TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 323, Number 2, February 1991

A COUNTABLY COMPACT TOPOLOGICAL GROUP H SUCH THAT $H \times H$ IS NOT COUNTABLY COMPACT

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ABSTRACT. Using $\mathbf{MA}_{\text{countable}}$ we construct a topological group with the properties mentioned in the title.

0. INTRODUCTION

In this paper we construct, assuming Martin's Axiom for countable posets ($\mathbf{MA}_{\text{countable}}$), a countably compact topological group H for which H^2 is not countably compact. The existence of such a group under $\mathbf{MA}$ was announced by van Douwen in [vD] but was never published.

Our construction is like van Douwen's construction in [vD] in that our group is a subgroup of ${}^{\mathfrak{c}}2$ and that we construct the points of our group in an induction of length $\mathfrak{c}$. But whereas van Douwen used $\mathbf{MA}$ to construct a countably compact subgroup of ${}^{\mathfrak{c}}2$ without convergent sequences and then constructed two countably compact subgroups of this group with a countable intersection, we construct our group all at once. Moreover, our group H can be written as $G + D$ with D a countable subgroup and G an ω-bounded subgroup, so that our group has many convergent sequences. Also van Douwen needed $\mathbf{MA}$ for certain uncountable posets, we get by using $\mathbf{MA}_{\text{countable}}$ only.

The interest in examples such as the ones in the present paper comes from the fact that by Comfort and Ross [CR] the product of an arbitrary family of pseudocompact topological groups is pseudocompact. (For more information see [vD].)

This paper is organized as follows: §1 contains some definitions and preliminaries. In §2 we use our technique to construct two countably compact subgroups H_0 and H_1 of ${}^{\mathfrak{c}}2$ such that $H_0 \times H_1$ is not countably compact, and in §3 we construct our main example H. The reason for doing this is that the construction in §3 is rather involved notationwise so that it may be useful to have a more transparant version available. Also, in §3, we essentially construct a countably compact subgroup H of ${}^{\mathfrak{c}}2 \times {}^{\mathfrak{c}}2$ which contains (modified

Received by the editors June 6, 1988 and, in revised form, February 8, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54B10, 54D30, 54G20, 22A99, 03E50.
Key words and phrases. Countably compact topological group, ω-independent family, Martin's Axiom for countable posets.

0002-9947/91 $1.00 + $.25 per page

versions of) $H_0 \times \{\underline{0}\}$ and $\{\underline{0}\} \times H_1$ so that H^2 contains $H_0 \times H_1$ is a closed noncountably compact subgroup.

1. Definitions and preliminaries

1.0. Notation. We follow the usual conventions regarding ordinals and cardinals. As always $\mathfrak{c} = 2^\omega$. For any set X we let

$$[X]^{\leq\omega} = \{A \subseteq X: |A| \leq \omega\}, \qquad [X]^{<\omega} = \{A \subseteq X: |A| < \omega\},$$
$$[X]^\omega = [X]^{\leq\omega} \setminus [X]^{<\omega}.$$

The ordinal $2 = \{0, 1\}$ is a topological group with $0 + 0 = 1 + 1 = 0$ and $0 + 1 = 1 + 0 = 1$, and when given the discrete topology. If X is any set then ${}^X2$ (the set of functions from X to 2) is a compact topological group when given the product group structure and the product topology. Its zero-element, the function $X \times \{0\}$, is written $\underline{0}$. If A is a subset of ${}^X2$ then $\langle\langle A \rangle\rangle$ denotes the subgroup generated by A. Note that $\langle\langle A \rangle\rangle$ is finite when A is finite (this follows from the fact that ${}^X2$ is Boolean, i.e., $\forall x: x + x = \underline{0}$).

1.1. Independent families and ω-bounded subgroups. In our construction we shall need an ω-independent family of size $\mathfrak{c}$ on $\mathfrak{c}$. This is a family $\mathscr{A}$ of subsets of $\mathfrak{c}$ satisfying: if $\mathscr{A}', \mathscr{A}'' \in [\mathscr{A}]^{\leq\omega}$ are disjoint then $|\bigcap \mathscr{A}' \setminus \mathscr{A}''| = \mathfrak{c}$. Such families exist; an easy example can be obtained as follows: for $\alpha \in \mathfrak{c}$ let

$$A_\alpha = \{F \in [\mathfrak{c} \times \mathfrak{c}]^{\leq\omega}: F \cap (\{\alpha\} \times \mathfrak{c}) \neq \varnothing\}.$$

Then $\mathscr{A} = \{A_\alpha: \alpha \in \mathfrak{c}\} \subseteq [\mathfrak{c} \times \mathfrak{c}]^{\leq\omega}$ satisfies the stronger condition

$$\text{if } K \in [\mathfrak{c}]^{\leq\omega} \text{ then } \left| \bigcap_{\alpha \in K} A_\alpha \setminus \bigcup_{\alpha \in \mathfrak{c} \setminus K} A_\alpha \right| = \mathfrak{c}.$$

Of course $\mathscr{A}$ lives not on $\mathfrak{c}$ but on $[\mathfrak{c} \times \mathfrak{c}]^{\leq\omega}$, a set of size $\mathfrak{c}$.

