

CIRCUIT PARTITIONS AND THE HOMFLY POLYNOMIAL OF CLOSED BRAIDS

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ABSTRACT. We present an expansion of the homfly polynomial $P(D, z, a)$ of a braid diagram D in terms of its circuit partitions. Another aspect of this result is an expression of $P(D, z, a)$ as the trace of a matrix associated to D in a simple way. We show how certain degree properties of the homfly polynomial can be derived easily from this model. In particular we obtain that if D is a positive braid diagram on n strings with w crossings, the maximum degree of $P(D, z, a)$ in the variable a equals $n - 1 - w$.

RESUME. Nous présentons une expansion pour le polynôme homfly $P(D, z, a)$ d'un diagramme de tresse D en termes de ses partitions en circuits. Un autre aspect de ce résultat consiste en une expression de $P(D, z, a)$ comme trace d'une matrice associée de façon simple à D . Nous montrons comment certaines propriétés de degré du polynôme homfly dérivent simplement de ce modèle. En particulier nous obtenons que pour un diagramme de tresse positif D à n brins et w croisements, le degré maximum de $P(D, z, a)$ en la variable a est égal à $n - 1 - w$.

1. INTRODUCTION

The homfly polynomial of an oriented link, whose discovery is presented in [7 and 22], can be defined inductively in a purely combinatorial way on link projections or *diagrams*. See [9, 18, 22]. This raises the question of the existence of a *combinatorial model* for this polynomial, similar to the models obtained by Kauffman for the special cases of the Alexander-Conway polynomial [14] and the Jones polynomial [15]. Such a model should lead to new, efficient proof techniques. This was already the case with Kauffman's "state model" for the Jones polynomial, as illustrated in [15, 21, and 23].

Important progress was made with the discovery of "state models" for an infinite sequence of one-variable specializations of the homfly polynomial [13, 16, 17, 24]. A state model valid only for braid diagrams was also described in [10]. We present here another state model for the homfly polynomial of braid diagrams that appears to be simpler than the previous model of [10], and that yields straightforward explanations for some degree properties such as the

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Morton-Franks-Williams inequalities [6, 19]. Quite recently, a model for the homfly polynomial of arbitrary diagrams that is similar to the one presented here was obtained in [11].

In §2 we introduce the basic concepts and terminology relative to link diagrams and the homfly polynomial. We present our main result in §3.1. This result is then reformulated in matrix terms in §3.2. Both formulations are used in the proof given in §3.3. Finally, some consequences of the model are derived in §4. For instance, for positive braids we show that equality holds in one of the Morton-Franks-Williams inequalities and we give a simple combinatorial interpretation of the coefficients of the Alexander-Conway polynomial.

2. BASIC CONCEPTS

2.1. Links, braids, and their diagrams. For the usual terminology of Graph Theory and Knot Theory, the reader is referred to [2–4 and 14]. By *link* we mean a tame, oriented link in the 3-dimensional Euclidean space. Links are represented by regular plane projections, which we call *diagrams*. We shall consider diagrams as 4-regular directed plane graphs whose vertices are signed. More precisely, a diagram is a triple $D = (V, E, s)$, where

- Some subset E' of E is the edge-set of a plane directed graph G in the usual sense (allowing loops and parallel edges) with vertex-set V .
- $E - E'$ is a set of simple, oriented, closed Jordan curves in the plane that are disjoint from G and mutually disjoint. We call these curves *free loops*. We consider (V, E) as a graph with vertex-set V and edge-set E , so that free loops are considered as edges without end-vertices.
- For each vertex v there are exactly two edges with terminal end v (they will be said *incident into* v) and exactly two edges with initial end v (they will be said *incident from* v). Moreover, the two edges incident into v form an angle at v .
- s is a mapping from V to $\{1, -1\}$ that associates to every vertex its *sign*. The sign mapping describes the crossing structure of the link represented by D as shown in Figure 1.

The *writhe* of a diagram D , denoted by $w(D)$, is the sum of the signs of its vertices.

Two diagrams will be said to be *isotopic* if they represent the same link up to ambient isotopy. The isotopy relation on diagrams is described combinatorially by Reidemeister's Theorem (see for instance, [4, Chapter 1]).

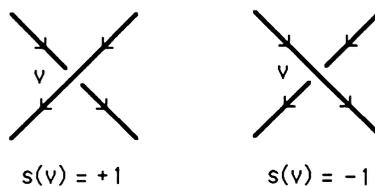


FIGURE 1

Let us consider an infinite alphabet $\{s_i, i \in \mathbb{N} - \{0\}\}$. Artin's braid group on n strings B_n ($n \geq 1$) is defined by the presentation

$$\langle s_1, \dots, s_{n-1} \mid s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, i = 1, \dots, n - 2; s_i s_j = s_j s_i, |i - j| \geq 2 \rangle.$$

Thus, B_n is the subgroup of B_{n+1} generated by s_1, \dots, s_{n-1} (which is trivial for $n = 1$). We call *braid* (on n strings) a pair (b, n) , where b belongs to B_n . We introduce a special symbol ε to represent the identity element of the groups B_n . We call a *braid word* on n strings any nonempty word on the alphabet $\{\varepsilon\} \cup \{s_i, s_i^{-1} \mid i = 1, \dots, n - 1\}$. Note that a braid word on n strings is also a braid word on n' strings for all $n' \geq n$. To every braid word $m = m_1 \cdots m_k$ of length k on n strings we associate a *braid diagram* on n strings as follows. For $i = 0, \dots, k$ and $j = 1, \dots, n$ let $p_{i,j}$ be the point in the (x, y) -plane with coordinates $(j, -i)$. To each letter m_i ($i = 1, \dots, k$) corresponds a portion of diagram situated in the rectangle with corners $p_{i-1,1}, p_{i-1,n}, p_{i,1}, p_{i,n}$ according to the rule described in Figure 2.

