ON HERMITE-FEJÉR INTERPOLATION IN A JORDAN DOMAIN

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Abstract. The Hermite-Fejér interpolation problem on a Jordan domain is studied. Under certain mild conditions on the smoothness of the boundary curve, we give both uniform and \( L^p \), \( 0 < p < \infty \), estimates on the rate of convergence. Our estimates are sharp even for the unit disk setting.

1. INTRODUCTION

Let \( D \) be a Jordan domain in the complex plane \( \mathbb{C} \) with boundary \( \Gamma \) and \( z_k = z_{nk}, k = 1, \ldots, n \), be sample points chosen on \( \Gamma \). Also, let \( q \) be a non-negative integer and \( N = N_n := (q + 1)n - 1 \). In this paper we will consider the interpolation problem:

\[
\begin{aligned}
\tilde{H}_N(f; z_k) &= f(z_k), & \tilde{H}_N^{(j)}(f; z_k) &= a_k^{(j)}, \\

k &= 1, \ldots, n \quad \text{and} \quad j = 1, \ldots, q,
\end{aligned}
\]

where \( f \) belongs to the class \( A(D) \) of functions analytic in \( D \) and continuous on \( \overline{D} = D \cup \Gamma \), and \( \tilde{H}_N(f; \cdot) \in \pi_N \), the space of all polynomials with degree at most \( N \). Note that since \( f \) is not necessarily differentiable at \( z_k \) relative to \( \overline{D} \) and the family of data values \( \{a_k^{(j)}\} \) is arbitrarily given, the problem under consideration is different from the Hermite interpolation problem. In particular, by choosing \( a_k^{(j)} = 0 \) for all \( k = 1, \ldots, q \), the problem

\[
\begin{aligned}
H_N(f; z_k) &= f(z_k), & H_N^{(j)}(f; z_k) &= 0,
\end{aligned}
\]

is usually called the \((0, 1, \ldots, q)\) Hermite-Fejér Interpolation Problem.

It is well known that even for the unit disk \( U = \{z : |z| < 1\} \), any \( q \), and \( z_k = e^{i2\pi k/n} \), there exists an \( f \in A(U) \) such that \( H_N(f; \cdot) \) does not converge uniformly on \( \overline{U} \) to \( f \) (see [13]). In this paper, under certain smoothness conditions on the Jordan curve \( \Gamma \), we will first give a necessary and sufficient...
condition on \( f \in \mathcal{A}(\overline{D}) \) that guarantees uniform convergence of \( \tilde{H}_N(f; \cdot) \) to \( f \) on \( \overline{D} \) for \( a_n^{(j)} = o(n^j / \ln n) \), and derive the order of uniform convergence on \( \overline{D} \) of the Hermite-Fejér interpolatory polynomials \( H_N(f; \cdot) \) to \( f \) in terms of the modulus continuity of \( f \). We will next show that for \( a_n^{(j)} = o(n^j) \), \( \tilde{H}_N(f; \cdot) \) always converges in \( L^p(\Gamma) \) to \( f \in \mathcal{A}(\overline{D}) \), \( 0 < p < \infty \), and, in fact, a sharp order of convergence of \( H_N(f; \cdot) \) in \( L^p(\Gamma) \), \( 0 < p < \infty \), will be given.

Of course, for \( q = 0 \), problems (1.1) and (1.2) become the Lagrange interpolation problem:

\[
L_N(f; z_k) = f(z_k),
\]

(1.3)

\( N = n - 1 \), \( k = 1, \ldots, n \), and \( L_N \in \pi_N \). For an analytic Jordan curve \( \Gamma \), Curtiss [3] has shown that \( \|L_N(f; \cdot) - f\|_2 \to 0 \) for all \( f \in \mathcal{A}(\overline{D}) \) by using the Fejér nodes \( z_k \) on \( \Gamma \). Here and throughout, \( \| \cdot \|_p \) denotes the \( L^p \)-norm on \( \Gamma \). Later, for a Jordan curve \( \Gamma \) of class \( C^{2+\delta} \), for some \( \delta > 0 \), Alper and Kalinogorskaja [2] improved the result in [3] by showing that

\[
\|L_N(f; \cdot) - f\|_p \to 0
\]

for any \( p \), \( 0 < p < \infty \). Recently, this result was further improved by the second author and Zhong [10] to a Jordan curve \( \Gamma \) of class \( C^{1+\delta} \) where the order of approximation \( O(\omega(f; \frac{1}{h})) \) is also given. Here and throughout, \( \omega(f; \delta) \) denotes the modulus of continuity of \( f \) on \( \Gamma \) using the uniform norm. We remark that the \( L^p \), \( 0 < p < \infty \), modulus of continuity cannot be used even for the \( L^p \) estimate of \( \|L_N(f; \cdot) - f\|_p \).

The only result in the literature for Hermite-Féjer interpolation on a Jordan curve different from the circle was obtained by Gaier [6], where an analytic curve \( \Gamma \) and \( q = 1 \) are considered and the convergence is only uniform on compact subsets of \( D \). Various recent results concerning convergence on the unit disk of Hermite-Féjer interpolatory polynomials at the \( n \)th roots of unity can be found in Szabados and Varma [11], Varma [12], and the second author [8, 9].

