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DISTAL FUNCTIONS AND UNIQUE ERGODICITY

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ABSTRACT. A. Knapp [5] has shown that the set, $D(S)$, of all distal functions on a group S is a norm closed subalgebra of $l^\infty(S)$ that contains the constants and is closed under the complex conjugation and left translation by elements of S. Also it is proved that [7] for any $k \in \mathbb{N}$ and any $\lambda \in \mathbb{R}$ the function $f: \mathbb{Z} \to \mathbb{C}$ defined by $f(n) = e^{i\lambda n^k}$ is distal on $\mathbb{Z}$. Now let $\mathbf{W}$ be the norm closure of the algebra generated by the set of functions

$$\{n \mapsto e^{i\lambda n^k} : k \in \mathbb{N},\ \lambda \in \mathbb{R}\},$$

which will be called the *Weyl algebra.* According to the facts mentioned above, all members of the Weyl Algebra are distal functions on $\mathbb{Z}$. In this paper, we will show that any element of $\mathbf{W}$ is uniquely ergodic (Theorem 2.13) and that the set $\mathbf{W}$ does not exhaust all the distal functions on $\mathbb{Z}$ (Theorem 2.14). The latter will answer the question that has been asked (to the best of my knowledge) by P. Milnes [6].

The term Weyl algebra is suggested by S. Glasner. I would like to express my warmest gratitude to S. Glasner for his helpful advise, and to my advisor Professor Namioka for his enormous helps and contributions.

1. PRELIMINARIES

Let S be a semigroup and $X = l^\infty(S)$ be the algebra of all bounded complex functions on S, equiped with the topology of pointwise convergence on S. Then $(S, l^\infty(S))$ forms a flow, where the action of S on $X = l^\infty(S)$ is defined by

$$(s, f) \mapsto sf = R_s f,$$

where $(sf)(t) = (R_s f)(t) = f(ts)$ for all $f \in X$ and $s, t \in S$. A member $f \in l^\infty(S)$ is called a *distal function* on S if it is, distal point relative to the flow $(S, l^\infty(S))$. For this flow and its distal functions we have the following results [7, 8]:

1.1. **Theorem.** *Let S be a semigroup and let $\Sigma(X)$ be the enveloping semigroup of the flow $(S, l^\infty(S))$. Then $\Sigma(X)$ is compact. Furthermore, each $\sigma \in \Sigma(X)$ is linear, multiplicative, and preserves the complex conjugation and the constant functions. Also $\|\sigma f\| \le \|f\|$ holds for all $f \in l^\infty(S)$.*

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1.2. **Theorem.** *Let D be the space of all distal functions on a semigroup S. Then D is a norm closed subalgebra of $l^{\infty}(S)$ containing constants and is closed under complex conjugation. Furthermore, D is invariant under each member σ of the enveloping semigroup of the flow $(S, l^{\infty}(S))$.*

Given two flows (S, X), and (S, Y) a continuous function $\varphi: X \to Y$ is called a *flow homomorphism* if $\varphi(sx) = s\varphi(x)$ holds for all $s \in S$ and $x \in X$. If, in addition, φ is onto (or 1-1), then it is called a *flow epimorphism* (or *monomorphism*). The following lemma facilitates a way to generate distal functions on S from distal points of a flow (S, X).

1.3. **Lemma.** *Let (S, X) be a flow, $F \in C(X)$, and $x_0 \in X$ be a distal point, and let the function $f: S \to \mathbb{C}$ be defined by $f(s) = F(sx_0)$. Then f is a distal on S.*

Proof. Consider the mapping $\varphi: X \to l^{\infty}(S)$ defined by

$$\varphi(x)(t) = F(tx) \quad \text{for all } x \in X, \ t \in S,$$

which is a flow homomorphism. Since the homomorphic image of a distal flow is again distal [2], φ maps any distal point of X to a distal point of $l^{\infty}(S)$. Thus, $f = \varphi(x_0)$ is a distal function on S.