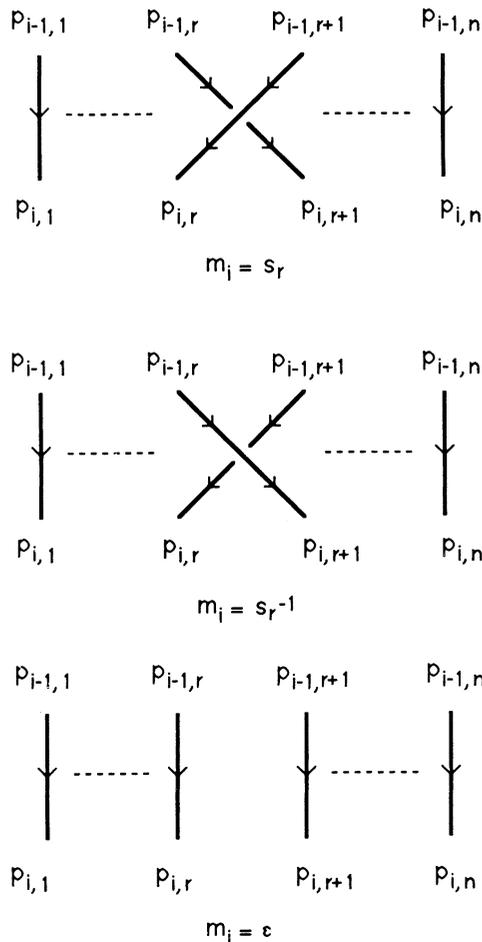


FIGURE 2

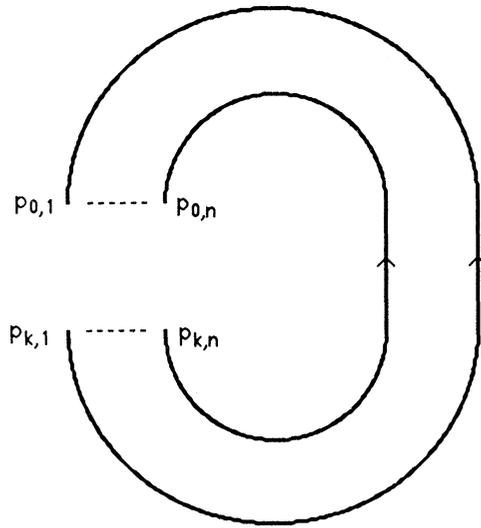


FIGURE 3

Now for each $j = 1, \dots, n$ we join the points $p_{k,j}$ and $p_{0,j}$ as shown in Figure 3 to form a diagram D . The edge of D which contains the points $p_{k,j}$ and $p_{0,j}$ will be called a *return edge* and will be assigned the *index* j .

Let (b, n) be a braid on n strings. Let us choose a braid word on n strings that represents b . The associated braid diagram on n strings will be called a *closure* of (b, n) . It follows from the definition of the braid group B_n that any two closures of (b, n) are isotopic.

There exists a homomorphism w from B_n to the additive group of integers such that $w(s_r) = 1$ and $w(s_r^{-1}) = -1$ ($r = 1, \dots, n-1$). Clearly if D is a closure of (b, n) , $w(b)$ equals $w(D)$, the writhe of D .

Two basic results on braid diagrams are the following.

Alexander's Theorem. *Every diagram is isotopic to some braid diagram.*

Markov's Theorem. *Two braids have isotopic closures if and only one can be obtained from the other by a finite sequence of moves of one of the following types:*

- *Markov move of type 1: Replace (b, n) by (cbc^{-1}, n) (for b, c in B_n).*
- *Markov move of type 2: For b in B_n , replace (b, n) by $(bs_n, n+1)$ or $(bs_n^{-1}, n+1)$, or perform the converse operation.*

2.2. The homfly polynomial. If D^+ , D^- , and D° are diagrams that are identical outside a small disk and behave as depicted in Figure 4 inside the disk, we shall say that (D^+, D^-, D°) form a *Conway triple*.

The following result appears in [7, 9, 18, and 22]; see also [8, 12].

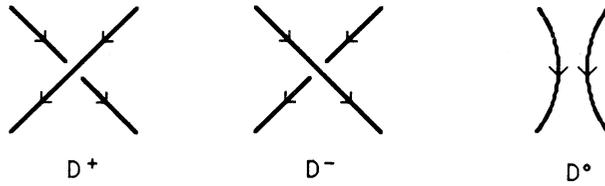


FIGURE 4

Proposition 1. *One can associate (in a unique way) to every diagram D a Laurent polynomial with integer coefficients in two variables, z and a , which we denote by $P(D, z, a)$ in such a way that the following properties hold:*

- (i) *If D and D' are isotopic, $P(D, z, a) = P(D', z, a)$.*
- (ii) *If (D^+, D^-, D°) form a Conway triple then*

$$aP(D^+, z, a) - a^{-1}P(D^-, z, a) = zP(D^\circ, z, a).$$

- (iii) *If D consists of a single free loop, $P(D, z, a) = 1$.*

The polynomial P is usually called the *homfly polynomial* or the *Jones-Conway polynomial*. We shall use the first terminology here.

