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GENERICITY OF NONTRIVIAL H-SUPERRECURRENT H-COCYCLES

KARMA DAJANI

ABSTRACT. We prove that most H-cocycles for a nonsingular ergodic transformation of type III_λ, $0 < \lambda < 1$, are H-superrecurrent. This is done by showing that the set of nontrivial H-superrecurrent H-cocycles form a dense G_δ set with respect to the topology of convergence in measure.

1. INTRODUCTION

In order to generalize the ergodic theorem to the case when the underlying transformation is nonsingular, Hurewicz considered what we call H-cocycles. An H-cocycle is a measurable function f_* on $Z \times X$ defined for positive integers n by $f_*(n, x) = \sum_{i=0}^{n-1} f(T^i x)\frac{d\mu\circ T^i}{d\mu}(x)$. (See below for a more general definition.) These are generalizations of cocycles f^* that were considered for the classical Birkhoff's ergodic theorem for measure preserving transformations and are defined for positive n as follows: $f^*(n, x) = \sum_{i=0}^{n-1} f(T^i x)$.

In his paper *On recurrence*, Klaus Schmidt [12] studied the simultaneous recurrence of a cocycle with the cocycle generated by the function $\rho(x) = \log\frac{d\mu\circ T}{d\mu}(x)$. This we call superrecurrence. He proved that recurrence is equivalent to superrecurrence. In [13] Ullman studied the recurrence of H-cocycles (which he calls H-recurrence) and in [4] their simultaneous recurrence with the cocycle of $\rho(x)$ was investigated (in the case of H-cocycles we call such a phenomenon H-superrecurrence). Also in [4] a class of H-cocycles called H-coboundaries were exhibited and were shown to be H-superrecurrent. These are the trivial H-cocycles as far as H-superrecurrence and H-cohomology are concerned. (Two H-cocycles are H-cohomologous if their difference is an H-coboundary.) Two natural questions are: What H-cocycles other than H-coboundaries are H-superrecurrent? How can we tell whether an H-cocycle is an H-coboundary or not? This lead us to borrow the notion of *essential range* defined for cocycles and extend it to H-cocycles. The essential range $\overline{E}(f_*)$ of an H-cocycle f_* is H-cohomology invariant so that H-coboundaries have

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essential range $\{0\}$. This tells us that an H-cocycle whose essential range contains a nonzero member is not an H-coboundary.

A common strategy adopted to show the "bigness" of a set is to use a category argument. More precisely, the set under consideration forms a dense G_δ. This approach serves two purposes: It shows the existense of objects with specified properties and that the number of such objects is large. We use the same idea to show that the set of H-superrecurrent H-cocycles with infinity in their essential range is a G_δ. For nonsingular transformations of type III_λ, $0 < \lambda < 1$, we show that this set is actually dense. The proof requires that a topology be defined on the set of H-cocycles. This is done by extending in a natural way the topology of convergence in measure on the set of all Borel measurable functions on X. We first prove that the set of H-superrecurrent H-cocycles and the set of H-cocycles with infinity in their essential range are G_δ sets. The denseness proof is a modified version of the methods employed by Parthasarathy and Schmidt [10] and Choksi, Hawkins, and Prasad [2] in proving that the set of cocycles with essential range $\overline{R}$ (the one-point compactification of R) is a dense G_δ. The proof uses an abelian countable group action Γ on $\{0, 1\}^N$ (see §6 for a definition) which is orbit equivalent to the odometer transformation of type III_λ [14]. For this, we investigate H-cohomology under orbit equivalence. First we generalize the notion of an H-cocycle for a Z action to that for an abelian countable group action. We then show that H-cocycles, H-coboundaries, the notions of H-superrecurrence and infinity in the essential range are all preserved under orbit equivalence. However, finite values of the essential range are preserved only when the orbit equivalence is measure preserving. This allows us to concentrate on the set of H-cocycles for the Γ action and then extend the results back to the case of a nonsingular transformation of type III_λ using orbit equivalence.

2. Definitions and preliminaries

Let $(X, \mathscr{B}, \mu)$ be a Lebesgue probability space. Let T: $X \to X$ be a nonsingular automorphism of X; that is, T is a measurable bijection of X such that for $A \in \mathscr{B}$, $\mu(TA) = 0$ if and only if $\mu(A) = 0$. We also assume that the transformation T is conservative: for all $B \in \mathscr{B}$ with $\mu(B) > 0$, there exists $n \neq 0$ such that $\mu(B \cap T^{-n}B) > 0$, and aperiodic:

$$\mu\left(\bigcup_{n>0}\{x : T^n x = x\}\right) = 0.$$

Let $\omega_n(x) = \frac{d\mu\circ T^n}{d\mu}(x)$ denote the Radon-Nikodým derivative of $\mu\circ T^n$ with respect to μ. Note that the function $\omega : Z\times X \to R$ given by $\omega(n, x) = \omega_n(x) = \frac{d\mu\circ T^n}{d\mu}(x)$ is a multiplicative cocycle, that is, $\omega(n+m, x) = \omega(n, x)\omega(m, T^n x)$, and that $\log\omega$: $X \times Z \to R$ given by $\log\omega(n, x) = \log\omega_n(x) = \log\frac{d\mu\circ T^n}{d\mu}(x)$ is an additive cocycle.

Let $f : X \to R$ be any measurable function.

Definition (2.1). The *H-cocycle* of f is defined to be the function $f_*: Z \times X \to R$ given by

$$f_*(n, x) = \begin{cases} \sum_{i=0}^{n-1} f(T^i x)\omega_i(x), & \text{if } n > 0, \\ 0, & \text{if } n = 0, \\ -\omega_n(x) f_*(-n, T^n x), & \text{if } n < 0, \end{cases}$$

where f_* satisfies the *H-cocycle* identity

$$f_*(n+m, x) = f_*(n, x) + \omega_n(x) f_*(m, T^n x),$$

for all $n, m \in Z$, and for almost all $x \in X$.

Definition (2.2). The H-cocycle f_* is said to be H-superrecurrent if for every $\varepsilon > 0$ and for every $B \subset X$ with $\mu(B) > 0$, there exists a nonzero integer n such that

$$\mu[B \cap T^{-n} B \cap \{x : |f_*(n, x)| + |\log \omega_n(x)| < \varepsilon\}] > 0.$$

An H-cocycle which is not H-superrecurrent is called H-supertransient. In [4] it is shown that H-supertransience is equivalent to the following: For every $B \subset X$ with $\mu(B) > 0$ and for every real number $M > 0$,

$$\mu\left[\limsup_{n\to\infty} B \cap T^{-n} B \cap \{x : |f_*(n, x)| + |\log \omega_n(x)| < M\}\right] = 0.$$

Definition (2.3). A measurable function f on X is said to be an *H-coboundary* if $f(x) = g(x) - \omega_1(x) g(Tx)$ for some measurable function g on X. Note that if f is an H-coboundary then $f_*(n, x) = g(x) - \omega_n(x) g(T^n x)$. Two functions f, g on X are said to be *H-cohomologous* if their difference is an H-coboundary. In this case we also say that the H-cocycles f_* and g_* are H-cohomologous. H-coboundaries are always H-superrecurrent and H-cohomologous H-cocycles are either both H-superrecurrent or both H-supertransient (see [4]).

Definition (2.4). Let $f: X \to R$ and let $\overline{R} = R \cup \{\infty\}$ be the one point compactification of R. For $r \in R$ let $N_\varepsilon(r) = \{s \in R : |r - s| < \varepsilon\}$ and let $N_\varepsilon(\infty) = \{s \in R : |s| > 1/\varepsilon\}$.

Define the essential range $\overline{E}(f_*)$ of an H-cocycle f_* to be the set of all $r \in \overline{R}$ such that, for every neighborhood $N_\varepsilon(r)$ of r, for every neighborhood $U_\varepsilon(1)$ of 1 and for every $B \subset X$ with $\mu(B) > 0$,

$$\mu\left[\bigcup_{n \in Z} B \cap T^{-n} B \cap \{x : f_*(n, x) \in N_\varepsilon(r)\} \cap \{x : \omega_n(x) \in U_\varepsilon(1)\}\right] > 0.$$

Set $E(f_*) = \overline{E}(f_*) \cap R$.

Note that since $f_*(0, x) = 0$ and $\omega_0(x) = 1$ it follows that $0 \in E(f_*)$ and hence $E(f_*) \neq \varnothing$. One can show that the essential range $\overline{E}(f_*)$ has the following properties (see [5]):

(1) $E(f_*)$ is a closed subgroup of R.

(2) $\overline{E}(f_*) = \overline{E}(g_*)$ whenever f_* is H-cohomologous to g_*.

(3) $\overline{E}(f_*) = \{0\}$ whenever f_* is an H-coboundary.

(4) $\overline{E}(f_*) = \{0, \infty\}$ whenever f_* is H-supertransient. The converse, however, is not true.

3. Orbit equivalence and H-cocycles for group actions

Let G be a countable group with identity e. Let $(X, \mathscr{B}, \mu)$ be a measure space. An action of G on $(X, \mathscr{B}, \mu)$ is a homomorphism from G into the group of nonsingular automorphisms of X. If $x \in X$ let $G(x) = \{gx : g \in G\}$. We call Gx the orbit of x under G. The action of G is said to be essentially free if $\mu(\bigcup_{g \neq e}\{x : gx = x\}) = 0$.

We define the group $Z(X, G, \mu)$ of H-cocycles on $(X, \mathscr{B}, \mu)$ (under pointwise addition) to be the set of all Borel maps a: $G \times X \to R$ satisfying $a(e, x) = 0$ and $a(g_1 g_2, x) = a(g_1, x) + \frac{d\mu\circ g_1}{d\mu}(x) a(g_2, g_1 x)$ for all $g_1, g_2 \in G$ and $x \in X$, where $\frac{d\mu\circ g}{d\mu}(x)$ denotes the Radon-Nikodým derivative of the measure $\mu \circ g$ defined by $\mu \circ g(A) = \mu(gA)$ with $gA = \{gx : x \in A\}$. The H-cocycle a is said to be an H-coboundary if there exists a Borel measurable function c: $X \to R$ such that $a(g, x) = c(x) - \frac{d\mu\circ g}{d\mu}(x)c(gx)$ for every $g \in G$ and almost every $x \in X$. The function c is called the transfer function of a.

