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LOCALLY FLAT 2-KNOTS IN $S^2 \times S^2$ WITH THE SAME FUNDAMENTAL GROUP

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ABSTRACT. We consider a locally flat 2-sphere in $S^2 \times S^2$ representing a primitive homology class ξ, which is referred to as a 2-knot in $S^2 \times S^2$ representing ξ. Then for any given primitive class ξ, there exists a 2-knot in $S^2 \times S^2$ representing ξ with simply-connected complement. In this paper, we consider the classification of 2-knots in $S^2 \times S^2$ whose complements have a fixed fundamental group. We show that if the complement of a 2-knot S in $S^2 \times S^2$ is simply connected, then the ambient isotopy type of S is determined. In the case of nontrivial π_1, however, we show that the ambient isotopy type of a 2-knot in $S^2 \times S^2$ with nontrivial π_1 is not always determined by π_1.

1. INTRODUCTION

Let ζ and η be natural generators of $H_2(S^2 \times S^2; \mathbb{Z})$ represented by the cross-section and fiber of the projection $S^2 \times S^2 \to S^2$ onto the first factor with $\zeta \cdot \zeta = \eta \cdot \eta = 0$ and $\zeta \cdot \eta = \eta \cdot \zeta = 1$. A 2-knot S in $S^2 \times S^2$ is a locally flat submanifold of $S^2 \times S^2$ homeomorphic to S^2. The fundamental group of the complement of S is referred to as the fundamental group of S. The exterior of S is the closure of the complement of a tubular neighborhood of S in $S^2 \times S^2$. Two 2-knots in $S^2 \times S^2$ are equivalent if they are ambient isotopic, that is, there exists an isotopic deformation $F: (S^2 \times S^2) \times I \to (S^2 \times S^2) \times I$ such that the homeomorphism F_1 takes one to the other. Kuga and Freedman have characterized those homology classes in $S^2 \times S^2$ that can be represented by 2-knots in $S^2 \times S^2$ as follows. Kuga has shown in [10] that the homology class $\xi = p\zeta + q\eta$, p, $q \in \mathbb{Z}$, can be represented by a smooth 2-knot in $S^2 \times S^2$ if and only if $|p| \leq 1$ or $|q| \leq 1$. Meanwhile, Freedman has shown in [6] that if p and q are relatively prime integers, then ξ can be represented by a 2-knot in $S^2 \times S^2$.

Since the problem of classifying 2-knots in $S^2 \times S^2$ is interesting, we consider in this paper the problem of whether the equivalence class of a 2-knot in $S^2 \times S^2$ is determined by its fundamental group. For any integer p, let $\rho_p: S^2 \to S^2$ be the canonical smooth map of degree p, and let $\phi_p: S^2 \to S^2 \times S^2$ be

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the embedding defined by $\phi_p(x) = (x, \rho_p(x))$. Then if we write Σ_p for the image $\phi_p(S^2)$, Σ_p is the standard smooth 2-knot in $S^2 \times S^2$ representing $\zeta + p\eta$. We obtained in [13] the following result: If the complement of a 2-knot S in $S^2 \times S^2$ representing $\zeta + p\eta$ is simply connected, then S and Σ_p are equivalent. In this paper we prove the unknotting theorem in more general cases: If the complement of a 2-knot S in $S^2 \times S^2$ representing $p\zeta + q\eta$ is simply connected, then the equivalence class of S is determined. Moreover, we prove that the equivalence class of a 2-knot in $S^2 \times S^2$ is not always determined by the fundamental group itself.

This paper is organized as follows. In §2, we consider the case that the fundamental group of a 2-knot is trivial. We show that for any relatively prime integers p and q, there is a 2-knot representing $p\zeta + q\eta$ with simply-connected complement, and prove the unknotting theorem. We consider in §3 the case that the fundamental group of a 2-knot is nontrivial. We prove that there exist distinct 2-knots with the same fundamental group. In §4, we consider the problem of whether a homology 3-sphere bounds a smooth acyclic 4-manifold or not, and by using Kuga's theorem and our technique in §2, we present a family of homology 3-spheres that cannot bound smooth acyclic 4-manifolds.

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2. 2-KNOTS IN $S^2 \times S^2$ WITH TRIVIAL π_1

It is easy to see that if the homology class represented by a 2-knot S is not primitive, then $H_1(S^2 \times S^2 - S; \mathbb{Z})$ is nonzero. We begin with the following proposition.