Proposition 1 can be proved purely combinatorially by working in the class of all diagrams (see [9, 18, 22]). It can also be proved by working in the restricted class of braid diagrams, which allows the efficient use of algebraic methods (see [8, 12, 25]). We now present a simple consequence of this second approach.

In the sequel we denote by K the ring of Laurent polynomials with integer coefficients in two variables, z and a .

Proposition 2. *For every positive integer n let F_n be a mapping from B_n to K and assume that the following properties hold:*

- (i) *For all b, b' in B_n , $F_{n+1}(bs_n b') = F_{n+1}(bs_n^{-1} b') = F_n(bb')$.*
- (ii) *For all b, b' in B_n and $i = 1, \dots, n-1$, $aF_n(bs_i b') - a^{-1}F_n(bs_i^{-1} b') = zF_n(bb')$.*
- (iii) $F_1(1) = 1$.

Then for every positive integer n and every closure D of the braid (b, n) , $F_n(b) = P(D, z, a)$.

Proof. For every $n \geq 1$ and b in B_n , let D be a closure of (b, n) and let $F'_n(b) = P(D, z, a)$. Since any two closures of (b, n) are isotopic, it follows from property (i) of Proposition 1 that $F'_n(b)$ does not depend on the choice of D . Thus, we have defined for every $n \geq 1$ a mapping F'_n from B_n to K . For all b, b' in B_n , $(bs_n b', n+1)$ and $(bs_n^{-1} b', n+1)$ can each be obtained from (bb', n) by one Markov move of type 2 inserted between two Markov moves of type 1. It then follows from Markov's Theorem and from property (i) of Proposition 1 that F'_{n+1} and F'_n together satisfy property (i) of Proposition 2. Similarly for all $i = 1, \dots, n-1$, we may choose closures of $(bs_i b', n)$, $(bs_i^{-1} b', n)$ and (bb', n) that form a Conway triple. It follows from Property

(ii) of Proposition 1 that F'_n satisfies property (ii) of Proposition 2. Finally, $F'_1(1) = 1$ is an immediate consequence of property (iii) of Proposition 1.

For every $n \geq 1$, let H_n be the quotient of the group algebra $K(B_n)$ by the two-sided ideal J_n generated by the elements $as_i - a^{-1}s_i^{-1} - z$ ($i = 1, \dots, n-1$) (H_n is isomorphic to the formal Hecke algebra H_n of [25] and to the Hecke algebra $H(q, n)$ of [12], the difference consisting only of the choice of the parameters a and z). The inclusion of B_n in B_{n+1} extends to an inclusion of H_n in H_{n+1} . Let us extend F_n and F'_n by linearity to $K(B_n)$. Then by property (ii) of Proposition 2 which is valid for F_n and F'_n , the kernels of F_n and F'_n contain J_n . Thus F_n and F'_n pass to the quotient and will be identified with the corresponding mappings from H_n to K .

We now prove by induction on n that $F_n = F'_n$. This is true for $n = 1$. Assume now that the result holds for the integer $n \geq 1$. We shall use the following classical result: H_{n+1} is generated as a K -algebra by $G_{n+1} = H_n \cup \{xs_n y/x, y \in H_n\}$ (see, for instance, [8, 12, 25]). Thus, it is enough to show that F_{n+1} and F'_{n+1} coincide on G_{n+1} . For x, y in H_n , it follows from property (i) of Proposition 2 and from the induction hypothesis that $F_{n+1}(xs_n y) = F_n(xy) = F'_n(xy) = F'_{n+1}(xs_n y)$. Finally properties (ii) and (i) yield $zF_{n+1}(x) = aF_{n+1}(xs_n) - a^{-1}F_{n+1}(xs_n^{-1}) = aF_n(x) - a^{-1}F_n(x)$ and the similar equality with F' instead of F . Then the equality $F_{n+1}(x) = F'_{n+1}(x)$ follows by induction. \square

3. A STATE MODEL FOR THE HOMFLY POLYNOMIAL OF BRAID DIAGRAMS

3.1. Admissible circuit partitions. Let σ be a permutation of a finite set X . The number of cycles of σ will be denoted by $c(\sigma)$. Let X' be a subset of X that contains at least one element of each cycle of σ . Such a subset X' together with a total ordering of its elements will be called an *ordered transversal* of σ . The *standard word* of σ with respect to its ordered transversal X' is obtained as follows. For each cycle C of σ we denote by $g(C)$ the minimum element of $X' \cap C$. Then we associate to C the word $W(C) = x_1 \cdots x_{|C|}$, where $x_i = \sigma^{i-1}(g(C))$ for $i = 1, \dots, |C|$. Now we denote the cycles of σ by $C_1, \dots, C_{c(\sigma)}$ in such a way that $g(C_i)$ is a smaller element of X' than $g(C_j)$ whenever $i < j$. The standard word of σ , denoted by $W(\sigma)$, is $W(C_1) \cdots W(C_{c(\sigma)})$.

Let D be a braid diagram and v be a vertex of D . Let us denote by e_1, e_2 the edges incident into v , and e_3, e_4 the edges incident from v as described in Figure 5. We call *left tangence at v* the ordered pair (e_1, e_3) and *right tangence at v* the ordered pair (e_2, e_4) . The pairs (e_1, e_4) and (e_2, e_3) will be called *crossings at v* . We call *passage* any ordered pair of edges that is either a tangence or a crossing at some vertex, or is of the form (e, e) for some free loop e .