Definition (3.1). Let G_i, $i = 1, 2$, be countable groups acting nonsingularly on $(X_i, \mathscr{B}_i, \mu_i)$, $i = 1, 2$, respectively. The actions G_1, G_2 are said to be *orbit equivalent* if there exists a measure-space isomorphism ϕ: $X_1 \to X_2$ such that

(1) $\mu_1 \circ \phi^{-1} \sim \mu_2$,

(2) $G_2\phi(x) = \phi G_1(x)$ a.e. $x \in X_1$ both hold.

Theorem (3.1). *Let G_i, $i = 1, 2$, be countable groups acting nonsingularly and which are essentially free on $(X_i, \mathscr{B}_i, \mu_i)$, $i = 1, 2$, respectively. Suppose the actions are orbit equivalent. Then, there is a group isomorphism*

$$\Phi: Z(X_1, G_1, \mu_1) \to Z(X_2, G_2, \mu_2)$$

which sends H-coboundaries to H-coboundaries.

This theorem is a special case of a result that appeared in [9] where they discuss the cohomology for a module over an equivalence relation. In this general situation, if the relations are induced by freely acting countable groups G_1, G_2 that are orbit equivalent, then given an abelian Polish group A, the set $U(X, A)$ of Borel maps from X to A can be given the structure of a G_1 and G_2 module. They prove that there is a natural isomorphism between the cohomology groups $H^n(G_1, U(X, A))$ and $H^n(G_2, U(X, A))$. In case the G_i module structure is given by $g \cdot F(x) = \frac{d\mu\circ g}{d\mu}(x)F(gx)$ and $n = 1$, the above reduces to Theorem (3.1). The proof in our situation will be given for completeness.

Proof. By hypothesis, X_1 and X_2 are orbit equivalent. Let ϕ: $X_1 \to X_2$ be the measure space isomorphism satisfying

(1) $\mu_1 \circ \phi^{-1} \sim \mu_2$,

(2) $G_2\phi(x) = \phi G_1(x)$ a.e. $x \in X_1$. Let $\nu = \mu_1 \circ \phi^{-1}$. Define

$$\Phi\colon Z(X_1, G_1, \mu_1) \to Z(X_2, G_2, \mu_2)$$

as follows. Let $a \in Z(X_1, G_1, \mu_1)$. For any $g_2 \in G_2$ and $x_2 \in X_2$ there exists $x_1 \in X_1$ such that $\phi x_1 = x_2$ and by (2) there exists $g_1 \in G_1$ such that $g_2\phi x_1 = g_2 x_2 = \phi g_1 x_1$. Set $\Phi a(g_2, x_2) = \frac{d\nu}{d\mu_2}(x_2)a(g_1, x_1)$. Then, Φa is an H-cocycle on $(X_2, \mathscr{B}_2, \mu_2)$. To see this, let $x_2 \in X_2$ and $g_2, g_2' \in G_2$. There exists $x_1 \in X_1$ such that $\phi x_1 = x_2$ and there exists $g_1, g_1' \in G_1$ such that $g_2\phi x_1 = g_2 x_2 = \phi g_1 x_1$ and $g_2'\phi x_1 = g_2' x_2 = \phi g_1' x_1$. Also ϕ is measure preserving with respect to the measures ν and μ_1 so that $\frac{d\nu\circ\phi}{d\mu_1}(x_1) = 1$ for almost every $x_1 \in X_1$. Now we know that $g_2\phi x_1 = g_2 x_2 = \phi g_1 x_1$ so that

$$\frac{d\nu \circ (\phi \circ g_1)}{d\mu_1}(x_1) = \frac{d\nu \circ \phi}{d\mu_1}(g_1 x_1) \cdot \frac{d\mu_1 \circ g_1}{d\mu_1}(x_1) = \frac{d\mu_1 \circ g_1}{d\mu_1}(x_1).$$

On the other hand,

$$\begin{aligned}\frac{d\nu \circ (\phi \circ g_1)}{d\mu_1}(x_1) &= \frac{d\nu \circ (g_2 \circ \phi)}{d\mu_1}(x_1)\\ &= \frac{d\nu \circ \phi}{d\mu_1}(x_1) \cdot \frac{d\nu \circ g_2}{d\nu}(\phi x_1) = \frac{d\nu \circ g_2}{d\nu}(\phi x_1).\end{aligned}$$

This implies that $\frac{d\mu_1\circ g_1}{d\mu_1}(x_1) = \frac{d\nu\circ g_2}{d\nu}(\phi x_1)$.

Furthermore, since ν is equivalent to μ_2 we have

$$\frac{d\nu \circ g_2}{d\nu}(x_2) \cdot \frac{d\nu}{d\mu_2}(x_2) = \frac{d\nu}{d\mu_2}(g_2 x_2) \cdot \frac{d\mu_2 \circ g_2}{d\mu_2}(x_2).$$

Thus,

$$\begin{aligned}\Phi(g_2 g_2', x_2) &= \frac{d\nu}{d\mu_2}(x_2) \cdot a(g_1 g_1', x_1)\\ &= \Phi a(g_2, x_2) + \frac{d\mu_2 \circ g_2}{d\mu_2}(x_2)\Phi a(g_2', g_2 x_2).\end{aligned}$$

That is, $\Phi a \in Z(X_2, G_2, \mu_2)$.

Now let a be an H-coboundary in $Z(X_1, G_1, \mu_1)$. Then there exists a measurable mapping c: $X \to R$ such that for almost every $x_1 \in X_1$ and for every $g_1 \in G_1$ we have

$$a(g_1, x_1) = c(x_1) - \frac{d\mu_1 \circ g_1}{d\mu_1}(x_1)c(g_1 x_1).$$

Let $g_2 \in G_2$ and $x_2 \in X_2$. There exists $g_1 \in G_1$ and $x_1 \in X_1$ such that $\phi x_1 = x_2$ and $g_2\phi x_1 = g_2 x_2 = \phi g_1 x_1$. Then,

$$\begin{aligned}\Phi a(g_2, x_2) &= \frac{d\nu}{d\mu_2}(x_2)a(g_1, x_1) = \frac{d\nu}{d\mu_2}(x_2)\left[c(x_1) - \frac{d\mu_1 \circ g_1}{d\mu_1}(x_1)c(g_1x_1)\right] \\ &= \frac{d\nu}{d\mu_2}(x_2)c\circ\phi^{-1}(x_2) - \frac{d\mu_2 \circ g_2}{d\mu_2}(x_2)\cdot\frac{d\nu}{d\mu_2}(g_2x_2)c\circ\phi^{-1}(g_2x_2).\end{aligned}$$

That is, Φa is an H-coboundary with respect to μ_2 with transfer function $\frac{d\nu}{d\mu_2}\cdot c\circ\phi^{-1}$. □

Proposition (3.1). *Φa is H-superrecurrent with respect to μ_2 if and only if a is H-superrecurrent with respect to μ_1.*

Proof. Assume that Φa is H-superrecurrent. Let $\varepsilon > 0$ be given and $B \subset X_1$ with $\mu_1(B) > 0$. Choose $M > 0$ sufficiently large so that the set $C = \{x_1 \in B : 1/M < \frac{d\mu_2}{d\nu}(\phi x_1) < M\}$ has positive measure. For $k \in Z$, let $C_k = \{x_1 \in C : k\varepsilon/2 < \log\frac{d\mu_2}{d\nu}(\phi x_1) \le (k+1)\varepsilon/2\}$. Then $C = \bigcup_{k\in Z} C_k$. Hence there exists $k \in Z$ such that $\mu_1(C_k) > 0$. Assume with no loss of generality that $k > 0$ (the proof of the other cases is similar). Then, $\nu(\phi C_k) = \mu_1(C_k) > 0$. But $\nu \sim \mu_2$ so that $\mu_2(C_k) > 0$. By H-superrecurrence of Φa there exists $g_2 \in G_2$ such that the set

$$\phi C_k \cap g_2^{-1}\phi C_k \cap \{x_2 \in X_2 : |\Phi a(g_2, x_2)| < \varepsilon/M\} \\ \cap\left\{x_2 \in X_2 : \left|\log\frac{d\mu_2 \circ g_2}{d\mu_2}(x_2)\right| < \varepsilon/2\right\}$$

has positive μ_2-measure. Fix $g_2 \in G_2$. For every $g_1 \in G_1$, let $C_k^{g_1} = \{x_1 \in C_k : \phi g_1 x_1 = g_2\phi x_1\}$. Then, $C_k = \bigcup_{g_1\in G_1} C_k^{g_1}$ and $\phi C_k = \bigcup_{g_1\in G_1}\phi C_k^{g_1}$. Now,

$$\phi C_k \cap g_2^{-1}\phi C_k = \bigcup_{g_1\in G_1}\{\phi x_1 \in \phi C_k^{g_1} : g_2\phi x_1 \in \phi C_k\} = \bigcup_{g_1\in G_1}\phi[C_k^{g_1}\cap g_1^{-1}C_k].$$

Thus,

$$\begin{aligned}&\phi C_k \cap g_2^{-1}\phi C_k \cap\left\{x_2 : |\Phi a(g_2, x_2)| < \frac{\varepsilon}{M}\right\}\cap\left\{x_2 : \left|\log\frac{d\mu_2\circ g_2}{d\mu_2}(x_2)\right| < \frac{\varepsilon}{2}\right\} \\ &\subset \bigcup_{g_1\in G_1}\phi\left[C_k^{g_1}\cap g_1^{-1}C_k \cap\{x_1 \in X_1 : |a(g_1, x_1)| < \varepsilon\}\right. \\ &\qquad\left.\cap\left\{x_1 \in X_1 : \left|\log\frac{d\mu_1\circ g_1}{d\mu_1}(x_1)\right| < \varepsilon\right\}\right].\end{aligned}$$

So that there exists $g_1 \in G_1$ such that

$$\mu_2\left\{\phi\left[C_k^{g_1}\cap g_1^{-1}C_k\cap\{x_1\in X_1 : |a(g_1, x_1)| < \varepsilon\}\right.\right. \\ \left.\left.\cap\left\{x_1 \in X_1 : \left|\log\frac{d\mu_1\circ g_1}{d\mu_1}(x_1)\right| < \varepsilon\right\}\right]\right\} > 0.$$

This implies that

$$\mu_1\left[C_k^{g_1}\cap g_1^{-1}C_k\cap\{x_1\in X_1 : |a(g_1, x_1)|<\varepsilon\}\right.$$
$$\left.\cap\left\{x_1\in X_1 : \left|\log\frac{d\mu_1\circ g_1}{d\mu_1}(x_1)\right|<\varepsilon\right\}\right]>0.$$

But $C_k^{g_1}\subset C_k\subset C\subset B$, hence

$$\mu_1\left[B\cap g_1^{-1}B\cap\{x_1\in X_1 : |a(g_1, x_1)|<\varepsilon\}\right.$$
$$\left.\cap\left\{x_1\in X_1 : \left|\log\frac{d\mu_1\circ g_1}{d\mu_1}(x_1)\right|<\varepsilon\right\}\right]>0.$$

That is, a is H-superrecurrent with respect to μ_1.

