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CONFORMAL AUTOMORPHISMS AND CONFORMALLY FLAT MANIFOLDS

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Dedicated to Professor Shôrô Araki on his sixtieth birthday

ABSTRACT. A geometric structure on a smooth n-manifold M is a maximal collection of distinguished charts modeled on a 1-connected n-dimensional homogeneous space X of a Lie group G where coordinate changes are restrictions of transformations from G. There exists a developing map $dev\colon wm \to X$ which is always locally a diffeomorphism. It is in general far from globally being a diffeomorphism. We study the rigid property of developing maps of (G, X)-manifolds. As an application we shall classify closed conformally flat manifolds M when the universal covering space $\widetilde{M}$ supports a one parameter group of conformal transformations.

This paper is the sequel to [G-K]. The main result of this paper concerns closed conformally flat manifolds of dimension greater than 2:

Theorem A. *Let M be a closed conformally flat n-manifold where $n > 2$. Suppose that the universal covering $\widetilde{M}$ posesses a complete conformal vector field. Then there exists a finite sheeted covering space of M which is conformally equivalent to a manifold in the following classes:*

(1) *the n-sphere S^n;*
(2) *a flat n-torus T^n;*
(3) *a hyperbolic n-manifold H^n/Γ;*
(4) *a Hopf manifold $S^1 \times S^{n-1}$;*
(5) *a product $S^1 \times H^{n-1}/\Gamma$ where H^{n-1}/Γ is a hyperbolic $(n-1)$-manifold;*
(6) *a quotient $S^n - \Lambda/\Gamma$ where $\Lambda \subset S^{n-2}$ is the limit set for the conformal holonomy group $\Gamma \subset \mathrm{Conf}(S^n)$.*

(*Conformally homogeneous domain in the sphere have been classified by Kimelfeld* [*Ki*]).

Recall that a geometric structure on a smooth n-manifold is a maximal collection of charts modeled on a simply connected n-dimensional homogeneous space X of a Lie group G whose coordinate changes are restrictions of transformations from G. We call such a structure a (G, X)-structure. A manifold with this structure is called a (G, X)-manifold.

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Suppose that a smooth connected n-manifold M admits a (G, X)-structure. Then there exists a developing pair (ψ, dev), where $dev: \widetilde{M} \to X$ is a "structure-preserving" immersion and $\psi: \pi_1(M) \to G$ is a homomorphism (both unique up to elements of G). The group $\Gamma = \psi(\pi_1(M))$ is called the holonomy group for M. Suppose that M is a closed connected (G, X)-manifold. In [G-K] (cf. also [G]) we have shown that

Lemma B. *Let A be a Γ-invariant closed subset in X. Suppose that in the complement of A in X there exists a component U which admits a Γ-invariant complete Riemannian metric. Then the developing map $dev: V \to U$ on each component V of $dev^{-1}(U)$ is a covering map.*

This lemma can be used to obtain the above theorem by finding a geometric k-sphere in S^n such as A. Indeed we can prove the topological rigidity of developing maps of (G, X)-manifolds as above when such A exists and is small enough. On the other hand the above rigidity does not hold for $k = n - 1$. For example let M be a closed conformally flat n-manifold with Fuchsian holonomy Γ (i.e., Γ is discrete and cocompact in $\mathrm{SO}(n, 1)$). Then we can show that its developing map is surjective unless M is conformally equivalent to a hyperbolic n-manifold. In [G] such an example of conformally flat structure has been constructed for a compact hyperbolic n-manifold. Consequently we have the following result in general. (See (2.3.1).)

Theorem C. *Let M be a closed conformally flat n-manifold. Suppose that the holonomy group Γ lies in the subgroup $\mathrm{SO}(n, 1)$ of $\mathrm{Conf}(S^n)$. Then M decomposes into a union of complete hyperbolic n-manifolds with ideal boundary.*

1. (G, X)-MANIFOLDS

1.1. Suppose that a closed connected n-manifold M admits a (G, X)-structure. Let (ψ, dev) be a developing pair and Γ the holonomy group as above.

Proof of Lemma B. Since dev is an immersion, N admits a $\pi_1(M)$-invariant Riemannian metric. Let $\Pi: \widetilde{M} \to M$ be the covering projection. Put $\Pi(N) = W$. There is an induced metric on W so that $\Pi: N \to W$ is a local isometry. It suffices to show that W is complete.

Let ρ, ρ_N, ρ_W be the distance functions on Y, N, W respectively. Let $\{w_i\}$ be a Cauchy sequence in W. Since M is compact, there exists an accumulation point w in M for the sequence $\{w_i\}$. Choose a point $\tilde{w} \in \widetilde{M}$ with $\Pi(\tilde{w}) = w$. There exists a neighborhood U of $\tilde{w}$ in $\widetilde{M}$ such that the restrictions

$$\Pi: U \to \Pi(U), \qquad (1)$$
$$dev: U \to dev(U)$$

are homeomorphisms. We can assume that there is a number $m > 0$ such that for $i \geq m$

$$w_i \in \Pi(U). \tag{2}$$

Let $B_r(p)$ (resp. $\overset{\circ}{B}_r(p)$) denote a closed (resp. open) metric ball of radius $r > 0$ about $p \in W$,

$$B_r(p) = \{x \in W \mid \rho_W(p, x) \leq r\}.$$

Since $\{w_i\}$ is Cauchy, to each integer $n > 0$ there exists an integer $\alpha(n)$ such that $\rho_W(w_i, w_j) < \frac{1}{n}$ for $i, j \geq \alpha(n)$. For $x = w_{\alpha(n)}$ and $r = \frac{1}{n}$,

$$w_i \in \overset{\circ}{B}_r(x) \tag{3}$$

for $i \geq \alpha(n)$.

By (2), for sufficiently small r (e.g., $\frac{1}{n}$)

$$\overset{\circ}{B}_r(x) \subset \Pi(U). \tag{4}$$

Now

$$\overset{\circ}{B}_r(\tilde{x}) \subset U. \tag{5}$$

For otherwise there is a point p in $\overset{\circ}{B}_r(\tilde{x}) \cap \partial U$ where ∂U is the frontier of U.

$$\rho_W(x, \Pi(p)) \leq \rho_N(\tilde{x}, p) < r.$$

By (4) it follows that $\Pi(p) \in \operatorname{Int}\Pi(U)$ while $\Pi(p) \in \Pi(\partial U) = \partial\Pi(U)$, because Π is a homeomorphism of U onto $\Pi(U)$. This contradiction proves (5).