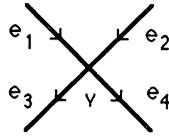


FIGURE 5

A *circuit partition* of D is a permutation π of its edge-set E such that for every edge e , $(e, \pi(e))$ is a passage. The cycles of π will be called *circuits*. A vertex v is a *tangence* (respectively, *crossing*) point of π if when e is an edge incident into v , $(e, \pi(e))$ is a tangence (respectively, crossing) at v . Note that a circuit partition is determined uniquely by the corresponding status of the different vertices (tangence or crossing point). We shall denote by $t(\pi)$ the number of tangence points of π and by $t'(\pi)$ the number of such points v with $s(v) = -1$.

The set of return edges of D , ordered by index, is an ordered transversal of every circuit partition. Thus, we may associate to every circuit partition π its standard word with respect to this ordered transversal, which we denote by $W(\pi)$. Then for a vertex v , let e be the leftmost letter of $W(\pi)$ that represents an edge incident into v . The *first passage* of π at v is $(e, \pi(e))$. The circuit partition π of D will be said to be *admissible* at the vertex v if $s(v) = 1$ and the first passage at v is not a right tangence, or $s(v) = -1$ and the first passage at v is not a left tangence. Finally, π will be said to be *admissible* if it is admissible at every vertex of D . We shall denote by $A(D)$ the set of admissible circuit partitions of D . Then we may state our main result as follows.

Proposition 3. *For any braid diagram on n strings D ,*

$$P(D, z, a) = a^{1-n-w(D)} \sum_{\pi \in A(D)} (-1)^{t'(\pi)} z^{t(\pi)} ((a^2 - 1)z^{-1})^{c(\pi)-1}.$$

Before proving this result, we shall introduce a reformulation that will be easier to study.

3.2. Labellings, states, and transfer matrices. Let $D = (V, E, s)$ be a braid diagram on n strings associated to the braid word on n strings $m = m_1 \cdots m_k$ as described in §2.1. Let $Q = \{p_{i,j} | i = 0, \dots, k; j = 1, \dots, n\}$. For $i = 0, \dots, k$ we denote by Q_i the set $\{p_{i,j} | j = 1, \dots, n\}$ and we define the *index* of $p_{i,j}$ as equal to j .

We define a *labelling* of D as a mapping f from Q to $\{1, \dots, n\}$ that satisfies the following properties:

- $f(Q_i) = \{1, \dots, n\}$ for $i = 0, \dots, k$.
- For $i = 1, \dots, k$, if m_i equals s_r or s_r^{-1} , $f(p_{i,j}) = f(p_{i-1,j})$ for $j \in \{1, \dots, n\} - \{r, r + 1\}$. If $m_i = \varepsilon$, $f(p_{i,j}) = f(p_{i-1,j})$ for $j = 1, \dots, n$.

We shall denote by $L(D)$ the set of labellings of D .

To every circuit partition π of D we associate a labelling f_π as follows. Assume that when we travel along the edge e of D from its initial end to its terminal end (or from an arbitrary point to the same point if e is a free loop), we encounter successively the points q_1, \dots, q_h of Q in this order. Then we denote by $q(e)$ the word $q_1 \cdots q_h$ on the alphabet Q . Let us define the permutation π' of Q , which will be said *associated to* π , as follows. Each letter of Q belongs to a unique word $q(e)$. Its image under π' is the next letter of $q(e)$ if it exists or the first letter of $q(\pi(e))$ otherwise. Clearly $c(\pi') = c(\pi)$. The set Q_0 , ordered by index, is an ordered transversal of π' . Let $W(\pi')$ be the standard word of π' with respect to this ordered transversal. This word has a unique factorization $Y_1 \cdots Y_n$ such that, for $j = 1, \dots, n$, the first letter of Y_j is in Q_0 , the last letter of Y_j is in Q_k , and the other letters of Y_j are neither in Q_0 nor in Q_k . Consider the mapping from Q to $\{1, \dots, n\}$ that for $j = 1, \dots, n$ assigns to every element of Y_j the value j . It is easy to check that this mapping is a labelling of D , which we denote by f_π .

Assume that m_i equals s_r or s_r^{-1} , and let v_i be the vertex of D corresponding to m_i . Clearly if v_i is a tangence (respectively, crossing) point of π , $(f_\pi(p_{i,r}), f_\pi(p_{i,r+1})) = (f_\pi(p_{i-1,r}), f_\pi(p_{i-1,r+1}))$ (respectively, $(f_\pi(p_{i,r}), f_\pi(p_{i,r+1})) = (f_\pi(p_{i-1,r+1}), f_\pi(p_{i-1,r}))$). It follows that the mapping $\pi \rightarrow f_\pi$ from the set of circuit partitions of D to $L(D)$ is one-to-one. For a labelling f let $\rho(f)$ be equal to zero if f is not in the image of this mapping, and to $((a^2 - 1)z^{-1})^{c(\pi)-1}$ if $f = f_\pi$. We call $\rho(f)$ the *return weight* of f .

We now observe that if the first passage of π at v_i is $(e, \pi(e))$, e contains the element of $\{p_{i-1,r}, p_{i-1,r+1}\}$ on which f_π takes the smallest value. This leads us to define the *interaction of the labelling f with the vertex v_i* , which we denote by $\langle v_i | f \rangle$, as follows. If $(f(p_{i,r}), f(p_{i,r+1})) = (f(p_{i-1,r+1}), f(p_{i-1,r}))$, then $\langle v_i | f \rangle = 1$. If $(f(p_{i,r}), f(p_{i,r+1})) = (f(p_{i-1,r}), f(p_{i-1,r+1}))$, we distinguish two cases.