(a) Let $\mathscr{B}$ be an ω-independent family on a set X of size $\mathfrak{c}$. We identify $\mathscr{B}$ with its set of characteristic functions. Thus, we can consider $G = \langle\langle \mathscr{B} \rangle\rangle$ the group generated by $\mathscr{B}$. We then let

$$\begin{aligned} G^+ &= \bigcup \{\overline{E}: E \in [G]^{\leq\omega}\} \\ &= \bigcup \{\overline{E}: E \in [G]^{\leq\omega} \text{ and } E \text{ is a subgroup of } G\}. \end{aligned}$$

Then G^+ is a subgroup of ${}^X2$ and it is easy to see that if $E \subseteq G^+$ is countable then $\overline{E} \subseteq G^+$. It follows that G^+ is ω-bounded (this means that if $E \subseteq G^+$ is countable then $\overline{E}$ (now closure in G^+) is compact) and in fact G^+ is the smallest ω-bounded subgroup of ${}^X2$ containing $\mathscr{B}$. We call G^+ the ω-bounded subgroup generated by $\mathscr{B}$.

(b) In (a) if $x \in G^+ \setminus \{\underline{0}\}$ then we can find $\mathscr{B}_0, \mathscr{B}_1, \mathscr{B}_2 \in [\mathscr{B}]^{\leq\omega}$ such that $\bigcap \mathscr{B}_0 \setminus \bigcup \mathscr{B}_1 \subseteq x^{\leftarrow}(1) \subseteq \bigcup \mathscr{B}_2$, and $\mathscr{B}_0 \cap \mathscr{B}_1 = \varnothing$.

Proof. Let $E \subseteq G$ be countable such that $x \in \overline{E}$. For each $e \in E$ we have a finite set $F_e \subseteq \mathscr{B}$ such that $e = \sum_{B \in F_e} B$. Put $\mathscr{B}_2 = \bigcup_{e \in E} F_e$. Now if $\alpha \notin \bigcup \mathscr{B}_2$ then $e(\alpha) = 0$ for all $e \in E$, as $x \in \overline{E}$ we get $x(\alpha) = 0$. Hence $x^{\leftarrow}(1) \subseteq \bigcup \mathscr{B}_2$. Next pick $\alpha \in x^{\leftarrow}(1)$. The set $U = \{y \in {}^{X}2 : y(\alpha) = 1\}$ is a neighborhood of x in ${}^{X}2$, hence $x \in \overline{U \cap E}$. For $e \in U \cap E$ put $F_e^+ = \{B \in F_e : \alpha \in B\}$ and $F_e^- = F_e \backslash F_e^+$. Clearly if $\mathscr{B}_0 = \bigcup \{F_e^+ : e \in U \cap E\}$ and $\mathscr{B}_1 = \bigcup \{F_e^- : e \in U \cap E\}$ then $\alpha \in \bigcap \mathscr{B}_0 \backslash \bigcup \mathscr{B}_1$.

Now let $\gamma \in \bigcap \mathscr{B}_0 \backslash \bigcup \mathscr{B}_1$ and $e \in U \cap E$. Then $B \in F_e^+$ implies $\gamma \in B$ and $B \in F_e^-$ implies $\gamma \notin B$. It follows that $e(\gamma) = e(\alpha) = 1$. So as $x \in \overline{U \cap E}$, also $x(\gamma) = 1$. Hence $\bigcap \mathscr{B}_0 \backslash \bigcup \mathscr{B}_1 \subseteq x^{\leftarrow}(1)$. □

(c) Let $\mathscr{B}$ be an ω-independent family on $\mathfrak{c}$. For every $B \in \mathscr{B}$ let X_B be such that $|X_B \triangle B| < \mathfrak{c}$. Then $\{X_B : B \in \mathscr{B}\}$ is also ω-independent.

Proof. If $\mathscr{B}', \mathscr{B}'' \subseteq \mathscr{B}$ are countable and disjoint, pick α such that for all $B \in \mathscr{B}' \cup \mathscr{B}''$, $X_B \triangle B \subseteq \alpha$. Then $(\bigcap_{B \in \mathscr{B}'} X_B \backslash \bigcup_{B \in \mathscr{B}''} X_B) \cap [\alpha, \mathfrak{c}) = (\bigcap \mathscr{B}' \backslash \bigcup \mathscr{B}'') \cap [\alpha, \mathfrak{c})$ has cardinality $\mathfrak{c}$. □

1.2. u-limits. In one case we will need the concept of a *u-limit* where $u \in \omega^*$ (the set of free ultrafilters on ω). If $\{x_n : n \in \omega\}$ is a sequence in a topological space X and $x \in X$ then we say $x = u\text{-}\lim x_n$ iff $\{n : x_n \in O\} \in u$ for every neighborhood O of x.

We need the following: if X is countably compact and if $\{x_n : n \in \omega\}$ is a sequence in X then for some $u \in \omega^*$, $u\text{-}\lim x_n$ exists. For let $x \in X \backslash \{x_n : n \in \omega\}$ be a point such that for every open neighborhood O of x, $\{n : x_n \in O\}$ is infinite. The set of all such subsets of ω is a free filter $\mathscr{F}$: for any $u \in \omega^*$ extending $\mathscr{F}$, $x = u\text{-}\lim x_n$.

1.3. Martin's Axiom for countable posets ($\mathbf{MA}_{\text{countable}}$). In topological language this statement says that the real line cannot be covered by fewer than $\mathfrak{c}$ nowhere dense sets. The partial order form, which is the form that we shall use, says the following:

> if $\mathbf{P}$ is a countable partially ordered set and if $\mathscr{D}$ is a collection of fewer than $\mathfrak{c}$ dense sets in $\mathbf{P}$ then there is a filter G on $\mathbf{P}$ intersecting every element of $\mathscr{D}$.