Let $dev(\tilde{x}) = y$. Since dev is a local isometry,

$$dev(B_r(\tilde{x})) \subset B_r(y). \tag{6}$$

Now $B_r(y)$ is a closed metric ball in Y and by hypothesis Y is complete. Thus $B_r(y)$ is compact (cf. [W, Theorem 2.8.8]) and by [W, Lemma 2.8.7], any two points in $\overset{\circ}{B}_r(y)$ can be joined by a minimizing geodesic. Hence $dev: B_r(\tilde{x}) \to dev(B_r(\tilde{x}))$ is an isometry, and it follows from (6) that

$$dev(B_r(\tilde{x})) = B_r(y). \tag{7}$$

In particular $B_r(\tilde{x})$ is compact. If $z \in \overset{\circ}{B}_r(\tilde{x})$ then there is a minimizing geodesic between z and $\tilde{x}$, again by [W, Lemma 2.8.7]. Now

$$\Pi(B_r(\tilde{x})) = B_r(x). \tag{8}$$

To prove (8), suppose there exists a point $z \in \overset{\circ}{B}_r(x)$ not in $\Pi(\overset{\circ}{B}_r(\tilde{x}))$. Let c be a (smooth) curve between x and z in W. Let c' be a subarc of c between x and the first contact to $\Pi(\partial B_r(\tilde{x}))$. Since the arc c' lies in $\Pi(B_r(\tilde{x}))$, there is a minimizing geodesic in $\Pi(B_r(\tilde{x}))$ between x and the endpoint of c' by the above remark. In particular the length $L(c') \geq r$. As $L(c) > L(c')$, it follows that

$$r > \rho_W(x, z) = \inf_c L(c) \geq r,$$

a contradiction. Therefore $\Pi(\overset{\circ}{B}_r(\tilde{x})) = \overset{\circ}{B}_r(x)$. By compactness of $B_r(\dot{\tilde{x}})$ and (4), we obtain (8). Consequently $B_r(x)$ is compact and by (3) it follows that the sequence $\{w_i\}$ converges to some point in W. This completes the proof of Lemma B. □

(1.1.1) **Proposition.** *Let A be a Γ-invariant closed subset of X with Hausdorff dimension $k < n-1$. Suppose that the complement $X - A$ admits a Γ-invariant complete Riemannian metric.*

(i) *if $k < n-2$, then $dev\colon \widetilde{M} \to X - \Lambda$ is a homeomorphism where Λ is the limit set for Γ.*

(ii) *for $n-2 \leq k < n-1$ assume that either $dev^{-1}(A) = \varnothing$ or $dev\colon \pi_1(\widetilde{M} - dev^{-1}(A)) \to \pi_1(X - A)$ is surjective. Then $dev\colon \widetilde{M} \to X - A$ is a covering map, or $dev\colon \widetilde{M} \to X - \Lambda$ is a homeomorphism.*

Proof. Recall that if the Hausdorff dimension k is less than $n-1$, then $\widetilde{M} - dev^{-1}(A)$ is connected [G-K]. Moreover if $k < n-2$, $X - A$ is 1-connected. Applying Lemma B, $dev\colon \widetilde{M} - dev^{-1}(A) \to X - A$ is a covering map. As above if $k < n-2$, $dev\colon \widetilde{M} - dev^{-1}(A) \to X - A$ is a homeomorphism. If $n-2 \leq k < n-1$ then according to that $dev^{-1}(A) = \varnothing$ or $dev^{-1}(A) \neq \varnothing$ with the surjectivity assumption it follows that $dev\colon \widetilde{M} \to X-A$ is a covering map or $dev\colon \widetilde{M} - dev^{-1}(A) \to X - A$ is a homeomorphism. Since *dev* is an immersion and $k < n-1$, for any point x in $\widetilde{M}$ there exists a neighborhood W of x in $\widetilde{M}$ such that $dev(W) \cap (X - A) \neq \varnothing$. This implies that $dev\colon \widetilde{M} \to dev(\widetilde{M})$ is injective. Hence Γ acts properly discontinuously on $dev(\widetilde{M})$ which shows that $dev(\widetilde{M}) \cap \Lambda = \varnothing$. Since it is known that Γ acts properly discontinuously on $X - \Lambda$ (cf. [Ku]), it follows that $dev(\widetilde{M}) = X - \Lambda$. □

(1.1.2) **Proposition.** *Let A be a codimension* 1*-submanifold of X such that $X - A$ consists of two simply connected components X_+, X_-. Suppose that the following conditions are satisfied:* (i) *each X_+ (resp. X_-) admits a Γ-invariant complete Riemannian metric,* (ii) *if $dev^{-1}(A) \neq \varnothing$, then there exists a connected set Y in $dev^{-1}(A)$ for which $dev(Y)$ is dense in A.*

Then one of the following is true.

(a) *dev is a homeomorphism of $\widetilde{M}$ onto either X_+ or X_-.*

(b) *dev is a homeomorphism of $\widetilde{M}$ onto the dense subset $dev(\widetilde{M})$ in X.*

Proof. Since the sets X_+, A, X_- are each Γ-invariant, we have a $\pi_1(M)$-invariant decomposition of $\widetilde{M}$; namely, $\widetilde{M} = dev^{-1}(X_+) \cup dev^{-1}(A) \cup dev^{-1}(X_-)$. We may assume that $dev^{-1}(X_+) \neq \varnothing$ If $dev^{-1}(A) = \varnothing$ then since $\widetilde{M}$ is connected $\widetilde{M} = dev^{-1}(X_+)$ and so $dev\colon \widetilde{M} \to X_+$ is a homeomorphism by the hypothesis (i) and Lemma B. Suppose that $dev^{-1}(A) \neq \varnothing$. Let V_+ be a component of $dev^{-1}(X_+)$. It follows from Lemma B that

$$(1.1.3) \qquad dev\colon V_+ \to X_+ \text{ is a homeomorphism.}$$

The boundary of V_+ in $\widetilde{M}$, ∂V_+ consists of a union of components in $dev^{-1}(A)$. Let Y be a connected set of $dev^{-1}(A)$ of the hypothesis (ii). We can assume that Y lies in a component of ∂V_*. The hypothesis that $\overline{dev(Y)} = A$ implies that ∂V_+ is a unique boundary component. Otherwise $dev\colon V_+ \to X_+$ fails to be injective, contradicting (1.1.3). As $dev^{-1}(X_-) \neq \varnothing$, there is a component V_- whose boundary contains Y. By the same argument as above ∂V_- is a unique boundary component of V_-. Since both ∂V_+ and ∂V_- are smooth manifolds containing Y, it follows that $\partial V_+ = \partial V_-$. Therefore we conclude that $\widetilde{M} = V_+ \cup \partial V_+ \cup V_-$ which is homeomorphic to $dev(\widetilde{M})$. □

1.2. Let N be a (G, X)-manifold. The automorphism group of N denoted by $\mathscr{A}(N)$ is the group of diffeomorphisms of N which preserve the given (G, X)-structure. If M is a (G, X)-manifold, then $\pi_1(M) \subset \mathscr{A}(\widetilde{M})$ and $\psi\colon \pi_1(M) \to G$ extends to a continuous homeomorphism $\psi\colon \mathscr{A}(\widetilde{M}) \to G$.