A (classical) *dynamical system* is a pair (X, T), where X is a nonempty compact T_2 topological space, and T is a continuous map of X into itself. Note that this is precisely the flow $(\mathbb{N}, X)$ with the action

$$(n, x) \mapsto T^n x,$$

and if $T: X \to X$ is a homeomorphism, then it is the flow $(\mathbb{Z}, X)$ with the same action. Let (X, T) and (Y, S) be two dynamical systems. Then a continuous function $\varphi: X \to Y$ is called a *homomorphism* if $\varphi(Tx) = S\varphi(x)$. If, in addition, φ is onto, then it is called an *epimorphism.* A measure μ on X is said to be *invariant* under T if $\mu(f) = \mu(f \circ T)$ holds for all $f \in C(X)$. Following H. Furstenberg [3] we shall refer to the triple (X, T, μ) as a *process*, where (X, T) is a dynamical system, and μ is an invariant probability measure on the Borel subsets of X. For any $f \in C(X)$ we define an associated sequence $\{f_n\}$ of functions on X by

$$f_n(x) = \frac{1}{n+1} \sum_{m=0}^{n} f(T^m x).$$

A point $x \in X$ is said to be *generic* for the process (X, T, μ) if for any $f \in C(X)$ the sequence $\{f_n(x)\}$ converges to $\mu(f)$. A dynamical system (X, T) is said to be *uniquely ergodic* if there is a unique invariant probability measure on the Borel subsets of X; if in addition, X is minimal, then (X, T) is called *strictly ergodic.*

The following theorem is a consequence of the Markov-Kakutani fixed point theorem [1, p. 45].

1.4. **Theorem.** *For any dynamical system* (X, T) *there is a probability measure* μ *on the Borel subsets of* X *that is invariant under* T.

And, for uniquely ergodic processes we have the following theorem [3, 9].

1.5. **Theorem.** *Let* (X, T, μ) *be a process. Then the following are equivalent:*

(a) (X, T) *is uniquely ergodic with invariant measure* μ;

(b) *For any* $f \in C(X)$ *its associated sequence* $\{f_n\}$ *converges to* $\mu(f)$ *uniformly on* X;

(c) *Every point of* X *is generic for the process* (X, T, μ).

1.6. **Corollary.** *Let* (X, T) *be a dynamical system in which for any* $f \in C(X)$ *its associated sequence* $\{f_n\}$ *converges pointwise to a constant on* X. *Then* (X, T) *is uniquely ergodic.*

1.7. **Corollary.** *Let* A *be a dense linear subspace of* $C(X)$, *and assume that for any* $f \in A$ *its associated sequence* $\{f_n\}$ *converges pointwise to a constant. Then* (X, T) *is uniquely ergodic.*

1.8. **Corollary.** *If the process* (X, T, μ) *is uniquely ergodic and* $\varphi: (X, T) \to (Y, S)$ *is an epimorphism, then* (Y, S) *is also uniquely ergodic.*

2. Ergodicity of elements of the Weyl algebra

In this section we would like to study the dynamical system (X, S), where $X = l^\infty(\mathbb{Z})$, and $S: X \to X$ is the shift operator defined by $Sf(n) = f(n+1)$. We note that this is exactly the flow $(\mathbb{Z}, l^\infty(\mathbb{Z}))$. For a function $f \in X$, let X_f be the orbit closure of f, that is, the closure of the set $\{S^n f; n \in \mathbb{Z}\}$. Since Σ, the enveloping semigroup of $(\mathbb{Z}, l^\infty(\mathbb{Z}))$, is compact, we have

$$X_f = \Sigma f = \{\sigma(f): \sigma \in \Sigma\}.$$

Also, if f is distal, then the X_f is minimal. Clearly, $S: X \to X$ is a homeomorphism, and $S(X_f) = X_f$. Thus the restriction of S to X_f, which will be denoted by S, is still a homeomorphism, and we can consider the dynamical system (X_f, S). By Theorem 1.4, there is a probability measure on the Borel subsets of X_f, which is invariant under S, and therefore invariant under all S^n $(n \in \mathbb{Z})$.

2.1. **Definition.** A function $f \in X$ is said to be *uniquely ergodic* if the system (X_f, S) is uniquely ergodic. If in addition X_f is also minimal, then we call f *strictly ergodic.*

For a distal function f, since X_f is minimal unique ergodicity is the same as strict ergodicity; therefore we do not make any distinction between them.

2.2. **Theorem.** *Suppose that* $f \in X$ *is uniquely ergodic and that the function* $F: \mathbb{C} \to \mathbb{C}$ *is continuous. Then* $F \circ f$ *is uniquely ergodic.*

Proof. Let $h = F \circ f$ and X_h be the orbit closure of h, and let $\varphi: X_f \to X_h$ be defined by $\varphi(g) = F \circ g$ $(g \in X_f)$. Then φ is an epimorphism, and thus by Corollary 1.8, h is uniquely ergodic.