Here a set $D \subseteq \mathbf{P}$ is *dense* iff $(\forall p \in \mathbf{P})(\exists q \leq p)(q \in D)$ and a set $G \subseteq \mathbf{P}$ is a *filter* iff

$$\forall p \in G \forall q \in \mathbf{P} : p \leq q \rightarrow q \in G \quad \text{and}$$

$$\forall p, q \in G \exists r \in G : r \leq p, q.$$

In this paper we shall apply $\mathbf{MA}_{\text{countable}}$ to posets which are subsets of posets of the form $\mathrm{Fn}(I, x)$, with I countable and x finite. Here

$$\mathrm{Fn}(I, x) = \{p : p \text{ is a finite function}, \ \mathrm{dom}(p) \subseteq I, \ \mathrm{ran}(p) \subseteq x\},$$

ordered by reverse inclusion: $p \leq q$ iff $p \supseteq q$. In this case a set D is dense if every p has an extension q such that $q \in D$, and a set G is a filter if it is closed under taking restrictions and if for any two elements p and q of G there is an $r \in G$ extending both p and q. If G is a filter then $\bigcup G$ is a function. Usually for every $i \in I$, $D_i = \{p : i \in \mathrm{dom}(p)\}$ is dense. If $G \cap D_i \neq \emptyset$ for every i then $\mathrm{dom}(\bigcup G) = I$. (For more information on Martin's Axiom see [We].)

2. Construction I

As mentioned in the introduction the construction in the next section needs a lot of notation and this might obscure the main idea. We therefore present an easier version giving two countably compact groups H_0 and H_1 with $H_0 \times H_1$ not countably compact. Also the construction §3 amounts to making a countably compact group H containing (versions of) H_0 and H_1 as closed subgroups and such that $H_0 \cap H_1 = \{\underline{0}\}$.

2.0. What we construct. Using $\mathbf{MA}_{\text{countable}}$ we shall construct three subgroups D, G_0, and G_1 of ${}^{\mathfrak{c}}2$ satisfying:

(i) D is countable,
(ii) G_0 and G_1 are ω-bounded,
(ii) $G_0 \cap G_1 = \{\underline{0}\} = D \cap (G_0 + G_1)$, and
(iv) if $E \subseteq D$ is infinite then E has accumulation points in G_0 and G_1.

Actually $\mathbf{MA}_{\text{countable}}$ is used only in the construction of D.

2.1. Why this works. If we have D, G_0, and G_1 as in 2.0 then we let

$$H_0 = D + G_0 \quad \text{and} \quad H_1 = D + G_1.$$

(a) H_0 and H_1 are countably compact.

Proof. Let $i \in \{0, 1\}$ and let $I = \{d_n + g_n : n \in \omega\} \subseteq H_i$ be infinite. If $\{d_n : n \in \omega\}$ is finite then, since G_i is ω-bounded, $\mathrm{Cl}_H I$ is compact. If $\{d_n : n \in \omega\}$ is infinite let $g \in G_i$ be an accumulation point of it. Pick $u \in \omega^*$ such that $g = u\text{-}\lim d_n$. Then $h = u\text{-}\lim g_n \in G_i$ again since G_i is ω-bounded. It follows that $g + h$ is an accumulation point of I in H. □

(b) $H_0 \times H_1$ is not countably compact.

Proof. By (iii) $H_0 \cap H_1 = D$ so that $\{\langle d, d\rangle : d \in D\}$ is a countable and closed subgroup of $H_0 \times H_1$. □

2.2. Explanation. To begin we take our ω-independent family $\mathscr{A}$ on $\mathfrak{c}$ from 1.1, and we split $\mathscr{A}$ into two families $\mathscr{A}_0$ and $\mathscr{A}_1$ of size $\mathfrak{c}$. We enumerate these as $\langle A_\alpha^0 : \omega \leq \alpha < \mathfrak{c}\rangle$ and $\langle A_\alpha^1 : \omega \leq \alpha < \mathfrak{c}\}$ respectively, without repetitions.

As we construct D we shall modify the sets A_α^0 and A_α^1 a bit to obtain sets B_α^0 and B_α^1 respectively such that $|B_\alpha^i \cap A_\alpha^i| < \mathfrak{c}$ for all α and i. We shall let $\mathscr{B}_i = \{B_\alpha^i : \omega \leq \alpha < \mathfrak{c}\}$ $(i \in 2)$ and $\mathscr{B} = \mathscr{B}_0 \cup \mathscr{B}_1$. By 1.1(c) $\mathscr{B}$ will still be ω-independent. Finally then G_0 and G_1 will be the ω-bounded subgroups of ${}^{\mathfrak{c}}2$ generated by $\mathscr{B}_0$ and $\mathscr{B}_1$ respectively.

Take $x \in G_0\backslash\{\underline{0}\}$ and $y \in G_1\backslash\{\underline{0}\}$. By 1.1(b) we find countable K, L, $M \subseteq [\omega, \mathfrak{c})$ such that

$$\bigcap_{\alpha\in K} B^0_\alpha \backslash \bigcup_{\alpha\in L} B^0_\alpha \subseteq x^{\leftarrow}(1) \quad \text{and} \quad y^{\leftarrow}(1) \subseteq \bigcup_{\alpha\in M} B^1_\alpha.$$

But then

$$\bigcap_{\alpha\in K} B^0_\alpha \backslash \left(\bigcup_{\alpha\in L} B^0_\alpha \cup \bigcup_{\alpha\in M} B^1_\alpha\right)$$

has cardinality $\mathfrak{c}$ and it is contained in $x^{\leftarrow}(1)\backslash y^{\leftarrow}(1)$. It follows that $G_0 \cap G_1 = \{\underline{0}\}$.