The converse is proved the same way using Φ^{-1} instead of Φ. □

Proposition (3.2). $\infty\in\overline{E}(\Phi a)$ *if and only if* $\infty\in\overline{E}(a)$.

The proof of the above proposition is similar to that of Proposition (3.1).

We have the following generalization of Proposition (3.2). The assumption that ϕ is measure preserving seems to be necessary to include finite values in the essential range.

Proposition (3.3). *If ϕ is measure preserving then $\overline{E}(\Phi a) = \overline{E}(a)$ for every $a\in Z(X_1, G_1, \mu_1)$.*

4. The full group

Let $[T]$ denote the full group of T; that is, $[T]$ is the set of all invertible bimeasurable nonsingular transformations V of X such that there exists a measurable function $n: X\to Z$ with $Vx = T^{n(x)}x$, for almost every $x\in X$. If $f: X\to R$ is measurable and $V\in[T]$ we define $f_*(V, x) = f_*(n(x), x)$. We also define $\omega_V(x) = \omega_{n(x)}(x) = \frac{d\mu\circ T^{n(x)}}{d\mu}(x)$. For $\lambda\in R$, denote by $N_\varepsilon(\lambda)$ the ball with center λ and radius ε, and let $N_\varepsilon(\infty) = \{r\in R : |r| > 1/\varepsilon\}$. The following proposition, a standard exhaustion argument, generalizes Lemma (4.1) of [10].

Proposition (4.1). *If $\lambda\in\overline{E}(f_*)$, then for any set $B\subset X$ with $\mu(B)>0$ and for any $\varepsilon>0$ there exists $V\in[T]$ with $\mu(VB\Delta B) = 0$ such that if $\lambda\in E(f_*)$, then $f_*(V, x)\in N_\varepsilon(\lambda)\cup N_\varepsilon(-\lambda)$ and $|\omega_V(x)-1|<\varepsilon$ a.e. on B and if $\lambda=\infty$, then $f_*(V, x)\in N_\varepsilon(\infty)$ and $|\omega_V(x)-1|<\varepsilon$ a.e. on B.*

Proof. We present the proof of the case $\lambda=\infty$, which will be needed in §§5 and 6. The proof for $\lambda\in\overline{E}(f_*)$ is similar. The proof is done in two steps:

Step 1. Assume with no loss of generality that $\varepsilon<1$. Let $\varepsilon_1 = \frac{\varepsilon}{3}$ and let B be any subset of X of positive measure. Since $\infty\in\overline{E}(f_*)$, there exists an integer n_1 such that

$$\mu[B\cap T^{-n_1}B\cap\{x : f(n_1, x)\in N_{\varepsilon_1}(\infty)\}\cap\{x : |\omega_{n_1}(x)-1|<\varepsilon_1\}]>0.$$

By Rokhlin's Lemma there exists a Borel set $B_1 \subset B$ with $\mu(B_1) > 0$ such that $B_1 \cup T^{n_1} B_1 \subset B$, $B_1 \cap T^{n_1} B_1 = \varnothing$, $f_*(n_1, x) \in N_{\varepsilon_1}(\infty)$, and $|\omega_{n_1}(x) - 1| < \varepsilon_1$. That is,

$$B_1 \subset B \cap T^{-n_1} B \cap \{x : f_*(n_1, x) \in N_{\varepsilon_1}(\infty)\} \cap \{x : |\omega_{n_1}(x) - 1| < \varepsilon_1\}.$$

If $\mu(B \backslash (B_1 \cup T^{n_1} B_1)) > 0$, proceeding as above we can find a set $B_2 \subset (B \backslash (B_1 \cup T^{n_1} B_1))$ and an integer n_2 such that $\mu(B_2) > 0$, $B_2 \cap T^{n_2} B_2 = \varnothing$, $f_*(n_2, x) \in N_{\varepsilon_1}(\infty)$, and $|\omega_{n_2}(x) - 1| < \varepsilon_1$ a.e. on B_2.

Continuing in this manner we can find a sequence of integers n_i and a sequence of measurable sets B_i, such that for each i, $\mu(B_i) > 0$, $B_i \subset B \setminus \bigcup_{j<i}(B_j \cup T^{n_j} B_j)$, $B_i \cap T^{n_i} B_i = B_i \cap T^{n_j} B_j = T^{n_i} B_i \cap T^{n_j} B_j = B_i \cap B_j = \varnothing$ for $i \neq j$. Also, for all $i \geq 1$, and a.e. $x \in B_i$ we have $f_*(n_i, x) \in N_{\varepsilon_1}(\infty)$ and $|\omega_{n_i}(x) - 1| < \varepsilon_1$. We let $\overline{B} = \bigcup_{i=1}^{\infty}(B_i \cup T^{n_i} B_i)$, and we define $V \in [T]$ as follows:

$$Vx = \begin{cases} T^{n_i} x, & \text{if } x \in B_i \text{ for some } i \geq 1, \\ T^{-n_i} x, & \text{if } x \in T^{n_i} B_i \text{ for some } i \geq 1, \\ x, & \text{otherwise.} \end{cases}$$

Thus for almost every $x \in \overline{B}$ either $x \in B_i$ or $x \in T^{n_i} B_i$ for some $i \in N$. If $x \in B_i$ for some i then

$$f_*(V, x) = f_*(n_i, x) \in N_{\varepsilon_1}(\infty) \subset N_{\varepsilon}(\infty),$$

and

$$|\omega_V(x) - 1| = |\omega_{n_i}(x) - 1| < \varepsilon_1 < \varepsilon.$$

If $x \in T^{n_i} B_i$ then $T^{-n_i} x \in B_i$ and

$$f_*(V, x) = f_*(-n_i, x) = -\omega_{-n_i}(x) f_*(n_i, T^{-n_i} x).$$

Also,

$$|f_*(n_i, T^{-n_i} x)| > \frac{1}{\varepsilon_1} \quad \text{and} \quad \frac{1}{1 + \varepsilon_1} < \omega_{-n_i}(x) < \frac{1}{1 - \varepsilon_1}.$$

It follows that

$$|f_*(V, x)| > \frac{1}{\varepsilon} \quad \text{and} \quad |\omega_V(x) - 1| = |\omega_{-n_i}(x) - 1| < \frac{\varepsilon_1}{1 - \varepsilon_1} < \varepsilon$$

a.e. on $\overline{B}$.

This shows that given any subset B of X with $\mu(B) > 0$, there exists a subset $\overline{B}$ (or $\overline{C}$) of B of positive measure with the desired property.

Step 2. Let $\mathscr{A}$ be the collection of all pairs $(\overline{B}, V)$ as defined in Step 1.

We define a partial order $<$ on $\mathscr{A}$ as follows: $(A, U) < (A', U')$ if and only if $A \subset A'$ and U is the restriction of U' onto A. Let $\mathscr{D} = \{(A_n, U_n)\}$ be a chain in $\mathscr{A}$. Let $D = \bigcup_n A_n$ and define U on D by: $Ux = U_n x$ where $x \in A_n$. Then (D, U) is an upper bound of $\mathscr{D}$, so that by Zorn's Lemma $\mathscr{A}$ has a maximal element, say (Y, S). We claim that $Y = B$. For if $\mu(B \setminus Y) > 0$

then $Z = B \setminus Y$ is a set of positive measure, so that by Step 1 there exist a subset Z_0 of Z with positive measure and a transformation $S' \in [T]$ as described in Step 1 such that $(Z_0, S') \in \mathscr{A}$. But then $V' \in [T]$ defined by

$$V'x = \begin{cases} Sx, & \text{if } x \in Y, \\ S'x, & \text{if } x \in Z_0, \\ x, & \text{otherwise,} \end{cases}$$

is such that $(Y \cup Z_0, V') \in \mathscr{A}$, which contradicts the maximality of (Y, S). □

Proposition (4.2). *If f_* is H-superrecurrent, then for every $\varepsilon > 0$ and for any subset B of X of positive measure there exists $V \in [T]$ with $\mu(B \Delta VB) = 0 = \mu(B \setminus \{x : Vx \neq x\})$, $f_*(V, x) \in N_\varepsilon(0)$, and $|\omega_V(x) - 1| < \varepsilon$ a.e. on B.*

Proof. The proof is similar to that of Proposition (4.1). □

Remark. Proposition (4.1) implies that if $\lambda \in \overline{E}(f_*)$ then for every $B \subset X$ of positive measure we have for $\lambda \in E(f_*)$

$$\sup_{V \in [T]} \mu[B \cap V^{-1}B \cap \{x : f_*(V, x) \in N_\varepsilon(\lambda) \cup N_\varepsilon(-\lambda)\} \cap \{x : |\omega_V(x) - 1| < \varepsilon\}] = \mu(B),$$

and for $\lambda = \infty$ we have

$$\sup_{V \in [T]} \mu\big[B \cap V^{-1}B \cap \{x : f_*(V, x) \in N_\varepsilon(\infty)\} \cap \{x : |\omega_V(x) - 1| < \varepsilon\}\big] = \mu(B).$$

Proposition (4.2) implies that if f_* is H-superrecurrent then for every $B \subset X$ with positive measure

$$\sup_{V \in [T]} \mu\big[B \cap V^{-1}B \cap \{x : f_*(V, x) \in N_\varepsilon(0)\} \cap \{x : Vx \neq x\} \cap \{x : |\omega_V(x) - 1| < \varepsilon\}\big] = \mu(B).$$

Consider the measure algebra $\mathscr{A}$, associated with the measure space $(X, \mathscr{B}, \mu)$. We can give $\mathscr{A}$ the Rokhlin topology that is described by the following metric: $d(A, B) = \mu(A \Delta B)$. The following proposition is a modified version of Lemma (2.1) of [2].

