

## MÖBIUS INVARIANT HILBERT SPACES OF HOLOMORPHIC FUNCTIONS IN THE UNIT BALL OF $\mathbb{C}^n$

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**ABSTRACT.** We prove that there exists a unique Hilbert space of holomorphic functions in the open unit ball of  $\mathbb{C}^n$  whose (semi-) inner product is invariant under Möbius transformations.

### 1. INTRODUCTION

Let  $B_n$  be the open unit ball in  $\mathbb{C}^n$  and  $\text{Aut}(B_n)$  be the Möbius group of biholomorphic mappings from  $B_n$  onto  $B_n$ . Let  $H$  be a Hilbert space of holomorphic functions in  $B_n$ . In this paper, a Hilbert space will be a linear space with a complete semi-inner product. Furthermore, we assume that  $H$  contains all polynomials and the polynomials are dense in  $H$ . We say that  $H$  is Möbius invariant if  $f \circ \varphi \in H$  and  $\|f \circ \varphi\| = \|f\|$  whenever  $f \in H$  and  $\varphi \in \text{Aut}(B_n)$ . The main result of the paper is the following

**Theorem.** *There exists a unique Hilbert space of holomorphic functions in  $B_n$  which is Möbius invariant.*

When  $n = 1$ , the above result was established in [1]. The unique Möbius invariant Hilbert space in this case is the Dirichlet space  $\mathcal{D}$  consisting of holomorphic functions  $f$  in the open unit disc  $\mathbb{D}$  of  $\mathbb{C}$  such that

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < +\infty,$$

where  $dA$  is the (normalized) area measure on  $\mathbb{D}$ . The (semi-) inner product in  $\mathcal{D}$  is given by

$$\langle f, g \rangle = \int_{\mathbb{D}} f'(z) \overline{g'(z)} dA(z).$$

We remark that the condition  $\|f \circ \varphi\| = \|f\|$  for all  $f \in H$  and  $\varphi \in \text{Aut}(B_n)$  is equivalent to the condition  $\langle f \circ \varphi, g \circ \varphi \rangle = \langle f, g \rangle$  for all  $f, g \in H$  and  $\varphi \in \text{Aut}(B_n)$ .

The organization of the paper is as follows: In the next section, we present some preliminary results which will be needed for the proof of the main theorem. In §3, we prove the uniqueness of invariant (semi-) inner products. In

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§4 we prove that the inner product obtained in the uniqueness proof is indeed Möbius invariant. This will establish the existence. §§5 and 6 discuss other possible ways of describing the invariant inner product and the invariant Hilbert space.

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### 2. PRELIMINARY RESULTS

For any ordered  $n$ -tuple  $\alpha = (\alpha_1, \dots, \alpha_n)$  of nonnegative integers, we use the following abbreviated notations:

$$\begin{aligned} |\alpha| &= \alpha_1 + \dots + \alpha_n, \\ \alpha! &= \alpha_1! \dots \alpha_n!, \\ z^\alpha &= z_1^{\alpha_1} \dots z_n^{\alpha_n}, \\ \frac{\partial^{|\alpha|} f}{\partial z^\alpha} &= \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}, \end{aligned}$$

where  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$  and  $f(z)$  is holomorphic in  $B_n$ . Recall that the inner product in  $\mathbb{C}^n$  is given by

$$\langle z, w \rangle = \sum_{k=1}^n z_k \bar{w}_k.$$

We simply write

$$|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \dots + |z_n|^2}$$

for any  $z$  in  $\mathbb{C}^n$ .

Let  $\mathcal{U}_n$  be the group of unitary operators on the Hilbert space  $\mathbb{C}^n$ . It is clear that  $\mathcal{U}_n$  is a subgroup of  $\text{Aut}(B_n)$ . In fact,  $\mathcal{U}_n$  is the isotropy subgroup of  $\text{Aut}(B_n)$  at 0 when we consider the natural action of  $\text{Aut}(B_n)$  on  $B_n$ . Therefore, for  $\varphi \in \text{Aut}(B_n)$ , we have  $\varphi \in \mathcal{U}_n$  iff  $\varphi(0) = 0$ .

For any  $a \in B_n$ , define  $\varphi_a \in \text{Aut}(B_n)$  as follows: If  $a = 0$ , then  $\varphi_a(z) \equiv -z$ . If  $a \neq 0$ , then

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1 - |a|^2} P_a^\perp z}{1 - \langle z, a \rangle},$$

where  $P_a$  is the orthogonal projection from  $\mathbb{C}^n$  onto the complex line  $[a]$  spanned in  $\mathbb{C}^n$  by  $a$ ,  $P_a^\perp$  is the orthogonal projection to the orthogonal complement of  $[a]$  in  $\mathbb{C}^n$ . It is easy to see that  $\varphi_a(0) = a$ ,  $\varphi_a(a) = 0$ , and  $\varphi_a \circ \varphi_a(z) \equiv z$ . For these and other properties of  $\varphi_a$ , see [6, 7, 4].

Given  $\varphi \in \text{Aut}(B_n)$ , let  $a = \varphi^{-1}(0)$  and  $U = \varphi \circ \varphi_a$ , then  $U \in \text{Aut}(B_n)$  and  $U(0) = \varphi(\varphi_a(0)) = \varphi(a) = 0$ . Thus  $U$  is a unitary. Since  $\varphi_a$  is involutive, we have  $\varphi = U \circ \varphi_a$ . This shows that  $\text{Aut}(B_n)$  is generated by  $\mathcal{U}_n$  and  $\{\varphi_a : a \in B_n\}$ . The following lemma will be needed in the proof of the main theorem.

**Lemma 1.** *Aut( $B_n$ ) is generated by  $\mathcal{U}_n$  and all  $\varphi_a$  with  $a = (r, 0, \dots, 0)$  and  $0 \leq r < 1$ .*

*Proof.* When  $a = (r, 0, \dots, 0)$ , we simply write  $\varphi_a = \varphi_r$ . Since  $\text{Aut}(B_n)$  is generated by  $\mathcal{U}_n$  and  $\{\varphi_a : a \in B_n\}$ , it suffices to show that each  $\varphi_a$  ( $a \in B_n$ ) can be written as a product of unitaries and some  $\varphi_r$  ( $0 \leq r < 1$ ).

Given  $a \in B_n$ . If  $a = 0$ , then  $\varphi_a$  is already a unitary. So we may assume  $a \neq 0$ . We show that there are  $U, V \in \mathcal{U}_n$  such that  $\varphi_a = \bigcup \varphi_{|a|} V$ . Suppose  $U \in \mathcal{U}_n$ , then  $UP_a U^*$  is a one-dimensional projection in  $\mathbb{C}^n$ . Since

$$UP_a U^*(Ua) = UP_a(a) = Ua,$$

$UP_a U^*$  must be the projection onto the complex line  $[Ua]$ . Thus  $UP_a U^* = P_{Ua}$ . It follows that  $UP_a^\perp U^* = P_{Ua}^\perp$  and  $U\varphi_a U^* = \varphi_{Ua}$ .