2.3. **Theorem.** *For a finite subset* $\Omega = \{\omega_1, \omega_2, \dots, \omega_j\}$ *of* $\mathbb{Z}$ *let the functions*

$$E(\Omega)\colon X \to \mathbb{C}, \qquad \overline{E}(\Omega)\colon X \to \mathbb{C},$$

defined by

$$E(\Omega)(f) = f(\omega_1)f(\omega_2)\cdots f(\omega_j), \quad \textit{and} \quad E(\varnothing)(f) = 1,$$
$$\overline{E}(\Omega)(f) = \overline{E(\Omega)(f)},$$

where $\overline{E(\Omega)(f)}$ *is the complex conjugate of* $E(\Omega)(f)$. *Then* f *is uniquely ergodic if and only if for any finite subsets* Ω, Δ *of* $\mathbb{Z}$,

$$\frac{1}{n+1}\sum_{m=0}^{n} E(\Omega)\overline{E}(\Delta)(S^m(\sigma f))$$

converges, as $n \to \infty$, *to a constant* $c(\Omega, \Delta)$ *that is independent of* $\sigma \in \Sigma$.

Proof. $\Rightarrow$ Let $\varphi = E(\Omega)\overline{E}(\Delta)$. Then $\varphi' \in C(X_f)$, and by 1.5(b), its associated sequence $\{\varphi_n\}$ converges to $\mu(\varphi)$ uniformly on X_f.

$\Leftarrow$ Let A be the linear subspace of $C(X_f)$ generated by all the functions of the form $E(\Omega)\overline{E}(\Delta)$ with Ω and Δ being any finite subsets of $\mathbb{Z}$. Then by the Stone-Weierstrass theorem, A is dense in $C(X_f)$, and the theorem follows from Corollary 1.7.

2.4. **Corollary.** *If* f *is uniquely ergodic, then the limit*

$$\lim_n \frac{1}{n+1}\sum_{m=0}^{n}[\sigma(f)(m)]$$

exists and is a constant independent of σ.

Proof. With the notation of the Theorem 2.3, let $\Omega = \{0\}$, and $\Delta = \varnothing$.

2.5. **Corollary.** *If the function* f *is uniquely ergodic, then so is* τf *for any* $\tau \in \Sigma$.

Proof. Let $\sigma \in \Sigma$, and $\nu = \sigma \circ \tau$. Then by Theorem 2.3,

$$\frac{1}{n+1}\sum_{m=0}^{n} E(\Omega)\overline{E}(\Delta)(S^m(\sigma(\tau f))) = \frac{1}{n+1}\sum_{m=0}^{n} E(\Omega)\overline{E}(\Delta)(S^m(\nu f)),$$

which converges uniformly to a constant independent of ν.

2.6. **Lemma.** *Let* $p(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ *be a real polynomial, and the function* $f\colon \mathbb{Z} \to \mathbb{C}$ *be defined by*

$$f(x) = e^{ip(x)} \qquad (x \in \mathbb{Z}).$$

Then for each $\sigma \in \Sigma$ *there is a real polynomial* p_σ *whose leading coefficient is* a_k, *and* $\sigma f(n) = e^{ip_\sigma(n)}$.

Proof. The conclusion follows obviously if $k = \deg p(x) = 0$. So let $k > 0$, and note that for any n, $s \in \mathbb{Z}$

$$p(n+s) = p(n) + \frac{s}{1!}p'(n) + \frac{s^2}{2!}p''(n) + \cdots + \frac{s^k}{k!}p^{(k)}(n).$$

Now suppose that $\{s_\alpha\}$ is a net in $\mathbb{Z}$ such that $\lim_\alpha s_\alpha = \sigma \in \Sigma$. By taking a subnet, if necessary, we may assume that

$$\lim_\alpha \exp\left(i\frac{s^l\alpha}{l!}\right) = \exp(i\theta_l) \qquad (l = 1, 2, \dots, k).$$

Consequently, for each $n \in \mathbb{Z}$

$$\sigma f(n) = \lim_\alpha f(n + s_\alpha) = \lim_\alpha e^{ip(n+s_\alpha)} = e^{ip_\sigma(n)},$$

where $p_\sigma(n) = p(n) + \theta_1 p'(n) + \theta_2 p''(n) + \cdots + \theta_k p^{(k)}(n)$ is a real polynomial in n, whose leading coefficient is the same as that of $p(n)$, i.e., a_k.