Proposition 2.1. *Let p and q be relatively prime integers. Then there exists a 2-knot in $S^2 \times S^2$ representing $p\zeta + q\eta$ with simply-connected complement.*

Proof. Since g.c.d.$(p, q) = 1$, there are two integers a, b such that $bp - aq = 1$. We consider the 3-manifold M obtained by surgery on the framed link L illustrated in Figure 1. The link L consists of two trivial knots K_1 and K_2. Since $|(2pq)(2ab) - (bp + aq)^2| = 1$, M is a homology 3-sphere, so that M bounds a topological contractible 4-manifold V. See [6]. Let W be the 4-manifold obtained by attaching two 2-handles h_1 and h_2 to the 4-disk D^4 along the framed link L (i.e., $W = D^4 \cup h_1 \cup h_2$). Set $X = W \cup_M V$, and X is a topological closed 4-manifold. Let B_i be a smooth 2-disk in D^4 which is the trivial knot K_i bounds, and let D_i be the core of h_i $(i = 1, 2)$. Then $S_i = B_i \cup D_i \subset W$ is diffeomorphic to S^2. Since the framing of K_1 is $2pq$, a closed tubular neighborhood of S_1 is the D^2-bundle $D(2pq)$ over S^2 with Euler number $2pq$. Since K_1 is trivial, W is the 4-manifold obtained by attaching the 2-handle h_2 to $D(2pq)$. Hence, by the duality of handle-

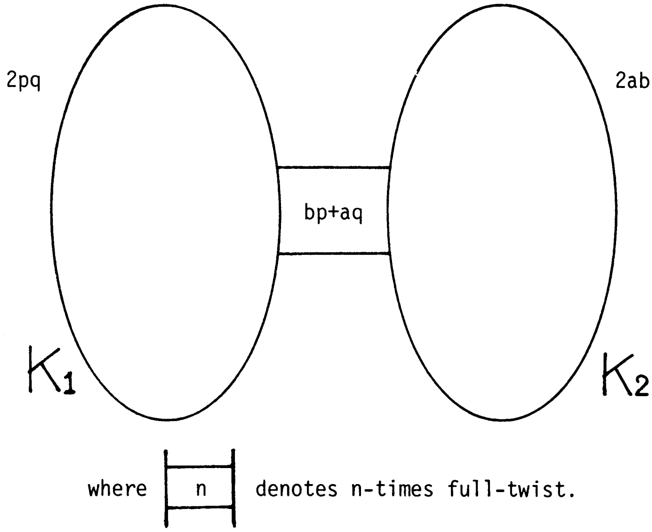

FIGURE 1

decompositions, we can view W as $(M \times I \cup h_2^*) \cup_\partial D(2pq)$, where h_2^* is the dual handle of h_2 and ∂ is the lens space $L(2pq, 2pq-1)$. Therefore, $X = Y \cup_\partial D(2pq)$, where $Y = V \cup_{M\times\{0\}} M \times I \cup h_2^*$. Then by van Kampen's theorem, $\pi_1(Y) = 1$ and $\pi_1(X) = 1$. Moreover, $\pi_1(X - S_1) \cong \pi_1(Y) = 1$. $H_2(X; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $[S_1]$ and $[S_2]$, and X has the intersection form

$$A = \begin{pmatrix} 2pq & bp+aq \\ bp+aq & 2ab \end{pmatrix}$$

with respect to these generators. Since the form A is even and indefinite, A is equivalent over $\mathbb{Z}$ to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In fact, if we let $(\mathbb{Z} \oplus \mathbb{Z}, A)$ and $(\mathbb{Z} \oplus \mathbb{Z}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$ be bilinear form spaces, then the matrix