2. Main results

Throughout this paper, \( w = \Phi(z) \) denotes the exterior conformal map from \( \mathbb{C} \setminus \overline{D} \) onto \( |w| > 1 \) such that \( \Phi(\infty) = \infty \) and \( \Phi'(\infty) > 0 \). Let \( \Psi = \Phi^{-1} \) and write

\[
z = \Psi(w) = dw + a_0 + a_1 w^{-1} + \cdots,
\]

(2.1)

where \( d = \Psi'(\infty) > 0 \). It will be clear that by a standard transformation, we may assume, without loss of generality, that \( d = 1 \). Extend \( \Psi \) to a continuous function on \( |w| \geq 1 \) and set \( z_k = z_{nk} = \Psi(w_{nk}) \) where \( w_{nk} = w_k = e^{i2\pi k/n} \). Recall that the \( z_{nk} \)'s are usually called the Fejér points on \( \Gamma = \partial D \). We need some assumptions on the smoothness of \( \Gamma \).
Definition. (i) \( \Gamma \) is said to be of class \( j_1 \) if \( \Psi'(w) \) exists and is continuous on \( |w| \geq 1 \), and its (uniform) modulus of continuity \( \sigma_1(t) \) on \( |w| = 1 \) satisfies the condition
\[
\int_0^a \frac{\sigma_1(t)}{t} |\ln t|^2 \, dt < \infty, \quad a > 0.
\]

(ii) \( \Gamma \) is said to be of class \( j_2 \) if \( \Psi''(w) \) exists and is continuous on \( |w| \geq 1 \), and its (uniform) modulus of continuity \( \sigma_2(t) \) on \( |w| = 1 \) satisfies the condition
\[
\int_0^a \frac{\sigma_2(t)}{t} \ln t \, dt < \infty, \quad a > 0.
\]

It is well known [1] that if \( \Gamma \) belongs to class \( j_1 \), then \( \Psi \) satisfies:
\[
0 < C_1 \leq \left| \frac{\Psi(w) - \Psi(u)}{w - u} \right| \leq C_2
\]
for all \( w \neq u \) and \( |w|, |u| \geq 1 \). We remark that in [1] it is shown that (2.4) already holds for those \( \Gamma \) with
\[
\int_0^a \frac{\sigma_1(t)}{t} \, dt < \infty.
\]
In addition, it is shown in the same paper that
\[
0 < C_1 \leq |\Psi'(w)| \leq C_2
\]
for all \( w, |w| \geq 1 \).

Let
\[
\omega_n(z) = \prod_{j=1}^n (z - z_j).
\]
Then for each \( k \), \((z - z_k)/\omega_n(z)\) is analytic at \( z_k \), so that we can write
\[
\left( \frac{z - z_k}{\omega_n(z)} \right)^{q+1} = \sum_{\nu=0}^\infty \alpha_{k,\nu}(z - z_k)^\nu,
\]
where \( \alpha_{k,\nu} = \alpha_{k,\nu}(q, n) \), \( q = 0, 1, \ldots \). In the following, we will give an asymptotic estimate of \( \alpha_{k,\nu} = \alpha_{k,\nu}(q, n) \) as \( n \to \infty \). We need the notation
\[
\Omega_n(w) = \prod_{k=1}^n \frac{z - z_k}{w - w_k}, \quad z = \Psi(w).
\]

**Theorem 1.** Let \( \Gamma \) belong to class \( j_2 \). Then for each \( \nu \) and \( q = 0, 1, \ldots \),
\[
\alpha_{k,\nu} = \alpha_{k,\nu}(q, n) = O \left( \frac{1}{n^{q+1-\nu}} \right)
\]
and the estimate is uniform in \( k \), \( 1 \leq k \leq n \), as \( n \to \infty \).

Here and throughout, \( \sum_{l \neq k} \) denotes the summation over all \( l = 1, \ldots, n \) with \( l \neq k \). To construct the interpolatory polynomials \( \tilde{H}_n(f; \cdot) \) and \( H_n(f; \cdot) \)
we introduce the fundamental functions:

\[(2.10) \quad A_{kj}(z) = \left( \frac{\omega_n(z)}{z - z_k} \right)^{q+1} \frac{(z - z_k)^j}{j!} \sum_{\nu=0}^{q-j} \alpha_{k\nu}(z - z_k)\nu,\]

where \(j = 0, \ldots, q\) and \(l = 1, \ldots, n\). It is obvious that \(A_{kj} \in \pi_N\) and we will verify that they satisfy

\[(2.11) \quad A_{kj}^{(\nu)}(z_l) = \delta_{kl}\delta_{\nu j}, \quad k, l = 1, \ldots, n; \quad \nu, j = 0, \ldots, q,\]

where, as usual, \(\delta_{ij}\) denotes the Kronecker delta.

**Theorem 2.** For any \(f \in A(D)\), any nonnegative integer \(q\), and arbitrary complex numbers \(a_k^{(j)}, k = 1, \ldots, n, j = 1, \ldots, q\), there exists a unique \(H_N(f; \cdot) \in \pi_N\) satisfying the interpolation conditions (1.1). Furthermore, \(H_N(f; \cdot)\) is given by

\[(2.12) \quad H_N(f; \cdot) = \sum_{k=1}^{n} f(z_k)A_{k0}(\cdot) + \sum_{k=1}^{n} \sum_{j=1}^{q} a_k^{(j)}A_{kj}(\cdot).\]

In addition, under the assumption that \(\Gamma\) belongs to the class \(j_2\), the fundamental functions \(A_{kj}\) satisfy the following estimates:

\[(2.13) \quad \max_{z \in D} \sum_{k=1}^{n} |A_{kj}(z)| = O\left(\frac{\ln n}{n^j}\right), \quad j = 0, \ldots, q,\]

and for \(1 < p < \infty\),

\[(2.14) \quad \left\| \sum_{k=1}^{n} b_k A_{kj}(\cdot) \right\|_p = O\left(\frac{1}{n^j}\right) \max_{1 \leq k \leq n} |b_k|, \quad j = 0, \ldots, q,\]

for any sequence \(\{b_k\}, k = 1, \ldots, n\).

Of course, if we choose \(a_k^{(j)} = 0\), then the polynomials \(H_N(f; \cdot)\) become \(H_N(f; \cdot)\) that satisfy the Hermite-Fejér interpolation condition (1.2). It is well known that even for the case \(D = U\), the unit disk, there exists an \(f \in A(\overline{U})\) such that \(H_N(f; \cdot)\) does not converge uniformly to \(f\) on \(\overline{U}\). We have the following result on the order of uniform approximation.