(1.2.1) **Proposition.** *Suppose $n \geq 3$ and let N be a simply connected n-manifold with a (G, X)-structure. If there is a connected subgroup H of $\mathscr{A}(N)$ such that the orbit $\psi(H) \cdot dev(x) = X - F$ where F is a finite set then $dev\colon N \to X - F$ is a homeomorphism.*

This follows easily if one notices the following. Let H be a connected subgroup of $\mathscr{A}(N)$. The restriction to the orbit, $dev\colon Hx \to \psi(H) \cdot dev(x)$ is a covering map since dev is a local homeomorphism. Also $X - F$ is simply connected as $n \geq 3$.

2. Applications to conformal geometry

2.1. The following gives a sufficient condition for the existence of complete metrics which we will use in the context of Lemma B (cf. [Ka]). The following gives a sufficient condition for the existence of complete metrics (cf. [Ka]).

(2.1.1) **Lemma.** *Let S^{k-1} be a geometric $(k-1)$-sphere in S^n for $1 \leq k \leq n$. Then the complement $S^n - S^{k-1}$ is conformally equivalent to the Riemannian product $\mathbf{H} \times S^{n-k}$ for which the group of conformal transformations $\mathrm{Conf}(S^n; S^{k-1})$ of S^n preserving S^{k-1} is isomorphic to the group of isometries $\mathrm{Iso}(\mathbf{H}^k) \times \mathrm{Iso}(S^{n-k}) = \mathrm{SO}(k, 1) \times \mathrm{O}(n-k+1)$.*

2.2. Discreteness criteria and structure of closed conformally flat manifolds.

(2.2.1) **Theorem.** *Suppose that the holonomy group Γ of a closed conformally flat n-manifold M leaves invariant a geometric $(k-1)$-sphere: S^{k-1} for $1 \leq k \leq n-1$. Then:*

(i) *if $k \leq n-2$ then Γ is discrete and M is conformally equivalent to $S^n - L(\Gamma)/\Gamma$ where $L(\Gamma) \subset S^{k-1}$ is the limit set for Γ.*

(ii) *if $k = n-1$ then according to whether $L(\Gamma)$ is a proper subset of S^{n-2} or $L(\Gamma) = S^{n-2}$, M is conformally equivalent to $S^n - L(\Gamma)/\Gamma$ or a Riemannian manifold of nonpositive sectional curvature $\mathbf{H}^{n-1} \times \mathbf{R}^1/\pi$. In the latter case $\pi_1(M) = \pi$ has a group extension $1 \to \mathbf{Z} \to \pi \to \Gamma \to 1$ and either Γ is discrete in $\mathrm{SO}(n-1, 1) \times \mathrm{O}(2)$ or $\overline{\Gamma} = \Gamma \cdot \mathrm{SO}(2)$.*

Proof. $S^n - S^{k-1}$ is connected for $1 \leq k \leq n-1$ and admits a Γ-invariant complete Riemannian metric by (2.1.1). For $k = n-1$, $S^n - S^{n-2} = \mathbf{H}^{n-1} \times S^1$. If $dev^{-1}(S^{n-2}) \neq \varnothing$ then $dev_*\colon \pi_1(\widetilde{M} - dev^{-1}(S^{n-2})) \to \pi_1(S^n - S^{n-2}) \approx \mathbf{Z}$ is surjective because dev is a local homeomorphism and (1.1.1) implies. When $dev^{-1}(S^{n-2}) = \varnothing$, $\widetilde{M}$ is isometric to a (complete) Riemannian product $\mathbf{H}^{n-1} \times \mathbf{R}^1$. It follows that $M \approx \mathbf{H}^{n-1} \times \mathbf{R}^1/\pi$ where $\pi_1(M) \approx \pi \subset \mathrm{SO}(n-1, 1) \times E(1)$. In this case it is easy to see that either Γ is discrete in $\mathrm{SO}(n-1, 1) \times \mathrm{O}(2)$ or indiscrete and $\overline{\Gamma} = \Gamma \cdot \mathrm{SO}(2)$ (cf. [Ka]). □

2.3. Conformally flat structures with special holonomy. The holonomy group Γ of a closed conformally flat n-manifold M is called special if Γ lies in the subgroup $\mathrm{SO}(n, 1)$ of $\mathrm{Conf}(S^n) \approx \mathrm{SO}(n+1, 1)$. As $S^n - S^{n-1} = \mathbf{H}^n \times S^0$, there is the $\mathrm{SO}(n, 1)$-invariant decomposition $S^n = \mathbf{H}^n_+ \cup S^{n-1} \cup \mathbf{H}^n_-$ where $\mathbf{H}^n_+$ and $\mathbf{H}^n_-$ are mutually conjugate. Let $P\colon \widetilde{M} \to M$ be the canonical projection. Put $M_+ = P(dev^{-1}(\mathbf{H}^n_+))$, $M_- = P(dev^{-1}(\mathbf{H}^n_-))$, $M_{\mathbf{R}} = P(dev^{-1}(S^{n-1}))$. Define the $\mathbf{R}^{n-1}$-decomposition: $M = M_+ \cup M_{\mathbf{R}} \cup M_-$. The following is derived from Lemma B and (2.1.1) (cf. [G]).

(2.3.1) **Theorem.** *Let M be a closed conformally flat n-manifold with special holonomy. Suppose that $M = M_+ \cup M_{\mathbf{R}} \cup M_-$ is the $\mathbf{R}^{n-1}$-decomposition. Then each component of M_+ or M_- is a complete hyperbolic n-manifold whose ideal boundary consists of a union of components of $M_{\mathbf{R}}$.*

For example if the holonomy group Γ of a closed conformally flat n-manifold M leaves S^{n-1} invariant, then either M or its two-fold covering has special holonomy.