2.7. **Lemma.** *Let $F\colon \mathbb{Z} \to \mathbb{C}$ be a periodic function with period $u > 0$, and $p(x) = a_q x^q + a_{q-1}x^{q-1} + \cdots + a_1 x + a_0$ be a real polynomial for which, if $q > 0$, the leading coefficient a_q is an irrational multiple of π. Then*

$$\lim_{n\to\infty}\left[\frac{1}{n+1}\sum_{m=0}^{n} F(m)e^{ip(m)}\right] = \begin{cases} 0, & \text{if } q > 0, \\ \frac{e^{ia_0}}{u}[F(0) + \cdots + F(u-1)], & \text{if } q = 0. \end{cases}$$

Proof. Observe that, for any $n \in \mathbb{N}$, there is a nonnegative integer k with $n = ku + r$ $(0 \le r < u)$. Hence

$$\frac{1}{n+1}\sum_{m=0}^{n} F(m)e^{ip(m)} = \frac{ku}{ku+r+1}\cdot\frac{1}{ku}\sum_{m=0}^{ku-1} F(m)e^{ip(m)} + \frac{1}{n+1}\sum_{m=ku}^{n} F(m)e^{ip(m)}.$$

If $F(m)e^{ip(m)}$ is bounded by M, then $|\sum_{m=ku}^{n} F(m)e^{ip(m)}| \le uM$. Therefore,

$$\lim_{n\to\infty}\left[\frac{1}{n+1}\sum_{m=ku}^{n} F(m)e^{ip(m)}\right] = 0.$$

Also, $\lim_{k\to\infty} ku/(ku+r+1) = 1$. Thus it is sufficient to calculate

$$\lim_{k\to\infty}\left[\frac{1}{ku}\sum_{m=0}^{ku-1} F(m)e^{ip(m)}\right].$$

But

$$\begin{aligned}\sum_{m=0}^{ku-1} F(m)e^{ip(m)} &= \sum_{j=0}^{k-1}\sum_{m=ju}^{(j+1)u-1} F(m)e^{ip(m)}\\ &= \sum_{j=0}^{k-1}\sum_{m=0}^{u-1} F(m+ju)e^{ip(m+ju)}\\ &= \sum_{j=0}^{k-1}\sum_{m=0}^{u-1} F(m)e^{ip(m+ju)}\\ &= \sum_{m=0}^{u-1}\sum_{j=0}^{k-1} F(m)e^{ip(m+ju)}\\ &= \sum_{m=0}^{u-1} F(m)\sum_{j=0}^{k-1} e^{ip(m+ju)}.\end{aligned}$$

If $q = \deg p(x) = 0$, then $e^{ip(m+ju)} = e^{ia_0}$, and

$$\begin{aligned}\frac{1}{ku}\sum_{m=0}^{ku-1} F(m)e^{ip(m)} &= \frac{1}{ku}\sum_{m=0}^{u-1} F(m)\sum_{j=0}^{k-1} e^{ip(m+ju)}\\ &= \frac{1}{ku}\sum_{m=0}^{u-1} F(m)\sum_{j=0}^{k-1} e^{ia_0}\\ &= \frac{e^{ia_0}}{u}[F(0)+\cdots+F(u-1)].\end{aligned}$$

For $q > 0$, note that $p(m+ju)$ is a polynomial in j, whose leading coefficient, $a_q u^q$, is an irrational multiple of π. Then, by the theorem of Weyl [10, Satz 9]

$$\lim_{k\to\infty}\left[\frac{1}{k}\sum_{j=0}^{k-1} e^{ip(m+ju)}\right] = 0.$$

Consequently

$$\lim_{k\to\infty}\left[\frac{1}{ku}\sum_{m=0}^{ku-1} F(m)e^{ip(m)}\right] = \sum_{m=0}^{u-1} F(m)\left\{\lim_{k\to\infty}\left[\frac{1}{ku}\sum_{j=0}^{k-1} e^{ip(m+ju)}\right]\right\} = 0,$$

and this proves the lemma.

2.8. **Theorem.** *Let $k \in \mathbb{N}$ be fixed and let*

$$p(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$$

be a real polynomial of degree k. Then the function $h\colon \mathbb{Z} \to \mathbb{C}$ defined by $h(x) = e^{ip(x)}$ is uniquely ergodic.