- If $s(v_i) = 1$, $\langle v_i | f \rangle = z$ if $f(p_{i-1,r}) < f(p_{i-1,r+1})$ and $\langle v_i | f \rangle = 0$ if $f(p_{i-1,r}) > f(p_{i-1,r+1})$.
- If $s(v_i) = -1$, $\langle v_i | f \rangle = 0$ if $f(p_{i-1,r}) < f(p_{i-1,r+1})$ and $\langle v_i | f \rangle = -z$ if $f(p_{i-1,r}) > f(p_{i-1,r+1})$.

We write $\langle D | f \rangle = \prod_{i=1, \dots, k} \langle v_i | f \rangle$ and we set this product as equal to 1 if D has no vertices.

Now we may reformulate Proposition 3 as follows.

Proposition 4. *For any braid diagram on n strings D ,*

$$P(D, z, a) = a^{1-n-w(D)} \sum_{f \in L(D)} \langle D | f \rangle \rho(f).$$

Proof of the equivalence with Proposition 3. The mapping $\pi \rightarrow f_\pi$ defines a bijection from $A(D)$ to the set of labellings f with $\langle D | f \rangle \rho(f) \neq 0$. Moreover for every π in $A(D)$, $\langle D | f_\pi \rangle = (-1)^{t'(\pi)} z^{t(\pi)}$ and $\rho(f_\pi) = ((a^2 - 1)z^{-1})^{c(\pi)-1}$. \square

We now reformulate Proposition 4 in matrix terms.

We first show that the return weight of a labelling f is determined by the restrictions of f to Q_0 and Q_k . The labelling f being given, let π' be the permutation of Q that for $i = 0, \dots, k - 1$ associates to every element of Q_i the unique element of Q_{i+1} with the same f -value, and associates to every element of Q_k the unique element of Q_0 with the same index. Clearly π' is associated to a unique circuit partition π of D , which has the property that $\rho(f) \neq 0$ if and only if $f = f_\pi$. This will be the case if and only if f and f_π coincide on Q_0 . Now let σ be the permutation of Q_0 that for $j = 1, \dots, n$ associates to $p_{0,j}$ the unique point $p_{0,h}$ such that $f(p_{k,h}) = f(p_{0,j})$. Note that $c(\sigma) = c(\pi') = c(\pi)$. The set Q_0 ordered by index is an ordered transversal of σ . We denote by $W(\sigma)$ the standard word of σ with respect to this ordered transversal. Then it is easy to see that $W(\sigma)$ can be obtained from $W(\pi') = Y_1 \cdots Y_n$ by retaining only the first letter of each factor Y_j . Hence, denoting by $f(W(\sigma))$ the word obtained from $W(\sigma)$ by replacing each letter by its image under f , $\rho(f) = ((a^2 - 1)z^{-1})^{c(\sigma)-1}$ if $f(W(\sigma))$ is equal to the word $1 \cdots n$, and $\rho(f) = 0$ otherwise.

Let us call *state* of D a sequence $g = (g_i, i = 0, \dots, k)$, where each g_i is a permutation of $\{1, \dots, n\}$, which satisfies the following properties for $i = 1, \dots, k$:

- If $m_i = \varepsilon$, $g_i = g_{i-1}$.
- If m_i equals s_r or s_r^{-1} , $g_i = g_{i-1}$ or $g_i = g_{i-1} \circ \tau_r$, where τ_r is the transposition $(r, r + 1)$.

We denote by $S(D)$ the set of states of D . For $i = 0, \dots, k$ let β_i be the bijection from $\{1, \dots, n\}$ to $Q_i = \{p_{i,j} | j = 1, \dots, n\}$ defined by $\beta_i(j) = p_{i,j}$ ($j = 1, \dots, n$). Let f be a labelling of D . The sequence $g = (g_i, i = 0, \dots, k)$, where $g_i = f \circ \beta_i$, is easily seen to be a state that we denote by $\gamma(f)$. Clearly the mapping γ is a bijection from $L(D)$ to $S(D)$. Let g be a state of D . We have seen that $\rho(\gamma^{-1}(g))$ is determined by the restrictions of $\gamma^{-1}(g)$ to Q_0 and Q_k . Hence, it is determined by g_0 and g_k in a similar fashion. More precisely, we introduce a *return weight matrix* R_n (also denoted by R if the context is clear) whose rows and columns are indexed by the group S_n of permutations of $\{1, \dots, n\}$. Let λ, μ be two permutations of $\{1, \dots, n\}$. Consider the permutation $\nu = \lambda^{-1} \circ \mu$. Let $W(\nu)$ be the standard word of ν with respect to its ordered transversal $\{1, \dots, n\}$ (with its natural order). We set $R(\lambda, \mu) = ((a^2 - 1)z^{-1})^{c(\nu)-1}$ if $W(\nu)$ is equal to the word $\mu^{-1}(1) \cdots \mu^{-1}(n)$, and $R(\lambda, \mu) = 0$ otherwise. Then it is easy to check that for every state $g = (g_i, i = 0, \dots, k)$, $\rho(\gamma^{-1}(g)) = R(g_k, g_0)$.

Now for $r = 1, \dots, n - 1$ we introduce *transfer matrices* $T_{n,r}^+$ and $T_{n,r}^-$ (or, more briefly, T_r^+ and T_r^- if this notation creates no ambiguity) whose rows and columns are indexed by S_n . For λ and μ in S_n , the corresponding matrix elements $T_r^+(\lambda, \mu)$ and $T_r^-(\lambda, \mu)$ are defined as follows. If $\mu = \lambda \circ \tau_r$, then

$T_r^+(\lambda, \mu) = T_r^-(\lambda, \mu) = 1$. If $\mu = \lambda$ and $\lambda(r) < \lambda(r + 1)$ then $T_r^+(\lambda, \mu) = z$ and $T_r^-(\lambda, \mu) = 0$. If $\mu = \lambda$ and $\lambda(r) > \lambda(r + 1)$, then $T_r^+(\lambda, \mu) = 0$ and $T_r^-(\lambda, \mu) = -z$. In all other cases, $T_r^+(\lambda, \mu) = T_r^-(\lambda, \mu) = 0$.