(2.3.2) **Proposition.** *Let $M = M_+ \cup M_{\mathbf{R}} \cup M_-$ be the $\mathbf{R}^{n-1}$-decomposition. Suppose that M is covered by neither S^n nor $S^1 \times S^{n-1}$.*

(1) *if $n \geq 3$ then each component of $M_{\mathbf{R}}$ is a (closed) conformally flat $(n-1)$-manifold whose holonomy group does not contain a solvable subgroup of finite index.*

(2) *if $n = 2$, each component of $M_{\mathbf{R}}$ is homeomorphic to a circle and its holonomy is generated by a hyperbolic element.*

Proof. If N is a component of $M_{\mathbf{R}}$, then the developing map for M restricts one for N; $(\psi, dev)\colon (\pi_1(N), \widetilde{N}) \to (\mathrm{SO}(n, 1), S^{n-1})$. Recall that *dev* is a homeomorphism of each component of $\widetilde{M}_+$ onto $\mathbf{H}^n_+$ (cf. Lemma B; see also (1.1.2)). Moreover since each component of $\widetilde{M}_R$ sits in the boundary of a component of $\widetilde{M}_+$, it follows that $dev\colon \widetilde{N} \to S^{n-1}$ is a homeomorphism onto its image. In particular $N \approx dev(\widetilde{N})/\Delta$ where $\Delta = \psi(\pi_1(N))$ is a discrete subgroup of $\mathrm{SO}(n, 1)$.

Suppose Δ is virtually solvable and $n \geq 3$. Since Δ is discrete, it is a virtually abelian subgroup of $\mathrm{SO}(n, 1)$ and is contained in $\mathrm{O}(n), \mathrm{E}(n-1)$ or $\mathrm{O}(n-1) \times \mathbf{R}^+$ up to conjugacy; the limit set $L(\Delta) = \varnothing$, $\{\infty\}$ or $\{0, \infty\}$ accordingly. In each case there is a Δ-invariant complete Riemannian metric on $S^{n-1} - L(\Delta)$ (cf. [Ka]). It follows from Lemma B that $dev(\widetilde{N}) = S^{n-1}$, $S^{n-1} - \{\infty\}$ or $S^{n-1} - \{0, \infty\}$ respectively. Also $S^{n-1} - \{0, \infty\}$ is simply connected as $n \geq 3$. Since $dev(\widetilde{N})$ is dense in S^{n-1}, (1.1.2) implies that $dev(\widetilde{M}) = S^n$, $S^n - \{\infty\}$ or $S^n - \{0, \infty\}$ respectively; thus $dev(\widetilde{M}) = S^n - \{\infty\}$. Then M cannot be compact because the holonomy group Γ leaves invariant S^{n-1} and (1) follows. For $n = 2$, N is necessarily homeomorphic to a circle. If a generator Δ is not hyperbolic then it is either elliptic or parabolic. It follows as above that $dev(\widetilde{N}) = S^1$ or $S^1 - \{\infty\}$ respectively and $dev(\widetilde{M}) = S^2$ or $S^2 - \{\infty\}$ respectively. This contradiction proves (2). □

2.4. Topological classification of closed conformally flat manifolds with positive dimensional automorphism groups. Denote by $\mathrm{Conf}(N)$ the group of confromal automorphisms of a conformally flat manifold N. In this section we prove the following.

(2.4.1) **Theorem.** *Let M be a closed conformally flat n-manifold. Suppose that the connected component $\mathrm{Conf}(\widetilde{M})^0$ of the automorphism group of the universal cover $\widetilde{M}$ is nontrivial. Then M is conformally equivalent to one of the following manifolds*: (i) *a spherical space form*; (ii) *a euclidean space form*; (iii) *a hyperbolic space form*; (vi) *a Riemannian manifold of nonpositive sectional curvature* $\mathbf{H}^{n-1} \times \mathbf{R}^1/\pi$, $\pi \subset \mathrm{SO}(n-1, 1) \times \mathrm{E}(1)$; (v) $S^n - L(\Gamma)/\Gamma$, $L(\Gamma) \subset S^k$ $(k = 0, \dots, n-2)$.

(2.4.2) **Corollary.** *Suppose that M is a closed conformally flat* 3*-manifold for which $\mathrm{Conf}(\widetilde{M})^0$ is nontrivial. Then M is finitely covered by one of the following manifolds*: (1) S^3 (2) T^3 (3) $\mathbf{H}^3/\Gamma$ (4) $S^1 \times$ (*compact surface of genus* $g \geq 2$) (5) $S^1 \times S^2$ (6) a connected sum $S^1 \times S^2 \sharp \cdots \sharp S^1 \times S^2$.

The rest of this section is devoted to the proof of the above results. The proof proceeds roughly as follows: Let $(\phi, dev)\colon (\mathrm{Conf}(\widetilde{M}), \widetilde{M}) \to (\mathrm{Conf}(S^n), S^n)$ be the developing pair for the conformally flat structure on M. We show that if the closure $\phi(\mathrm{Conf}(\widetilde{M})^0)$ is noncompact, then $\phi(\mathrm{Conf}(\widetilde{M}))$ leaves invariant S^{k-1} $(k = 1, \dots, n)$ or a point $\{\infty\}$ in S^n. Then the result follows from

(2.2.1) for $1 \leq k \leq n-1$. On the other hand, by using a couple of lemmas we can prove that if $\phi(\mathrm{Conf}(\widetilde{M}))$ leaves invariant S^{n-1} or $\{\infty\}$ of S^n, then M is a hyperbolic space form or a euclidean space form respectively.

2.5. Lemmas concerning subgroups of $\mathrm{Conf}(S^n)$. Let G be a connected subgroup of $\mathrm{Conf}(S^n) \approx \mathrm{SO}(n+1, 1)$. Denote $\overline{G}^0$ the identity component of the closure of G in $\mathrm{Conf}(S^n)$. If $\overline{G}^0$ is compact, then up to conjugacy $\overline{G}^0$ is contained in $\mathrm{O}(n+1)$. If $\overline{G}^0$ has a noncompact radical, it is shown that $\overline{G}^0$ fixes a point ∞ or exactly two points $\{0, \infty\}$ in S^n (cf. [C-G]). Then it follows that $\overline{G}^0 \subset Sim(\mathbf{R}^n)$ where $\mathbf{R}^n = S^n - \{\infty\}$.