$$B = \begin{pmatrix} p & a \\ q & b \end{pmatrix}$$

gives an isomorphism between them. Let $\{u, v\}$ and $\{\zeta, \eta\}$ be bases for the bilinear form spaces $(\mathbb{Z} \oplus \mathbb{Z}, A)$ and $(\mathbb{Z} \oplus \mathbb{Z}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$, respectively. The matrix B takes u to $p\zeta + q\eta$. Thus X has the intersection form $(\mathbb{Z} \oplus \mathbb{Z}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$, and the homology class of S_1, $[S_1]$, is $p\zeta + q\eta$. By Freedman's theorem, there is a homeomorphism $h: X \to S^2 \times S^2$. Then the induced isomorphism $h_*: H_2(X; \mathbb{Z}) \to H_2(S^2 \times S^2; \mathbb{Z})$ gives an automorphism of $(\mathbb{Z} \oplus \mathbb{Z}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$. Since the automorphism group of this form space is $\{C \in GL(2, \mathbb{Z}); {}^tC\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)C = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\} = \{\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \pm\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}$, $h_* = \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\pm\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus the image $h(S_1)$ is a locally flat 2-sphere in $S^2 \times S^2$ representing $\pm(p\zeta + q\eta)$ or $\pm(q\zeta + p\eta)$, and $\pi_1(S^2 \times S^2 - h(S_1)) \cong \pi_1(X - S_1) = 1$. After changing the orientation of $S^2 \times S^2$ and/or the orientation of ζ and η (if necessary), $h(S_1)$ may represent $p\zeta + q\eta$. Therefore, $h(S_1)$ is a required 2-knot in $S^2 \times S^2$.

Our key lemma in this section is the following.

Lemma 2.2. *Let p and q be relatively prime integers, and let S_1 and S_2 be 2-knots in $S^2 \times S^2$ representing $p\zeta + q\eta$. If the complements of S_1 and S_2 are simply connected, then there exists a homeomorphism of $S^2 \times S^2$ taking S_1 to S_2.*

Since we can prove this lemma in the same manner as [13], we only sketch the proof.

Proof (sketch). Let N_i be a closed tubular neighborhood of S_i and E_i the exterior of S_i $(i = 1, 2)$. Then N_i is homeomorphic to $D(2pq)$, and so the boundary ∂E_i of E_i is the lens space $L(2pq, 2pq-1)$, where $L(0, -1) = S^2 \times S^1$. Hence $(S^2 \times S^2, S_i)$ is pairwise homeomorphic to $(D(2pq) \cup_{\gamma_i} E_i, \nu(S^2))$, where $\gamma_i \colon L(2pq, 2pq-1) \to L(2pq, 2pq-1)$ is some gluing homeomorphism and $\nu \colon S^2 \to D(2pq)$ is the zero section. By the isotopy extension theorem, it is easily seen that the homeomorphism type of 2-knots with exterior E_i depends only on the isotopy class of the homeomorphism γ_i. To prove Lemma 2.2, we need the following lemma.

Lemma 2.3. *Suppose E_1 and E_2 are simply connected. Then E_1 is homeomorphic to E_2. In particular if $(p, q) = (\pm 1, 0)$ or $(0, \pm 1)$, then E_1 and E_2 are homeomorphic to $S^2 \times D^2$.*

Proof. We give E_i the orientation opposite to the one inherited from $S^2 \times S^2$. It follows that the intersection form $(H_2(E_i; \mathbb{Z}), \cdot)$ is isomorphic to $(\mathbb{Z}, (2pq))$, where $(2pq) \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is the bilinear form defined by $(2pq)(1, 1) = 2pq$. Hence, E_i is a simply-connected compact 4-manifold with boundary $L(2pq, 2pq - 1)$ and the intersection form $(\mathbb{Z}, (2pq))$. In [2], Boyer calculated the set of all oriented homeomorphism types of simply-connected compact 4-manifolds with given boundary and given intersection form. In the case of $(p, q) = (\pm 1, 0)$ or $(0, \pm 1)$, Remarks (5.3) of [2] say that E_1 and E_2 are homeomorphic to $S^2 \times D^2$. Next we consider the case of $pq \neq 0$. Since g.c.d$(p, q) = 1$, there are two integers a, b such that $bp - aq = 1$. If we set $u_i = [S_i] = p\zeta + q\eta$ and $v = a\zeta + b\eta$, then u_i and v generate $H_2(S^2 \times S^2; \mathbb{Z})$. Let w_i be a generator of $H_2(E_i, \partial E_i; \mathbb{Z}) \cong \mathbb{Z}$. Since $u_i \cdot v = bp + aq$, $\partial w_i \in H_i(\partial E_i; \mathbb{Z}) = H_1(L(2pq, 2pq-1); \mathbb{Z})$ is represented by $(bp+aq)$-times the ∂D^2-fiber of the D^2-bundle N_i over S_i. Since $bp - aq = 1$, $(bp+aq)^2 \equiv 1 \pmod{2pq}$. Hence, it follows from Example 5.4 and Remarks 5.6 of [2] that E_1 is homeomorphic to E_2. □