Let now f be a labelling of D and let $\gamma(f) = g = (g_i, i = 0, \dots, k)$ be the corresponding state. It is easy to check that if m_i equals s_r (respectively, s_r^{-1}), denoting by v_i the vertex of D corresponding to m_i , the interaction $\langle v_i | f \rangle$ is equal to $T_r^+(g_{i-1}, g_i)$ (respectively, $T_r^-(g_{i-1}, g_i)$). Let I be the identity matrix with rows and columns indexed by S_n . For all $i = 1, \dots, k$, let M_i be the matrix I if $m_i = \varepsilon$, T_r^+ if $m_i = s_r$, T_r^- if $m_i = s_r^{-1}$. Then if $g = (g_i, i = 0, \dots, k)$ is a state, $\prod_{i=1, \dots, k} M_i(g_{i-1}, g_i) = \langle D | \gamma^{-1}(g) \rangle$. Moreover if the sequence $g = (g_i, i = 0, \dots, k)$, where the g_i belong to S_n , is not a state, $\prod_{i=1, \dots, k} M_i(g_{i-1}, g_i)$ equals 0. We now obtain the following matrix form of Proposition 4.

Proposition 5. *For any braid diagram on n strings D associated to the braid word on n strings $m = m_1 \cdots m_k$,*

$$P(D, z, a) = a^{1-n-w(D)} \text{Trace} \left(\left(\prod_{i=1, \dots, k} M_i \right) R \right).$$

Proof of the equivalence with Proposition 4. Since γ is a bijection from $L(D)$ to $S(D)$, writing $g = (g_i, i = 0, \dots, k)$ for every state g , we have

$$\begin{aligned} \sum_{f \in L(D)} \langle D | f \rangle \rho(f) &= \sum_{g \in S(D)} \langle D | \gamma^{-1}(g) \rangle \rho(\gamma^{-1}(g)) \\ &= \sum_{g \in S(D)} \left(\prod_{i=1, \dots, k} M_i(g_{i-1}, g_i) \right) R(g_k, g_0). \end{aligned}$$

The similar summation extended to all sequences $g = (g_i, i = 0, \dots, k)$, where the g_i belong to S_n , has the same value, since $\prod_{i=1, \dots, k} M_i(g_{i-1}, g_i)$ is zero if g is not a state. This last expression is the trace of the matrix $((\prod_{i=1, \dots, k} M_i)R)$. \square

3.3. Proof of Propositions 3, 4, and 5. Let D be a braid diagram on n strings associated to the braid word on n strings $m = m_1 \cdots m_k$. We want to show that

$$P(D, z, a) = a^{1-n-w(D)} \text{Trace} \left(\left(\prod_{i=1, \dots, k} M_i \right) R \right).$$

Let $GL(S_n, K)$ be the multiplicative group of invertible square matrices with coefficients in K whose rows and columns are indexed by S_n . It is easy to check that for $r = 1, \dots, n - 1$, $T_r^+ T_r^- = I$ (it is enough to do this for $n = 2$). Similarly one can easily check that for $r = 1, \dots, n - 2$, $T_r^+ T_{r+1}^+ T_r^+ = T_{r+1}^+ T_r^+ T_{r+1}^+$ (clearly we can restrict our attention to the case $n = 3$). The

commutation relations $T_r^+ T_r^+ = T_r^+ T_r^+$ for $|r' - r| \geq 2$ are immediate. It follows that for every integer $n \geq 1$, there exists a representation $\varphi_n: B_n \rightarrow GL(S_n, K)$ such that $\varphi_n(s_r) = T_{n,r}^+$ and $\varphi_n(s_r^{-1}) = T_{n,r}^-$ (these representations will easily be seen to be essentially identical to the classical representations of Hecke algebras described, for instance, in [8].) For every positive integer n , let us define a mapping F_n from B_n to K as follows: for every b in B_n , $F_n(b) = a^{1-n-w(b)} \text{Trace}(\varphi_n(b)R_n)$. It remains to show that for every braid diagram on n strings D that is a closure of the braid (b, n) , $F_n(b) = P(D, z, a)$. This will be done by proving that properties (i), (ii), and (iii) of Proposition 2 hold for F_n . Property (iii) reduces to $\text{Trace}(R_1) = 1$, which is immediate. One easily checks that $T^+ - T^- = zI$. Hence for $i = 1, \dots, n - 1$, $\varphi_n(bs_i b') - \varphi_n(bs_i^{-1} b') = z\varphi_n(bb')$. Since $w(bb') = w(bs_i b') - 1 = w(bs_i^{-1} b') + 1$, property (ii) follows.

Now for the study of property (i) it is convenient to come back to the formulation in terms of circuit partitions. Clearly property (i) is equivalent to the following:

(i') Let m and m' be two braid words on n strings and let D be the braid diagram on n strings associated to mm' . Let D^1 (respectively, D^2) be the braid diagram on $n + 1$ strings associated with $ms_n m'$ (respectively, $ms_n^{-1} m'$). Then

$$\sum_{\pi \in A(D)} C(\pi) = a^{-2} \sum_{\pi \in A(D^1)} C(\pi) = \sum_{\pi \in A(D^2)} C(\pi),$$

where $C(\pi) = (-1)^{l'(\pi)} z^{l(\pi)} ((a^2 - 1)z^{-1})^{c(\pi)-1}$.