(2.5.1) **Lemma.** *Suppose $\overline{G}^0$ is noncompact but has compact radical. Then $G = H \cdot K$ where K is a compact Lie group and H is a Lie subgroup that acts simply transitively on a totally geodesic subspace $\mathbf{H}^k$ in $\mathbf{H}^{n+1}$ for $2 \leq k \leq n+1$.*

Proof. It suffices to check that G does not have a fixed point and $L(G)$ contains more than two points; the lemma then follows from [C-G, Lemma 4.4.5]. If G has a fixed point, then G is conjugate to a subgroup of $Sim(\mathbf{R}^n)$. Since $Sim(\mathbf{R}^n)$ is amenable (and so is any closed connected subgroup), $\overline{G}^0$ is an amenable Lie subgroup of $Sim(\mathbf{R}^n)$ and thus an extension of a solvable Lie group by a compact Lie group. Since $\overline{G}^0$ has compact radical, $\overline{G}^0$ is itself compact, contradicting $\overline{G}^0$ being noncompact. If $L(G)$ consists of less than three points, then either $L(G) = \phi$ or $\overline{G}^0$ fixes a point. Since $L(\overline{G}) = L(G)$, $L(G) = \phi$ implies that $\overline{G}$ is compact. □

(2.5.2) **Corollary.** *If G is a connected subgroup of* $\mathrm{Conf}(S^n)$, *then G satisfies either one of the following*:

(1) *G is conjugate to a subgroup of $Sim(\mathbf{R}^n)$ and $\overline{G}$ is noncompact.*
(2) *G has a unique fixed point in $\mathbf{H}^{n+1}$, or a conjugate of G leaves S^k fixed for $0 \leq k \leq n-2$.*
(3) *G acts transitively on S^{k-1} for $2 \leq k \leq n+1$.*

(2.5.3) **Definition.** If a subgroup of $\mathrm{Conf}(S^n)$ satisfies (1), (2) or (3), then G is said to be of parabolic, elliptic, or loxodromic type respectively.

2.6. Lemma for parabolic case.

(2.6.1) **Lemma.** *Let N be a simply connected noncompact conformally flat n-manifold and (ψ, dev): $(\mathrm{Conf}(N), N) \to (Sim(\mathbf{R}^n), \mathbf{R}^n)$ a developing pair. Suppose there is a closed subgroup G of* $\mathrm{Conf}(N)$ *for which* (i) *$\psi(G)$ contains a one parameter group of transformations which is mapped onto $\mathbf{R}^+$*; (ii) *for each $x \in \mathbf{R}^n$, the orbit $\psi(G)x$ is dense. Then $dev: N \to \mathbf{R}^n$ is a homeomorphism.*

Proof. Consider the exact sequence $1 \to \mathrm{E}(n) \to Sim(\mathbf{R}^n) \to \mathbf{R}^+ \to 1$. Put $H = \mathrm{E}(n) \cap \psi(G^0)$, $H' = \mathrm{E}(n) \cap \psi(G)$. We recall that

I. If an element T of $Sim(\mathbf{R}^n)$ fixes exactly one point $\{\infty\}$, then the eigenvalues of T restricted to the hyperbolic plane are one, i.e., $T \in \mathrm{E}(n)$ (cf. [C-G, 3.4]). In particular a one parameter group R' which is mapped onto $\mathbf{R}^+$ is conjugate to $O(n)\times\mathbf{R}^+$ by an element of $\mathbf{R}^n$. It then follows that $\psi(G^0) = H \rtimes R'$, $\psi(G) = H\times R'$. Since the developing pair is unique up to elements of $Sim(\mathbf{R}^n)$, we may assume that $R' \subset O(n)\times\mathbf{R}^+$.

Consider the following exact sequences.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{R}^n\cap\psi(G) & \longrightarrow & \psi(G) & \longrightarrow & \mu(\psi(G)) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{R}^n & \longrightarrow & Sim(\mathbf{R}^n) & \xrightarrow{\mu} & O(n)\times\mathbf{R}^n+ & \longrightarrow & 1
\end{array}$$

Let V be a connected component of $\mathbf{R}^n\cap\psi(G)$.

II. Suppose that V is trivial. We prove that the closure $\overline{H}$ is compact in $E(n)$, contradicting (ii). Now $A = \mathbf{R}^n\cap H$ is a discrete subgroup of H. Then H is a covering group posited in the exact sequence: $1 \to A \to H \to \mu(H) \subset O(n)$. It is in general true that the connected group $\mu(H) = K\times W$ where K is a compact group and W is a vector group (cf. [H, Theorem 2.7, XIII]). Since $\pi_1(K)$ is finitely generated and A is free abelian, it follows that $\pi_1(K) = \pi_1(H)\times A'$ where $A'\approx A$. If $\widetilde{K}$ is a universal covering group of K, then it follows from the structure theorem [H, Theorem 1.3, XIII] that there is a splitting $\widetilde{K} = \widetilde{K}'\times\widetilde{A}'$ according to $\pi_1(K) = \pi_1(H)\times A'$ in which $K' = \widetilde{K}'/\pi_1(H)$ is compact and $\widetilde{A}'/A'$ is a torus T'. Since $H = \widetilde{K}/\pi_1(H)\times W$, we obtain

$$(*) \qquad H = K'\times\widetilde{A}'\times W.$$

Let S be the radical of H. As $\psi(G^0) = H\rtimes R'$, $\psi(G^0)$ normalizes S. The subgroups R', S generate a solvable subgroup $\langle R', S\rangle$ in $\psi(G^0)$. If we note that $R'\subset O(n)\times\mathbf{R}^+$, it follows that $\mu(\langle R', S\rangle)\subset O(n)\times\mathbf{R}^+$. let $P\colon O(n)\times\mathbf{R}^+\to O(n)$ be the projection. Then $P(\mu(\langle R', S\rangle))$ is solvable in $O(n)$. The identity component of its closure $\overline{P(\mu(\langle R', S\rangle))}^0$ is abelian. And so the commutator $P(\mu[R', S]) = \{1\}$, i.e., $\mu[R', S]\subset\mathbf{R}^+$. On the other hand, since $[R', S]\subset H$ it follows that $\mu([R', S])\subset O(n)$ and hence $\mu([R', S]) = \{1\}$. Since the commutator $[R', S]$ lies in A and A is totally disconnected, we have that $[R', S] = \{1\}$. We prove that $S\subset O(n)$. An element T of R' has the form $(0, \lambda\cdot A)$ in $Sim(\mathbf{R}^n)$. Let $g = (t, B)$ in $E(n)$ for an element $g\in S$. The equality $TgT^{-1} = g$ implies that $\lambda At = t$. As $|\lambda\cdot At| = \lambda|t|$ and $\lambda\neq 1$, it must be that $t = 0$ or $g\in O(n)$. The radial S contains $\widetilde{A}\times W$. By $(*)$ we obtain that $H\subset K'\times O(n)$. Therefore $\overline{H}$ is compact in $E(n)$.

The fixed point set for $\overline{H}$ is a linear subspace V in $\mathbf{R}^n$. If $V = \mathbf{R}^n$ then $\overline{H} = \{1\}$ or $\psi(G^0) = R'$. $\psi(G^0)$ has the origin 0 as a unique fixed point. Since $\psi(G^0)$ is normal in $\psi(G)$, $\psi(G)\cdot 0 = 0$. If $\dim V < n$, $\psi(G)$ leaves V invariant. And so $\psi(G)\cdot x\subset V$ for $x\in V$. In each case it contradicts the hypothesis (ii).