Proposition (4.3). *$\lambda \in \overline{E}(f_*)$ if and only if there exists $0 < K < 1$ such that for every $\varepsilon > 0$ and for every set A in a countable dense collection of sets of the measure algebra $\mathscr{A}$, we have*

$$\sup_{V \in [T]} \mu\big[A \cap V^{-1}A \cap \{x : f_*(V, x) \in N_\varepsilon(\lambda) \cup N_\varepsilon(-\lambda)\} \cap \{x : |\omega_V(x) - 1| < \varepsilon\}\big] > K\mu(A)$$

in case $\lambda \in E(f_*)$, and

$$\sup_{V \in [T]} \mu\big[A \cap V^{-1}A \cap \{x : f_*(V, x) \in N_\varepsilon(\infty)\} \cap \{x : |\omega_V(x) - 1| < \varepsilon\}\big] > K\mu(A)$$

in case $\lambda = \infty$.

Proof. The sufficiency of the condition follows immediately from Proposition (4.1). We now show the necessity of the condition. The proof is done only for the case $\lambda = \infty$ (the other case is proved similarly). Let $0 < \varepsilon < 1/2$ and let $B \subset X$ with $\mu(B) > 0$. Choose A from the dense sequence of sets from the measure algebra $\mathscr{A}$, such that $\mu(A\Delta B) < \frac{K}{2(K+3)}\mu(B)$. By hypothesis, there exists $V \in [T]$ such that

$$\mu[A \cap V^{-1}A \cap \{x : f_*(V, x) \in N_\varepsilon(\infty)\} \cap \{x : |\omega_V(x) - 1| < \varepsilon\}] > K\mu(A).$$

Let

$$\overline{A} = A \cap V^{-1}A \cap \{x : f_*(V, x) \in N_\varepsilon(\infty)\} \cap \{x : |\omega_V(x) - 1| < \varepsilon\}.$$

Then $\mu(\overline{A}) > K\mu(A)$. Let $\overline{B} = \overline{A} \cap B$. Since $\mu(B) \leq \mu(A \cap B) + \mu(A\Delta B) < \mu(A \cap B) + \frac{K}{2(K+3)}\mu(B)$, it follows that $\mu(A \cap B) > [1 - \frac{K}{2(K+3)}]\mu(B)$. Thus $\mu(\overline{A}) > K\mu(A \cap B) > \frac{(K^2+6K)}{2(K+3)}\mu(B)$. Then $\mu(\overline{B}) = \mu(\overline{A} \cap B) > \mu(\overline{A}) - \mu(A\Delta B) > \frac{(K^2+5K)}{2(K+3)}\mu(B)$. We claim that $\mu(\overline{A} \cap B \cap V^{-1}B) > 0$. For this we first show that $\mu(V\overline{B} \cap B) > 0$. To this end, observe that

$$\begin{aligned} \mu(V\overline{B}) &\geq (1 - \varepsilon)\mu(\overline{B}) > (1/2)(K^2 + 5K)/2(K + 3)\mu(B) \\ &\geq (K^2 + 5K)/4(K + 3)\mu(B). \end{aligned}$$

Since,

$$(K^2 + 5K)/4(K + 3) - K/2(K + 3) = K/4 > 0,$$

$$\mu(A\Delta B) < K/2(K + 3)\mu(B), \quad \text{and} \quad V\overline{B} \subset V\overline{A} \subset A,$$

it follows that $\mu(V\overline{B} \cap B) > 0$, but $\mu(V\overline{A} \cap VB \cap B) = \mu(V\overline{B} \cap B) > 0$. Since V is nonsingular we have $\mu(\overline{A} \cap B \cap V^{-1}B) > 0$. That is,

$$\mu[(A \cap B) \cap V^{-1}(A \cap B) \cap \{x : f_*(V, x) \in N_\varepsilon(\infty)\} \cap \{x : |\omega_V(x) - 1| < \varepsilon\}] > 0.$$

Thus $\infty \in \overline{E}(f_*)$. □

Proposition (4.4). *If there exists* $0 < K < 1$ *such that for every* $\varepsilon > 0$ *and for every set* A *in a dense sequence of elements of the measure algebra* $\mathscr{A}$, *we have*

$$\begin{aligned} \sup_{V\in[T]} \mu[A \cap V^{-1}A &\cap \{x : f_*(V, x) \in N_\varepsilon(0)\} \\ &\cap \{x : |\omega_V(x) - 1| < \varepsilon\} \cap \{x \in X : Vx \neq x\}] > K\mu(A) \end{aligned}$$

then f_* *is H-superrecurrent.*

Proof. The proof of this proposition is similar to that of Proposition (4.3). □

5. The topology on the set of H-cocycles

Let $\mathscr{C}(X, R, \mu)$ be the set of all Borel maps $f : X \to R$. Define an equivalence relation $\sim$ on $\mathscr{C}(X, R, \mu)$ as follows: $f \sim g$ if and only if $f(x) = g(x)$ for a.e. $x \in X$.

Let $\overline{\mathscr{E}}(X, R, \mu)$ be the set of equivalence classes of $\sim$. We consider the topology on $\overline{\mathscr{E}}(X, R, \mu)$ that is induced by the following metric: $d(f, g) = \int_X \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} d\mu(x)$. As is well known, convergence with respect to d is convergence in measure.

Let $Z(X, R, \mu)$ be the set of H-cocycles. We extend the equivalence relation $\sim$ to $Z(X, R, \mu)$ as follows: $f_* \sim g_*$ if and only if $f_*(l, x) = g_*(l, x)$ a.e. $x \in X$ and for all $l \in Z$. Let $\overline{Z}(X, R, \mu)$ be the set of equivalence classes of H-cocycles. We make $\overline{Z}(X, R, \mu)$ into a topological space by defining the following notion of convergence: $f_*^{(n)}$ converges to f_* if and only if $f_*^{(n)}(l, \cdot)$ converges in measure to $f_*(l, \cdot)$ for all $l \in Z$.

Fix $0 < K < 1$. Let $\{C_k\}_{k=1}^{\infty}$ be a dense sequence in the measure algebra.

Lemma (5.1). *The statement $f_*^{(n)}(l, \cdot)$ converges to $f_*(l, \cdot)$ in measure for all $l \in Z$ is equivalent to the statement $f_*^{(n)}(V, \cdot)$ converges in measure to $f_*(V, \cdot)$ for all $V \in [T]$.*

Proof. Assume that $f_*^{(n)}(l, \cdot) \to f_*(l, \cdot)$ for all $l \in Z$. Let $\varepsilon > 0$ and let $V \in [T]$. There exists a function $n : X \to Z$ such that $Vx = T^{n(x)}x$ for a.e. $x \in X$. Choose $M > 0$ sufficiently large so that the set $Y = \{x \in X : |n(x)| < M\}$ has measure greater than $1 - \varepsilon$. For each $|l| < M$, let $N_l > 0$ be such that $\mu[\{x \in X : |f_*^{(n)}(l, x) - f_*(l, x)| > \varepsilon\}] < \varepsilon/2^{|l|+3}$ for all $l \geq N_l$. Let $N = \max_{|l|<M} N_l$. Then for $n \geq N$ we have

$$\begin{aligned}
&\{x \in X : |f_*^{(n)}(V, x) - f_*(V, x)| > \varepsilon\} \\
&\quad \subset \{x \in Y : |f_*^{(n)}(V, x) - f_*(V, x)| > \varepsilon\} \cup X \setminus Y \\
&\quad = \{x \in Y : |f_*^{(n)}(n(x), x) - f_*(n(x), x)| > \varepsilon\} \cup X \setminus Y \\
&\quad \subset \bigcup_{|l|<M} \{x \in X : |f_*^{(n)}(l, x) - f_*(l, x)| > \varepsilon\} \cup X \setminus Y.
\end{aligned}$$

Thus,

$$\mu[\{x \in X : |f_*^{(n)}(V, x) - f_*(V, x)| > \varepsilon\}] < \sum_{|l|<M} \varepsilon/2^{|l|+3} + \varepsilon/2 < \varepsilon.$$

Thus $f_*^{(n)}(V, \cdot)$ converges to $f_*(V, \cdot)$ in measure. The converse is clear since $T \in [T]$. □

Lemma (5.2). *If for all $V \in [T]$, $f_*^{(n)}(V, \cdot) \to f_*(V, \cdot)$ in measure then for any $V \in [T]$ and any $p \in N$ we have:*

$$\lim_{n\to\infty} \mu[\{x : f_*^{(n)}(V, x) \in N_{1/p}(\infty)\}\Delta\{x : f_*(V, x) \in N_{1/p}(\infty)\}] = 0$$

and

$$\lim_{n\to\infty} \mu[\{x : f_*^{(n)}(V, x) \in N_{1/p}(0)\}\Delta\{x : f_*(V, x) \in N_{1/p}(0)\}] = 0.$$

Proof. Let $\varepsilon > 0$ be given. There exists an $N > 0$ such that

$$\mu[\{x : |f_*^{(n)}(V, x) - f_*(V, x)| > \tfrac{\varepsilon}{2}\}] < \tfrac{\varepsilon}{2}.$$