We denote by v^1 (respectively, v^2) the vertex of D^1 (respectively, D^2) corresponding to s_n (respectively, s_n^{-1}). Let V^i be the vertex-set of D^i ($i = 1, 2$). We identify $V^i - \{v^i\}$ with the vertex-set V of D . Let π be a circuit partition of D . A circuit partition of D^i will be said to be *compatible* with π if every vertex of V that is a tangence (respectively, crossing) point of this partition is also a tangence (respectively, crossing) point of π . We denote by $\tau_i(\pi)$ (respectively, $\kappa_i(\pi)$) the unique circuit partition of D^i compatible with π that has v^i as a tangence (respectively, crossing) point. Every circuit partition of D^i is equal either to $\tau_i(\pi)$ or to $\kappa_i(\pi)$ for a unique circuit partition π of D . Moreover, it is easy to see that $\tau_1(\pi), \kappa_1(\pi), \kappa_2(\pi)$ are admissible if and only if π is admissible, and that $\tau_2(\pi)$ is never admissible. Hence

$$\sum_{\pi \in A(D^1)} C(\pi) = \sum_{\pi \in A(D)} (C(\tau_1(\pi)) + C(\kappa_1(\pi)))$$

and

$$\sum_{\pi \in A(D^2)} C(\pi) = \sum_{\pi \in A(D)} C(\kappa_2(\pi)).$$

Thus, to prove (i') it is enough to prove that for every circuit partition π of D , $C(\pi) = C(\kappa_2(\pi))$ and $C(\pi) = a^{-2}(C(\tau_1(\pi)) + C(\kappa_1(\pi)))$. The first equality is

immediate since the parameters t , t' , and c take the same values for π and $\kappa_2(\pi)$. For the second equality, we note that $t(\tau_1(\pi)) = t(\pi) + 1$, $t'(\tau_1(\pi)) = t'(\pi)$, $c(\tau_1(\pi)) = c(\pi) + 1$, $t(\kappa_1(\pi)) = t(\pi)$, $t'(\kappa_1(\pi)) = t'(\pi)$, and $c(\kappa_1(\pi)) = c(\pi)$. Hence, $C(\tau_1(\pi)) = z((a^2 - 1)z^{-1})C(\pi)$ and $C(\kappa_1(\pi)) = C(\pi)$. The result follows immediately. \square

4. SOME CONSEQUENCES

4.1. Degrees in the variable a . It follows from Proposition 3 that $P(D, z, a)$ is a polynomial (with no negative exponents) in the variables z and $(a^2 - 1)z^{-1}$, multiplied by a factor $a^{1-n-w(D)}$. A similar result (Theorem 1.8) is proved by inductive methods in [6]. See also [19] for a related work on arbitrary diagrams. An important consequence is that if we denote by $e_{\min}(D)$ (respectively, $e_{\max}(D)$) the minimum (respectively, maximum) degree of $P(D, z, a)$ in the variable a , the following ‘‘Morton-Franks-Williams inequalities’’ hold:

- (i) $1 - n - w(D) \leq e_{\min}(D) \leq e_{\max}(D) \leq n - 1 - w(D)$,
- (ii) $n \geq 1 + ((e_{\max}(D) - e_{\min}(D))/2)$.

The inequalities (i) follow from Proposition 3, since $1 \leq c(\pi) \leq n$ for every circuit partition π . The inequality (ii) is an immediate consequence of (i) and has proved to be useful in the study of the braid index. It has been observed (see, for instance, [12]) that equality occurs frequently in (ii). This is the case if and only if both inequalities in (i) hold with equality. The following easy consequence of Proposition 3 characterizes this situation in terms of circuit partitions.

Proposition 6. *For any braid diagram on n strings D : (1) $1 - n - w(D) \leq e_{\min}(D)$ with equality iff $\sum_{\pi \in A(D)} (-1)^{t(\pi)-c(\pi)+1} z^{t(\pi)-c(\pi)+1} \neq 0$. (2) $e_{\max}(D) \leq n - 1 - w(D)$ with equality iff $\sum_{\pi \in A(D), c(\pi)=n} (-1)^{t(\pi)} z^{t(\pi)} \neq 0$.*

A braid diagram is said to be *positive* (respectively, *negative*) if every vertex has positive (respectively, negative) sign. In answer to a question of Franks and Williams, Morton and Short exhibit in [20] a positive braid for which the inequality (ii) is strict. However, we have the following result.

Proposition 7. (1) *For any negative braid diagram on n strings D , $e_{\min}(D) = 1 - n - w(D)$. (2) For any positive braid diagram on n strings D , $e_{\max}(D) = n - 1 - w(D)$.*

Proof. (1) If D is negative, for every circuit partition π of D , $t'(\pi) = t(\pi)$. Since the circuit partition with no tangence points is admissible, the result follows from Proposition 6, part (1).

(2) If D is positive, for every circuit partition π of D , $t'(\pi) = 0$. Moreover, the circuit partition for which all vertices are tangence points is admissible and has n circuits. The result now follows from Proposition 6, part (2). \square

Remark. Let D^\sim denote the mirror image of D , that is, the braid diagram obtained from D by changing the signs of all vertices. It follows immediately

from Proposition 1 that $P(D^\sim, z, a) = P(D, -z, a^{-1})$. Hence, $e_{\min}(D^\sim) = -e_{\max}(D)$ and we obtain directly the equivalence of parts (1) and (2) of Proposition 7.