Next we consider how to make sure that $D \cap (G_0 + G_1) = \{\underline{0}\}$. For this observe that $G_0 + G_1$ is in fact the ω-bounded subgroup of ${}^{\mathfrak{c}}2$ generated by $\mathscr{B}$. Now if $x \in (G_0 + G_1)\backslash\{\underline{0}\}$ then by 1.1(b) there is a countable $\mathscr{B}' \subseteq \mathscr{B}$ with $x^{\leftarrow}(1) \subseteq \mathscr{B}'$. Thus a sufficient condition for $D \cap (G_0 + G_1) = \{\underline{0}\}$ would be

$$(\forall d \in D\backslash\{\underline{0}\})(\forall \mathscr{B}' \in [\mathscr{B}]^{\omega})(d^{\leftarrow}(1)\backslash \textstyle\bigcup \mathscr{B}') \neq \varnothing.$$

However we know $\mathscr{B}$ only at the end of the construction. But we do know $\mathscr{A}$ in advance and a moments reflection shows that the following is sufficient:

(v) $$(\forall d \in D\backslash\{\underline{0}\})(\forall \mathscr{A}' \in [\mathscr{A}]^{\leq\omega})(|d^{\leftarrow}(1)\backslash \textstyle\bigcup \mathscr{A}'| = \mathfrak{c}).$$

Finally we describe what D will look like. We put $I = \{x \in {}^{\omega}2 : x^{\leftarrow}(1) \text{ is finite}\}$. We will have $D = \{d_x : x \in I\}$ such that for all $x \in I$, $d_x \restriction \omega = x$ and for all $x, y \in I$, $d_x + d_y = d_{x+y}$. We already have 2.0(i) and we indicated how we shall take care of 2.0(ii) and 2.0(iii). For 2.0(iv) we enumerate $[I]^{\omega}$ as $\{E_\alpha : \omega \leq \alpha < \mathfrak{c}\}$. During our construction we will take at step α an accumulation point y_α of $\{d_x \restriction \alpha : x \in E_\alpha\}$ and put $B^i_\alpha = y_\alpha \cup A^i_\alpha \restriction [\alpha, \mathfrak{c})$ $(i \in 2)$. For the rest of the construction we will make sure that B^0_α and B^1_α will be accumulation points of $\{d_x : x \in E_\alpha\}$. We also have to take care of (v). For this we let $\langle\langle x_\alpha, \mathscr{C}_\alpha\rangle : \omega \leq \alpha < \mathfrak{c}\rangle$ enumerative

$$\{\langle x, \mathscr{C}\rangle : x \in I\backslash\{\underline{0}\},\ \mathscr{C} \in [\mathscr{A}]^{\leq\omega}\}$$

in such a way that for every $\langle x, \mathscr{C}\rangle$ the set $\{\alpha : \langle x_\alpha, \mathscr{C}_\alpha\rangle = \langle x, \mathscr{C}\rangle$ and $\alpha \notin \bigcup \mathscr{C}\}$ has cardinality $\mathfrak{c}$. Then (v) follows once we have $d_{x_\alpha}(\alpha) = 1$ for all α.

Now that we know what to do, it only remains to do it.

2.3. The construction of D. We construct $D \restriction \alpha = \{d_x \restriction \alpha : x \in I\}$ by induction on $\alpha \in [\omega, \mathfrak{c}]$. In the end we put $d_x = d_x \restriction \mathfrak{c}$ and $D = \{d_x : x \in I\}$. At every stage once $D \restriction \alpha$ is found we let $y_\alpha \in {}^{\alpha}2$ be an accumulation point of $\{d_x \restriction \alpha : x \in E_\alpha\}$ and we put, as promised,

$$B^i_\alpha = y_\alpha \cup A^i_\alpha \restriction [\alpha, \mathfrak{c}) \qquad (i \in 2).$$

Our inductive hypotheses are

(0) $\forall x \in I : d_x \restriction \omega = x$,

(1) $\forall x, y \in I : d_x \restriction \alpha + d_y \restriction \alpha = d_{x+y} \restriction \alpha$,
(2) if $\omega \leq \beta < \alpha$ then $d_{x_\beta}(\beta) = 1$ and $\forall x \in I \ \ d_x \restriction \beta \subseteq d_x \restriction \alpha$, and
(3) if $\omega \leq \beta < \alpha$ then $B^0_\beta \restriction \alpha$ and $B^1_\beta \restriction \alpha$ are accumulation points of $\{d_x \restriction \alpha : x \in E_\beta\}$.

If $\alpha = \omega$ then $d_x \restriction \omega = x$ $(x \in I)$ and $D \restriction \omega = \{d_x \restriction \omega : x \in I\}$. If $\alpha > \omega$ is a limit then we are forced to let

$$d_x \restriction \alpha = \bigcup_{\omega \leq \beta < \alpha} d_x \restriction \beta \quad \text{for } x \in I.$$

One readily checks that (1), (2), and (3) hold in this case.