**Theorem 3.** Let \(\Gamma\) belong to class \(j_2\) and \(f \in A(D)\). Then for any nonnegative integer \(q\),

\[(2.15) \quad \max_{z \in D} |f(z) - H_N(f; z)| = O\left(\omega\left(f; \frac{1}{n}\right) \ln n\right).\]

We remark that this result is sharp as shown by the second author in [9] for \(D = U\). For nonzero \(a_k^{(j)}\), we have the following result.
Theorem 4. Let $\Gamma$ belong to class $j_2$, $f \in A(\overline{D})$, and $q$ be any nonnegative integer. Suppose that

$$\max_{1 \leq k \leq n} a_k^{(j)} = o\left(\frac{n^j}{\ln n}\right), \quad j = 1, \ldots, q,$$

and

$$\lim_{\delta \to 0} \omega(f; \delta) \ln \delta = 0.$$

Then

$$\lim_{N \to \infty} \max_{z \in \overline{D}} \|f(z) - \tilde{H}_N(f; z)\| = 0.$$

For $L^p$ convergence, $0 < p < \infty$, we no longer need $\ln n$ in (2.15) as in the following

Theorem 5. Let $\Gamma$ belong to class $j_2$, $f \in A(\overline{D})$, $q$ be any nonnegative integer, and $0 < p < \infty$. Then

$$\|f - H_N(f; \cdot)\|_p = O\left(\omega\left(f; \frac{1}{n}\right)\right).$$

Again, this result is sharp even for $D = U$ as shown in [10]. For nonzero $a_k^{(j)}$, we have the following result.

Theorem 6. Let $\Gamma$ belong to class $j_2$, $f \in A(\overline{D})$, $q$ be any nonnegative integer, and $\{a_k^{(j)}\}$ satisfy

$$\max_{1 \leq k \leq n} \left| a_k^{(j)} \right| = o(n^j), \quad j = 1, \ldots, q.$$

Then

$$\lim_{N \to \infty} \|f - \tilde{H}_N(f; \cdot)\|_p = 0, \quad 0 < p < \infty.$$

3. PROOF OF THEOREM 1

To establish Theorem 1, we need three lemmas.

Lemma 1. Let $\Psi''$ be continuous on $|w| \geq 1$. Then for each $k = 1, \ldots, n$,

$$\omega''_n(z_k) = n \frac{\Omega'_n(w_k)}{\Psi'(w_k) \Psi''(w_k)}$$

and

$$\sum_{l \neq k} \frac{1}{z_k - z_l} = \frac{1}{2} \omega''_n(z_k) = \frac{1}{2 \Psi'(w_k)} \left[ \frac{n - 1}{w_k} + \frac{2 \Omega'_n(w_k)}{\Omega_n(w_k)} - \frac{\Psi''(w_k)}{\Psi'(w_k)} \right].$$

Proof. From (2.6) and (2.8), we have

$$\omega_n(z) = (w^n - 1) \Omega_n(w),$$
so that
\[ (3.3) \quad \omega'_n(z) = \left[ nw^{n-1} \Omega_n'(w) + (w^n - 1) \Omega'_n(w) \right] \frac{1}{\Psi'(w)}, \]
from which (3.1) follows. To establish the two identities in (3.2), we first use logarithmic derivatives to obtain
\[ \frac{\beta'_k(z)}{\beta_k(z)} = \sum_{l \neq k} \frac{1}{z - z_l} \]
with \( \beta_k(z) := \omega_n(z)/(z - z_k) \). Since \( \beta_k(z_k) = w'_n(z_k) \) and
\[ \beta'_k(z_k) = \lim_{z \to z_k} \frac{\omega'_n(z)(z - z_k) - \omega_n(z)}{(z - z_k)^2} = \lim_{z \to z_k} \frac{\omega'_n(z_k)}{(z - z_k)^2} \]
we have established the first identity in (3.2). To derive the second identity in (3.2), we first observe that
\[ \omega''_n(z_k) = n(n - 1)w_k^{-2} \Omega_n(w_k) + 2nw_k^{-1} \Omega'_n(w_k) \frac{1}{[\Psi'(w_k)]^2} \]
\[ - nw_k^{-1} \Omega''_n(w_k) \frac{\Psi''(w_k)}{[\Psi'(w_k)]^3} \]
by using (3.3) and the fact that \( w''_k = 1 \). By substituting this quantity and the quantity in (3.1) into \( \omega''_n(z_k)/\omega'_n(z_k) \), we arrive at the second identity in (3.2). \( \square \)

In the following, we give certain estimates on \( \Omega_N \) and its relation with \( \Omega'_N \).

**Lemma 2.** If \( \Gamma \) belongs to class \( j_1 \), then
\[ (3.4) \quad \max_{|w| \geq 1} \left| \Omega_n(w) - 1 \right| = o \left( \frac{1}{\ln n} \right). \]
Furthermore, if \( \Gamma \) belongs to class \( j_2 \), then
\[ (3.5) \quad \max_{|w| \geq 1} \left| \frac{\Omega'_n(w)}{\Omega_n(w)} \right| = o(1). \]