Return to the proof of Lemma 2.2. Since the complements of S_1 and S_2 are simply connected, there is a homeomorphism $h \colon E_1 \to E_2$. Let $\tilde{h}$ be the restriction of h to ∂E_1. If the homeomorphism $\gamma_2^{-1}\tilde{h}\gamma_1 : \partial D(2pq) \to \partial D(2pq)$ extends to a homeomorphism g of $(D(2pq), \nu(S^2))$, we have the following required homeomorphism:

$$\varphi : (D(2pq) \cup_{\gamma_1} E_1, \nu(S^2)) \to (D(2pq) \cup_{\gamma_2} E_2, \nu(S^2))$$

by setting

$$\varphi = \begin{cases} g & \text{on } D(2pq), \\ h & \text{on } E_1. \end{cases}$$

Now we remark that in the case of $pq = 0$, $\gamma_2^{-1}\tilde{h}\gamma_1 : S^2 \times S^1 \to S^2 \times S^1$ is not isotopic to the twist $\tau : S^2 \times S^1 \to S^2 \times S^1$ defined by $\tau((\theta, \phi), \psi) = ((\theta + \psi, \phi), \psi)$, since E_1 and E_2 are homeomorphic to $S^2 \times D^2$ and the second Stiefel-Whitney class of $S^2 \times S^2$ is trivial. Hence, by investigating the homeotopy group of $L(2pq, 2pq - 1)$, it follows that there is an extension g as the above. See [1], [8] and [9]. This completes the proof. □

Theorem 2.4. *Let S_1 and S_2 be 2-knots in $S^2 \times S^2$ as in Lemma 2.2. If the complements of S_1 and S_2 are simply connected, then S_1 and S_2 are equivalent, i.e., ambient isotopic.*

Proof. In the case when $|p| = 1$ or $|q| = 1$, we proved in [13]. We may assume that $|p| \geq 2$ and $|q| \geq 2$. It follows from [14] that the homeotopy group of $S^2 \times S^2$ corresponds to the subgroup of $GL(2; \mathbb{Z})$ consisting of

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with respect to generators ζ and η.

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are orientation preserving, while

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are orientation reversing. By Lemma 2.2, there is a homeomorphism ϕ of $S^2 \times S^2$ taking S_1 to S_2. Since $\phi_*(p\zeta + q\eta) = \phi_*([S_1]) = \pm[S_2] = \pm(p\zeta + q\eta)$, $\phi_* = \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We consider the case of $\phi_* = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then $\phi|_{S_1}$ is orientation reversing. Let $-S_2$ be S_2 with opposite orientation. We decompose $(S^2 \times S^2, \pm S_2)$ as in Lemma 2.2: $(S^2 \times S^2, \pm S_2) = (D(2pq) \cup_{\gamma^\pm} E^\pm, \nu(S^2))$. Here we may assume that $\gamma^\pm$ is the identity map. Let $g : D(2pq) \to D(2pq)$ be the orientation-preserving homeomorphism such that its restriction to $\nu(S^2)$ is the antipodal map and its restriction to each fiber is the map induced on the unit disk in the complex plane by complex conjugation. Then $g' = g|\partial D(2pq)$ is a homeomorphism of $\partial D(2pq)$ such that $g'_*(\partial w_+) = \pm\partial w_-$, where $w_\pm$ is a generator of $H_2(E^\pm, \partial E^\pm; \mathbb{Z}) \cong \mathbb{Z}$. Since Boyer's results are based on a theorem that gives necessary and sufficient conditions for the existence of a homeomorphism between simply-connected 4-manifolds extending a given homeomorphism of their boundaries, the fact that $g'_*(\partial w_+) = \pm\partial w_-$ implies that there is an orientation-preserving homeomorphism $h\colon E^+ \to E^-$ such that $h|_{\partial E} = g'$. See [2]. Let $\psi\colon S^2 \times S^2 \to S^2 \times S^2$ be the orientation-preserving homeomorphism defined from g and h. From the definition of g, it is easily seen that $\psi(S_2) = -S_2$. Hence $\psi \cdot \phi$ is a homeomorphism of $S^2 \times S^2$ taking S_1 to S_2 such that $(\psi \cdot \phi)_*$ is the identity map.