Now we consider the successor step $\alpha \to \alpha + 1$. We have to define $d_x(\alpha)$ for $x \in I$. Condition (1) tells us that the map $\phi : x \to d_x(\alpha)$ is to be a homomorphism from I to 2. Condition (2) tells us that we must have $\phi(x_\alpha) = 1$. For condition (3) we need the following:

$$\begin{array}{l} \text{if } \beta \in [\omega, \alpha],\ F \in [\alpha + 1]^{<\omega},\ \text{and } i \in 2 \\ \text{then } \{x \in E_\beta : d_x \restriction F = B^i_\beta\} \text{ is infinite.} \end{array}$$

If $\alpha \notin F$ this is satisfied because $B^i_\beta \restriction \alpha$ is an accumulation point of $\{d_x \restriction \alpha : x \in E_\beta\}$. So let $F \in [\alpha]^{<\omega}$ and consider $F \cup \{\alpha\}$.Then $d_x \restriction (F \cup \{\alpha\}) = B^i_\beta \restriction (F \cup \{\alpha\})$ means

$$d_x \restriction F = B^i_\beta \restriction F \quad \text{and} \quad \phi(x) = B^i_\beta(\alpha).$$

So what we need is

$$(\oplus) \qquad \begin{array}{l} \text{if } \beta \in [\omega, \alpha],\ F \in [\alpha]^{<\omega},\ \text{and } i \in 2 \\ \text{then } \{x \in E_\beta : d_x \restriction F = B^i_\beta \restriction F \text{ and } \phi(x) = B^i_\beta(\alpha)\} \text{ is infinite.} \end{array}$$

This we achieve by applying **MA** to the countable poset

$$\begin{aligned} \mathbf{P} = \{p \in \mathrm{Fn}(I, 2) : {} & \mathrm{dom}(p) \text{ is a finite subgroup of } I, \\ & p \text{ is a homomorphism}, \ x_\alpha \in \mathrm{dom}(p) \text{ and } p(x_\alpha) = 1\}, \end{aligned}$$

ordered by reverse inclusion. For every $x \in I$, $A_x = \{p : x \in \mathrm{dom}(p)\}$ is dense in $\mathbf{P}$: if $x \notin \mathrm{dom}(p)$ then we can make $p' \leq p$ with $\mathrm{dom}(p') = \langle\langle \mathrm{dom}(p) \cup \{x\}\rangle\rangle$ and $p'(x) = 0$. Thus, if $\mathscr{G}$ is a filter on $\mathbf{P}$ intersecting every A_x then $\phi = \bigcup \mathscr{G}$ is a homomorphism on I satisfying $\phi(x_\alpha) = 1$. For $(\oplus)$ we need more dense sets: for $\beta \in [\omega, \alpha]$, $F \in [\alpha]^{<\omega}$, $i \in 2$, and $n \in \omega$ put

$$\begin{aligned} A(\beta, F, i, n) = \{p : |\{x \in \mathrm{dom}(p) : {} & x \in E_\beta; \\ & d_x \restriction F = B^i_\beta \restriction F; p(x) = B^i_\beta(\alpha)\}| \geq n\}. \end{aligned}$$

Note that $A(\alpha, F, i, 0) = \mathbf{P}$ and that if $p \in A(\beta, F, i, n)$ then there is a $p' \leq p$ with $p' \in A(\beta, F, i, n+1)$. For let $p \in A(\beta, F, i, n)$. Since $\{x \in E_\beta : d_x \restriction F = B^i_\beta \restriction F\}$ is infinite we can pick x from this set with $x \notin \mathrm{dom}(p)$. Then we can make $p' \leq p$ with $\mathrm{dom}(p') = \langle\langle \mathrm{dom}(p) \cup \{x\} \rangle\rangle$ and $p'(x) = B^i_\beta(\alpha)$. But then $p' \in A(\beta, F, i, n+1)$. It follows that every $A(\beta, F, i, n+1)$ is dense. We have only $|\alpha| < \mathfrak{c}$ such dense sets so by $\mathbf{MA}_{\text{countable}}$ we can take a filter $\mathscr{G}$ on $\mathbf{P}$ intersecting every A_x and every $A(\beta, i, F, n)$. Then $\phi = \bigcup \mathscr{G}$ is a homomorphism and it satisfies $(\oplus)$. This completes the construction of D and also of G_0 and G_1 and hence of H_0 and H_1.

3. Construction II

In this section we construct a single countably compact group H such that H^2 is not countably compact. The construction is basically the same as the one in §2 but with more notation. We shall describe what H will look like and then concentrate on the actual construction, trusting that the reader who made it this far will have no problem seeing what is going on.

3.0. A picture of H. For any set of ordinals x we put $\overline{x} = x \times 2$, and we let $\rho: \overline{\mathfrak{c}} \to \overline{\mathfrak{c}}$ be defined by $\rho(\alpha, i) = \langle \alpha, 1-i \rangle$. Then ρ induces an automorphism (also denoted by ρ) of ${}^{\overline{\mathfrak{c}}}2$. We mentioned that H can be considered to be a subgroup of ${}^{\mathfrak{c}}2 \times {}^{\mathfrak{c}}2$ containing $H_0 \times \{\underline{0}\}$ and $\{\underline{0}\} \times H_1$ where H_0 and H_1 are much like the H_0 and H_1 from §2.

The notation becomes a bit more manageable if we work in ${}^{\overline{\mathfrak{c}}}2$ rather than ${}^{\mathfrak{c}}2 \times {}^{\mathfrak{c}}2$. For $i \in 2$ put

$$K_i = \{x \in {}^{\overline{\mathfrak{c}}}2 : x^{\leftarrow}(1) \subseteq \mathfrak{c} \times \{i\}\}.$$

Now H will be $D + G_0 + G_1 + G_2$ with D countable and G_0, G_1, and G_2 ω-bounded. We will let $D_i = D \cap K_i$ for $i \in 2$, and we will have $D = D_0 + D_1$ and $D_1 = \rho(D_0)$. Furthermore we will have $G_i \subseteq K_i$ for $i \in 2$. Then for $i \in 2$ every infinite $E \subseteq D_i$ will have a limit point in G_i so that $H_i = D_i + G_i$ will be countably compact. We will also have $G_1 \cap \rho(G_0) = \{\underline{0}\}$ and $D \cap (G_0 + G_1) = \{\underline{0}\}$, and this will imply that $H_0 \times H_1$ is not countably compact. We will use G_2 to provide accumulation points for the infinite subsets of $D \setminus (D_0 \cup D_1)$.