Proof. If $a_1, a_2, \ldots, a_k$, are all rational multiples of π, then h is periodic and X_h is finite and minimal. Hence it is uniquely ergodic. So, we may assume that at least one of the coefficients $a_1, a_2, \dots, a_k$, is an irrational multiple of π. Suppose that a_q $(0 < q \le k)$ is the last such number, and let

$$Q(n) = a_k x^k + \cdots + a_{q+1} x^{q+1},$$
$$R(n) = a_q x^q + \cdots + a_1 x + a_0.$$

If $f(n) = e^{iQ(n)}$ and $g(n) = e^{iR(n)}$, then $h(n) = f(n)g(n)$, and the function f is a periodic function (say, with period u). By Lemma 2.6, for any σ, there is a real polynomial R_σ with the same leading coefficient as that of R (i.e., a_q), such that $\sigma g(n) = e^{iR_\sigma(n)}$. Now let $\Omega = \{\omega_1, \omega_2, \dots, \omega_j\}$, and $\Delta = \{\delta_1, \delta_2, \dots, \delta_l\}$ be any two finite subsets of $\mathbb{Z}$. Then by Theorem 2.3, we must show that

$$\lim_{n\to\infty} \left[\frac{1}{n+1}\sum_{m=0}^{n} E(\Omega)\overline{E}(\Delta)(S^m(\sigma h))\right]$$

exists and its value is independent of σ. Using the properties of σ, S, E, and $\overline{E}$ we note that

$$\begin{aligned} E(\Omega)\overline{E}(\Delta)(S^m\sigma h) &= E(\Omega)\overline{E}(\Delta)(S^m(\sigma f \sigma g)) \\ &= [E(\Omega)\overline{E}(\Delta)(S^m(\sigma f))]\cdot[E(m+\Omega)\overline{E}(m+\Delta)(\sigma g)] \\ &= [E(\Omega)\overline{E}(\Delta)(S^m(\sigma f))]\cdot[E(m+\Omega)\overline{E}(m+\Delta)e^{iR_\sigma}] \\ &= E(\Omega)\overline{E}(\Delta)(S^m(\sigma f))]\cdot e^{iG_\sigma(m)}, \end{aligned}$$

where $G_\sigma(m) = \sum_{\omega\in\Omega} R_\sigma(m+\omega) - \sum_{\delta\in\Delta} R_\sigma(m+\delta)$ is a polynomial in m. Now let $F\colon \mathbb{Z} \to \mathbb{C}$ be defined by

$$F(m) = E(\Omega)\overline{E}(\Delta)(S^m f).$$

Then F is periodic function of period u, and a straightforward calculation shows that

$$E(\Omega)\overline{E}(\Delta)(S^m\sigma f) = (\sigma F)(m).$$

Consequently,

$$\frac{1}{n+1}\sum_{m=0}^{n} E(\Omega)\overline{E}(\Delta)(S^m(\sigma h)) = \frac{1}{n+1}\sum_{m=0}^{n}(\sigma F)(m)e^{iG_\sigma(m)}.$$

Also, if $\{s_\alpha\}$ is a net in $\mathbb{Z}$ such that $\lim_\alpha s_\alpha = \sigma \in \Sigma$, then

$$\begin{aligned} \sigma F(0) + \cdots + \sigma F(u-1) &= \lim_\alpha[F(0+s_\alpha) + \cdots + F(u-1+s_\alpha)] \\ &= \lim_\alpha[F(0) + \cdots + F(u-1)] = F(0) + \cdots + F(u-1), \end{aligned}$$

which shows that $\sigma F(0) + \cdots + \sigma F(u-1)$ is a constant independent of $\sigma \in \Sigma$. Similarly, one can show that σF is also a periodic function of period u. On the other hand, by Taylor's theorem

$$\begin{aligned} G_\sigma(m) &= \sum_{\omega\in\Omega} R_\sigma(m+\omega) - \sum_{\delta\in\Delta} R_\sigma(m+\delta) \\ &= \lambda_0 R_\sigma(m) + \lambda_1 R'_\sigma(m) + \cdots + \lambda_q R^{(q)}_\sigma(m), \end{aligned}$$

where $\lambda_\nu = \frac{1}{\nu!}(\omega_1^\nu + \cdots + \omega_j^\nu - \delta_1^\nu - \cdots - \delta_l^\nu)$ $(\nu = 0, 1, \ldots, q)$ is a rational number independent of σ.