4.2. Degrees in the variable z . Let D be a braid diagram on n strings with vertex-set V . By Proposition 3,

$$P(D, z, a) = a^{1-n-w(D)} \sum_{\pi \in A(D)} (-1)^{t'(\pi)} (a^2 - 1)^{c(\pi)-1} z^{t(\pi)-c(\pi)+1}.$$

Let π_0 (respectively, π_1) be the circuit partition of D for which all vertices are crossing (respectively, tangence) points. Then $t(\pi_0) = 0$, $t(\pi_1) = |V|$, $c(\pi_1) = n$ and $c(\pi_0)$ is the number of components of D , which we denote by $k(D)$. If the circuit partition π' is obtained from the circuit partition π by changing exactly one crossing point into a tangence point, $t(\pi') = t(\pi) + 1$ and $c(\pi') \leq c(\pi) + 1$, so that $t(\pi') - c(\pi') \geq t(\pi) - c(\pi)$. It follows that for every circuit partition π of D , $1 - k(D) \leq t(\pi) - c(\pi) + 1 \leq |V| - n + 1$. Moreover, the coefficient of $z^{1-k(D)}$ in $P(D, z, a)$ is, up to the multiplicative factor $a^{1-n-w(D)}(a^2 - 1)^{k(D)-1}$, equal to

$$\sum_{\pi \in A(D), c(\pi)-t(\pi)=k(D)} (-1)^{t'(\pi)} (a^2 - 1)^{t(\pi)}.$$

The constant term of this polynomial in the variable $a^2 - 1$ is nonzero, since π_0 is admissible. Hence, we obtain the following result.

Proposition 8. *Let D be a diagram.*

- (1) *The minimum degree of $P(D, z, a)$ in the variable z equals $1 - k(D)$.*
- (2) *If D is a braid diagram on n strings with vertex-set V , the maximum degree of $P(D, z, a)$ in the variable z is not greater than $|V| - n + 1$.*

Part (1) of Proposition 8 appears in [18], and a generalization of part (2) to all diagrams is given in [19]. In both cases the method of proof is by induction using the definition of Proposition 1.

4.3. A simple model for the Alexander-Conway polynomial. The Alexander-Conway polynomial of D [1, 5], which we denote by $\nabla(D, z)$, is equal to $P(D, z, 1)$. Thus, the following result is a special case of Proposition 3.

Proposition 9. *For any braid diagram D ,*

$$\nabla(D, z) = \sum_{\pi \in A(D), c(\pi)=1} (-1)^{t'(\pi)} z^{t(\pi)}.$$

As a consequence, if D is a positive braid diagram, $\nabla(D, z)$ has positive integer coefficients. More precisely, the coefficient of z^k in $\nabla(D, z)$ is the number of admissible circuit partitions of D with exactly one circuit and exactly k tangence points.

4.4. On the number of admissible circuit partitions. The author is grateful to the referee for proposing the study of the number of admissible circuit partitions of a braid diagram as a function of its number of vertices. This suggestion is motivated by the observation that for the Jones polynomial, Kauffman's state model [15] associates to a diagram with p vertices an expansion involving 2^p terms, whereas the circuit partition expansion would involve fewer terms in general.

A first remark is that by Proposition 3, for any positive braid diagram D on n strings with p vertices, $|A(D)| = (\sqrt{2})^{n+p-1} P(D, 1, \sqrt{2})$. This equality was used in some of the computations reported here.

It is not difficult to see that the average number of admissible circuit partitions of a braid diagram with p vertices is $(3/2)^p$. More precisely, let $D = (V, E, s)$ be a braid diagram with $|V| = p$, and for $S \subseteq V$ let D_S be the diagram obtained from D by changing the signs of the vertices of S . Since a circuit partition of the graph (V, E) with exactly x crossing points is admissible for exactly 2^x diagrams of the form D_S , it is clear that $\sum_{S \subseteq V} |A(D_S)| = \sum_{X \subseteq V} 2^{|X|} = 3^p$, and hence the average number of admissible circuit partitions of a diagram D_S is $(3/2)^p$. We may interpret this number as a theoretical measure of the average time complexity of the computation of the homfly polynomial of a closed braid with p vertices using the circuit partition model.

A similar measure of the worst case time complexity of this computation would be given by the maximum possible number $A(n, p)$ of admissible circuit partitions of a braid diagram on n strings with p vertices. Let f_p be the p th Fibonacci number defined by the recurrence relation $f_p = f_{p-1} + f_{p-2}$ with the initial conditions $f_0 = f_1 = 1$, or equivalently by the formula

$$f_p = (1/\sqrt{5})((1 + \sqrt{5})/2)^{p+1} - (1/\sqrt{5})((1 - \sqrt{5})/2)^{p+1}.$$

It is not difficult to show that $A(2, p) = f_{p+1}$ (this maximum being given by the positive braid diagram corresponding to the braid word s_1^p). In general, for $p \leq n - 1$, $A(n, p) = 2^p$ (this is given by the braid words $s_1 \cdots s_p$). For $p \geq n$, the braid word $s_1^{p-n+2} s_2 \cdots s_{n-1}$ gives $A(n, p) \geq 2^{n-2} f_{p-n+3}$, and there is experimental evidence that equality might hold. Thus it seems reasonable to conjecture that for every fixed n , $A(n, p)$ is equivalent when $p \rightarrow \infty$ to $c_n((1 + \sqrt{5})/2)^p$ for some constant c_n .

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