It is easy to see that there exists a unitary  $U \in \mathcal{U}_n$  such that  $U(|a|, 0, \dots, 0) = a$ . The above argument now implies that  $\varphi_a = U\varphi_{|a|} U^*$ , completing the proof of Lemma 1.  $\square$

Note that by the definition of  $\varphi_a$ , we have

$$\varphi_r(z) = \left( \frac{r - z_1}{1 - rz_1}, -\frac{\sqrt{1 - r^2} z_2}{1 - rz_1}, \dots, -\frac{\sqrt{1 - r^2} z_n}{1 - rz_1} \right)$$

for all  $z \in B_n$  and  $r \in [0, 1)$ .

In this paper, we'll be assuming that all Hilbert spaces of holomorphic functions in  $B_n$  contain the polynomials and the polynomials are dense in them. It is natural to ask if this condition is always satisfied. For invariant function spaces, we have

**Lemma 2.** *Suppose  $H$  is a linear space of holomorphic functions in  $B_n$  with a complete semi-inner product which is invariant under  $\text{Aut}(B_n)$ . Then  $H$  contains all the polynomials and the polynomials are dense in it iff  $H$  contains a nonconstant function and  $\mathcal{U}_n$  acts on  $H$  continuously.*

*Proof.* The “only if” part will follow from our main result. The “if” part has a proof similar to that of the one-dimensional case. See Proposition 2 in [2] and Lemma 3 in [8].  $\square$

*Remark.* We will not need Lemma 2 in the proof of the main theorem.

Let  $dm_n = dx_1 dy_1 \dots dx_n dy_n$  be the Euclidean measure in  $\mathbb{C}^n = R^{2n}$ . For any  $r > 0$ , the volume of any Euclidean ball in  $\mathbb{C}^n$  with radius  $r$  is  $\pi^n r^{2n} / n!$ . Let  $dV_n = \frac{n!}{\pi^n} dm_n$  be the normalized volume measure on  $B_n$  so that  $\int_{B_n} dV_n(z) = 1$ . When  $n = 1$ , we write  $dA = dV_1$ . Clearly,  $dA = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta$  in polar coordinates. The following lemma will be needed in our later discussions.

**Lemma 3.** *If  $n \geq 1$  and  $f(z) = f(z_1)$  only depends on  $z_1$ , then*

$$\int_{B_n} f(z) dV_n(z) = n \int_{\mathbb{D}} (1 - |z_1|^2)^{n-1} f(z_1) dA(z_1),$$

where  $\mathbb{D} = B_1$  is the open unit disc in  $\mathbb{C}$ .

*Proof.*

$$\begin{aligned}
 \int_{B_n} f(z) dV_n(z) &= \frac{n!}{\pi^n} \int_{B_n} f(z) dm_n(z) \\
 &= \frac{n!}{\pi^n} \int_{\mathbb{D}} f(z_1) dx_1 dy_1 \int_{|z_2|^2 + \dots + |z_n|^2 < 1 - |z_1|^2} dx_2 dy_2 \cdots dx_n dy_n \\
 &= \frac{n!}{\pi^n} \int_{\mathbb{D}} f(z_1) \frac{\pi^{n-1} (1 - |z_1|^2)^{n-1}}{(n-1)!} dx_1 dy_1 \\
 &= n \int_{\mathbb{D}} (1 - |z_1|^2)^{n-1} f(z_1) \frac{dx_1 dy_1}{\pi} \\
 &= n \int_{\mathbb{D}} (1 - |z_1|^2)^{n-1} f(z_1) dA(z_1). \quad \square
 \end{aligned}$$

### 3. THE UNIQUENESS

Suppose  $H$  is a Hilbert space of holomorphic functions in  $B_n$  with (semi-) inner product  $\langle \cdot, \cdot \rangle$ . For any  $f, g \in H$ , we write

$$f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}, \quad g(z) = \sum_{\beta} b_{\beta} z^{\beta}.$$

Since the polynomials are dense in  $H$ , we have

$$\langle f, g \rangle = \sum_{\alpha, \beta} a_{\alpha} \bar{b}_{\beta} \langle z^{\alpha}, z^{\beta} \rangle.$$

We compute the inner product  $\langle z^{\alpha}, z^{\beta} \rangle$  in this section under the assumption that  $\langle \cdot, \cdot \rangle$  be Möbius invariant, that is,  $\langle f \circ \varphi, g \circ \varphi \rangle = \langle f, g \rangle$  if  $f, g \in H$  and  $\varphi \in \text{Aut}(B_n)$ . The main result of this section is

**Theorem 4.** *If  $H$  is a Hilbert space of holomorphic functions in  $B_n$  with a nonzero Möbius invariant (semi-) inner product  $\langle \cdot, \cdot \rangle$ , then*

$$\langle f, g \rangle = c \sum_{\alpha} a_{\alpha} \bar{b}_{\alpha} \frac{\alpha!}{|\alpha|!} |\alpha|$$

for some constant  $c > 0$  and all  $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$ ,  $g(z) = \sum_{\alpha} b_{\alpha} z^{\alpha}$  in  $H$ .

It follows from the above theorem that if  $H$  is an invariant Hilbert space of holomorphic functions in  $B_n$ , then

$$H = \left\{ f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha} : \sum_{\alpha} |a_{\alpha}|^2 \frac{\alpha!}{|\alpha|!} |\alpha| < +\infty \right\}.$$

Thus  $H$  is unique. Moreover, the above theorem also implies that the inner product in  $H$  is unique up to a positive multiple. The canonical inner product in  $H$  is

$$\langle f, g \rangle = \sum_{\alpha} a_{\alpha} \bar{b}_{\alpha} \frac{\alpha!}{|\alpha|!} |\alpha|.$$

It is not clear at all that the above inner product is Möbius invariant. This will be proved in the next section, thus establishing the existence of invariant Hilbert spaces of holomorphic functions in  $B_n$ .

In order to prove Theorem 4, it suffices to prove the following three equalities:

- (1)  $\langle z^\alpha, z^\beta \rangle = 0$  if  $\alpha \neq \beta$ ,
- (2)  $\langle z^\alpha, z^\alpha \rangle = \frac{\alpha!}{|\alpha|!} \langle z_1^{|\alpha|}, z_1^{|\alpha|} \rangle$ ,
- (3)  $\langle z_1^k, z_1^k \rangle = k \langle z_1, z_1 \rangle$ .

The constant  $c$  in Theorem 4 is then  $\langle z_1, z_1 \rangle$ .

The proof of (1) is almost trivial. Assume that  $\alpha \neq \beta$ , then there exists some  $1 \leq k \leq n$  such that  $\alpha_k \neq \beta_k$ . We may as well assume  $\alpha_1 \neq \beta_1$ . Let  $U$  be the unitary operator on  $\mathbb{C}^n$  defined by

$$U(z_1, z_2, \dots, z_n) = (z_1 e^{i\theta}, z_2, \dots, z_n),$$

where  $\theta$  is any real number. Since  $H$  is invariant under  $\mathcal{U}_n$ , we must have

$$\langle z^\alpha, z^\beta \rangle = \langle z^\alpha \circ U, z^\beta \circ U \rangle = e^{i(\alpha_1 - \beta_1)\theta} \langle z^\alpha, z^\beta \rangle.$$

Since  $\theta$  is arbitrary and  $\alpha_1 \neq \beta_1$ , we must have  $\langle z^\alpha, z^\beta \rangle = 0$ .

The proof of (3) is similar to that of Theorem 1 in [1]. We reproduce it here for completeness.