Now, if $\lambda_0 = \lambda_1 = \cdots = \lambda_{q-1} = 0$, then $G_\sigma(m) = \lambda_q a_q q!$ and, by Lemma 2.7,

$$\lim_{n\to\infty}\left[\frac{1}{n+1}\sum_{m=0}^{n}(\sigma F)(m)e^{iG_\sigma(m)}\right] = \frac{e^{i\lambda_q a_q q!}}{u}[\sigma F(0) + \cdots + \sigma F(u-1)]$$

$$= \frac{e^{i\lambda_q a_q q!}}{u}[F(0) + \cdots + F(u-1)],$$

which is independent of σ. If one of $\lambda_0, \lambda_1, \ldots, \lambda_{q-1}$ is nonzero, say $\lambda_r \neq 0$, but $\lambda_0 = \lambda_1 = \cdots = \lambda_{r-1} = 0$ $(0 \le r < q)$, then $\deg G_\sigma(m) = r$, and its leading coefficient is the same as that of $\lambda_r R_\sigma^{(r)}(m)$, which is an irrational multiple of π. Therefore, by Lemma 2.7,

$$\lim_{n\to\infty}\left[\frac{1}{n+1}\sum_{m=0}^{n}(\sigma F)(m)e^{iG_\sigma(m)}\right] = 0.$$

Thus, in either case, the limit

$$\lim_{n\to\infty}\left[\frac{1}{n+1}\sum_{m=0}^{n}E(\Omega)\overline{E}(\Delta)(S^m\sigma h)\right] = \lim_{n\to\infty}\left[\frac{1}{n+1}\sum_{m=0}^{n}(\sigma F)(m)e^{iG(m)}\right]$$

exists and is independent of $\sigma \in \Sigma$. This completes the proof.

2.9. **Theorem.** *Let the system (X, T) be strictly ergodic, with $T\colon X \to X$ a homeomorphism, and let the function $\Lambda\colon C(X) \times X \to l^\infty(\mathbb{Z})$ be defined by $\Lambda(F, x)(n) = F(T^n x)$. Then any element in the range of Λ is strictly ergodic.*

Proof. Let $f = \Lambda(F, x_0)$ be fixed, and define the mapping $\varphi\colon X \to l^\infty(\mathbb{Z})$ by $\varphi(x)(n) = F(T^n x)$. Then $\varphi(x_0) = f$, and

$$\varphi(Tx)(n) = F(T^n(Tx)) = F(T^{n+1}x) = \varphi(x)(n+1) = (S\varphi(x))(n)$$

or $\varphi(Tx) = S\varphi(x)$, which shows that φ is a flow homomorphism. Since X is minimal, $\varphi[X] = X_f$, the orbit closure of f. Therefore, by Corollary 1.7, f is strictly ergodic.

Next we would like to show that any function $h\colon \mathbb{Z} \to \mathbb{C}$ of the form $h(n) = \sum_{j=1}^{r} A_j e^{iP_j(n)}$ is uniquely ergodic, where $A_j \in \mathbb{C}$ and $P_j(n)$ is a real polynomial. The proof is presented for the case $r = 2$, but it can be applied to any $r \in \mathbb{N}$. First we need the following theorem.

2.10. **Theorem.** *Let f, g be two uniquely ergodic functions on $\mathbb{Z}$ such that for any four finite subsets Ω_1, Ω_2, Δ_1, and Δ_2 of $\mathbb{Z}$, the function*

$$F(\Omega_1, \Omega_2, \Delta_1, \Delta_2) = \prod_{\omega\in\Omega_1} S^\omega f \prod_{\omega\in\Omega_2} S^\omega g \prod_{\delta\in\Delta_1} S^\delta \overline{f} \prod_{\delta\in\Delta_2} S^\delta \overline{g}$$

is uniquely ergodic. Then $f + g$ is uniquely ergodic.

Proof. First we consider $Y = \Sigma(f, g)$, the orbit closure of (f, g), and we wish to show that it is uniquely ergodic. By Corollary 1.6, we must prove that for any $\varphi \in C(\Sigma(f, g))$, and any element $\sigma(f, g) = (\sigma f, \sigma g) \in \Sigma(f, g)$ the limit

$$\lim_{n\to\infty}\left[\frac{1}{n+1}\sum_{m=0}^{n}\varphi(S^m(\sigma f, \sigma g))\right]$$

exists and is a constant independent of σ. For this we use the fact that $\Sigma(f, g) \subset \Sigma f \times \Sigma g = X_f \times X_g$. Following the notation of Theorem 2.3, for a finite subset Ω of $\mathbb{Z}$, let the functions