III. Suppose that V is nontrivial. We prove by induction on $\dim N$. Choose a euclidean metric on $\mathbf{R}^n$ so that $dev\colon N\to\mathbf{R}^n$ is a local isometry. Since V

is closed in $\mathrm{E}(n)$, so is its inverse $\psi^{-1}(V)$ in $Iso(N)$. If L is a connected component of $\psi^{-1}(V)$, then it follows that $\psi\colon L \to V$ is an isomorphism. The group L is a closed subgroup of $Iso(N)$ without compact factors and thus L acts properly and freely on N. We have the following diagram of principal bundles:

$$\begin{array}{ccc} (G, N) & \xrightarrow{(\psi, dev)} & (\psi(G), \mathbf{R}^n) \\ \downarrow & & \downarrow \\ (G/L, N/L) & \xrightarrow{(\widehat{\psi}, \widehat{dev})} & (\psi(G)/V, \mathbf{R}^n/V). \end{array}$$

When $\mathbf{R}^n = V \oplus V^{\perp}$ is the orthogonal direct sum, each element of $\psi(G)$ has the following form:

$$\psi(g) = \left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right)$$

where $(a, A) \in Sim(V)$, $(b, B) \in Sim(V^{\perp})$. As $V^{\perp}$ is identified with $\mathbf{R}^n/V$, the quotient group $\psi(G)/V$ acts on $V^{\perp}$ by similarity transformations. Therefore it induces a representation $\rho\colon \psi(G)/V \to Sim(V^{\perp})$ where its kernel lies in $\mathrm{O}(V)$. We note that $\psi(G)/V = H'/V \rtimes R'$ and every orbit of $\rho(\psi(G)/V)$ is dense in $V^{\perp}$. Therefore the manifold N/L, equipped with the group G/L, is a conformally flat manifold for which $(\rho\widehat{\psi}, dev)$ is a developing pair satisfying (i), (ii). The induction hypothesis implies that $\widehat{dev}\colon N/L \to \mathbf{R}^n/V$ is a homemorphism. By the above diagram it follows that $dev\colon N \to \mathbf{R}^n$ is a homeomorphism. □

(2.6.2) **Corollary.** *Let N be a simply connected noncompact conformally flat manifold and $(\psi, dev)\colon (\mathrm{Conf}(N), N) \to (\mathrm{Conf}(S^n), S^n)$ a developing pair. Suppose that there is a closed subgroup G of* $\mathrm{Conf}(N)$ *for which* (i) *$\psi(G)$ fixes a point ∞ in S^n*; (ii) *$\psi(G)$ contains a one parameter group of transformations which is mapped onto $\mathbf{R}^+$. Then one of the following is true.*

(a) *$dev\colon N \to \mathbf{R}^n$ is a homeomorphism.*
(b) *$\psi(G)$ leaves invariant a geometric sphere S^m of S^n for $0 \le m \le n-2$.*
(c) *$dev\colon N \to \mathbf{H}^n$ is a homeomorphism.*

Proof. Since $\psi(G)$ belongs to $Sim(\mathbf{R}^n)$, it follows that $\psi(G^0) = H \rtimes R'$ by Remark I. Let $\overline{H}^0$ be the identity component of the closure of H in $\mathrm{E}(n)$, K its maximal compact subgroup and $\dim \overline{H}^0/K = m$. If $m = n$ then $\overline{H}^0/K = \mathbf{R}^n$ and every $\psi(G)$ orbit is dense in $\mathbf{R}^n$. It follows that $dev\colon N - dev^{-1}(\infty) \to \mathbf{R}^n$ is a homeomorphism. If $dev^{-1}(\infty) \neq \emptyset$ then $dev\colon N \to S^n$ is a homeomorphism. As we assumed that N is noncompact, $dev\colon N \to \mathbf{R}^n$ is a homeomorphism. Suppose that $m \le n-1$. $\overline{H}^0/K$ is a totally geodesic m-subspace in $\mathbf{R}^n$. Recall that the group $\overline{\psi(G^0)} = \overline{H} \rtimes R'$ belongs to the group of isometries $Iso(\mathbf{H}^{n+1})$ and $\overline{H}^0 \rtimes R'$ is a transitive group of a totally geodesic subspace $\mathbf{H}^{m+1}$ where $\mathbf{H}^{m+1}$ is viewed as the warped product. Therefore the limit set $L(\overline{(\psi(G)})=L(\overline{\psi(G^0)})=L(\overline{\psi(G^0)}^0)=S^m$. $\psi(G)$ leaves S^m invariant for $m \leq n-1$. If $m=n-1$ then $\overline{H}^0 \cdot 0=\mathbf{R}^{n-1}$ and so the connected subset $\psi(G^0)\cdot 0$ is dense in S^{n-1}. It follows from (1.1.2) that $dev\colon N \to \mathbf{H}^n$ is a homeomorphism or $dev\colon N \to dev(N)$ is homeomorphic where $dev(N)$ is dense in S^n. The latter case implies that $dev(N)=\mathbf{R}^n$. For if $S^n - dev(N)$ contains more than one point, then $L(\psi(G)) \subset S^n - dev(N)$. Since $L(\psi(G)) = L(\overline{\psi(G)}) = S^{n-1}$, $dev(N)$ cannot be dense in S^n. Thus $dev(N)=\mathbf{R}^n$. □

(2.6.3) **Proposition.** *Let M be a closed conformally flat manifold and $(\psi, dev)\colon (\mathrm{Conf}(\widetilde{M}), \widetilde{M}) \to (\mathrm{Conf}(S^n), S^n)$ be a developing pair. Suppose that $\psi(\mathrm{Conf}(\widetilde{M}))$ fixes a point in ∞ in S^n and $\psi(\mathrm{Conf}(\widetilde{M})^0) \subset \mathrm{E}(n)$ whose closure $\psi(\mathrm{Conf}(\widetilde{M})^0)$ is noncompact. Then one of the following is true.*

(i) *the holonomy group $\psi(\pi_1(M))$ has a subgroup of finite index in* $\mathrm{E}(n)$.
(ii) *$\psi(\mathrm{Conf}(\widetilde{M}))$ leaves invariant a geometric sphere S^m of S^n for $0 < m \leq n-2$.*

Proof. Put $G=\psi(\mathrm{Conf}(\widetilde{M})^0)$ and $\Gamma = \psi(\pi_1(M))$. Consider the exact sequences.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & h(G) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{R}^n & \longrightarrow & \mathrm{E}(n) & \xrightarrow{h} & \mathrm{O}(n) & \longrightarrow & 1 \end{array}$$

where $A=\mathbf{R}\cap G$. Let V be a connected component of A.