Recall that for  $r \in [0, 1)$ ,

$$\varphi_r(z) = \left( \frac{r - z_1}{1 - rz_1}, -\frac{\sqrt{1 - r^2} z_2}{1 - rz_1}, \dots, -\frac{\sqrt{1 - r^2} z_n}{1 - rz_1} \right).$$

The invariance of  $\langle \cdot, \cdot \rangle$  gives  $\langle f, f \rangle = \langle f \circ \varphi_r, f \circ \varphi_r \rangle$  for all  $f$  in  $H$ . Let  $f(z) = 1 - rz_1$ , then

$$\langle f, f \rangle = \langle 1, 1 \rangle + r^2 \langle z_1, z_1 \rangle$$

by (1) and

$$f \circ \varphi_r(z) = 1 - r \frac{r - z_1}{1 - rz_1} = \frac{1 - rz_1 - r^2 + rz_1}{1 - rz_1} = \frac{1 - r^2}{1 - rz_1}.$$

Therefore,

$$\begin{aligned} \langle 1, 1 \rangle + r^2 \langle z_1, z_1 \rangle &= \left\langle \frac{1 - r^2}{1 - rz_1}, \frac{1 - r^2}{1 - rz_1} \right\rangle \\ &= (1 - r^2)^2 \sum_{k, j=0}^{\infty} \langle r^k z_1^k, r^j z_1^j \rangle \\ &= (1 - r^2)^2 \sum_{k=0}^{\infty} r^{2k} \langle z_1^k, z_1^k \rangle \\ &= \langle 1, 1 \rangle + r^2 \langle z_1, z_1 \rangle - 2r^2 \langle 1, 1 \rangle \\ &\quad + \sum_{k=2}^{\infty} (\langle z_1^k, z_1^k \rangle - 2 \langle z_1^{k-1}, z_1^{k-1} \rangle + \langle z_1^{k-2}, z_1^{k-2} \rangle) r^{2k}. \end{aligned}$$

Since  $r \in [0, 1)$  is arbitrary, we must have  $\langle 1, 1 \rangle = 0$  and

$$\langle z_1^k, z_1^k \rangle - 2\langle z_1^{k-1}, z_1^{k-1} \rangle + \langle z_1^{k-2}, z_1^{k-2} \rangle = 0, \quad k \geq 2.$$

Now it is easy to show by induction that

$$\langle z_1^k, z_1^k \rangle = k\langle z_1, z_1 \rangle$$

for all  $k = 0, 1, 2, \dots$ , completing the proof of (3).

In order to prove (2), we need the following

**Lemma 5.** *Suppose  $k \geq 0$  is an integer and  $\lambda_\alpha$  ( $|\alpha| = k$ ) are complex numbers such that*

$$\sum_{|\alpha|=k} \frac{k!}{\alpha_1! \cdots \alpha_n!} t_1^{\alpha_1} \cdots t_n^{\alpha_n} \lambda_\alpha = 1$$

for all  $t_1 + \cdots + t_n = 1$  and  $t_j > 0$  ( $1 \leq j \leq n$ ), then  $\lambda_\alpha = 1$  for all  $|\alpha| = k$ .

*Proof.* The assertion is clearly true for  $n = 1$ . We may as well assume that  $n \geq 2$ . since

$$\sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} t_1^{\alpha_1} \cdots t_n^{\alpha_n} = 1$$

for all  $t_1 + \cdots + t_n = 1$ , we have

$$\sum_{|\alpha|=k} a_\alpha t_1^{\alpha_1} \cdots t_n^{\alpha_n} = 0$$

for all  $t_1 + \cdots + t_n = 1$  and  $t_j > 0$  ( $1 \leq j \leq n$ ), where

$$a_\alpha = \frac{|\alpha|!}{\alpha!} (\lambda_\alpha - 1), \quad |\alpha| = k.$$

For  $|\alpha| = k$  and  $t_1 + \cdots + t_n = 1$ , we can write

$$\begin{aligned} & \sum_{|\alpha|=k} a_\alpha t_1^{\alpha_1} \cdots t_n^{\alpha_n} \\ &= \sum_{|\alpha|=k} a_\alpha t_1^{\alpha_1} \cdots t_{n-1}^{\alpha_{n-1}} (1 - t_1 - \cdots - t_{n-1})^{k - \alpha_1 - \cdots - \alpha_{n-1}} \\ &= (1 - t_1 - \cdots - t_{n-1})^k \\ & \quad \times \sum_{|\alpha|=k} a_\alpha \left( \frac{t_1}{1 - t_1 - \cdots - t_{n-1}} \right)^{\alpha_1} \cdots \left( \frac{t_{n-1}}{1 - t_1 - \cdots - t_{n-1}} \right)^{\alpha_{n-1}}. \end{aligned}$$

It follows that

$$\sum_{|\alpha|=k} a_\alpha x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} = 0$$

for all  $x_j > 0$  ( $1 \leq j \leq n - 1$ ). This implies that  $a_\alpha = 0$  ( $|\alpha| = k$ ). Hence  $\lambda_\alpha = 1$  for all  $|\alpha| = k$ , completing the proof of Lemma 5.  $\square$

We can now prove the main equation (2).

Fix any  $k \geq 1$  and let  $U = (u_{ij})_{n \times n}$  be any unitary matrix. We assume that  $U$  acts on  $\mathbb{C}^n$  by the usual matrix multiplication  $Uz$ , where  $z \in \mathbb{C}^n$  is considered a column vector. It follows that

$$z_1 \circ U(z) = u_{11}z_1 + \dots + u_{1n}z_n.$$

Therefore, the invariance of  $\langle \cdot, \cdot \rangle$  implies that

$$\begin{aligned} \langle z_1^k, z_1^k \rangle &= \langle (u_{11}z_1 + \dots + u_{1n}z_n)^k, (u_{11}z_1 + \dots + u_{1n}z_n)^k \rangle \\ &= \sum_{|\alpha|=k} \sum_{|\beta|=k} \frac{|\alpha|! |\beta|!}{\alpha! \beta!} u_{11}^{\alpha_1} \dots u_{1n}^{\alpha_n} \bar{u}_{11}^{\beta_1} \dots \bar{u}_{1n}^{\beta_n} \langle z^\alpha, z^\beta \rangle \\ &= \sum_{|\alpha|=k} \left( \frac{|\alpha|!}{\alpha!} \right)^2 |u_{11}|^{2\alpha_1} \dots |u_{1n}|^{2\alpha_n} \langle z^\alpha, z^\alpha \rangle. \end{aligned}$$

Since  $U$  is unitary, we have  $|u_{11}|^2 + \dots + |u_{1n}|^2 = 1$ . Also  $U$  is arbitrary, thus we have

$$\langle z_1^k, z_1^k \rangle = \sum_{|\alpha|=k} \left( \frac{|\alpha|!}{\alpha!} \right)^2 t_1^{\alpha_1} \dots t_n^{\alpha_n} \langle z^\alpha, z^\alpha \rangle$$

for all  $t_1 + \dots + t_n = 1$  and  $t_j > 0$  ( $1 \leq j \leq n$ ). Now if  $\langle z_1^k, z_1^k \rangle = 0$ , then clearly all  $\langle z^\alpha, z^\alpha \rangle = 0$  and so  $\langle f, f \rangle = 0$  for all  $f \in H$ , contradicting the fact that  $\langle \cdot, \cdot \rangle$  is nonzero. Thus  $\langle z_1^k, z_1^k \rangle \neq 0$  for all  $k \geq 1$ . Now for any fixed  $k \geq 1$ , let

$$\lambda_\alpha = \frac{|\alpha|! \langle z^\alpha, z^\alpha \rangle}{\alpha! \langle z_1^k, z_1^k \rangle}, \quad |\alpha| = k,$$

then

$$\sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} t_1^{\alpha_1} \dots t_n^{\alpha_n} \lambda_\alpha = 1$$

for all  $t_1 + \dots + t_n = 1$  and  $t_n > 0$  ( $1 \leq j \leq n$ ). By Lemma 5, all  $\lambda_\alpha = 1$ , thus

$$\langle z^\alpha, z^\alpha \rangle = \frac{\alpha!}{|\alpha|!} \langle z_1^{|\alpha|}, z_1^{|\alpha|} \rangle.$$