Thus, we have a homeomorphism ϕ' of $S^2 \times S^2$ taking S_1 to S_2 such that ϕ'_* is the identity map, so ϕ' is isotopic to the identity map. Therefore, S_1 and S_2 are equivalent. This completes the proof. □

Remark 2.5. Let K be a 2-knot in S^4 and S a 2-knot in $S^2 \times S^2$. Then we obtain another 2-knot in $S^2 \times S^2$ by forming the connected sum of pairs $(S^2 \times S^2, S)$ and (S^4, K). However, we do not always get a new 2-knot in $S^2 \times S^2$ in this manner. In fact, Theorem 2.2 says that if $\pi_1(S^2 \times S^2 - S) = 1$, then the connected sum of S with any 2-knot in S^4 is always equivalent to the original 2-knot S. See [13].

Remark 2.6. Let S_1 and S_2 be 2-knots in $S^2 \times S^2$ representing $p\zeta + q\eta$, where $p \neq q$ and $pq \neq 0$. If there is a homeomorphism g of $S^2 \times S^2$ taking S_1 to S_2 such that $g|_{S_1}$ is orientation preserving, then S_1 and S_2 are equivalent.

3. 2-KNOTS IN $S^2 \times S^2$ WITH NONTRIVIAL π_1

We describe a construction of 2-knots in $S^2 \times S^2$ from [11] and [13]. Let K be a 2-knot in S^4 and C a smoothly embedded circle in $S^4 - K$. Since we may assume that C is standardly embedded in S^4 up to ambient isotopy, the closure of the complement of a tubular neighborhood of C in S^4 is $S^2 \times D^2$. Then K is contained in $S^2 \times D^2$, so that this gives us a 2-knot S in $S^2 \times S^2 = S^2 \times D^2 \cup S^2 \times D^2$. If C is homologous in $S^4 - K$ to a meridian of K, then the 2-knot S represents ζ [13]. Moreover, by van Kampen's theorem $\pi_1(S^2 \times S^2 - S)$ is isomorphic to $\pi_1(S^4 - K)/H$, where H is the normal closure of the element represented by C in $\pi_1(S^4 - K)$.

We are concerned with the following two 2-knots in $S^2 \times S^2$ representing ζ. Let $K \subset S^4$ be the 5-twist spun 2-knot of the trefoil [15]. Then $\pi_1(S^4 - K) \cong \mathscr{D} \times \mathbb{Z}$, where $\mathscr{D}$ is the binary dodecahedral group

$$\langle a, b;\, a^3 = b^5 = (ab)^2 \rangle$$

and $\mathbb{Z}$ is generated by μ which is homologous to a meridian of K. The group $\mathscr{D}$ is perfect and of order 120. The center of $\mathscr{D}$ is generated by $c = a^3$ in $\mathscr{D}$, and it is of order 2. Let C_1 and C_2 be embedded circles representing μ and μc^{-1} in $\pi_1(S^4 - K)$, respectively. Let S_1 be the 2-knot in $S^2 \times S^2$ constructed from K and C_1, and let S_2 be the 2-knot in $S^2 \times S^2$ constructed from K and C_2. Let E_1 and E_2 be exteriors of S_1 and S_2, respectively. Then both S_1 and S_2 represent ζ, and $\pi_1(S^2 \times S^2 - S_1) \cong \pi_1(S^2 \times S^2 - S_2) \cong \mathscr{D}$. Thus S_1 and S_2 are 2-knots in $S^2 \times S^2$ that represent ζ and whose fundamental groups are isomorphic to $\mathscr{D}$.

Now we investigate meridian elements in $\mathscr{D}$ of the preceding 2-knots in $S^2 \times S^2$. We note that the group of the 5-twist spun 2-knot of the trefoil, $\pi_1(S^4 - K)$, has the following presentation:

$$\pi_1(S^4 - K) = \langle u, v; uvu = vuv, v = u^{-5}vu^5 \rangle,$$

where u is a meridian and the second relation comes from the 5-twisting. Zeeman showed in [15] that $\pi_1(S^4 - K)$ is isomorphic to

$$\langle x, y, z; x^5 = (xy)^3 = (xyx)^2, z^{-1}xz = y, z^{-1}yz = yx^{-1} \rangle,$$

by making the substitution $u \to z$, $v \to xz$. Then z is a meridian. By making the substitution $x \to b$, $xy \to a$, this group is isomorphic to

$$\begin{aligned}&\langle a, b, z; a^3 = b^5 = (ab)^2, z^{-1}bz = b^{-1}a, z^{-1}b^{-1}az = b^{-1}ab^{-1} \rangle \\ &\qquad \cong \langle a, b, z, \mu; a^3 = b^5 = (ab)^2, \mu = ab^{-1}z, [\mu, a] = [\mu, b] = 1 \rangle \\ &\qquad \cong \mathscr{D} \times \mathbb{Z}.\end{aligned}$$