For $n \geq N$, let $S_n = \{x : |f_*^{(n)}(V, x) - f_*(V, x)| > \frac{\varepsilon}{2}\}$; then $\mu(S_n) < \frac{\varepsilon}{2}$. Let $x \in \{x : f_*(V, x) \in N_{1/(p+\varepsilon)}(\infty)\} \cap (X \setminus S_n)$; then $|f_*^{(n)}(V, x)| \geq |f_*(V, x)| - |f_*(V, x) - f_*^{(n)}(V, x)| > p + \frac{\varepsilon}{2} > p$. Thus $\{x : f_*(V, x) \in N_{1/(p+\varepsilon)}(\infty)\} \cap (X \setminus S_n) \subset \{x : f_*^{(n)}(V, x) \in N_{1/p}(\infty)\}$ and hence

$$\{x : f_*(V, x) \in N_{1/(p+\varepsilon)}(\infty)\} \setminus \{x : f_*^{(n)}(V, x) \in N_{1/p}(\infty)\} \subset S_n .$$

Thus

$$\mu\big[\{x : f_*(V, x) \in N_{1/(p+\varepsilon)}(\infty)\} \setminus \{x : f_*^{(n)}(V, x) \in N_{1/p}(\infty)\}\big] < \tfrac{\varepsilon}{2}$$

for all $n \geq N$. This implies that

$$\lim_{n\to\infty} \mu\big[\{x : f_*(V, x) \in N_{1/p}(\infty)\} \setminus \{x : f_*^{(n)}(V, x) \in N_{1/p}(\infty)\}\big] = 0.$$

Similarly,

$$\lim_{n\to\infty} \mu\big[\{x : f_*^{(n)}(V, x) \in N_{1/p}(\infty)\} \setminus \{x : f_*(V, x) \in N_{1/p}(\infty)\}\big] = 0.$$

Therefore,

$$\lim_{n\to\infty} \mu\big[\{x : f_*^{(n)}(V, x) \in N_{1/p}(\infty)\}\Delta\{x : f_*(V, x) \in N_{1/p}(\infty)\}\big] = 0.$$

The proof of the second statement is similar. □

Lemma (5.3). *For any* $V \in [T]$ *and* $k, p, q \in N$ *the mappings*

$$f_* \to \mu\big[C_k \cap V^{-l}C_k \cap \{x : f_*(V, x) \in N_{1/p}(\infty)\} \cap \{x : \omega_V(x) \in N_{1/q}(1)\}\big]$$

and

$$f_* \to \mu\big[C_k \cap V^{-l}C_k \cap \{x : f_*(V, x) \in N_{1/p}(0)\} \cap \{x : Vx \neq x\} \cap\{x : \omega_V(x) \in N_{1/q}(1)\}\big]$$

are continuous.

Proof. We shall only prove that the first function is continuous since the proof for the second function is similar. For any $f_* \in \overline{Z}(X, R, \mu)$ let

$$C^k_{f_*} = C_k \cap V^{-1}C_k \cap \{x : f_*(V, x) \in N_{1/p}(\infty)\} \cap \{x : \omega_V(x) \in N_{1/q}(1)\}.$$

Observe that for $f_*, g_* \in \overline{Z}(X, R, \mu)$ we have

$$C^k_{f_*}\Delta C^k_{g_*} \subset \{x : f_*(V, x) \in N_{1/p}(\infty)\}\Delta\{x : g(V, x) \in N_{1/p}(\infty)\}.$$

Thus,

$$\begin{aligned} |\mu(C^k_{f_*}) - \mu(C^k_{g_*})| &\leq \mu(C^k_{f_*}\Delta C^k_{g_*}) \\ &\leq \mu\big[\{x : f_*(V, x) \in N_{1/p}(\infty)\}\Delta\{x : g(V, x) \in N_{1/p}(\infty)\}\big]. \end{aligned}$$

The result follows from Lemma (5.2). □

Lemma (5.4). *For any fixed* $k, p, q \in N$ *the mappings* $\overline{\phi}, \overline{\lambda} : \overline{Z}(X, R, \mu) \to R$ *given by*

$$\overline{\phi}(f_*) = \sup_{V\in[T]} \mu[C_k \cap V^{-l} C_k \cap \{x : f_*(V, x) \in N_{1/p}(\infty)\} \cap \{x : \omega_V(x) \in N_{1/q}(1)\}]$$

and

$$\overline{\lambda}(f_*) = \sup_{V\in[T]} \mu[C_k \cap V^{-l} C_k \cap \{x : f_*(V, x) \in N_{1/p}(0)\} \cap \{x : Vx \neq x\} \cap \{x : \omega_V(x) \in N_{1/q}(1)\}]$$

respectively are lower semicontinuous.

Proof. Choose a sequence $V_n \in [T]$ such that

$$\mu[C_k \cap V_n^{-l} C_k \cap \{x : f_*(V_n, x) \in N_{1/p}(\infty)\} \cap \{x : \omega_{V_n}(x) \in N_{1/q}(1)\}]$$

converges to

$$\sup_{V\in[T]} \mu[C_k \cap V^{-l} C_k \cap \{x : f_*(V, x) \in N_{1/p}(\infty)\} \cap \{x : \omega_V(x) \in N_{1/q}(1)\}].$$

For each n, let

$$\overline{\phi}_n(f_*) = \mu[C_k \cap V_n^{-l} C_k \cap \{x : f_*(V_n, x) \in N_{1/p}(\infty)\} \cap \{x : \omega_{V_n}(x) \in N_{1/q}(1)\}].$$

Then $\sup_n \overline{\phi}_n(f_*) = \lim_{n\to\infty} \overline{\phi}_n(f_*) = \overline{\phi}(f_*)$. But by Lemma (5.3) $\overline{\phi}_n$ is a continuous function so that $\overline{\phi}$ is lower semicontinuous. The proof that $\overline{\lambda}$ is lower semicontinuous is similar. □

For each $k, p, q \in N$ let $N_T(k, p, q)$ be the set of all H-cocycles f_* such that

$$\sup_{V\in[T]} \mu[C_k \cap V^{-l} C_k \cap \{x : f_*(V, x) \in N_{1/p}(\infty)\} \cap \{x : \omega_V(x) \in N_{1/q}(1)\}],$$

and let $M_T(k, p, q)$ be the set of all H-cocycles f_* such that

$$\sup_{V\in[T]} \mu[C_k \cap V^{-l} C_k \cap \{x : f_*(V, x) \in N_{1/p}(0)\} \cap \{x : Vx \neq x\} \cap \{x : \omega_V(x) \in N_{1/q}(1)\}].$$

Theorem (5.1). *For every* $k, p, q \in N$ *the sets* $N_T(k, p, q)$ *and* $M_T(k, p, q)$ *are open.*

Proof. Let $\overline{\phi}$ and $\overline{\lambda}$ be the mappings defined in Lemma (5.4). Observe that $N_T(k, p, q) = \overline{\phi}^{-1}[(K\mu(C_k), \infty)]$ and $M_T(k, p, q) = \overline{\lambda}^{-1}[(k\mu(C_k), \infty)]$. Since $\overline{\phi}$ and $\overline{\lambda}$ are lower semicontinuous it follows that $N_T(k, p, q)$ and $M_T(k, p, q)$ are open. □

Proposition (5.1).

$$\{f_* : \infty \in \overline{E}(f_*)\} = \bigcap_{k=1}^{\infty} \bigcap_{p=1}^{\infty} \bigcap_{q=1}^{\infty} N_T(k, p, q)$$

and $\{f_* : f_*$ *is H-superrecurrent*$\} = \bigcap_{k=1}^{\infty} \bigcap_{p=1}^{\infty} \bigcap_{q=1}^{\infty} M_T(k, p, q)$.

Proof. Follows from Propositions (4.3) and (4.4). □

Theorem (5.2). *The set of H-superrecurrent H-cocycles which are not H-coboundaries contains a G_δ set.*

Proof. Since H-coboundaries have essential range the set $\{0\}$, an H-cocycle f_* with $\infty \in \overline{E}(f_*)$ is not an H-coboundary. Thus the set

$$\begin{aligned}\{f_* : f_* \text{ is H-superrecurrent and } \infty \in \overline{E}(f_*)\} \\ = \bigcap_{k=1}^{\infty} \bigcap_{p=1}^{\infty} \bigcap_{q=1}^{\infty} N_T(k, p, q) \cap M_T(k, p, q)\end{aligned}$$

is a G_δ. □

Remarks. (a) It is possible a priori that this G_δ set might be empty.

(b) With respect to this topology the group isomorphism of Theorem (3.1) is in fact a topological group isomorphism.

6. Properties of generic H-cocycles

In §3 we showed that H-cocycles, H-coboundaries, the notions of H-superrecurrence and $\infty \in \overline{E}(f_*)$ are preserved under orbit equivalence. But by Krieger's Theorem (see [14]) every nonsingular transformation is orbit equivalent to an odometer transformation (defined below). We want to show that the set of H-superrecurrent H-cocycles, for a transformation T of type III_λ, $0 < \lambda < 1$, which are not H-coboundaries form a dense G_δ with respect to the topology induced from the topology of convergence in measure as described in §5. To do this we only need to consider H-cocycles for a countable abelian group Γ acting on the underlying space of the dyadic odometer S of type III_λ, where $0 < \lambda < 1$, which is orbit equivalent to S. We show that the set of H-superrecurrent H-cocycles with ∞ in their essential range for the Γ action form a dense G_δ. Then using Krieger's Theorem we extend the result to the set of H-cocycles for any nonsingular transformation of type III_λ, $0 < \lambda < 1$.

Let $X = \{0, 1\}^N$ and $\mathscr{D}$ be the σ-algebra generated by the cylinder sets. Fix $0 < \lambda < 1$ and consider the measure $\mu = \prod_{i=1}^{\infty} \mu_i$ such that for $i \geq 1$, $\mu_i(0) = 1/(1+\lambda)$ and $\mu_i(1) = \lambda/(1+\lambda)$.