This completes the proof of Theorem 4.  $\square$

#### 4. THE EXISTENCE

By Theorem 4 in the last section, if  $H$  is an invariant Hilbert space of holomorphic functions in  $B_n$  with semi-inner product  $\langle \cdot, \cdot \rangle$ , then

$$\langle f, g \rangle = c \sum_\alpha a_\alpha \bar{b}_\alpha \frac{\alpha!}{|\alpha|!} |\alpha|$$

for all  $f(z) = \sum_\alpha a_\alpha z^\alpha$  and  $g(z) = \sum_\alpha b_\alpha z^\alpha$  in  $H$ . The purpose of this section is to show that

$$\langle f, g \rangle = \sum_\alpha a_\alpha \bar{b}_\alpha \frac{\alpha!}{|\alpha|!} |\alpha|$$

is indeed a Möbius invariant inner product.

**Theorem 6.** Let  $H$  be the space of holomorphic functions  $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$  in  $B_n$  such that

$$\sum_{\alpha} |a_{\alpha}|^2 \frac{\alpha!}{|\alpha|!} |\alpha| < +\infty.$$

Then  $H$  is a Möbius invariant Hilbert space with (semi-) inner product

$$\langle f, g \rangle = \sum_{\alpha} a_{\alpha} \bar{b}_{\alpha} \frac{\alpha!}{|\alpha|!} |\alpha|$$

for all  $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$ ,  $g(z) = \sum_{\alpha} b_{\alpha} z^{\alpha}$  in  $H$ .

*Proof.* Clearly  $H$  is a nontrivial Hilbert space and the polynomials are dense in  $H$ . It only remains to show that its inner product is invariant under  $\text{Aut}(B_n)$ . By Lemma 1, it suffices to prove the following identities:

$$(1) \quad \langle f \circ U, g \circ U \rangle = \langle f, g \rangle,$$

$$(2) \quad \langle f \circ \varphi_r, g \circ \varphi_r \rangle = \langle f, g \rangle,$$

for all  $f, g$  in  $H$  and  $U \in \mathcal{U}_n$ ,  $r \in [0, 1)$ .

In order to prove (1), we recall the homogeneous expansion of holomorphic functions in  $B_n$ . Suppose  $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$ . For any  $k \geq 0$ , let

$$f_k(z) = \sum_{|\alpha|=k} a_{\alpha} z^{\alpha},$$

then  $f_k$  is a homogeneous polynomial of degree  $k$ , and

$$f(z) = \sum_{k=0}^{\infty} f_k(z).$$

This is called the homogeneous expansion of  $f$ . It is easy to see that the homogeneous expansion of  $f$  is unique and invariant under linear transformations of  $\mathbb{C}^n$ . Thus if  $A$  is a linear transformation on  $\mathbb{C}^n$ , then

$$(f \circ A)_k = f_k \circ A.$$

In particular, for all  $U \in \mathcal{U}_n$ , we have

$$(f \circ U)_k = f_k \circ U.$$

Equation (1) of this section will be a consequence of the following

**Lemma 7.** For any  $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$ ,  $g(z) = \sum_{\alpha} b_{\alpha} z^{\alpha}$  in  $H$ , we have

$$\sum_{\alpha} a_{\alpha} \bar{b}_{\alpha} \frac{\alpha!}{|\alpha|!} |\alpha| = \sum_{k=0}^{\infty} \frac{(n+k)!}{n!k!} k \int_{B_n} f_k(z) \bar{g}_k(z) dV(z),$$

where  $dV$  is the normalized volume measure on  $B_n$ .

*Proof.* Since the volume measure is rotation invariant, we have

$$\int_{B_n} f_k(z) \bar{g}_k(z) dV(z) = \sum_{|\alpha|=k} a_{\alpha} \bar{b}_{\alpha} \int_{B_n} |z^{\alpha}|^2 dV(z).$$



By 1.4.9(2) of [7],

$$\int_{B_n} |z^\alpha|^2 dV(z) = \frac{n!\alpha!}{(n + |\alpha|)!}.$$

Therefore,

$$\int_{B_n} f_k(z) \bar{g}_k(z) dV(z) = \sum_{|\alpha|=k} \frac{n!\alpha!}{(n + k)!} a_\alpha \bar{b}_\alpha.$$

It follows that

$$\begin{aligned} & \sum_{k=0}^\infty \frac{(n + k)!}{n!k!} k \int_{B_n} f_k(z) \bar{g}_k(z) dV(z) \\ &= \sum_{k=0}^\infty \frac{(n + k)!}{n!k!} k \sum_{|\alpha|=k} \frac{n!\alpha!}{(n + k)!} a_\alpha \bar{b}_\alpha \\ &= \sum_{k=0}^\infty \sum_{|\alpha|=k} \frac{\alpha!}{k!} k a_\alpha \bar{b}_\alpha = \sum_\alpha \frac{\alpha!}{|\alpha|!} |\alpha| a_\alpha \bar{b}_\alpha, \end{aligned}$$

and Lemma 7 is proved.  $\square$

Now equation (1) of this section follows from the above lemma and the facts that homogeneous expansions are invariant under linear transformations and the volume measure is invariant under unitary transformations of  $\mathbb{C}^n$ .

In order to prove equation (2) of this section, take  $f, g \in H$  and  $r \in [0, 1)$ . We have

$$\begin{aligned} \langle f \circ \varphi_r, g \circ \varphi_r \rangle &= \left\langle \sum_\alpha a_\alpha z^\alpha \circ \varphi_r, \sum_\beta b_\beta z^\beta \circ \varphi_r \right\rangle \\ &= \sum_\alpha \sum_\beta a_\alpha \bar{b}_\beta \langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle. \end{aligned}$$

So it suffices to prove

$$\langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle = \langle z^\alpha, z^\beta \rangle = \begin{cases} 0, & \alpha \neq \beta, \\ \frac{\alpha!}{|\alpha|!} |\alpha|, & \alpha = \beta. \end{cases}$$

Note that

$$z^\alpha \circ \varphi_r(z) = \left( \frac{r - z_1}{1 - rz_1} \right)^{\alpha_1} \left( -\frac{\sqrt{1 - r^2} z_2}{1 - rz_1} \right)^{\alpha_2} \cdots \left( -\frac{\sqrt{1 - r^2} z_n}{1 - rz_1} \right)^{\alpha_n}.$$