Case 1. Suppose that V is trivial. It follows similarly to the proof of II of (2.6.1) that $G=K' \times M$ where K' is a compact group and M is a vecotr group. If $A(G)$ is the real algebraic closure of G in $\mathrm{E}(n)$ then $A(G)=\mathbf{R}^l \times K$. Note that K is a nontrivial compact subgroup and $0<l<n$. For this if $l=n$ then K is trivial so that $M \subset \mathbf{R}^n$ contradicting that V is trivial, and $l \neq 0$ because $\overline{G}$ is noncompact. Let W be the fixed point set of K in $\mathbf{R}^n$. The set W contains $\mathbf{R}^l$ and $\dim W \leq n-2$ since K is a semisimple subgroup of $\mathrm{O}(n)$. Since $\psi(\mathrm{Conf}(\widetilde{M}))$ normalizes $A(G)$, it follows that $\psi(\mathrm{Conf}(\widetilde{M}))$ normalizes K by maximality. And so $\psi(\mathrm{Conf}(\widetilde{M}))$ leaves W invariant. If we note that the union $W\cup\{\infty\}$ is a geometric sphere S^m of S^n for $0<m\leq n-2$ then $\psi(\mathrm{Conf}(\widetilde{M}))$ leaves S^m invariant.

Case 2. Suppose that V is nontrivial (in $\psi(\mathrm{Conf}(\widetilde{M}))$. First note that $\Gamma \subset Sim(\mathbf{R}^n)$. Choose a subgroup L of $\psi^{-1}(V)$ such that $\psi\colon L \to V$ is an isomorphism. If $L \cap \pi_1(M)$ is a nontrivial subgroup Δ then $\psi(\Delta) \subset V$. The group Δ acts properly discontinuously in the orbit Lx for each $x \in \widetilde{M}$. Since $dev\colon Lx \to Vdev(x)$ is a homeomorphism, $\psi(\Delta)$ acts properly discontinuously in the orbit $Vdev(x)$. in particular $\psi(\Delta)$ is discrete in $Sim(\mathbf{R}^n)$. Recall that each element of $\mathbf{R}^+$ in $Sim(\mathbf{R}^n)$ acts as expansion or contraction of $\mathbf{R}^n$. Since $\psi(\Delta)$ is normal in $\Gamma \subset Sim(\mathbf{R}^n)$, we conclude that $\Gamma \subset \mathrm{E}(n)$. Therefore it suffices to show that Δ is nontrivial. Put $N = \widetilde{M} - dev^{-1}(\infty)$. Choose a euclidean metric on $\mathbf{R}^n$ so that $dev\colon N \to \mathbf{R}^n$ is a local isometry. Then $L \subset Iso(N)$ and by the fact that $V \subset \mathbf{R}^n$, the group L acts as parallel translations of N with respect to the flat metric. Note that since $rLx = Lrx$ for $r \in \pi_1(M)$, each orbit rLx is a totally geodesic subspace of N parallel to the orbit Lx.

Suppose that $\Delta = L \cap \pi_1(M)$ is trivial. Then L has a dense orbit in M because $M = \widetilde{M}/\pi_1(M)$ is compact. This implies that

(2.6.4) there exists a nontrivial element $r \in \pi_1(M)$ such that the orbit rLx is arbitrarily close to Lx.

And so there exists a nontrivial element r and a geodesic l between x and rx. We may assume that $dev(x)$ is the origin 0 in $\mathbf{R}^n$. The image $dev(l)$ is a geodesic (straight line) from the origin 0 to $\psi(r)0$ in $\mathbf{R}^n$. Let C_r be the union of geodesic segments $\{\psi(r)^i dev(l),\ i \geq 0\}$. For each integer $i > 0$, the totally geodesic subspace $\psi(r)^i \cdot V0$ is mutually parallel and since $\psi(r)$ is conformal (i.e., angle-preserving), it follows that C_r is a straight line segment starting at the origin. Let $\psi(r) = (a, \lambda \cdot A)$ in $Sim(\mathbf{R}^n)$. We may assume that $\lambda < 1$. As $\psi(r)0 = a$, $\psi(r)^2 0 = a + \lambda A a$, which implies that $Aa = a$. It is now easy to see that C_r has an endpoint $p_r \in \mathbf{R}^n$. Since $dev\colon Lx \to V0$ is a homeomorphism, there exists a tubular neighborhood U of Lx such that $dev\colon U \to dev(U)$ is a homeomorphism. By (2.6.4) we can assume that the endpoint p_r is sufficiently close to the orbit $V0$ and so $p_r \in dev(U)$. Choose $z \in U$ such that $dev(z) = p_r$ and a geodesic α from x to z inside U. Let C'_r be the union of geodesics $\{r^i l,\ i \geq 0\}$ in N. Since dev maps C'_r onto C_r, C'_r is also a geodesic segment starting at x in N. Note that C'_r is invariant under the element r. Thus both $dev(\alpha)$ and C_r are straight lines with the same endpoint. This implies that C'_r coincides with α as the set. Therefore we have that $\lim r^i x = z \in N$. Since C'_r is invariant under r, z is a fixed pint of r. This contradicts the assumption that $\pi_1(M)$ acts freely. □

2.7. Proof of Theorem 2.4.1. Let M be a closed conformally flat n-manifold and $(\psi, dev)\colon (\mathrm{Conf}(\widetilde{M}), \widetilde{M}) \to (\mathrm{Conf}(S^n), S^n)$ a developing pair. Recall that $\pi_1(M) \subset \mathrm{Conf}(\widetilde{M})$ and $\mathrm{Conf}(\widetilde{M})^0$ is its identity component.

(2.7.1) **Proposition.** *Suppose that* $\mathrm{Conf}(\widetilde{M})^0$ *is nontrivial. If* $\psi(\mathrm{Conf}(\widetilde{M})^0)$ *is of elliptic type, then* M *is conformally equivalent to* $S^n - L(\Gamma)/\Gamma$. *Moreover* (a) *the limit set* $L(\Gamma)$ *is a proper subset of* S^k *for* $1 \leq k \leq n-2$. (b) $\mathrm{Conf}(\widetilde{M})^0$ *is isomorphic to* $\psi(\mathrm{Conf}(\widetilde{M})^0) = \mathrm{SO}(n-k)$ *for which* S^k *is the fixed point set.*

Proof. $\psi(\mathrm{Conf}(\widetilde{M}))$ is of elliptic type because it normalizes $\psi(\mathrm{Conf}(\widetilde{M})^0)$. We then prove that $\psi(\mathrm{Conf}(\widetilde{M}))$ leaves S^k fixed for $1 \leq k \leq n-2$. By (2.5.2), if it has a unique fixed point in $\mathbf{H}^{n+1}$, then $\overline{\psi(\mathrm{Conf}(\widetilde{M}))}$ is compact. Since the holonomy group Γ is contained in $\psi(\mathrm{Conf}(\widetilde{M}))$, it follows that $dev: \widetilde{M} \to S^n$ is a homeomorphism and hence $\psi(\mathrm{Conf}(\widetilde{M})) = \mathrm{SO}(n+1, 1)$. This contradicts that $\psi(\mathrm{Conf}(\widetilde{M}))$ is of elliptic type. Then $\psi(\mathrm{Conf}(\widetilde{M})) \subset \mathrm{SO}(k+1, 1) \times \mathrm{O}(n-k)$ for $0 \leq k \leq n-2$ and:

(2.7.2) $L(\Gamma)$ is a proper subset of S^k if and only if $\mathrm{Conf}(S^n - L(\Gamma))^0 = \mathrm{SO}(n-k)$.