3.1. The construction.

3.1.0. Let $\mathscr{A}$ be the ω-independent family on $\mathfrak{c}$ from 1.1. We split $\mathscr{A}$ into four disjoint pieces $\mathscr{A}_0$, $\mathscr{A}_1$, $\mathscr{A}_2$, and $\mathscr{A}_3$ of size $\mathfrak{c}$. We define $\mathscr{A}_0^+ = \{A \times \{0\} : A \in \mathscr{A}_0\}$, $\mathscr{A}_1^+ = \{A \times \{1\} : A \in \mathscr{A}_1\}$, and $\mathscr{A}_i^+ = \{A \times 2 : A \in \mathscr{A}_i\}$ $(i = 2, 3)$. As before we identify every $A \in \mathscr{A}_0^+ \cup \mathscr{A}_1^+ \cup \mathscr{A}_2^+ \cup \mathscr{A}_3^+$ with its characteristic function in ${}^{\overline{\mathfrak{c}}}2$.

3.1.1. Let $I = \{x \in {}^{\overline{\omega}}2 : x^{\leftarrow}(1)$ is finite$\}$ and for $i \in 2$ let $I_i = \{x \in I : x^{\leftarrow}(1) \subseteq \omega \times \{i\}\}$. Then $I = I_0 + I_1$ and each $x \in I$ is written (uniquely) as $x_0 + x_1$ with $x_0 \in I_0$ and $x_1 \in I_1$. We will have $D = \{d_x : x \in I\}$ and $D_i = \{d_x : x \in I_i\}$ $(i \in 2)$.

3.1.2. To see for what infinite sets $E \subseteq D$ we have to provide accumulation points let $E \subseteq I$ be infinite. We can have (i) $E \cap I_0$ is infinite, but $\{d_x : x \in E \cap I_0\}$ will have an accumulation point in G_0; (ii) $E \cap I_1$, is infinite, now G_1

provides an accumulation point; (iii) for infinitely many $x \in E$, x is symmetric (i.e., $x = \rho(x)$), in this case we will provide an accumulation point in G_2. If none of the above cases occur then without loss of generality for all $x \in E$, $x_0, x_1, x + \rho(x) \neq \underline{0}$. In this case we can have an infinite $E' \subseteq E$ and an $a \in I$ such that (iv) $\forall x \in E'$, $x_0 = a$, (v) $\forall x \in E'$, $x_1 = a$ or (vi) $\forall x \in E'$, $x + \rho(x) = a$.

In cases (iv) or (v) $d_a + g$ will be an accumulation point of $\{d_x : x \in E'\}$ when g is an accumulation point of $\{d_x + d_a : x \in E'\}$ (from (i) or (ii)). In case (v) take $E'' \subseteq E'$ infinite and $b \in I$ such that $\forall x \in E''$, $x \cap (x + \rho(x)) = b$. Then $x = b + (x \cap \rho(x))$ for $x \in E''$ and $d_b + g$ is an accumulation point of $\{d_x : x \in E''\}$ (from (iii)).

Finally if (iv), (v), and (vi) do not occur either, we get an infinite $E' \subseteq E$ such that

$$(*) \qquad \forall x, y \in E' : x \neq y \to x_0 \neq y_0,\ x_1 \neq y_1, \text{ and } x + \rho(x) \neq y + \rho(y).$$

We conclude that it suffices to provide accumulation points for $\{d_x : x \in E\}$ in case $E \subseteq I_0$, $E \subseteq I_1$, every $x \in E$ is symmetric, or E satisfies $(*)$.

3.1.3. We enumerate everything in sight [Ru].

$-\langle A^i_\alpha : \omega \leq \alpha < \mathfrak{c}\rangle$ enumerates $\mathscr{A}_i^+$ without repetitions $(i \in 4)$,

$-\langle E^i_\alpha : \omega \leq \alpha < \mathfrak{c}\rangle$ enumerates $[I_i]^\omega$ $(i \in 2)$,

$-\langle E^2_\alpha : \omega \leq \alpha < \mathfrak{c}\rangle$ enumerates $\{E \in [I]^\omega : E$ is symmetric$\}$,

$-\langle E^3_\alpha : \omega \leq \alpha < \mathfrak{c}\rangle$ enumerates $\{E \in [I]^\omega : E$ satisfies $(*)\}$, and

$-\langle\langle x_\alpha, \mathscr{C}_\alpha\rangle : \omega \leq \alpha < \mathfrak{c}\rangle$ enumerates $(I \backslash \{\underline{0}\}) \times [\mathscr{A}]^\omega$ in such a way that for every $\langle x, \mathscr{C}\rangle$ the set $\{\alpha : \langle x_\alpha, \mathscr{C}_\alpha\rangle = \langle x, \mathscr{C}\rangle$ and $\alpha \notin \bigcup \mathscr{C}\}$ has cardinality $\mathfrak{c}$.