When  $\alpha_j \neq \beta_j$  for some  $2 \leq j \leq n$ , then

$$\langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle = 0$$

since no monomial in the Taylor expansion of  $z^\alpha \circ \varphi_r$  is equal to any monomial in the Taylor expansion of  $z^\beta \circ \varphi_r$  (just looking at the powers of  $z_j$ ). So we may as well assume that

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \quad \beta = (\beta_1, \alpha_2, \dots, \alpha_n).$$

We will prove in this case that

$$\langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle = \begin{cases} 0, & \alpha_1 \neq \beta_1, \\ \langle z^\alpha, z^\alpha \rangle, & \alpha_1 = \beta_1. \end{cases}$$

Let  $N = \alpha_2 + \dots + \alpha_n$ , then

$$z^\alpha \circ \varphi_r(z) = (-1)^N (1 - r^2)^{\frac{N}{2}} \frac{(r - z_1)^{\alpha_1}}{(1 - rz_1)^{\alpha_1 + N}} z_2^{\alpha_2} \dots z_n^{\alpha_n},$$

$$z^\beta \circ \varphi_r(z) = (-1)^N (1 - r^2)^{\frac{N}{2}} \frac{(r - z_1)^{\beta_1}}{(1 - rz_1)^{\beta_1 + N}} z_2^{\alpha_2} \dots z_n^{\alpha_n}.$$

Let

$$F(z_1) = \frac{(r - z_1)^{\alpha_1}}{(1 - rz_1)^{\alpha_1 + N}} = \sum_{k=0}^{\infty} c_k z_1^k,$$

$$G(z_1) = \frac{(r - z_1)^{\beta_1}}{(1 - rz_1)^{\beta_1 + N}} = \sum_{k=0}^{\infty} d_k z_1^k,$$

then

$$\begin{aligned} \langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle &= (1 - r^2)^N \sum_{k=0}^{\infty} c_k \bar{d}_k \langle z_1^k z_2^{\alpha_2} \dots z_n^{\alpha_n}, z_1^k z_2^{\alpha_2} \dots z_n^{\alpha_n} \rangle \\ &= (1 - r^2)^N \sum_{k=0}^{\infty} c_k \bar{d}_k \frac{k! \alpha_2! \dots \alpha_n!}{(k + N)!} (k + N). \end{aligned}$$

We first settle the cases  $N = 0, 1$ .

When  $N = 0$ , we have

$$\begin{aligned} \langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle &= \sum_{k=0}^{\infty} k c_k \bar{d}_k = \int_{\mathbb{D}} F'(z_1) \overline{G'(z_1)} dA(z_1) \\ &= \int_{\mathbb{D}} \left[ \left( \frac{r - z_1}{1 - rz_1} \right)^{\alpha_1} \right]' \left[ \left( \frac{r - z_1}{1 - rz_1} \right)^{\beta_1} \right]' dA(z_1). \end{aligned}$$

A change of variable  $z_1 \rightarrow \frac{r - z_1}{1 - rz_1}$  leads to

$$\begin{aligned} \langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle &= \int_{\mathbb{D}} (z_1^{\alpha_1})' \overline{(z_1^{\beta_1})'} dA(z_1) \\ &= \begin{cases} 0 & \text{if } \alpha_1 \neq \beta_1, \\ \alpha_1 & \text{if } \alpha_1 = \beta_1, \end{cases} \\ &= \langle z^\alpha, z^\beta \rangle \end{aligned}$$

for  $\alpha_2 = \dots = \alpha_n = 0$ . This is actually the one-dimensional result. See [1].

When  $N = 1$ , we have

$$\begin{aligned} \langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle &= (1 - r^2) \sum_{k=0}^\infty c_k \bar{d}_k \\ &= (1 - r^2) \frac{1}{2\pi} \int_0^{2\pi} F(e^{i\theta}) \overline{G(e^{i\theta})} d\theta \\ &= (1 - r^2) \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{r - e^{i\theta}}{1 - re^{i\theta}} \right)^{\alpha_1} \left( \frac{r - e^{-i\theta}}{1 - re^{-i\theta}} \right)^{\beta_1} d\theta. \end{aligned}$$

By the change of variable  $\frac{r - e^{i\theta}}{1 - re^{i\theta}} \rightarrow e^{it}$ , we get

$$\begin{aligned} \langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos t} e^{i(\alpha_1 - \beta_1)t} dt \\ &= \begin{cases} 0 & \text{if } \alpha_1 \neq \beta_1, \\ 1 & \text{if } \alpha_1 = \beta_1, \end{cases} \\ &= \langle z^\alpha, z^\beta \rangle \end{aligned}$$

for  $\alpha_2 + \dots + \alpha_n = 1$ .

Now if  $N \geq 2$ , then

$$\begin{aligned} \langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle &= (1 - r^2)^N \sum_{k=0}^\infty c_k \bar{d}_k \frac{k! \alpha_2! \dots \alpha_n!}{(k + N)!} (k + N) \\ &= (1 - r^2)^N \frac{\alpha_2! \dots \alpha_n!}{(N - 1)!} \sum_{k=0}^\infty c_k \bar{d}_k \frac{k!(N - 1)!}{(k + N - 1)!}. \end{aligned}$$

By 1.4.9(2) of [7],

$$\frac{k!(N - 1)!}{(k + N - 1)!} = \int_{B_{N-1}} |z_1^k|^2 dV(z),$$

where  $dV$  is the normalized volume measure on  $B_{N-1}$ . Therefore,

$$\begin{aligned} \langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle &= (1 - r^2)^N \frac{\alpha_2! \dots \alpha_n!}{(N - 1)!} \sum_{k=0}^\infty c_k \bar{d}_k \int_{B_{N-1}} |z_1^k|^2 dV(z) \\ &= (1 - r^2)^N \frac{\alpha_2! \dots \alpha_n!}{(N - 1)!} \int_{B_{N-1}} F(z_1) \overline{G(z_1)} dV(z). \end{aligned}$$

By Lemma 3,

$$\int_{B_{N-1}} F(z_1) \overline{G(z_1)} dV(z) = (N - 1) \int_{\mathbb{D}} F(z_1) \overline{G(z_1)} (1 - |z_1|^2)^{N-2} dA(z_1).$$

Thus

$$\begin{aligned} \langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle &= \frac{\alpha_2! \dots \alpha_n!}{(N - 2)!} \int_{\mathbb{D}} \left( \frac{r - z_1}{1 - rz_1} \right)^{\alpha_1} \overline{\left( \frac{r - z_1}{1 - rz_1} \right)^{\beta_1}} \\ &\quad \times \frac{(1 - r^2)^N (1 - |z_1|^2)^N}{|1 - rz_1|^{2N}} \frac{dA(z_1)}{(1 - |z_1|^2)^2}. \end{aligned}$$

Note that

$$\frac{(1-r^2)^N(1-|z_1|^2)^N}{|1-rz_1|^{2N}} = \left(1 - \left|\frac{r-z_1}{1-rz_1}\right|^2\right)^N$$

and the measure  $dA(z_1)/(1-|z_1|^2)^2$  is invariant under Möbius transformations of  $\mathbb{D}$ , thus a change of variable gives

$$\langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle = \frac{\alpha_2! \cdots \alpha_n!}{(N-2)!} \int_{\mathbb{D}} z_1^{\alpha_1} \bar{z}_1^{\beta_1} (1-|z_1|^2)^N \frac{dA(z_1)}{(1-|z_1|^2)^2}.$$