$$E_1(\Omega)\colon \Sigma f \times \Sigma g \to \mathbb{C}, \qquad E_2(\Omega)\colon \Sigma f \times \Sigma g \to \mathbb{C}$$

be defined by $E_1(\Omega)(\sigma f, \tau g) = E(\Omega)(\sigma f)$, and $E_2(\Omega)(\sigma f, \tau g) = E(\Omega)(\tau g)$, and let $\overline{E_j}(\Omega)$ be the complex conjugate of $E_j(\Omega)$, $j = 1, 2$. Then $E_j(\Omega)$ is continuous and the set of linear combinations of the functions of the form

$$E_1(\Omega_1)E_2(\Omega_2)\overline{E_1}(\Delta_1)\overline{E_2}(\Delta_2)$$

is dense in $C(\Sigma f \times \Sigma g) = C(X_f \times X_g)$. Thus by Corollary 1.7, is enough to consider $\varphi = E_1(\Omega_1)E_2(\Omega_2)\overline{E_1}(\Delta_1)\overline{E_2}(\Delta_2)$. With this choice

$$\varphi(S^m(\sigma f, \sigma g)) = F(\Omega_1, \Omega_2, \Delta_1, \Delta_2)(m).$$

Since $F(\Omega_1, \Omega_2, \Delta_1, \Delta_2)(m)$ is uniquely ergodic, by Theorem 2.3, the limit

$$\lim_{n}\left[\frac{1}{n+1}\sum_{m=0}^{n}\varphi(S^m(\sigma f, \sigma g))\right]$$

exists and is independent of σ. This proves that $Y = \Sigma(f, g)$ is uniquely ergodic. Next by Theorem 2.9, the mapping $\Lambda\colon C(Y) \times Y \to l^\infty(\mathbb{Z})$ defined by $\Lambda(G, y)(n) = G(S^n y)$ produces uniquely ergodic functions. Now let $y_0 = (f, g)$, and $G = E_1(0) + E_2(0)$. Then

$$\Lambda(G, y_0)(n) = G(S^n y_0) = G(S^n(f, g)) = G(S^n f, S^n g) = f(n) + g(n)$$

is uniquely ergodic.

2.11. **Corollary.** *Let $p(n)$, $q(n)$ be any two real polynomials, and $f(n) = e^{ip(n)}$, and $g(n) = e^{iq(n)}$. Then $f + g$ is uniquely ergodic.*

Proof. It is sufficient to show that, for any four finite subsets Ω_1, Ω_2, Δ_1, and Δ_2 of integers the function

$$F(\Omega_1, \Omega_2, \Delta_1, \Delta_2) = \prod_{\omega\in\Omega_1} S^\omega f \prod_{\omega\in\Omega_2} S^\omega g \prod_{\delta\in\Delta_1} S^\delta \overline{f} \prod_{\delta\in\Delta_2} S^\delta \overline{g}$$

is uniquely ergodic. But $F(\Omega_1, \Omega_2, \Delta_1, \Delta_2)(n) = e^{iQ(n)}$, where

$$Q(n) = \sum_{\omega\in\Omega_1} p(n+\omega) + \sum_{\omega\in\Omega_2} q(n+\omega) - \sum_{\delta\in\Delta_1} p(n+\delta) - \sum_{\delta\in\Delta_2} q(n+\delta)$$

is a real polynomial in n. Therefore, by Theorem 2.8, $F(\Omega_1, \Omega_2, \Delta_1, \Delta_2)$ is uniquely ergodic.

So far, it has been shown that any element of the algebra $\mathbf{A}$ generated by the set $\{n \mapsto e^{i\lambda n^k} : \lambda \in \mathbb{R}, k \in \mathbb{N}\}$ is strictly ergodic. Next, we would like to demonstrate the main result of this paper; that is, any element of the Weyl algebra $\mathbf{W}$ is strictly ergodic. To accomplish this result, we need the following theorem.