3.1.4. By induction on $\alpha \in [\omega, \mathfrak{c}]$ we construct $D \restriction \overline{\alpha} = \{d_x \restriction \overline{\alpha} : x \in I\}$ satisfying:

(0) $\forall x \in I : d_x \restriction \overline{\omega} = x$ and if $\omega \leq \beta < \alpha$ then $d_x \restriction \overline{\beta} \subseteq d_x \restriction \overline{\alpha}$,

(1) $\forall x \in I : d_{\rho(x)} \restriction \overline{\alpha} = \rho(d_x \restriction \overline{\alpha})$ and $\forall x, y \in I : d_x \restriction \overline{\alpha} + d_y \restriction \overline{\alpha} = d_{x+y} \restriction \overline{\alpha}$,

(2) $\forall x \in I_i : d_x^{\leftarrow}(1) \cap \overline{\alpha} \subseteq \alpha \times \{i\}$ $(i \in 2)$, and

(3) if $\omega \leq \beta < \alpha$ then $d_{x_\beta}(\beta, 0) = 1$ if $x_\beta \notin I_1$, and $d_{x_\beta}(\beta, 1) = 1$ if $x_\beta \notin I_0$.

Once $D \restriction \overline{\alpha}$ is found we pick for $i = 0, 1, 2, 3$ an accumulation point x^i_α for $\{d_x \restriction \overline{\alpha} : x \in E^i_\alpha\}$ in ${}^{\overline{\alpha}}2$. Then for $i \in 4$ we put $B^i_\alpha = x^i_\alpha \cup A^i_\alpha \restriction \overline{[\alpha, \mathfrak{c})}$; note that $B^i_\alpha \in K_i$ for $i \in 2$. We require for $i = 0, 1, 2, 3$:

(4) if $\omega \leq \beta < \alpha$ then $B^i_\beta \restriction \overline{\alpha}$ is an accumulation point of $\{d_x \restriction \overline{\alpha} : x \in E^i_\beta\}$.

3.1.5. If $\alpha = \omega$ we put $d_x \restriction \overline{\omega} = x$ $(x \in I)$ and $D \restriction \overline{\omega} = I$. If α is a limit we put

$$d_x \restriction \overline{\alpha} = \bigcup_{\omega \leq \beta < \alpha} d_x \restriction \overline{\beta} \qquad (x \in I)$$

and an easy check gives (0)–(4).

3.1.6. Now consider the successor step $\alpha \to \alpha+1$. As in §2 we use a homomorphism ϕ to define $d_x(\alpha, 0)$ and $d_x(\alpha, 1)$ for all $x \in I$.

Now we will have $\phi: I \to {}^2 2$. We need

$$\begin{aligned}
&\phi(\rho(x))(0) = \phi(x)(1) \quad \text{and} \quad \phi(\rho(x))(1) = \phi(x)(0) \quad \text{for (1)},\\
&\phi(x)(1) = 0\ (x \in I_0) \quad \text{and} \quad \phi(x)(0) = 0 \quad (x \in I_1) \quad \text{for (2)},\\
&\phi(x_\alpha)(0) = 1 \text{ if } x_\alpha \notin I_1 \quad \text{and} \quad \phi(x_\alpha)(1) = 1 \text{ if } x_\alpha \notin I_0 \quad \text{for (3)}.
\end{aligned}$$

We use the countable poset

$$\begin{aligned}
\mathbf{P} = \{&\rho \in \mathrm{Fn}(I, {}^2 2) : \mathrm{dom}(\rho) \text{ is a finite subgroup of } I;\ \rho \text{ is a homomorphism};\\
&\text{if } x \in \mathrm{dom}(\rho) \text{ then } x_0, x_1, \rho(x) \in \mathrm{dom}(\rho) \text{ and } \rho(x_0)(1) = \rho(x_1)(0)\\
&= 0 \text{ and } \rho(\phi(x))(i) = \rho(x)(1-i)\ (i \in 2);\ x_\alpha \in \mathrm{dom}(\rho) \text{ and}\\
&\qquad x_\alpha \notin I_0 \to \rho(x_\alpha)(1) = 1,\ x_\alpha \notin I_1 \to \rho(x_\alpha)(0) = 1\},
\end{aligned}$$

ordered by reverse inclusion.

Let $x \in I$ and let $\mathscr{A}_x = \{p : x \in \mathrm{dom}(p)\}$. To show $\mathscr{A}_x$ is dense consider the group F_x generated by $\{x_0, x_1, \rho(x_0), \rho(x_1)\}$. We have to define $p' \le p$ with $\mathrm{dom}(p') = \mathrm{dom}(p) + F_x$. The special demands on $\mathrm{dom}(p)$ lead to the following four possibilities for $F_x \cap \mathrm{dom}(p)$:

- $\mathrm{dom}(p) \cap F_x = \{\underline{0}\}$,
  in this case $p'(x_i)(0) = p'(x_i)(1) = 0\ \ (i \in 2)$ works;
- $\mathrm{dom}(p) \cap F_x = \{\underline{0}, x_0, \rho(x_0), x_1 + \rho(x_1)\}$,
  in this case $p'(x_1)(0) = p'(x_1)(1) = 0$ works;
- $\mathrm{dom}(p) \cap F_x = \{\underline{0}, x_1, \rho(x_1), x_1 + \rho(x_1)\}$,
  in this case $p'(x_0)(0) = p'(x_0)(1) = 0$ works;
- $\mathrm{dom}(p) \cap F_x = \{\underline{0}, x_0 + \rho(x_1), x_1 + \rho(x_0), x + \rho(x)\}$,
  in this case $\varepsilon = p(x + \rho(x))$ is symmetric: $\varepsilon(0) = \varepsilon(1) = i$. Now let $p'(x_0)(0) = i$, $p'(x_0)(1) = 0$ and $p'(x_1)(1) = 0$; this works.