**Proof.** To prove (3.4), let
\[ (3.6) \quad g(w, u) = \begin{cases} \frac{(\Psi(w) - \Psi(u))/(w - u)}{\Psi'(w)} & \text{for } u \neq w, \\ \Psi'(w) & \text{for } u = w, \end{cases} \]
where \(|u|, |w| \geq 1\). Hence, from the definition of \(\Omega_N(w)\) and \(g(w, u)\), we have

\[
\ln \Omega_N(w) = \sum_{k=1}^{n} \ln g(w, w_k),
\]

where the branch of the logarithm is taken so that \(\ln 1 = 0\). On the other hand, it is clear that

\[
\frac{\partial \ln g(w, u)}{\partial u} = -\frac{\Psi'(u)(w - u) + (\Psi(w) - \Psi(u))}{(w - u)(\Psi'(w) - \Psi'(u))}
\]

and

\[
\|\Psi(w) - \Psi(u) - \Psi'(u)(w - u)\| = \left| \int_{\gamma} [\Psi'(\xi) - \Psi'(u)] d\xi \right| \\
\leq C_1 \sigma_1(|w - u|) \int_{\gamma} |d\xi| \leq C_2 |w - u| \sigma_1(|w - u|),
\]

where \(\gamma\) is a contour joining \(u\) to \(w\) on \(|\xi| \geq 1\) with length bounded by \(\frac{x}{2}|u - w|\) and \(\sigma_1\) denotes the modulus of continuity of \(\Psi'\). By using (2.4), (3.8), and (3.9), we have

\[
\left| \frac{\partial \ln g(w, u)}{\partial u} \right| \leq C \sigma_1(|w - u|),
\]

for \(|u|, |w| \geq 1\). Hence, from the hypothesis that \(\Gamma\) belongs to class \(j_1\), as a function of \(u\) on \(|u| = 1\), the function \(\ln g(w, u)\) satisfies the Dini condition uniformly on \(|w| = 1\). It follows that

\[
\ln g(w, u) = \sum_{j=1}^{\infty} \frac{a_j(w)}{u^j}
\]

uniformly on \(|u|, |w| \geq 1\). From the property

\[
\sum_{k=1}^{n} w_k^{-j} = \begin{cases} 0 & \text{if } n \nmid j, \\ n & \text{if } n \mid j \end{cases}
\]

of the \(n\)th roots of unity, we have, from (3.7),

\[
\ln \Omega_n(w) = n \sum_{l=1}^{\infty} a_{ln}(w)
\]

uniformly on \(|w| \geq 1\). To estimate \(a_j(w)\), since \(\Gamma\) belongs to class \(j_1\), we may use the Hardy-Littlewood inequality (cf. [4, p. 100])

\[
|\Psi''(u)| \leq C \frac{\sigma_1(|u| - 1)}{|u| - 1}, \quad |u| > 1.
\]
Indeed, letting \( 1 < \rho \leq \frac{3}{2} \), we have from (3.8), for \( |w| = 1 \),

\[

t_\rho^{2} \ln g(w, u) \left| du \right| \\
= \int_{|u| = \rho} \left| \frac{\Psi''(u)}{\Psi(u) - \Psi(w)} + \frac{(\Psi'(u))^2}{(\Psi(u) - \Psi(w))^2} - \frac{1}{(w - u)^2} \right| \left| du \right|
\]

\[
\leq \int_{|u| = \rho} \left| \frac{\Psi''(u)}{\Psi(u) - \Psi(w)} \right| \left| du \right|
+ \int_{|u| = \rho} \left| \frac{\Psi'(u)}{\Psi(u) - \Psi(w)} - \frac{1}{w - u} \right| \left| \frac{\Psi'(u)}{\Psi(u) - \Psi(w)} + \frac{1}{w - u} \right| \left| du \right|
:= I_1 + I_2,
\]

where by applying (2.4) and (3.12), we have

\[
I_1 \leq C_1 \frac{\sigma_1(\rho - 1)}{\rho - 1} \int_{|u| = \rho} \frac{du}{w - u} \leq C_2 \frac{\sigma_1(\rho - 1)}{\rho - 1} \ln \frac{1}{\rho - 1},
\]

and by using (3.9) and (2.4), we also have

\[
I_2 \leq C_1 \int_{|u| = \rho} \frac{\sigma_1(|w - u|)}{|w - u|^2} |du| \leq C_2 \frac{\sigma_1(\rho - 1)}{\rho - 1} \ln \frac{1}{\rho - 1}.
\]

That is,

\[
(3.13) \quad \int_{|u| = \rho} \left| \frac{\partial^2 \ln g(w, u)}{\partial u^2} \right| |du| \leq C \frac{\sigma_1(\rho - 1)}{\rho - 1} \ln \frac{1}{\rho - 1}, \quad \rho > 1.
\]

By taking the second derivative of the power series (3.10) and applying the estimate in (3.13), we have, for \( j = 2, 3, \ldots \),

\[
|a_j(w)| = \left| \frac{1}{2\pi i} \int_{|u| = \rho} \frac{1}{j(j + 1)} \frac{\partial^2 \ln g(w, u)}{\partial u^2} u^{j+1} du \right| \\
\leq \rho^{j+1} \frac{1}{2\pi j^2} \int_{|u| = \rho} \left| \frac{\partial^2 \ln g(w, u)}{\partial u^2} \right| |du| \leq C \rho^{j+1} \frac{\sigma_1(\rho - 1)}{j^2(\rho - 1)} \ln \frac{1}{\rho - 1}.
\]

By taking \( \rho = 1 + \frac{1}{j} \), it follows that

\[
\max_{|w| = 1} |a_j(w)| \leq C \frac{\sigma_1(j^{-1})}{j} \ln j.
\]

We now apply this estimate to (3.11), yielding

\[
\max_{|w| = 1} |\ln \Omega_n(w)| \leq C n \sum_{i=1}^{\infty} \frac{\sigma_1(1/ln)}{ln} \ln(ln) \\
\leq C \int_{n}^{\infty} \frac{\sigma_1(1/t)}{t} \ln t dt = C \int_{0}^{1/n} \frac{\sigma_1(s)}{s} |\ln s| ds \\
\leq C \frac{1}{ln n} \int_{0}^{1/n} \frac{\sigma_1(s)}{s} |\ln s|^2 ds = o\left( \frac{1}{ln n} \right).
\]
where \((2.2)\) has been used. This estimate is equivalent to \((3.4)\).

