certain $r \in \mathbf{N}$ (see Theorem (4.8)). Let x be a system of parameters of A and M an MCM A-module. By [M_2, (14.11)] we have

$$\operatorname{length}_A(M/xM) = e(xA, M)$$

because x is an M-regular sequence. Let $u \in \mathbf{N}$ be such that $\mathfrak{m}^u \subseteq xA$. By [$M_2$, (14.3), (14.4)] we get

$$e(xA, M) \leq e(\mathfrak{m}^u, M) = e(M)u^d,$$

where $d = \dim A$. Choosing x in $\mathfrak{m}^r$ it follows that

$$\operatorname{length}_A(M/\mathfrak{m}^r M) \leq e(M)u^d. \tag{1}$$

Let $\mathscr{M}$ be the class of all MCM A-modules whose isomorphic classes are vertices in Γ^0. Using (1) we get

$$\operatorname{length}_A(M/\mathfrak{m}^r M) \leq s = nu^d = \text{constant for every } M \in \mathscr{M}. \tag{2}$$

Let M, N be two indecomposable MCM A-modules and $f\colon M \to N$ a linear A-map such that $(A/\mathfrak{m}^r) \otimes f \neq 0$. If $M \in \mathscr{M}$ then there is a chain of

irreducible maps from M to N of length $< t := 2^s$ which is nontrivial modulo $\mathfrak{m}^r$. Otherwise there exists a chain of irreducible maps as in Lemma (5.3)(i)

$$M = M_0 \xrightarrow{f_1} M_1 \to \cdots \xrightarrow{f_t} M_t$$

and a linear A-map $g\colon M_t \to N$ such that $A/\mathfrak{m}^r \otimes (g \circ f_t \circ \cdots \circ f_1) \neq 0$. Then $A/\mathfrak{m}^r \otimes (f_t \circ \cdots \circ f_1) \neq 0$ which contradicts Lemma (5.2) (the M_i are all in Γ^0 because Γ^0 is conex and apply (2)). In particular we get $N \in \mathscr{M}$. Conversely if $N \in \mathscr{M}$ then a dual argument (using (5.3)(ii) instead of (i)) shows that $M \in \mathscr{M}$ and there exists a nontrivial chain of irreducible maps from M to N of length $< t$.

If M is a finitely generated A-module there exists a linear A-map $f\colon A \to M$ such that $(A/\mathfrak{m}^r) \otimes f \neq 0$ (choose $x \in M \backslash \mathfrak{m}M$ and take $f(a) = ax$). If $M \in \mathscr{M}$ then $A \in \mathscr{M}$. Moreover if M is an indecomposable A-module then $M \in \mathscr{M}$ because $A \in \mathscr{M}$. Thus $\Gamma^0 = \Gamma$. Since Γ is locally finite and every module from $\mathscr{M}$ can be connected with A by a chain of irreducible maps of length $< t$ we conclude that Γ is finite. □

(5.5) *Remark.* When A is Artinian then our theorem is a consequence of [R, Au_1]. When A is complete, $pA = 0$ and k is perfect then our theorem follows from [Y, (1.1)].

(5.6) *Remark.* Another possible approach to study the CM-type is to use Artin approximation theory (see [Ar, Po]). Let $(A, \mathfrak{m})$ be a Noetherian local ring with the property of Artin approximation (shortly A is an AP-ring), i.e. for every finite system of polynomial equations f over A, every $s \in \mathbf{N}$ and every formal solution $\hat{y}$ of f in the completion $\hat{A}$ of A there exists a solution y of f in A such that $y \equiv \hat{y} \bmod \mathfrak{m}^s\hat{A}$. Let M, N be two finitely generated A-modules. If A is an AP-ring then

(i) M is indecomposable iff $\hat{A} \otimes_A M$ is so,

(ii) $M \cong N$ iff $\hat{A} \otimes_A M \cong \hat{A} \otimes_A N$.

For the proof note that the question can be expressed by the compatibility of some systems of polynomial equations over A (as in the proof of (4.2); but this time the equations are not linear). In particular the CM-type of A is finite if the CM-type of $\hat{A}$ is so. Since excellent henselian local rings are AP (see [Po, Theorem (1.3)] we note that our Theorem (5.4) follows from [Y, (1.1)] when k is perfect and $pA = 0$.

Added in proof. The inequalities from (4.9) and (4.10) are in fact equalities by Elkik's algebraization theorem and (4.11) holds also when $\operatorname{char} k = 2$ (for details see, INCREST Preprint 57/1988).

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