Therefore, ba^{-1} and ba^2 in $\mathscr{D}$ are meridian elements of 2-knots S_1 and S_2, respectively. Since a^3 in $\mathscr{D}$ is of order 2, ba^{-1} is of order 10. Also, since $ba^2 = a^3(ba^{-1})$ and a^3 is an element in the center of $\mathscr{D}$, ba^2 is of order 5. Thus the order of a meridian element of S_1 is different from that of S_2, so that there is not a ∂-preserving homotopy equivalence $f: (E_1, \partial E_1) \to (E_2, \partial E_2)$, that is, two 2-knots S_1 and S_2 are inequivalent. Thus we have

Theorem 3.1. *There exists 2-knots in $S^2 \times S^2$ representing ζ with fundamental group isomorphic to the binary dodecahedral group, but whose exteriors are not ∂-preserving homotopy equivalent.*

Remark 3.2. The complements of 2-knots S_1 and S_2 in $S^2 \times S^2$ as given earlier are not $K(\pi, 1)$. In fact, $\pi_2(S^2 \times S^2 - S_i) \neq 0$ $(i = 1, 2)$. Let S be either S_1 or S_2, and let X be the complement of S. Then, since S represents $\zeta \in H_2(S^2 \times S^2; \mathbb{Z})$, $H_2(X; \mathbb{Z}) \cong \mathbb{Z}$. If we let $p: \widetilde{X} \to X$ be the universal covering, then we have a homomorphism $\tau: H_2(x; \mathbb{Z}) \to H_2(\widetilde{X}; \mathbb{Z})$ such that $p_*\tau(\alpha) = 120\alpha$. Here p_* is the homomorphism $H_2(\widetilde{X}; \mathbb{Z}) \to H_2(X; \mathbb{Z})$ induced by the projection p, and α is a generator $H_2(X; \mathbb{Z}) \cong \mathbb{Z}$. Hence, $\pi_2(X) \cong \pi_2(\widetilde{X}) \cong H_2(\widetilde{X}; \mathbb{Z})$ is not trivial.

4. Concluding remarks

We consider in this section the problem of whether or not a given homology 3-sphere bounds a smooth acyclic 4-manifold. We have the Rohlin invariant

$\mu: H^3 \to \mathbb{Z}/2\mathbb{Z}$, where H^3 is the homology cobordism group of homology 3-spheres. If a homology 3-sphere M bounds a smooth acyclic 4-manifold, then $\mu(M) = 0$. Some families of homology 3-spheres that bound smooth acyclic (or contractible) 4-manifolds are known. Meanwhile, the celebrated work of Donaldson [4] implies that if a homology 3-sphere M bounds a smooth 4-manifold with nonstandard definite intersection form, then M cannot bound a smooth acyclic 4-manifold. Also, Fintushel and Stern showed that if the invariant $R(a_1, \dots, a_n)$ defined in [5] is positive, then the Seifert fibered homology 3-sphere $\Sigma(a_1, \dots, a_n)$ cannot bound a smooth $\mathbb{Z}/2\mathbb{Z}$-acyclic 4-manifold. However, we note that every homology 3-sphere bounds a topological contractible 4-manifold. See [6].

Definition 4.1. Let L be the following framed link in S^3 consisting of two knots J and K with linking number t and with framing m and n.

M(L;t:m,n) =

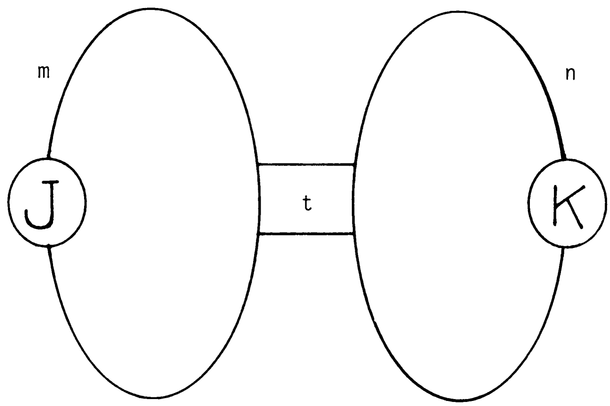

FIGURE 2

Then $M(L; t: m, n)$ is defined as a 3-manifold obtained by Dehn surgery on the framed link L.