To prove \((3.5)\) for \(\Gamma\) belonging to class \(i_2\), we will apply the inequality of Hardy-Littlewood

\[
|\Psi''(u)| \leq C \frac{\sigma_2(|u| - 1)}{|u| - 1}, \quad |u| > 1,
\]

where \(\sigma_2\) denotes the modulus of continuity of \(\Psi''(u)\). Set

\[
g_1(w, u) = \frac{\Psi'(u)}{\Psi(w) - \Psi(u)} - \frac{1}{w - u}.
\]

Then by taking the logarithmic derivative of \(\Omega_n\) and \(g(w, w_k)\) in \((2.8)\) and \((3.6)\), respectively, we have

\[
\frac{\Omega'_n(w)}{\Omega_n(w)} = \sum_{k=1}^{n} \frac{g'(w, w_k)}{g(w, w_k)} = \sum_{k=1}^{n} g_1(w, w_k),
\]

where \(g_1(w, u) = \Psi'(u)/(\Psi(w) - \Psi(u)) - 1/(w - u)\), hence, in view of \(\Gamma\) belonging to class \(i_2\), we have

\[
g_1(w, u) = \sum_{j=1}^{\infty} \frac{b_j(w)}{w^j}
\]

uniformly on \(|u|, |w| \geq 1\). By applying \((3.14)\) we may obtain an estimate similar to that of \((3.13)\), namely,

\[
\int_{|u|=\rho} \left| \frac{\partial^2 g_1(w, u)}{\partial u^2} \right| |du| \leq C \frac{\sigma_2(\rho - 1)}{\rho - 1} \frac{1}{\ln \frac{1}{\rho - 1}},
\]

where \(\rho > 1\). Hence, as before, we have

\[
\max_{|u|=1} \left| b_j(w) \right| \leq C \frac{\sigma_2(j^{-1})}{j} \ln j
\]

and

\[
\frac{\Omega'_n(w)}{\Omega_n(w)} = \sum_{l=1}^{\infty} b_{ln}(w), \quad |w| = 1,
\]

so that

\[
\max_{|w| \geq 1} \left| \frac{\Omega'_n(w)}{\Omega_n(w)} \right| \leq C n \sum_{l=1}^{\infty} \frac{\sigma_2(1/ln)}{ln} \ln(ln) \leq \int_0^{1/n} \frac{\sigma_2(t)}{t} |\ln t| dt
\]

which is \(o(1)\) by \((2.3)\). This completes the proof of the lemma.

**Remark.** In [10], where Lagrange interpolation (or \(q = 0\)) was considered, the Jordan curve \(\Gamma\) was assumed to belong to \(C^{1+\delta}, \delta > 0\). However, from our estimate \((3.4)\) and the procedure in [10], it can be shown that the result there also holds for \(\Gamma\) belonging to class \(j_1\).

As a consequence of estimates \((3.5)\) in Lemma 2, the identity \((3.2)\) in Lemma 1 yields the following result.
Corollary 1. Let $\Gamma$ belong to class $j_2$. Then

$$
\sum_{l \neq k} \frac{1}{z_k - z_l} = \frac{(n - 1)}{2w_k \Psi'(w_k)} - \frac{\Psi''(w_k)}{2[\Psi'(w_k)]^2} + o(1)
$$

uniformly in $k$, $1 \leq k \leq n$.

In the proof of Theorem 1, the following estimates will also be used.

Lemma 3. Let $\Gamma$ belong to $j_2$. Then for $k = 1, 2, \ldots, n$ and $r = 0, 1, \ldots$

$$
\sum_{l \neq k} \frac{1}{(z_k - z_l)^{r+1}} = O(n^{r+1})
$$

and

$$
\frac{d^r}{dz^r} \left( \frac{z - z_k}{\omega_n(z)} \right) \bigg|_{z=z_k} = O(n^{r-1})
$$

uniformly in $k$, $1 \leq k \leq n$.

Proof. The estimate (3.18) for $r = 0$ can easily be deduced by (3.17). For $r \geq 1$, by using (2.4) we have

$$
\left| \sum_{l \neq k} \frac{1}{(z_k - z_l)^{r+1}} \right| \leq \sum_{l \neq k} \left| \frac{1}{z_k - z_l} \right|^{r+1} \leq C_2 \sum_{l \neq k} \left| \frac{1}{w_k - w_l} \right|^{r+1}
$$

$$
\leq 2C_2 \sum_{l=1}^{[(n-1)/2]} \frac{1}{(2 \sin l \pi n)^{r+1}}
$$

$$
\leq 2C_2 \sum_{l=1}^{[(n-1)/2]} \frac{1}{(4l/n)^{r+1}} = O(n^{r+1}).
$$

We are going to verify (3.19) by induction. For $r = 0$, by using (3.1), (3.4), and (2.5), we have (3.19). For $r \geq 1$, by the induction hypothesis and using (3.18), we obtain

$$
\left( \frac{z - z_k}{\omega_n(z)} \right)^{(s+1)} \bigg|_{z=z_k} = - \left( \frac{z - z_k}{\omega_n(z)} \sum_{l \neq k} \frac{1}{z - z_l} \right)^{(s)}
$$

$$
= - \sum_{\nu=0}^{s} \binom{s}{\nu} \left( \frac{z - z_k}{\omega_n(z)} \right)^{(\nu)} \left( \sum_{l \neq k} \frac{1}{z - z_l} \right)^{(s-\nu)}
$$

$$
= - \sum_{\nu=0}^{s} \binom{s}{\nu} \left( \frac{z - z_k}{\omega_n(z)} \right)^{(\nu)} \sum_{l \neq k} \frac{(-1)^{s-\nu}(s-\nu)!}{(z - z_l)^{s-\nu+1}}
$$

$$
= \sum_{\nu=0}^{s} \binom{s}{\nu} O(n^{n-1}) \cdot O(n^{s-\nu+1}) = O(n^s).
$$

This completes the proof of the lemma. \[\square\]
We are now ready to prove Theorem 1.