2.12. **Theorem.** *The set of all uniquely ergodic functions in $X = l^\infty(\mathbb{Z})$ is closed under uniform convergence.*

Proof. Let $f_\alpha \in X$ be a sequence of uniquely ergodic functions that converge uniformly to f. We choose Ω, Δ to be any two finite subsets of $\mathbb{Z}$, and define the function $F: l^\infty(\mathbb{Z}) \to \mathbb{C}$ by $F(f) = E(\Omega)\overline{E}(\Delta)(f)$. For each α, the function f_α is uniquely ergodic, and therefore there is a number c_α independent of $\sigma \in \Sigma$ such that

$$\begin{aligned}\lim_{n\to\infty}&\left[\frac{1}{n+1}\sum_{m=0}^{n} E(\Omega)\overline{E}(\Delta)(S^m(\sigma f_\alpha))\right]\\ &= \lim_{n\to\infty}\left[\frac{1}{n+1}\sum_{m=0}^{n} F(S^m\sigma f_\alpha)\right] = c_\alpha .\end{aligned}$$

Let $M = \sup_\alpha \|f_\alpha\|_\infty$. Then $S^m(\sigma(f_\alpha))$ and $S^m(\sigma(f))$ are in the set

$$B = \{h \in l^\infty(\mathbb{Z}) : \|h\|_\infty \le M\}.$$

Since F is uniformly continuous on B, $\{c_\alpha\}$ is Cauchy, and it converges to a number $c \in \mathbb{C}$. Now we would like to show that for any $\sigma \in \Sigma$

$$\lim_{n\to\infty}\left[\frac{1}{n+1}\sum_{m=0}^{n} F(S^m(\sigma f))\right] = c .$$

Given $\varepsilon > 0$, by uniform continuity of F on B, there is a $\delta > 0$ such that

$$h_1, h_2 \in B \text{ and } \|h_1 - h_2\| < \delta \Rightarrow |F(h_1) - F(h_2)| < \varepsilon/3 .$$

Let α be chosen so that $|c_\alpha - c| < \varepsilon/3$, and $\|f_\alpha - f\|_\infty < \delta$, which implies that $|F(f_\alpha) - F(f)| < \varepsilon/3$. Therefore

$$\begin{aligned}\left|\frac{1}{n+1}\sum_{m=0}^{n} F(S^m(\sigma f)) - c\right| &\le \frac{1}{n+1}\sum_{m=0}^{n} |F(S^m(\sigma f)) - F(S^m(\sigma f_\alpha))|\\ &+ \left|\frac{1}{n+1}\sum_{m=0}^{n} F(S^m(\sigma f_\alpha)) - c_\alpha\right| + |c_\alpha - c| .\end{aligned}$$

Now, since f_α is uniquely ergodic, there is a number $M_0 \in \mathbb{N}$ such that

$$n \ge M_0 \Rightarrow \left|\frac{1}{n+1}\sum_{m=0}^{n} F(S^m(\sigma f_\alpha)) - c_\alpha\right| < \frac{\varepsilon}{3};$$

therefore

$$n \geq M_0 \Rightarrow \left| \frac{1}{n+1} \sum_{m=0}^{n} F(S^m(\sigma f)) - c \right| < \varepsilon,$$

which completes the proof.

2.13. **Theorem.** *Any element of the Weyl algebra is uniquely ergodic.*

Proof. This follows from Theorem 2.12 and the fact that the Weyl algebra is the uniform closure of a set of uniquely ergodic functions.

2.14. **Theorem.** *The Weyl algebra does not exhaust all distal functions.*

Proof. By Theorem 2.13, it is enough to introduce a distal function on $\mathbb{Z}$ that is not uniquely ergodic. Let $\mathbb{T} = \{z \in \mathbb{C}: |z| = 1\}$ be the unit circle, $\lambda \in \mathbb{T}$, and $g: \mathbb{T} \to \mathbb{T}$ be any continuous function, and let $\varphi: \mathbb{T}^2 \to \mathbb{T}^2$ be defined by

$$\varphi(u, v) = (\lambda u, g(u)v).$$

Then the flow $(\mathbb{Z}, \mathbb{T}^2)$ is distal, where the action is $(n, \xi) \mapsto \varphi^n(\xi)$ [4]. In [3], Furstenberg has introduced a continuous function $g: \mathbb{T} \to \mathbb{T}$ and an element $\lambda \in \mathbb{T}$ such that not all ergodic averages do exist; that is, there is a $\xi \in \mathbb{T}^2$ and $h \in C(\mathbb{T}^2)$ such that

$$\lim_{n\to\infty} \left[\frac{1}{n+1} \sum_{m=0}^{n} h(\varphi^m(\xi)) \right]$$

does not exist. Now if we define $f: \mathbb{Z} \to \mathbb{C}$ by $f(m) = h(\varphi^m(\xi))$, then, by Lemma 1.3, f is distal, while, by Corollary 2.4 (with $\sigma = R_0$, the right translation by 0); it is not uniquely ergodic $(f \notin \mathbf{W})$.

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