The order of $H_1(M(L; t: m, n); \mathbb{Z})$ is $|mn - t^2|$. Hence, if $|mn - t^2| = 1$, then $M(L; t: m, n)$ is a homology 3-sphere.

Before stating the main result in this section, we notice the following. Since Donaldson's result in [3] extends without change to 4-manifolds with arbitrary fundamental groups [4], Kuga's result in [10] also extends to such 4-manifolds, that is,

Theorem 4.2. *Let X be a closed smooth 4-manifold with the intersection form*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to ζ and η of $H_2(X; \mathbb{Z})$ /torsion $\cong \mathbb{Z} \oplus \mathbb{Z}$. Then the homology class $p\zeta + q\eta$ cannot be represented by a smoothly embedded 2-sphere in X provided $|p| \geq 2$ and $|q| \geq 2$.

Proof. This follows in the same manner as in [10].

Our main result in this section is the following.

Theorem 4.3. *Let t be a positive odd integer. Let J and K be slice knots. Suppose that m and n are positive even integers such that $mn - t^2 = -1$. If $|m-t| > 1$ or $|n-t| > 1$, then $M = M(L; t\colon m, n)$ cannot bound a smooth compact 4-manifold V with $\widetilde{H}_*(V; \mathbb{Q}) = 0$.*

Hence, such an M does not bound a smooth acyclic 4-manifold.

Proof. Suppose that there is such a smooth 4-manifold V. Let W be the smooth 4-manifold obtained by attaching two 2-handles to D^4 along the framed link $L = J \cup K$. Then $X = W \cup_M V$ is a closed smooth 4-manifold with the intersection form

$$A = \begin{pmatrix} m & t \\ t & n \end{pmatrix}$$

with respect to some generators of $H_2(X; \mathbb{Z})$ /torsion $\cong \mathbb{Z} \oplus \mathbb{Z}$. Then there are x and y in $H_2(X; \mathbb{Z})$ such that $x^2 = m$, $y^2 = n$ and $x \cdot y = t$, and both x and y are represented by smoothly embedded 2-spheres in X. Since m and n are even integers with $mn - t^2 = -1$, A is equivalent over $\mathbb{Z}$ to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Hence, X has the intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to generators ζ and η of $H_2(X; \mathbb{Z})$ /torsion. For some integers p, q, r and s, $x = p\zeta + q\eta$ and $y = r\zeta + s\eta$. Since $|m-t| > 1$ or $|n-t| > 1$, it is seen that either $\min(|p|, |q|)$ or $\min(|r|, |s|)$ is greater than 1. Hence, there is a smoothly embedded 2-sphere in X representing $a\zeta + b\eta$ with $|a| \geq 2$, and $|b| \geq 2$, contradicting Theorem 4.2. This completes the proof. □

Remark 4.4. (1) Let J and K be any knots, and let m and n be even integers with $mn - t^2 = -1$. Then $\mu(M(L; t\colon m, n)) = 0$. (2) When J and K are trivial knots, $M(L; t\colon m, n)$ is the Brieskorn homology 3-sphere $\Sigma(t, |m-t|, |n-t|)$ if $|m-t| > 1$ and $|n-t| > 1$. Moreover,

$$R(t, |m-t|, |n-t|) = 1.$$

(3) If J and K are slices, then $M = M(L; \pm 1\colon 0, 0)$ is embedded smoothly in S^4. See [7]. Hence, M bounds a smooth acyclic 4-manifold.

We can find the following lemma in [12].

Lemma 4.5. *If a homology* 3*-sphere M is embedded smoothly in $S^2 \times S^2$, then M bounds a smooth acyclic* 4*-manifold.*

Since every homology 3-sphere admits a locally flat embedding into S^4, it also admits such an embedding into $S^2 \times S^2$. However, Theorem 4.3 and Lemma 4.5 imply the following proposition.

Proposition 4.6. *There exists a* μ*-invariant* 0 *homology* 3*-sphere that cannot be embedded smoothly in* $S^2 \times S^2$.

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