**Proof of Theorem 1.** For $q = 0$, from (2.7), it is clear that

\[
\sigma_{k\nu}(0) = \frac{1}{\nu!} \frac{d^\nu}{dz^\nu} \left( \frac{z - z_k}{\omega_n(z)} \right) \bigg|_{z=z_k}
\]

which yields (2.9) by using (3.19) in Lemma 3. We will now use induction in $q$. Indeed, by the induction hypothesis and using (3.19) in Lemma 3 it follows that

\[
\begin{align*}
\sigma_{k\nu}(q + 1) &= \frac{1}{\nu!} \frac{d^\nu}{dz^\nu} \left( \frac{z - z_k}{\omega_n(z)} \right)^{q+2} \bigg|_{z=z_k} \\
&= \frac{1}{\nu!} \sum_{j=0}^{\nu} \binom{\nu}{j} j! \left( \frac{z - z_k}{\omega_n(z)} \right)^{q+1} \frac{d^{\nu-j}}{dz^{\nu-j}} \left( \frac{z - z_k}{\omega_n(z)} \right) \bigg|_{z=z_k} \\
&= \frac{1}{\nu!} \sum_{j=0}^{\nu} \binom{\nu}{j} j! O \left( \frac{1}{n^{q-j+1}} \right) O \left( n^{\nu-j-1} \right) = O \left( \frac{1}{n^{q+2-\nu}} \right).
\end{align*}
\]

This completes the proof of Theorem 1. \(\square\)

4. **Proof of Theorem 2**

We first establish the existence and uniqueness of $\tilde{H}_N(f; \cdot)$ for any given $f \in A(D)$. Since $N = (q + 1)n - 1$, it follows from the definition (2.11) that $A_{k_i} \in \pi_N$. Hence, $\tilde{H}_N(f; \cdot) \in \pi_N$ also. Next, we will establish (2.11). For $l \neq k$, it is clear from the first factor that $A_{k_l}^{(\nu)}(z_i) = 0$ for all $\nu, j = 0, \ldots, q$. We now consider the case $l = k$. From (2.10) and (2.7), it follows that

\[
A_{kj}(z) = \left( \frac{\omega_n(z)}{z - z_k} \right)^{q+1} \left( \frac{z - z_k}{\omega_n(z)} \right)^{q+1} \left[ \frac{1}{j!} \sum_{\mu=q-j+1}^{\infty} \alpha_{kj}(z - z_k)\mu \right]
\]

Hence, for $\nu < j$, we have $A_{kj}^{(\nu)}(z_k) = 0$. For $\nu = j$, then $A_{kj}^{(j)}(z_k) = 1$. Finally, for $\nu > j$, we also have $A_{kj}^{(\nu)}(z_k) = 0$. This establishes the interpolatory property of $A_{kj}$ in (2.11). Thus, by defining $\tilde{H}_N(f; \cdot)$ as in (2.12), $\tilde{H}_N(f; \cdot)$ solves the interpolation problem (1.1). The uniqueness of $\tilde{H}_N(f; \cdot)$ is trivial. \(\square\)

In order to establish the estimates (2.13) and (2.14), we need the following lemma.
Lemma 4. Let $\Gamma$ belong to class $j_1$. Then

\begin{equation}
\max_{z \in \mathbb{D}} \sum_{k=1}^{n} \left| \frac{\omega_n(z)}{(z - z_k)\omega'_n(z_k)} \right| = O(\ln n),
\end{equation}

\begin{equation}
\max_{z \in \mathbb{D}} \sum_{k=1}^{n} \left| \frac{\omega_n(z)}{(z - z_k)\omega'_n(z_k)} \right|^{1+\delta} = O(1), \quad \delta > 0,
\end{equation}

and for $1 < p < \infty$,

\begin{equation}
\left\| \sum_{k=1}^{n} b_k \frac{\omega_n(z)}{(z - z_k)\omega'_n(z_k)} \right\|_p = O \left( \max_{1 \leq k \leq n} |b_k| \right).
\end{equation}

Proof. From (2.8) and (3.1), we have

\begin{equation}
\frac{\omega_n(z)}{(z - z_k)\omega'_n(z_k)} = \frac{w_k \Psi'(w_k)(w^n - 1)}{n(\Psi(w) - \Psi(w_k)) \Omega_n(w)},
\end{equation}

where $z = \Psi(w)$ and $z_k = \Psi(w_k)$. For the unit disk, it is well known (cf. Gaier [6, pp. 80–81]) that

\[ \max_{|w| \leq 1} \sum_{k=1}^{n} \left| \frac{w_k}{n} \frac{w^n - 1}{w - w_k} \right| = O(\ln n). \]

In addition, for $\delta > 0$ it is also well known (cf. [9]) that

\[ \max_{|w| \leq 1} \sum_{k=1}^{n} \left| \frac{w_k}{n} \frac{w^n - 1}{w - w_k} \right|^{1+\delta} = O(1). \]

Hence, by applying (2.4), (2.5), and (3.4) in Lemma 2 to (4.4), we have both (4.1) and (4.2). Next, by (4.4), it follows that

\[ \sum_{k=1}^{n} b_k \frac{\omega_n(z)}{(z - z_k)\omega'_n(z_k)} = \sum_{k=1}^{n} b_k \frac{w_k \Psi'(w_k)(w^n - 1)}{n(\Psi(w) - \Psi(w_k)) \Omega_n(w)} + \sum_{k=1}^{n} b_k \frac{w_k}{n} \frac{w^n - 1}{w - w_k} \]

\[ = I_5 + I_6 + I_7. \]