Since the measure  $(1-|z_1|^2)^{N-2} dA(z_1)$  is rotation invariant, we have

$$\int_{\mathbb{D}} z_1^{\alpha_1} \bar{z}_1^{\beta_1} (1-|z_1|^2)^{N-2} dA(z_1) = 0$$

if  $\alpha_1 \neq \beta_1$ . Thus

$$\langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle = \langle z^\alpha, z^\beta \rangle$$

if  $\alpha \neq \beta$ . If  $\alpha_1 = \beta_1$ , then  $\alpha = \beta$  and

$$\begin{aligned} \langle z^\alpha \circ \varphi_r, z^\alpha \circ \varphi_r \rangle &= \frac{\alpha_2! \cdots \alpha_n!}{(N-2)!} \int_{\mathbb{D}} |z_1|^{2\alpha_1} (1-|z_1|^2)^{N-2} dA(z_1) \\ &= \frac{\alpha_2! \cdots \alpha_n!}{(N-2)!} \int_0^1 t^{\alpha_1} (1-t)^{N-2} dt. \end{aligned}$$

Using the classical  $B$ -function and  $\Gamma$ -function, we have

$$\int_0^1 t^{\alpha_1} (1-t)^{N-2} dt = B(\alpha_1 + 1, N-1) = \frac{\Gamma(\alpha_1 + 1)\Gamma(N-1)}{\Gamma(\alpha_1 + N)} = \frac{\alpha_1!(N-2)!}{(\alpha_1 + N-1)!}.$$

Therefore,

$$\langle z^\alpha \circ \varphi_r, z^\alpha \circ \varphi_r \rangle = \frac{\alpha_1! \cdots \alpha_n!}{(\alpha_1 + N-1)!} = \frac{\alpha!}{|\alpha|!} |\alpha| = \langle z^\alpha, z^\alpha \rangle.$$

This completes the proof of Theorem 6.  $\square$

### 5. OTHER DESCRIPTIONS OF THE INVARIANT INNER PRODUCT

The results in this section are somewhat negative. We point out many other ways of constructing the invariant inner product on the Dirichlet space  $\mathcal{D}$  of the open unit disc  $\mathbb{D}$ . Then we show one-by-one that these constructions fail in higher dimensions.

The Dirichlet pairing

$$\langle f, g \rangle_{\mathcal{D}} = \int_{\mathbb{D}} f'(z) \overline{g'(z)} dA(z)$$

is clearly an invariant inner product on the Dirichlet space  $\mathcal{D}$ . Our problem here is to try to find a natural analog of this inner product in higher dimensions.

As the first trial, one may be tempted to look at the pairing

$$\langle f, g \rangle = \int_{B_n} \langle \nabla f(z), \nabla g(z) \rangle_{\mathbb{C}^n} dV(z),$$

where  $\nabla f(z)$  is the complex gradient of  $f$  at  $z$ . This is indeed an inner product, but it is easy to see that it is not invariant when  $n \geq 2$ .

Recall that the invariant Laplacian  $\tilde{\Delta}$  of  $B_n$  is defined by

$$\tilde{\Delta}f(z) = \Delta(f \circ \varphi_z)(0),$$

where  $\Delta$  is the usual Laplacian in  $\mathbb{C}^n$ .  $\tilde{\Delta}$  is invariant in the sense that

$$\tilde{\Delta}(f \circ \varphi)(z) = (\tilde{\Delta}f) \circ \varphi(z)$$

for all  $\varphi \in \text{Aut}(B_n)$ . On the unit disc  $\mathbb{D}$ ,

$$\tilde{\Delta}(|f|^2)(z) = 4(1 - |z|^2)^2 |f'(z)|^2$$

for all holomorphic functions  $f$ . It follows that

$$\langle f, g \rangle_{\mathcal{D}} = \int_{\mathbb{D}} \tilde{\Delta}(f\bar{g})(z)K(z, z) dA(z),$$

where  $K(z, w)$  is Bergman kernel of  $\mathbb{D}$ :

$$K(z, w) = \frac{1}{(1 - z\bar{w})^2}.$$

The above inner product is invariant because  $\tilde{\Delta}$  is invariant and the Bergman kernel is invariant. Naturally we look at the generalization to  $B_n$ :

$$\langle f, g \rangle = \int_{B_n} \tilde{\Delta}(f\bar{g})(z)K(z, z) dV(z),$$

where  $K(z, w)$  is the Bergman kernel of  $B_n$ :

$$K(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1}}.$$

The pairing is clearly Möbius invariant since both  $\tilde{\Delta}$  and the Bergman kernel are invariant. Unfortunately, when  $n \geq 2$ , the only holomorphic functions  $f$  in  $B_n$  with

$$\int_{B_n} \tilde{\Delta}(|f|^2)(z)K(z, z) dV(z) < +\infty$$

are the constant functions. See [8].

Let  $P$  be the Bergman projection defined by

$$Pf(z) = \int_{B_n} K(z, w)f(w) dV(w).$$

Given a function  $g \in L^2(B_n, dV)$ , the Hankel operator  $H_g$  on the Bergman space

$$L_a^2(B_n) = \{f \in L^2(B_n, dV) : f \text{ is holomorphic}\}$$

is defined by

$$H_g f = (I - P)(fg),$$

where  $I$  is the identity operator on  $L^2(B_n, dV)$ . Basic properties of Hankel operators can be found in [3–5]. Hankel operators depend on its symbol invariantly, that is, if  $\varphi \in \text{Aut}(B_n)$ , then

$$H_{g \circ \varphi} = U_\varphi H_g U_\varphi^*$$

where  $U_\varphi$  is the unitary operator on  $L^2(B_n, dV)$  defined by

$$U_\varphi f(z) = J_\varphi(z) f(\varphi(z)),$$

$J_\varphi(z)$  is the complex Jacobian determinant of  $\varphi$  at  $z$ . It was shown in [3] that if  $f$  is holomorphic in  $\mathbb{D}$ , then  $H_f$  is Hilbert-Schmidt iff  $f \in \mathcal{D}$ . Moreover,

$$\langle f, g \rangle_{\mathcal{D}} = \text{tr}(H_f^* H_g)$$

for all  $f, g$  in  $\mathcal{D}$ , where  $\text{tr}$  denotes the trace of an operator. Once again, this nice result does not generalize to higher dimensions. As shown in [8], when  $n \geq 2$ , the only Hilbert-Schmidt Hankel operator  $H_f$  with  $f$  holomorphic in  $B_n$  is the zero operator (when  $f$  is a constant).

The so-called Berezin transform is an invariant transform which has attracted much attention lately in function theory and operator theory [4, 5]. We have a brief discussion of it here because of its connection with invariant inner products.

Given a function  $f$  in  $L^1(B_n, dV)$ , let

$$\tilde{f}(z) = \int_{B_n} f(w) \frac{|K(z, w)|^2}{K(z, z)} dV(w), \quad z \in B_n.$$

The function  $\tilde{f}$  is called the Berezin transform of  $f$  (see [4, 5]). Clearly,  $\tilde{\tilde{f}} = \tilde{f}$ , and the reproducing property of  $K(z, w)$  gives  $\tilde{f} = f$  if  $f$  is holomorphic. The invariance of the Bergman kernel  $K(z, w)$  implies the invariance of the Berezin transform namely,

$$\tilde{f \circ \varphi}(z) = \tilde{f}(\varphi(z))$$

for all  $\varphi \in \text{Aut}(B_n)$ .