Let $T : X \to X$ be the left shift. That is $T(x_1, x_2, \dots) = (x_2, x_3, \dots)$. Let $S : X \to X$ be the odometer transformation. That is, the transformation S is defined as follows: Let $x = (x_1, x_2, \dots) \in X$ where $x_i \in \{0, 1\}$ for all $i \geq 1$. Let $m(x)$ be the least positive integer such that $x_{m(x)} = 0$. That is, $x_i = 1$ for $i < m(x)$. Set $Sx = S(x_1, x_2, \dots) = (y_1, y_2, \dots)$ where

$$y_i = \begin{cases} 0, & \text{if } i < m(x), \\ 1, & \text{if } i = m(x), \\ x_i, & \text{if } i > m(x). \end{cases}$$

For $k \geq 1$, define $\gamma^k : X \to X$ by

$$(\gamma^k x)_i = \begin{cases} x_i, & \text{if } i \neq k, \\ x_i + 1 \pmod 2, & \text{if } i = k. \end{cases}$$

We identify γ^k with the element $(\gamma_1, \gamma_2, \dots)$ of X where

$$\gamma_i = \begin{cases} 0, & \text{if } i \neq k, \\ 1, & \text{if } i = k. \end{cases}$$

We write $\gamma^k = (\gamma_1, \gamma_2, \dots)$; then $\gamma^k x = \gamma^k + x$ where $+$ denotes coordinate-wise addition mod 2. Let Γ denote the group generated by the γ^k's. That is for $\gamma \in \Gamma$, $\gamma_i = 0$ for all i except finitely many. Γ acts on X by coordinate-wise addition. In other words, we have the action $\Gamma \times X \to X$ given by $(\gamma, x) \to \gamma + x$. Let Γ_n denote the subgroup of Γ whose elements consist of all $\gamma \in \Gamma$ such that $\gamma_m = 0$ for all $m \geq n$. Let $\overline{0} = (0, 0, \dots)$ denote the zero element of Γ. Let $\Gamma_0 = \{\overline{0}\}$. Then $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n$. The action Γ is orbit equivalent to the odometer described above.

For $\gamma \in \Gamma$ there exists a least positive integer n such that $\gamma_m = 0$ for all $m \geq n$. Then

$$\omega_\gamma(x) = \frac{\prod_{i=1}^n \mu_i((\gamma + x)_i)}{\prod_{i=1}^n \mu_i(x_i)}.$$

An H-cocycle with respect to Γ is a Borel mapping $a : \Gamma \times X \to R$ satisfying: $a(\overline{0}, x) = 0$ and $a(\gamma + \delta, x) = a(\gamma, x) + \omega_\gamma(x) a(\delta, \gamma + x)$.

Lemma (6.1). *Let $\alpha : X \to X$ be any Borel map. Then $\alpha(\gamma + x)\omega_\gamma(x) = \alpha(x)$ for all $\gamma \in \Gamma_n$ if and only if there exists a Borel map $\beta : X \to R$ such that $\alpha(x) = \frac{1}{\prod_{i=1}^n \mu(x_i)} (\beta \circ T^n)(x)$.*

Proof. Suppose $\alpha(\gamma + x)\omega_\gamma(x) = \alpha(x)$ for all $\gamma \in \Gamma_n$. Then

$$\alpha(\gamma + x) \prod_{i=1}^n \mu_i((\gamma + x)_i) = \alpha(x) \prod_{i=1}^n \mu_i(x_i)$$

for all $\gamma \in \Gamma_n$.

Let $\beta'(x) = \alpha(x) \prod_{i=1}^n \mu_i(x_i)$. We shall show that $\beta'(x)$ is independent of the first n-coordinates of x. To this end, let $x, y \in X$ be such that $T^n x = T^n y$; that is, $x_i = y_i$ for all $i > n$. Then there exists a $\gamma \in \Gamma_n$ such that $y = \gamma + x$. Then

$$\begin{aligned} \beta'(y) &= \alpha(y) \prod_{i=1}^n \mu_i(y_i) = \alpha(\gamma + x) \prod_{i=1}^n \mu_i((\gamma + x)_i) \\ &= \alpha(x) \prod_{i=1}^n \mu_i(x_i) = \beta'(x). \end{aligned}$$

Thus there exists a Borel map $\beta : X \to R$ such that $\beta'(x) = \beta \circ T^n(x)$. That is

$$\alpha(x) = \frac{1}{\prod_{i=1}^n \mu_i(x_i)} (\beta \circ T^n)(x).$$

Conversely, if $\alpha(x) = \frac{1}{\prod_{i=1}^{n}\mu_i(x_i)}(\beta\circ T^n)(x)$ for some Borel function β, then for $\gamma\in\Gamma_n$ we have $\omega_\gamma(x) = \frac{\prod_{i=1}^{n}\mu_i(\gamma_i+x_i)}{\prod_{i=1}^{n}\mu_i(x_i)}$ so that $\alpha(\gamma+x)\omega_\gamma(x) = \alpha(x)$. □

Notation. For every $n\ge 0$ define the Borel map $\rho_n : X\to R$ as follows:

$$\rho_n(x) = \begin{cases} \frac{1}{\prod_{i=1}^{n}\mu_i(x_i)}, & \text{if } n>1,\\ 1, & \text{if } n=0.\end{cases}$$

Then for $\gamma\in\Gamma_n$, $\omega_\gamma(x) = \rho_n(x)/\rho_n(\gamma+x)$.

Proposition (6.1). *Let $a : \Gamma\times X\to R$ be an H-cocycle. Then there exists a sequence of Borel maps $\{\beta_n : n\ge 0\}$ such that*

$$a(\gamma, x) = \sum_{k=0}^{\infty}\rho_k(\gamma+x)(\beta_k\circ T^k)(\gamma+x)\omega_\gamma(x) - \rho_k(x)(\beta_k\circ T^k)(x).$$

Conversely, if $\{\beta_n : n\ge 0\}$ is any sequence of Borel maps, then

$$a(\gamma, x) = \sum_{k=0}^{\infty}\rho_k(\gamma+x)(\beta_k\circ T^k)(\gamma+x)\omega_\gamma(x) - \rho_k(x)(\beta_k\circ T^k)(x).$$

defines an H-cocycle.

Proof. Let a be any H-cocycle. For $n\ge 1$ and $x\in X$, set $x^{(n)} = (x_1, x_2, \dots, x_n, 0, 0, \dots)$ and $x_{(n)} = (0, 0, \dots, 0, x_{n+1}, x_{n+2}, \dots)$. Then, $x = x^{(n)} + x_{(n)}$ where $x^{(n)}\in\Gamma_n$. Thus for $\gamma\in\Gamma_n$ we have $\gamma^{(n)} = \gamma$ and $\gamma_{(n)} = \overline{0}$.

For $n\ge 1$ we define the map $\alpha_n : X\to R$ as follows:

$$\alpha_n(x) = a(x^{(n)}, x_{(n)})\frac{\prod_{i=1}^{n}\mu_i(0)}{\prod_{i=1}^{n}\mu_i(x_i)} = a(x^{(n)}, x_{(n)})\frac{\rho_n(x)}{\rho_n(\overline{0})}.$$

Then for any $\gamma\in\Gamma_n$ we have

$$\alpha_{n+1}(\gamma+x)\omega_\gamma(x) - \alpha_{n+1}(x) = a(\gamma, x) = \alpha_n(\gamma+x)\omega_\gamma(x) - \alpha_n(x)$$

or

$$[\alpha_{n+1}(\gamma+x) - \alpha_n(\gamma+x)]\omega_\gamma(x) = \alpha_{n+1}(x) - \alpha_n(x).$$

For $n\ge 1$, set $\beta_n'(x) = \alpha_{n+1}(x) - \alpha_n(x)$. Then for all $n\ge 1$ we have

$$\beta_n'(\gamma+x)\omega_\gamma(x) = \beta_n'(x),$$

so that by Lemma (6.1) there exists a sequence of Borel maps $\{\beta_n\}$ such that

$$\beta_n'(x) = \rho_n(x)(\beta_n\circ T^n)(x).$$

Let $\beta_0(x) = \alpha_1(x)$. Then

$$a(\gamma, x) = \sum_{k=0}^{\infty}\rho_k(\gamma+x)(\beta_k\circ T^k)(\gamma+x)\omega_\gamma(x) - \rho_k(x)(\beta_k\circ T^k)(x).$$

Conversely, let $\{\beta_n\}$ be a sequence of Borel maps on X. Define a on $\Gamma\times X$ by

$$a(\gamma, x) = \sum_{k=0}^{\infty} \rho_k(\gamma + x)(\beta_k \circ T^k)(\gamma + x)\omega_\gamma(x) - \rho_k(x)(\beta_k \circ T^k)(x).$$

Then by Lemma (6.1) for each $\gamma \in \Gamma_n$, we have

$$a(\gamma, x) = \sum_{k=0}^{n-1} \rho_k(\gamma + x)(\beta_k \circ T^k)(\gamma + x)\omega_\gamma(x) - \rho_k(x)(\beta_k \circ T^k)(x).$$

Let $\alpha_k(x) = \rho_k(x)(\beta_k \circ T^k)(x)$ and let $\gamma_1, \gamma_2 \in \Gamma$. There exists $n \geq 1$ such that $\gamma_1, \gamma_2 \in \Gamma_n$. Then

$$\begin{aligned} a(\gamma_1 + \gamma_2, x) &= \sum_{k=0}^{n-1} \alpha_k(\gamma_1 + \gamma_2 + x)\omega_{\gamma_1+\gamma_2}(x) - \alpha_k(x) \\ &= \sum_{k=0}^{n-1} \alpha_k(\gamma_1 + \gamma_2 + x)\omega_{\gamma_2}(\gamma_1 + x)\omega_{\gamma_1}(x) - \alpha_k(x) \\ &= a(\gamma_1, x) + \omega_{\gamma_1}(x)a(\gamma_2, \gamma_1 + x). \end{aligned}$$

That is, a is an H-cocycle. □

Corollary (6.1). *The set of H-coboundaries is dense in the set of H-cocyles under the action of the group* Γ.