By applying (4.1) and (3.4) in Lemma 2, we obtain

\[ \max_{z \in \mathbb{D}} |I_5| = O \left( \max_{1 \leq k \leq n} |b_k| \right). \]

In order to estimate $I_7$, we may assume, without loss of generality, that $|w - 1|$ is not greater than $|w - w_k|, k = 1, 2, \ldots, n - 1$, so that

\[ |\arg w| \leq \frac{\pi}{n}, \quad \left| \frac{w^n - 1}{w - 1} \right| \leq n, \]
and 
\[ \left| \frac{1}{w - w_k} \right| \leq \begin{cases} \pi/n, & 1 \leq k \leq n/2, \\ (n - k)/k, & n/2 < k \leq n - 1. \end{cases} \]

Hence from (3.9), we have
\begin{equation}
(4.5) \quad \left| b_n \frac{w^n}{n} (w^n - 1) \left[ \frac{\Psi(1)}{\Psi(w) - \Psi(1)} - \frac{1}{w - 1} \right] \right| \leq C |b_n| \tag{4.5}
\end{equation}

and
\begin{equation}
(4.6) \quad \left| \sum_{k=1}^{n-1} b_k \frac{w^k}{n} (w^n - 1) \left[ \frac{\Psi'(w_k)}{\Psi(w) - \Psi(w_k)} - \frac{1}{w - w_k} \right] \right| \leq C \max_{1 \leq k \leq n-1} |b_k| \frac{1}{n} \sum_{k=1}^{n-1} \frac{\sigma_1(|w - w_k|)}{|w - w_k|} \tag{4.6}
\end{equation}

\[ \leq C \max_{1 \leq k \leq n-1} |b_k| \frac{1}{n} \sum_{k=1}^{n-1} \frac{\sigma_1(k/n)}{k/n} \]

\[ \leq C \max_{1 \leq k \leq n-1} |b_k| \int_0^1 \frac{\sigma_1(t)}{t} \, dt \leq C \max_{1 \leq k \leq n-1} |b_k|, \]

where the condition in (2.2) is used. Thus combining (4.5) and (4.6), we obtain
\[ \max_{|z| \in D} |I_j| = O \left( \max_{1 \leq k \leq n} |b_k| \right). \]

Finally, by the Marcinkiewicz-Zygmund inequality for \( 1 < p < \infty \) (cf. [15]), we have
\[ \|I_6\|_p = O \left( \frac{1}{n} \sum_{k=1}^{n} |b_k|^p \right)^{1/p} = O \left( \max_{1 \leq k \leq n} |b_k| \right). \]

This completes the proof of the lemma. \( \square \)

We now return to the estimates of (2.13) and (2.14) and obtain
\[ A_{k,j}(z) = \left( \frac{\omega_n(z)}{z - z_k} \right)^{q+1} \frac{1}{j!} \alpha_k q-j (z - z_k)^q + \left( \frac{\omega_n(z)}{z - z_k} \right)^{q+1} \frac{1}{j!} \alpha_k q-j (z - z_k)^q \sum_{\nu=0}^{q-j-1} \alpha_{k,\nu} (z - z_k)^\nu \]
\[ := I_8(z) + I_9(z). \]

Hence, by (3.1) in Lemma 1, (3.4) in Lemma 2, (2.5), (2.9) in Theorem 1, and (4.1), (4.3) in Lemma 4, we have
\[ \sum_{k=1}^{n} |I_8(z)| = O \left( \frac{1}{n^j} \right) \max_{z \in D} \sum_{k=1}^{n} \left| \frac{\omega_n(z)}{z - z_k} \right| = O \left( \ln n \right) \]
and for \( 1 < p < +\infty \)
\[ \left\| \sum_{k=1}^{n} b_k I_q(z) \right\|_j = O(n) \max_{1 \leq k \leq n} |b_k| \alpha_k q-j = O \left( \frac{1}{n^j} \right) \max_{1 \leq k \leq n} |b_k|. \]
Similarly, by (3.1) in Lemma 1, (3.4) in Lemma 2, (2.5), (2.9) in Theorem 1, and (4.2) in Lemma 4, we have
\[
\sum_{k=1}^{n} |I_g(z)| = O \left( \frac{1}{n^l} \right).
\]

By combining these estimates, we have established both (2.13) and (2.14). \(\square\)

5. PROOF OF THEOREMS 3–6

Let \( \Gamma \) belong to class \( j_2 \) and \( f \in A(\overline{D}) \). It is known (cf. Theorems 1 and 6 in [5, Chapter 9]) that there exists \( P_N \in \pi_N \) such that
\[
\max_{z \in \overline{D}} |f(z) - P_N(z)| = O \left( \omega \left( f; \frac{1}{N} \right) \right)
\]
and
\[
\max_{z \in \overline{D}} |P_N^{(m)}(z)| = O \left( N^m \omega \left( f; \frac{1}{N} \right) \right), \quad m = 1, 2, \ldots.
\]

By using the first part of Theorem 2, we have
\[
f(z) - H_N(f; z) = f(z) - P_N(z) + \sum_{k=1}^{N} \left( P_N(z_k) - f(z_k) \right) A_k(z) + \sum_{k=1}^{n} \sum_{j=1}^{q} P_N^{(j)}(z_k) A_{kj}(z).
\]
Hence, by applying (2.13) and (2.14) of Theorem 2 and (5.1), (5.2) above, we have completed the proof of Theorem 3. Next, we write
\[
f(z) - \tilde{H}_N(f; z) = f(z) - H_N(f; z) + \sum_{k=1}^{N} \sum_{j=1}^{q} a_k^{(j)} A_{kj}(z).
\]
Here, by using the hypothesis (2.17) and Theorem 3, we have
\[
\max_{z \in \overline{D}} |f(z) - H_N(f; z)| \to 0,
\]
and by using the hypothesis (2.16) and applying (2.13) in Theorem 2, we also have
\[
\max_{z \in \overline{D}} \left| \sum_{k=1}^{N} \sum_{j=1}^{q} a_k^{(j)} A_{kj}(z) \right| \to 0.
\]
This completes the proof of Theorem 4. The proofs of Theorems 5 and 6 are similar simply by applying (2.14) in Theorem 2, noting that Hölder's inequality can be applied for \( 0 < p \leq 1 \) and using the result for \( p = 2 \). \(\square\)