Now consider the pairing

$$\langle f, g \rangle = \int_{B_n} (\tilde{f} \tilde{g}(z) - \tilde{f}(z) \tilde{g}(z)) K(z, z) dV(z)$$

and the space

$$H_n = \left\{ f: \int_{B_n} (|\tilde{f}|^2(z) - |\tilde{f}(z)|^2) K(z, z) dV(z) < +\infty, \right. \\ \left. f \text{ holomorphic in } B_n \right\}.$$

Since the Berezin transform is invariant and the measure  $K(z, z) dV(z)$  is also invariant,  $H_n$  is an invariant space. It is also easy to see that  $\mathcal{U}_n$  acts on  $H_n$

continuously, thus by Lemma 2,  $H_n$  will be an invariant Hilbert space if  $H_n$  contains a nonconstant function.

When  $n = 1$ , we show that the function  $f(z) = z$  is in  $H_1$ . In fact, for  $f(z) = z$  in  $\mathbb{D}$ , we have

$$\begin{aligned} |\widetilde{f}|^2(z) - |\tilde{f}(z)|^2 &= \int_{\mathbb{D}} |f \circ \varphi_z(w) - f(z)|^2 dA(w) \quad (\text{see [4]}) \\ &= \int_{\mathbb{D}} \left| \frac{z-w}{1-\bar{z}w} - z \right|^2 dA(w) \\ &= \frac{(1-|z|^2)^2}{|z|^4} \left[ \log \frac{1}{1-|z|^2} - |z|^2 \right]. \end{aligned}$$

It follows that for  $f(z) = z$  in  $\mathbb{D}$ ,

$$\int_{\mathbb{D}} (|\widetilde{f}|^2(z) - |\tilde{f}(z)|^2) K(z, z) dA(z) = \int_{\mathbb{D}} \frac{1}{|z|^4} \left[ \log \frac{1}{1-|z|^2} - |z|^2 \right] dA(z) < +\infty.$$

By the uniqueness of invariant Hilbert spaces, we must have  $\mathcal{D} = H_1$  and

$$\langle f, g \rangle_{\mathcal{D}} = c^{-1} \int_{\mathbb{D}} (\tilde{f}\tilde{g}(z) - \widetilde{f\tilde{g}}(z)) K(z, z) dA(z),$$

where

$$c = \int_{\mathbb{D}} \frac{1}{|z|^4} \left[ \log \frac{1}{1-|z|^2} - |z|^2 \right] dA(z) = 1.$$

Once again this construction fails when  $n \geq 2$ . The proof is essentially contained in [8].

Finally in this section, we describe a general method of constructing the invariant inner product on the Dirichlet space  $\mathcal{D}$ .

Suppose  $F$  is a positive continuous function defined on  $[0, +\infty)$ . Consider the pairing

$$\langle f, g \rangle_F = \int_{B_n} \int_{B_n} (f(z) - f(w))(g(z) - g(w)) |K(z, w)|^2 F(|\varphi_z(w)|) dV(z) dV(w).$$

Let  $\rho(z, w) = |\varphi_z(w)|$  be the pseudohyperbolic distance on  $B_n$ .  $\rho$  is invariant in the sense that

$$\rho(\varphi(z), \varphi(w)) = \rho(z, w), \quad \varphi \in \text{Aut}(B_n).$$

The invariance of  $K(z, w)$  and  $\rho$  implies that  $\langle \cdot, \cdot \rangle_F$  is an invariant pairing; that is,

$$\langle f \circ \varphi, g \circ \varphi \rangle_F = \langle f, g \rangle_F$$

for all  $f$  and  $g$  in the space

$$H_F = \{f: \langle f, f \rangle_F < +\infty, f \text{ holomorphic}\}.$$

It is easy to check that  $\mathcal{U}_n$  acts on  $H_F$  continuously. Thus by Lemma 2,  $H_F$  is an invariant Hilbert space of holomorphic functions in  $B_n$  iff  $H_F$  contains a nonconstant function.

**Proposition 8.** *When  $n = 1$ , we have  $\mathcal{D} = H_F$  iff  $\int_0^1 F(\sqrt{r}) \log \frac{1}{1-r} dr < +\infty$ .*

*Proof.* By the above remarks, we have  $\mathcal{D} = H_F$  iff  $z \in H_F$ . First observe that

$$\begin{aligned} \langle f, f \rangle_F &= \int_{B_n} \int_{B_n} |f(z) - f(w)|^2 |K(z, w)|^2 F(|\varphi_z(w)|) dV(z) dV(w) \\ &= \int_{B_n} K(z, z) dV(z) \int_{B_n} |f(z) - f(w)|^2 F(|\varphi_z(w)|) \frac{|K(z, w)|^2}{K(z, z)} dV(w). \end{aligned}$$

Since the real Jacobian determinant of  $\varphi_z$  at  $w$  is precisely  $|K(z, w)|^2 / K(z, z)$  (see 2.2.6 of [7]), a change of variable now gives

$$\begin{aligned} \langle f, f \rangle_F &= \int_{B_n} K(z, z) dV(z) \int_{B_n} |f(z) - f \circ \varphi_z(w)|^2 F(|w|) dV(w) \\ &= \int_{B_n} F(|w|) dV(w) \int_{B_n} |f(z) - f \circ \varphi_z(w)|^2 K(z, z) dV(z). \end{aligned}$$

When  $n = 1$  and  $f(z) = z$ , we have

$$f(z) - f \circ \varphi_z(w) = \frac{w(1 - |z|^2)}{1 - \bar{z}w},$$

thus

$$\int_{\mathbb{D}} |f(z) - f \circ \varphi_z(w)|^2 K(z, z) dA(z) = \int_{\mathbb{D}} \frac{|w|^2}{|1 - \bar{z}w|^2} dA(z) = \log \frac{1}{1 - |w|^2}.$$

Therefore,

$$\begin{aligned} \langle z, z \rangle_F &= \int_{\mathbb{D}} F(|w|) \log \frac{1}{1 - |w|^2} dA(w) \\ &= 2 \int_0^1 r F(r) \log \frac{1}{1 - r^2} dr \\ &= \int_0^1 F(\sqrt{r}) \log \frac{1}{1 - r} dr. \end{aligned}$$

Hence  $\mathcal{D} = H_F$  iff  $z \in H_F$  iff  $\int_0^1 F(\sqrt{r}) \log \frac{1}{1-r} dr < +\infty$ . This completes the proof of Proposition 8.  $\square$

**Corollary 9.** *For any  $f$  holomorphic in  $\mathbb{D}$  and  $F: [0, \infty) \rightarrow (0, +\infty)$  continuous, we have*

$$\begin{aligned} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\bar{w}|^4} F\left(\left|\frac{z-w}{1-z\bar{w}}\right|\right) dA(z) dA(w) \\ = \int_{\mathbb{D}} |f'(z)|^2 dA(z) \cdot \int_0^1 F(\sqrt{r}) \log \frac{1}{1-r} dr. \end{aligned}$$

Note that when  $F \equiv 1$ , we have

$$\int_0^1 \log \frac{1}{1-r} dr = 1$$



and so

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\bar{w}|^4} dA(z) dA(w) = \int_{\mathbb{D}} |f'(z)|^2 dA(z)$$

for any holomorphic function  $f$  in  $\mathbb{D}$ . By choosing other special functions  $f$ , we can obtain many other interesting integral formulae.