Proof. Let a be an H-cocyle. By Proposition (6.1) there exists a sequence $\{\beta_n\}$ of Borel maps such that

$$a(\gamma, x) = \sum_{k=0}^{\infty} \rho_k(\gamma + x)(\beta_k \circ T^k)(\gamma + x)\omega_\gamma(x) - \rho_k(x)(\beta_k \circ T^k)(x)$$

for every $\gamma \in \Gamma$. For each $n \geq 0$ let $\alpha_n(x) = \sum_{k=0}^{n} \rho_k(x)(\beta_k \circ T^k)(x)$. Then $a(\gamma, x) = \lim_{n\to\infty} \alpha_n(\gamma + x)\omega_\gamma(x) - \alpha_n(x)$. This implies that the set of H-coboundaries is dense. □

Proposition (6.2). *Let $F(X, R, \mu)$ be the set of all equivalences classes of Borel maps from X to R that depend on finitely many coordinates only. Let $\overline{Z}'(X, R, \mu) = \overline{Z}'$ be the set of equivalence classes of H-cocycles a such that: $a(\gamma, x) = \sum_{k=0}^{\infty} \rho_k(\gamma+x)(\beta_k \circ T^k)(\gamma+x)\omega_\gamma(x) - \rho_k(x)(\beta_k \circ T^k)(x)$ where $\beta_n \in F(X, R, \mu)$ for all $n \geq 0$. Then $\overline{Z}'$ is dense in $\overline{Z}$ and every H-cocycle in $\overline{Z}$ is H-cohomologous to an H-cocycle in $\overline{Z}'$.*

Proof. Let $a \in \overline{Z}$. By Proposition (6.1) there exists a sequence of Borel maps $\{\beta_n\colon n \geq 0\}$ such that $a(\gamma, x) = \sum_{k=0}^{\infty} \rho_k(\gamma + x)(\beta_k \circ T^k)(\gamma + x)\omega_\gamma(x) - \rho_k(x)(\beta_k \circ T^k)(x)$. Now, for each $k \geq 1$, $\rho_k(x) = \frac{1}{\prod_{i=1}^{k} \mu_i(x_i)}$ depends only on the first k coordinates of $x \in X$ for $k \geq 1$. Let $N_k = \min_{x\in X} \rho_k(x)$

and $M_k = \max_{x\in X} \rho_k(x)$. Since $\rho_0(x) = 1$ for all $x \in X$ we let $N_0 = M_0 = 1$. Then, for all $k \geq 0$, $\rho_k(x) \geq 1$, and $N_{n_0}/M_{n_0} \leq 1$. For every $\{\gamma^{(1)}, \dots, \gamma^{(l)}\} \subset \Gamma$ there exists $n_0 \in N$ such that $\gamma^{(i)} \in \Gamma_{n_0}$ for all $1 \leq i \leq l$. Thus, $\omega_{\gamma^{(i)}}(x) = \frac{\prod_{j=1}^{n_0} \mu_j((\gamma^{(i)}+x)_j)}{\prod_{j=1}^{n_0} \mu_j(x_j)}$, which implies that for every $1 \leq i \leq l$ and a.e. $x \in X$ we have $N_{n_0}/M_{n_0} \leq \omega_{\gamma^{(i)}}(x) \leq M_{n_0}/N_{n_0}$. Since $F(X, R, \mu)$ is dense in the set of Borel maps, then for every $\varepsilon > 0$ and for every $\{\gamma^{(1)}, \dots, \gamma^{(l)}\} \subset \Gamma$ there exists a sequence of Borel maps $\{\beta'_k : k \geq 0\}$ in $F(X, R, \mu)$ such that

$$d(\beta_k \circ T^k, \beta'_k \circ T^k) = \int_X \frac{|\beta_k \circ T^k(x) - \beta'_k \circ T^k(x)|}{1 + |\beta_k \circ T^k(x) - \beta'_k \circ T^k(x)|} d\mu(x) < \frac{\varepsilon N_{n_0} 2^{-k-2}}{M_k M_{n_0}}.$$

Then,

$$\begin{aligned} d(\rho_k \cdot \beta_k \circ T^k, \rho_k \cdot \beta'_k \circ T^k) &= \int_X \frac{\rho_k(x)|\beta_k \circ T^k(x) - \beta'_k \circ T^k(x)|}{1 + \rho_k(x)|\beta_k \circ T^k(x) - \beta'_k \circ T^k(x)|} d\mu(x) \\ &< M_k \int_X \frac{|\beta_k \circ T^k(x) - \beta'_k \circ T^k(x)|}{1 + |\beta_k \circ T^k(x) - \beta'_k \circ T^k(x)|} d\mu(x) \\ &< \frac{M_k \cdot \varepsilon N_{n_0} 2^{-k-2}}{M_k M_{n_0}} < \varepsilon \cdot 2^{-k-2}. \end{aligned}$$

Also, for each $k \geq 0$ and all $1 \leq i \leq l$ we have

$$1 + \rho_k(x)\omega_{\gamma^{(i)}}(x)|\beta_k \circ T^k(x) - \beta'_k \circ T^k(x)| \geq \frac{N_{n_0}}{M_{n_0}}(1 + |\beta_k \circ T^k(x) - \beta'_k \circ T^k(x)|).$$

Thus,

$$\begin{aligned} &d(\rho_k \cdot \omega_{\gamma^{(i)}} \cdot \beta_k \circ T^k \circ \gamma^{(i)}, \rho_k \cdot \omega_{\gamma^{(i)}} \cdot \beta'_k \circ T^k \circ \gamma^{(i)}) \\ &= \int_X \frac{\rho_k(x) \cdot \omega_{\gamma^{(i)}}(x)|\beta_k \circ T^k(\gamma^{(i)} + x) - \beta'_k \circ T^k(\gamma^{(i)} + x)|}{1 + \rho_k(\gamma^{(i)} + x) \cdot \omega_{\gamma^{(i)}}(\gamma^{(i)} + x)|\beta_k \circ T^k(\gamma^{(i)} + x) - \beta'_k \circ T^k(\gamma^{(i)} + x)|} d\mu(x) \\ &\leq \frac{M_k \cdot M_{n_0}}{N_{n_0}} \int_X \frac{|\beta_k \circ T^k(\gamma^{(i)} + x) - \beta'_k \circ T^k(\gamma^{(i)} + x)|}{1 + |\beta_k \circ T^k(\gamma^{(i)} + x) - \beta'_k \circ T^k(\gamma^{(i)} + x)|} \omega_{\gamma^{(i)}}(x) d\mu(x) \\ &= \frac{M_k \cdot M_{n_0}}{N_{n_0}} \int_X \frac{|\beta_k \circ T^k(x) - \beta'_k \circ T^k(x)|}{1 + |\beta_k \circ T^k(x) - \beta'_k \circ T^k(x)|} d\mu(x) \\ &< \frac{M_k \cdot M_{n_0}}{N_{n_0}} \cdot \frac{\varepsilon N_{n_0} 2^{-k-2}}{M_k M_{n_0}} = \varepsilon 2^{-k-2}. \end{aligned}$$

Let $c(x) = \sum_{k=0}^{\infty} \rho_k(x)(\beta_k \circ T^k(x) - \beta'_k \circ T^k(x))$. Then c is a well-defined element of $B(X, R, \mu)$. Let

$$a'(\gamma, x) = \sum_{k=0}^{\infty} \rho_k(\gamma + x)(\beta'_k \circ T^k)(\gamma + x)\omega_\gamma(x) - \rho_k(x)(\beta'_k \circ T^k)(\gamma + x).$$

Then, $a(\gamma+x) = a'(\gamma+x) + c(\gamma+x)\omega_\gamma(x) - c(x)$ and $a' \in \overline{Z}'$, so that the H-cocycle a is H-cohomologous to an element in $\overline{Z}'(X, R, \mu)$. Also for $1 \le i \le l$ $d(a(\gamma^{(i)}, \cdot), a'(\gamma^{(i)}, \cdot)) = d(c(\gamma^{(i)} + x)\omega_{\gamma^{(i)}}(x), c(x)) < \varepsilon$. □

Now, consider the sets $N_S(k, p, q)$ and $M_S(k, p, q)$ as defined at the end of §5. Since $[\Gamma] = [S]$, in the definition of $N_S(k, p, q)$ and $M_S(k, p, q)$ the supremum can be taken over all $V \in [\Gamma]$.

Theorem (6.1). *For every $k, p, q \in N$ the sets $N_S(k, p, q)$ and $M_S(k, p, q)$ are dense in $\overline{Z}(X, R, \mu)$.*