6. FINAL REMARKS

In this section, we give examples of the domain \( D \) whose boundary curve \( \Gamma \) belongs to classes \( j_1 \) and \( j_2 \). Let \( \Gamma \) be of class \( C^1 \) and denote its angle of inclination as a function of arc length \( s \) by \( \theta(s), \ 0 \leq s \leq |\Gamma| \), the length of \( \Gamma \).
Proposition 1. If \( \Gamma \) satisfies
\[
\int_{0}^{a} \frac{\omega(\theta'; t)}{t} |\ln t|^3 \, dt < \infty, \quad a > 0,
\]
then \( \Gamma \) belongs to class \( J_1 \).

Of course, every \( \Gamma \) of class \( C^{1+\delta} \) for some \( \delta > 0 \) satisfies (6.1). We also have the following

Proposition 2. If \( \Gamma \) satisfies
\[
\int_{0}^{a} \frac{\omega(\theta'; t)}{t} |\ln t|^2 \, dt < \infty, \quad a > 0,
\]
then \( \Gamma \) belongs to class \( J_2 \).

Of course, every \( \Gamma \) of class \( C^{2+\delta} \) for some \( \delta > 0 \) satisfies (6.2).

To prove these results, we need the following result in [14]: If
\[
\int_{0}^{a} \frac{\omega(\theta^{(n)}; t)}{t} \, dt < \infty, \quad a > 0,
\]
then \( \Psi^{(n+1)} \) is continuous on \( |w| \geq 1 \) and
\[
\omega \left( \Psi^{(n+1)}; t \right) = O \left( \int_{0}^{t} \frac{\omega(\theta^{(n)}; \tau)}{\tau} \, d\tau \right)
\]
\[
\quad + t \int_{t}^{a} \frac{\omega(\theta^{(n)}; \tau)}{\tau^2} \, d\tau + t \ln \frac{1}{t}, \quad a > 0.
\]

Let \( n = 0 \). If (6.1) is satisfied, so is (6.3), and hence \( \Psi' \) is continuous on \( |w| \geq 1 \). Using (6.4) for \( n = 0 \), we have
\[
\int_{0}^{a} \frac{\omega(\Psi'; t)}{t} |\ln t|^2 \, dt
\]
\[
\begin{align*}
= & \; O \left( \int_{0}^{a} \left( \int_{0}^{t} \frac{\omega(\theta'; \tau)}{\tau} \, d\tau \right) \frac{|\ln t|^2}{t} \, dt \\
+ & \; \int_{0}^{a} \left( t \int_{t}^{a} \frac{\omega(\theta'; \tau)}{\tau^2} \, d\tau \right) \frac{|\ln t|^2}{t} \, dt + \int_{0}^{a} t \ln \frac{1}{t} |\ln t|^2 \, dt \right) \\
= & \; O \left( \int_{0}^{a} \left( \int_{\tau}^{t} \frac{|\ln t|^2}{t} \, dt \right) \frac{\omega(\theta'; \tau)}{\tau} \, d\tau \right.
\end{align*}
\]
\[
\quad + \int_{0}^{a} \left( \int_{0}^{\tau} |\ln t|^2 \, dt \right) \frac{\omega(\theta'; \tau)}{\tau^2} \, d\tau \right) + O(1)
\]
\[
= O \left( \int_{0}^{a} \frac{\omega(\theta'; t)}{t} |\ln t|^3 \, dt \right) + O(1) < \infty.
\]
That is, \( r \) is of class \( j_j \). This completes the proof of Proposition 1. The proof of Proposition 2 is similar by applying \( n = 1 \) in (6.3) and (6.4), using the condition (6.2). \( \square \)

We conclude this paper by posing three open problems.

1. In this paper, we consider the \((0, 1, \ldots, q)\) Hermite-Fejér interpolation problem where the interpolatory polynomials \( H_N(f; z_k) \) satisfy \( H_N^{(j)}(f; z_k) = 0 \),\( j = 1, \ldots, q \) and \( k = 1, \ldots, n \). It is interesting to study if the convergence and estimates in this paper are still valid if we impose a more general interpolatory condition:

\[
H_N^{(j)}(f; z_k) = 0 \quad \text{for } j = 1, \ldots, q, \quad k = 1, \ldots, n,
\]

where \( q_k = q_k(n) \) satisfies \( \max_{1 \leq k \leq n} q_k(n) \leq M < \infty \) for all \( n \).

2. How much can the Fejér points \( z_k = z_{nk} \) be perturbed on \( \Gamma \) so that the convergence and estimates in this paper are still valid?

3. If \( D \) is different from the unit disk, do there exist \((0, m_1, \ldots, m_q)\) Birkhoff-Fejér interpolants \( B_N(f; \cdot) \); that is,

\[
B_N(f; z_k) = f(z_k) \quad \text{and} \quad B_N^{(m)}(f; z_k) = 0,
\]

for \( j = 1, \ldots, q \) and \( k = 1, \ldots, n \)? If \( B_N(f; \cdot) \) exist, do they converge to \( f \) in \( L^p \), \( 0 < p < \infty \)? For the unit disk, results on convergence and estimates have been obtained in [10].

REFERENCES


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