We remark that the above construction on  $\mathbb{D}$  does not generalize to  $B_n$  when  $n \geq 2$ . Namely, when  $n \geq 2$ , the only way to make

$$\int_{B_n} \int_{B_n} |f(z) - f(w)|^2 |K(z, w)|^2 F(|\varphi_z(w)|) dV(z) dV(w) < +\infty$$

is to choose  $f$  to be a constant or  $F$  identically zero. The argument is elementary and it reduces to showing that

$$\int_{B_n} |z - \varphi_z(w)|^2 K(z, z) dV(z) = +\infty$$

for all  $n \geq 2$  and  $w \neq 0$ . The details are omitted here.

### 6. OTHER DESCRIPTIONS OF THE INVARIANT HILBERT SPACE

Although we have been unsuccessful in finding a simple intrinsic description of the invariant inner product, there is a way of describing the invariant Hilbert space in terms of the (generalized) radial derivative. What follows is essentially suggested by the referee. The author thanks the referee for his useful comments.

Suppose  $f(z)$  is holomorphic in  $B_n$ . The radial derivative of  $f$  is

$$Rf(z) = \sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}(z).$$

If  $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$ , then it is easy to see that

$$Rf(z) = \sum_{\alpha} |\alpha| a_{\alpha} z^{\alpha}.$$

Given any real number  $t$ , define an operator  $R^t$  on the space of holomorphic functions in  $B_n$  as follows:

$$R^t \left( \sum_{\alpha} a_{\alpha} z^{\alpha} \right) = \sum_{\alpha \neq 0} |\alpha|^t a_{\alpha} z^{\alpha}.$$

We can think of these operators as generalized radial derivatives.

**Theorem 10.** *Suppose  $t > \frac{n}{2}$  and  $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$  is holomorphic in  $B_n$ , then*

$$\sum_{\alpha} |a_{\alpha}|^2 \frac{\alpha!}{|\alpha|!} |\alpha| < +\infty$$

*if and only if*

$$(1 - |z|^2)^t R^t f(z) \in L^2(B_n, d\lambda),$$

where

$$d\lambda(z) = K(z, z) dV(z) = \frac{dV(z)}{(1 - |z|^2)^{n+1}}$$

is the Möbius invariant measure on  $B_n$ .

*Proof.* Fix  $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$  holomorphic in  $B_n$ . Since  $R^t f(z) = \sum |\alpha|^t a_{\alpha} z^{\alpha}$  and  $(1 - |z|^2)^{2t} d\lambda(z)$  is invariant under unitary transformations of  $\mathbb{C}^n$ , we have

$$\begin{aligned} & \int_{B_n} (1 - |z|^2)^{2t} |R^t f(z)|^2 d\lambda(z) \\ &= \sum_{\alpha} |\alpha|^{2t} |a_{\alpha}|^2 \int_{B_n} (1 - |z|^2)^{2t} |z^{\alpha}|^2 d\lambda(z) \\ &= \sum_{\alpha} |\alpha|^{2t} |a_{\alpha}|^2 \int_{B_n} (1 - |z|^2)^{2t-(n+1)} |z^{\alpha}|^2 dV(z). \end{aligned}$$

Using polar coordinates on  $B_n$  (1.4.3 of [7]), we get

$$\begin{aligned} & \int_{B_n} (1 - |z|^2)^{2t-(n+1)} |z^{\alpha}|^2 dV(z) \\ &= 2n \int_0^1 r^{2n-1} (1 - r^2)^{2t-(n+1)} r^{2|\alpha|} dr \int_{\partial B_n} |w^{\alpha}|^2 d\sigma(w), \end{aligned}$$

where  $d\sigma$  is the normalized area measure on  $\partial B_n$ . By 1.4.9 of [7],

$$\int_{\partial B_n} |w^{\alpha}|^2 d\sigma(w) = \frac{(n - 1)! \alpha!}{(n - 1 + |\alpha|)!}.$$

Using  $\Gamma$ -functions, we have

$$\begin{aligned} & 2 \int_0^1 r^{2n-1} (1 - r^2)^{2t-(n+1)} r^{2|\alpha|} dr \\ &= \int_0^1 t^{n-1} (1 - r)^{2t-(n+1)} r^{|\alpha|} dr \\ &= \int_0^1 r^{n+|\alpha|-1} (1 - r)^{2t-n-1} dr \\ &= \frac{\Gamma(n + |\alpha|) \Gamma(2t - n)}{\gamma(|\alpha| + 2t)} = \frac{(n + |\alpha| - 1)! \Gamma(2t - n)}{\Gamma(|\alpha| + 2t)}. \end{aligned}$$

It follows that

$$\int_{B_n} (1 - |z|^2)^{2t-(n+1)} |z^{\alpha}|^2 dV(z) = \frac{n! \alpha! \Gamma(2t - n)}{\Gamma(|\alpha| + 2t)}$$

and

$$\begin{aligned} & \int_{B_n} (1 - |z|^2)^{2t} |R^t f(z)|^2 d\lambda(z) \\ &= n! \Gamma(2t - n) \sum_{\alpha} |a_{\alpha}|^2 |\alpha|^{2t} \frac{\alpha!}{\Gamma(|\alpha| + 2t)} \\ &= n! \Gamma(2t - n) \sum_{\alpha} |a_{\alpha}|^2 \frac{\alpha!}{|\alpha|!} |\alpha| \frac{|\alpha|! |\alpha|^{2t-1}}{\Gamma(|\alpha| + 2t)}. \end{aligned}$$

Stirling's formula shows that

$$\lim_{|\alpha| \rightarrow +\infty} \frac{|\alpha|! |\alpha|^{2t-1}}{\Gamma(|\alpha| + 2t)} = 1.$$

Thus

$$\int_{B_n} (1 - |z|^2)^{2t} |R^t f(z)|^2 d\lambda(z) < +\infty$$

if and only if

$$\sum_{\alpha} |a_{\alpha}|^2 \frac{\alpha!}{|\alpha|!} |\alpha| < +\infty,$$

completing the proof of Theorem 10.  $\square$

**Corollary 11.** *If  $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$  is holomorphic in  $B_n$ , then  $\sum_{\alpha} |a_{\alpha}|^2 \frac{\alpha!}{|\alpha|!} |\alpha| < +\infty$  if and only if*

$$\int_{B_n} |R^{(n+1)/2} f(z)|^2 dV(z) < +\infty.$$

Finally we point out that it is also possible to characterize the invariant Hilbert space in terms of the (generalized) radial derivative on the boundary of  $B_n$ . In fact, the referee computed that for  $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$  holomorphic in  $B_n$ ,  $\sum_{\alpha} |a_{\alpha}|^2 \frac{\alpha!}{|\alpha|!} |\alpha| < +\infty$  if and only if

$$\int_{\partial B_n} |R^{n/2} f(w)|^2 d\sigma(w) < +\infty.$$

The proof is similar to that of Theorem 10, we omit the details here.

**Note added in proof.** After the present paper was accepted for publication, the author was informed that Jaak Peetre obtained essentially the same results (uniqueness and existence) in 1984 in an unpublished manuscript. However, Peetre's method was quite different from the computational approach here.

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