Proof. Assume not. Then there exist open sets U, W in $\overline{Z}(X, R, \mu)$ such that $N_S(k, p, q) \cap U = \varnothing$ and $M_S(k, p, q) \cap W = \varnothing$. By Proposition (6.2) there exist

$$a_0(\gamma, x) = \sum_{k=0}^{\infty} \rho_k(\gamma+x)(\beta_k \circ T^k)(\gamma+x)\omega_\gamma(x) - \rho_k(x)(\beta_k \circ T^k)(x)$$

and

$$b_0(\gamma, x) = \sum_{k=0}^{\infty} \rho_k(\gamma+x)(\eta_k \circ T^k)(\gamma+x)\omega_\gamma(x) - \rho_k(x)(\eta_k \circ T^k)(x)$$

in $\overline{Z}'(X, R, \mu)$ with $a_0 \in U$ and $b_0 \in W$. Since U, W are open and a_0, b_0 are interior points of U, W respectively it follows that there exist $\varepsilon_1, \varepsilon_2 > 0$ and finite sets $\{\gamma^{(1)}, \dots, \gamma^{(l)}\}$ and $\{\zeta^{(1)}, \dots, \zeta^{(r)}\}$ contained in Γ such that $Y = \{a \in \overline{Z}(X, R, \mu) : d(a(\gamma^{(i)}, \cdot), a_0(\gamma^{(i)}, \cdot)) < \varepsilon_1, 1 \le i \le l\} \subset U$ and $Y' = \{a \in \overline{Z}(X, R, \mu) : d(a(\zeta^{(i)}, \cdot), b_0(\zeta^{(i)}, \cdot)) < \varepsilon_2, 1 \le i \le r\} \subset W$. Since for each $1 \le i \le l$, $1 \le j \le r$, $\gamma^{(i)}$, $\zeta^{(j)}$, have only finitely many non-zero coordinates, and for each $k \ge 0$, β_k and η_k depend on only finitely many coordinates, it follows that there exist $M_2' > M_1' > 2$ and $M_2 > M_1 > 2$ such that

$$a_0(\gamma^{(i)}, x) = \sum_{k=0}^{M_1} \rho_k(\gamma^{(i)}+x)(\beta_k \circ T^k)(\gamma^{(i)}+x)\omega_{\gamma^{(i)}}(x) - \rho_k(x)(\beta_k \circ T^k)(x),$$

for $1 \le i \le l$,

$$b_0(\zeta^{(j)}, x) = \sum_{p=0}^{M_1'} \rho_p(\zeta^{(j)}+x)(\eta_p \circ T^p)(\zeta^{(j)}+x)\omega_{\zeta^{(j)}}(x) - \rho_p(x)(\eta_p \circ T^p)(x),$$

for $1 \le j \le r$, and for $1 \le k \le M_1$, $1 \le p \le M_1'$, $\beta_k \circ T^k$, $\zeta_p \circ T^p$ depend on the first M_2, M_2' coordinates respectively. Using the recurrence of the Radon-Nikodým derivative we can find $\delta^{(1)}, \sigma^{(1)} \in \Gamma$ both different from the identity and subsets $B_1, B_1' \subset C_k$ such that

(1) $\delta_i^{(1)}, \sigma_j^{(1)} = 0$ for $i \le M_1 + 2$ and $j \le M_1' + 2$,

(2) $B_1 \cap \delta^{(1)} B_1 = B_1' \cap \sigma^{(1)} B_1' = \emptyset$ and $B_1 \cup \delta^{(1)} B_1, B_1' \cup \sigma^{(1)} B_1' \subset C_k$,

(3) $\omega_{\delta^{(1)}}(x) = 1$ a.e. on $B_1 \cup \delta^{(1)} B_1$ and $\omega_{\sigma^{(1)}}(x) = 1$ a.e. on $B_1' \cup \sigma^{(1)} B_1'$.

Since $\delta^{(1)}, \sigma^{(1)} \neq \bar{0}$ it follows that there exist $m_1 \ge k_1 > M_1+2$ and $m_1' \ge k_1' > M_1'+2$ such that $\delta_{k_1}^{(1)} = \sigma_{k_1'}^{(1)} = 1$ and $\delta_n^{(1)} = \sigma_{n'}^{(1)} = 0$ for all $n > m_1$ or $n < k_1$, and $n' > m_1'$ or $n' < k_1'$. We apply a standard exhaustive argument similar to that of Proposition (4.1) to find sequences $\delta^{(i)}, \sigma^{(i)} \in \Gamma$, $B_i, B_i' \subset C_k$, and $\{k_i\}, \{k_i'\}, \{m_i\}, \{m_i'\} \in N$ such that $m_i \ge k_i > m_{i-1} + 2 \ge k_{i-1} + 2$, $m_i' \ge k_i' > m_{i-1}' + 2 \ge k_{i-1}' + 2$ and $\delta_{k_i}^{(i)} = \sigma_{k_i'}^{(i)} = 1$, $\delta_n^{(i)} = \sigma_{n'}^{(i)} = 0$ for all $n > m_i$ or $n < k_i$, and $n' > m_i'$ or $n' < k_i'$. Also, $B_i \cap \delta^{(i)} B_i = B_j \cap \delta^{(j)} B_j = B_i \cap \delta^{(j)} B_j = B_i \cap B_j = B_i' \cap \sigma^{(i)} B_i' = B_j' \cap \sigma^{(j)} B_j' = B_i' \cap B_j' = \emptyset$, and $B_i \cup \delta^{(i)} B_i \subset C_k \setminus \bigcup_{j<i} B_j \cup \delta^{(j)} B_j$, $B_i' \cup \sigma^{(i)} B_i' \subset C_k \setminus \bigcup_{j<i} B_j' \cup \sigma^{(j)} B_j'$, and $\mu(C_k \setminus \bigcup_{i \ge 1} B_i \cup \delta^{(i)} B_i) = \mu(C_k \setminus \bigcup_{i \ge 1} B_i' \cup \sigma^{(i)} B_i') = \emptyset$.

Define $R, R' \in [\Gamma]$ as follows:

$$Rx = \begin{cases} x, & \text{if } x \notin \bigcup_{i \ge 1} B_i \cup \delta^{(i)} B_i, \\ \delta^{(i)} + x, & \text{if } x \in B_i \cup \delta^{(i)} B_i, \end{cases}$$

and

$$R'x = \begin{cases} x, & \text{if } x \notin \bigcup_{i \ge 1} B_i' \cup \sigma^{(i)} B_i', \\ \sigma^{(i)} + x, & \text{if } x \in B_i' \cup \sigma^{(i)} B_i'. \end{cases}$$

Then $\omega_R(x) = \omega_R'(x) = 1$ a.e. on X, and $\mu(C_k \Delta R C_k) = \mu(C_k \Delta R' C_k) = 0$. Define $\beta', \eta' : X \to R$ by

$$\beta'(x) = \begin{cases} p, & \text{if } x_1 = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\eta'(x) = \begin{cases} 1, & \text{if } x_1 = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$b(\gamma, x) = \sum_{k=0}^{\infty} \rho_{k_i - 1}(\gamma + x)(\beta' \circ T^{k_i - 1})(\gamma + x)\omega_\gamma(x) - \rho_{k_i - 1}(x)(\beta' \circ T^{k_i - 1})(x),$$

$$b'(\gamma, x) = \sum_{k=0}^{\infty} \rho_{k_i - 2}(\gamma + x)(\eta' \circ T^{k_i - 2})(\gamma + x)\omega_\gamma(x) - \rho_{k_i - 2}(x)(\eta' \circ T^{k_i - 2})(x).$$

Let

$$a(\gamma, x) = \sum_{k=0}^{M_1} \rho_k(\gamma + x)(\beta_k \circ T^k)(\gamma + x)\omega_\gamma(x) - \rho_k(x)(\beta_k \circ T^k)(x) + b(\gamma, x)$$

and

$$a'(\gamma, x) = \sum_{k=0}^{M_1} \rho_k(\gamma + x)(\eta_k \circ T^k)(\gamma + x)\omega_\gamma(x) - \rho_k(x)(\eta_k \circ T^k)(x) + b'(\gamma, x).$$

Observe that for $1 \le i \le l$ and $1 \le j \le r$ we have $b(\gamma^{(i)}, x) = b'(\zeta^{(j)}, x) = 0$, so that

$$a(\gamma^{(i)}, x) = a_0(\gamma^{(i)}, x)$$

and

$$a'(\zeta^{(j)}, x) = b_0(\zeta^{(j)}, x).$$

Thus, $a \in Y$ and $a' \in Y'$. Also for $x \in B$ there exist $i, j \ge 1$ such that $x \in B_i \cup \delta^{(i)} B_i$ and $x \in B'_j \cup \sigma^{(j)} B'_j$ so that

$$a(R, x) = \rho_{k_i - 1}(x)\left[\beta' \circ T^{k_i - 1}(\delta^{(i)} + x) - \beta' \circ T^{k_i - 1}(x)\right] \in N_S(k, p, q)$$

and

$$a(R', x) = \rho_{k_j - 2}(x)[\eta' \circ T^{k_j - 2}(\sigma^{(j)} + x) - \eta' \circ T^{k_j - 2}(x)] \in M_S(k, p, q).$$

That is, $a(R, x) \in N_S(k, p, q) \cap U$, and $a'(R', x) \in M_S(k, p, q) \cap W$, which is a contradiction. □

Theorem (6.2). *The set of H-superrecurrent H-cocycles for the* Γ *action that are not H-coboundaries contains a dense* G_δ *set.*

Proof. By Proposition (5.1) we have

$$\{f_* : f_* \text{ is H-superrecurrent }\} = \bigcap_{k=1}^{\infty} \bigcap_{p=1}^{\infty} \bigcap_{q=1}^{\infty} M_S(k, p, q)$$

and $\{f_* : \infty \in \overline{E}(f_*)\} = \bigcap_{k=1}^{\infty} \bigcap_{p=1}^{\infty} \bigcap_{q=1}^{\infty} N_S(k, p, q)$. Also, by Theorem (6.1) for each (k, p, q), $M_S(k, p, q)$ and $N_S(k, p, q)$ are dense. Therefore, the set $\{f_* : f_* \text{ is H-superrecurrent and } \infty \in \overline{E}(f_*)\}$ is a nonempty dense G_δ set. □

Theorem (6.3). *Let* R *be an arbitrary nonsingular transformation of type* III_λ *where* $0 < \lambda < 1$. *Then the set of H-superrecurrent H-cocycles of* R *which are not H-coboundaries contains a dense* G_δ *set.*

Proof. Let R be a nonsingular transformation of type III_λ where $0 < \lambda < 1$. Then R is orbit equivalent to the odometer S as described at the beginning of this section. Let Γ be the countable abelian group defined above which is orbit equivalent to the odometer S. By orbit equivalence the set of H-superrecurrent H-cocycles with ∞ in their essential range for the odometer transformation is also a dense G_δ set and hence the same is true for any nonsingular transformation of type III_λ. □

Question. Let R be a nonsingular transformation of type III_λ. Does a dense G_δ set of H-cocycles under R satisfy $\overline{E}(f_*) = \{0, \infty\}$?

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DEPARTMENT OF MATHEMATICS, THE GEORGE WASHINGTON UNIVERSITY, WASHINGTON, D.C. 20052

Current address: Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, North Carolina 27599-3250