Thus if $\mathscr{G}$ is a filter on $\mathbf{P}$ intersecting all A_x then $\phi = \bigcup \mathscr{G}$ is a homomorphism taking care of everything except possibly (4). For (4) we consider the following sets: for $\beta \in [\omega_0, \alpha]$, $F \in [\overline{\alpha}]^{<\omega}$, $i \in 4$, and $n \in \omega$ let

$$\begin{aligned}
A(\beta, F, i, n) = \{p : |x \in \mathrm{dom}(p) : x \in E_\beta, d_x \restriction F = B^i_\beta \restriction F,&\\
(x)(j) = B^i_\beta(\alpha, j)\ (j \in 2)\}| \ge n\}.&
\end{aligned}$$

Then $A(\beta, F, i, 0) = \mathbf{P}$ and if $p \in A(\beta, F, i, n)$ then there is a $p' \le p$ with $p' \in A(\beta, F, i, n+1)$:

$i = 0, 1$: Pick $x \in E^i_\beta$ with $d_x \restriction F = B^i_\beta \restriction F$ and $x \notin \mathrm{dom}(p)$. Then $F_x = \{\underline{0}, x, \rho(x), x + \rho(x)\}$ and $F_x \cap \mathrm{dom}(p) = \{\underline{0}\}$. Define $p' \le p$ by $\mathrm{dom}(p') = \mathrm{dom}(p) + F_x$ and $p'(x)(j) = B^i_\beta(\alpha, j)\ \ (j \in 2)$. Note that $B^i_\beta(\alpha, 1-i) = 0$, so that this definition is in order.

$i = 2$: Again pick $x \in E^2_\beta$ with $d_x \restriction F = B^i_\beta \restriction F$ and $x \notin \mathrm{dom}(p)$. Then $F_x = \{\underline{0}, x_0, x_1, x\}$ and $F_x \cap \mathrm{dom}(p) = \{\underline{0}\}$. Note that $B^1_\beta(\alpha, 0) = B^2_\beta(\alpha, 1)$, since $\alpha \geq \beta$, so that the following safely defines $p' \leq p$ with $\mathrm{dom}(p') = \mathrm{dom}(p) + F_x$: $p'(x_0)(0) = p'(x_1)(1) = B^2_\beta(\alpha, 0)$ and $p'(x_0)(1) = p'(x_1)(0) = 0$.

$i = 3$: Now we can pick $x \in E^3_\beta$ with $d_x \restriction F = B^3_\beta \restriction F$ and $x_0, x_1, x_0 + \rho(x_1) \notin \mathrm{dom}(p)$. Again $F_x \cap \mathrm{dom}(p) = \{\underline{0}\}$, and since $B^3_\beta(\alpha, 0) = B^3_\beta(\alpha, 1)$ we can (again) safely define $p' \leq p$ by $\mathrm{dom}(p') = \mathrm{dom}(p) + F_x$, and $p'(x_0)(0) = p'(x_1)(1) = B^3_\beta(\alpha, 0)$ and $p'(x_0)(1) = p'(x_1)(0) = 0$.

In the end we have $d_x = d_x \restriction \bar{\mathfrak{c}}$ $(x \in I)$ and $D = D \restriction \bar{\mathfrak{c}}$. We let G_0 be the ω-bounded subgroup of ${}^{\bar{\mathfrak{c}}}2$ generated by $\mathscr{B}_0$ and similarly $\mathscr{B}_1$ generates G_1 and $\mathscr{B}_2 \cup \mathscr{B}_3$ generates G_2.

3.1.7. We put $H_0 = D_0 + G_0$, $H_1 = D_1 + G_1$, and $H = D + G_0 + G_1 + G_2$. By (3) we know that for every countable $\mathscr{C} \subseteq \mathscr{A}$ and for every $d \in D$,

$$|d^{\leftarrow}(1) \backslash (\bigcup \mathscr{C} \times 2)| = \mathfrak{c}.$$

This readily implies that $D \cap (G_0 + G_1 + G_2)| = \{\underline{0}\}$. The considerations given in 3.1.1 now give us, using (4), that every infinite subset of D has an accumulation point in H. Together with the ω-boundedness of $G_0 + G_1 + G_2$ this yields that H is countably compact. Next because $G_0 \cap \rho(G_1) = \{\underline{0}\}$ we see that $\Delta = \{\langle d, \rho(d)\rangle : d \in D_0\}$ is closed in $H_0 \times H_1$ whence $H_0 \times H_1$ is not countably compact. Finally $H_0 = H \cap K_0$ and $H_1 = H \cap K_1$ so that $H_0 \times H_1$ is closed in H^2 so that H^2 is not countably compact. (The equalities $G_0 \cap \rho(G_1) = \{\underline{0}\}$, $H_0 = H \cap K_0$, and $H_1 = H \cap K_1$ follow easily from 1.1(b), just as in 2.2.)

3.2. Remark. It seems that our example needs $\mathbf{MA}_{\text{countable}}$ in an essential way for the same reasons as mentioned in [vD]. $\mathbf{MA}_{\text{countable}}$ implies that every $u \in \omega^*$ has character $\mathfrak{c}$. Assume $2^\omega = 2^{\omega_1} = \mathfrak{c}$. Then we can choose an ω_1-independent family on $\mathfrak{c}$ and make the group $G_0 + G_1 + G_2$ ω_1-bounded (these concepts have the obvious meanings). Then H will be initially ω_1-compact (meaning that every open cover of size $\leq \omega_1$ has a finite subcover). But this variation does not work without $\mathbf{MA}_{\text{countable}}$: if there is an ultrafilter of character ω_1 on ω then the product of any family of initially ω_1-compact spaces is countably compact.

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