It follows from (2.2.1) that M is conformally equivalent to $S^n - L(\Gamma)/\Gamma$ and so $\psi(\mathrm{Conf}(M)^0) = Conf(S^n - L(\Gamma))^0$. $l(\Gamma)$ must be proper by (2.7.2). In particular the case that $k = 0$, i.e., $L(\Gamma) = S^0$ does not occur. □

(2.7.3) **Proposition.** *Suppose that* $\mathrm{Conf}(\widetilde{M})^0$ *is nontrivial. If* $\psi(\mathrm{Conf}(\widetilde{M})^0)$ *is of loxodromic type, then* M *is conformally equivalent to one of the following manifolds: a spherical space form, a hyperbolic space form, a Riemannian manifold of nonpositive sectional curvature* $\mathbf{H}^{n-1} \times \mathbf{R}^1/\pi$, *or a sphere bundle over a hyperbolic orbifold* $\mathbf{H}^k \times S^{n-k}/\Gamma$ $(k = 2, \ldots, n-2)$.

Proof. $\psi(\mathrm{Conf}(\widetilde{M})^0)$ is transitive on S^{k-1} for $2 \leq k \leq n+1$. If $k = n+1$ then $dev: \widetilde{M} \to S^n$ is a homeomorphism by (1.2.2). Hence M is a spherical space form S^n/Γ where $\pi_1(M) \approx \Gamma \subset \mathrm{O}(n+1)$ is a finite group. For $2 \leq k \leq n$, it follows that $L(\psi(\mathrm{Conf}(\widetilde{M}))) = L(\psi(\mathrm{Conf}(\widetilde{M})^0)) = S^{k-1}$. Since $\psi(\mathrm{Conf}(\widetilde{M}))$ is of loxodromic type and by (2.7.2), we obtain that $L(\Gamma) = S^{k-1}$. Then the result follows from (2.2.1) and (2.1.2) for $k \leq n-1$. Finally we have to show that M is a hyperbolic space form $\mathbf{H}^n/\Gamma$ for $k = n$. Let $L(\Gamma) = S^{n-1}$. We apply (1.1.2) to M or its twofold cover. If $dev^{-1}(S^{n-1}) = \varnothing$ then it follows from (a) that $dev: \widetilde{M} \to \mathbf{H}^n$ is a homeomorphism, i.e., M is a hyperbolic space form $\mathbf{H}^n/\Gamma$. Suppose that $dev^{-1}(S^{n-1}) \neq \varnothing$. Then the orbit $\mathrm{Conf}(\widetilde{M})^0 \cdot x \subset dev^{-1}(S^{n-1})$ for each $x \in dev^{-1}(S^{n-1})$ and $dev(\mathrm{Conf}(\widetilde{M})^0 \cdot x) = S^{n-1}$. This implies that $dev: \widetilde{M} \to S^n$ is a homeomorphism by (b) of (1.1.2) and Γ is a finite group contradicting $L(\Gamma) = S^{n-1}$. □

(2.7.4) **Proposition.** *Suppose that* $\mathrm{Conf}(\widetilde{M})^0$ *is nontrivial. If* $\psi(\mathrm{Conf}(\widetilde{M})^0)$ *is of parabolic type, then* M *is conformally equivalent to either a euclidean space form or Hopf manifold* $S^1 \times S^{n-1}/F$ *where* F *is a finite subgroup of* $\mathrm{O}(2) \times \mathrm{O}(n)$.

Proof. Since $\mathrm{Conf}(\widetilde{M})^0$ is normal in $\mathrm{Conf}(\widetilde{M})$, we may assume that $\psi(\mathrm{Conf}(\widetilde{M}))$ fixes ∞ of S^n. Then either $\psi(\mathrm{Conf}(\widetilde{M})^0)$ is nontrivial in $\mathbf{R}^+$ or $\psi(\mathrm{Conf}(\widetilde{M})^0) \subset \mathrm{E}(n)$ each of which satisfies the hypothesis of (2.6.2) or (2.6.3) respectively. Suppose either $\psi(\pi_1(M)) \subset \mathrm{E}(n)$ or $dev: \widetilde{M} \to \mathbf{R}^n$ is a homeomorphism (i.e., (i) of (2.6.3), (a) of (2.6.2)). It follows that the holonomy group Γ is discrete and M is a euclidean space form $\mathbf{R}^n/\Gamma$. In particular as M is compact, the case (c) that $dev: \widetilde{M} \approx \mathbf{H}^n$ does not occur. Suppose that $\psi(\mathrm{Conf}(\widetilde{M}))$ leaves S^m invariant for $m \leq n-2$ ((b) of (2.6.2), (ii) of (2.6.3)). If $m = 0$, it follows from (2.2.1) that M is a Hopf manifold $S^n - S^0/\Gamma = S^1 \times S^{n-1}/F$. It suffices to show that the case $0 < m \leq n-2$

does not occur. As the holonomy group Γ leaves a proper subspace $\mathbf{R}^m$ invariant, it then follows from (2.2.1) that either Γ is discrete in $Sim(\mathbf{R}^n)$ or $\psi(\mathrm{Conf}(\widetilde{M})) = \mathrm{SO}(n-1, 1) \times \mathrm{O}(2)$. In the former case the Bieberbach group Γ cannot leave invariant any proper subspace of $\mathbf{R}^n$, while Γ leaves $\mathbf{R}^m$ invariant. In the latter case $\mathrm{SO}(n-1, 1) \times \mathrm{O}(2)$ is not amenable. This is impossible because $\psi(\mathrm{Conf})(\widetilde{M}))$ is a closed subgroup of $Sim(\mathbf{R}^n)$. $\square$

(2.7.5) *Remark.* Let M be a closed conformally flat manifold whose holonomy group is amenable, (i.e., it lies in $Sim(\mathbf{R}^n)$ up to conjugacy). Then it follows from the main result of Schoen and Yau [S-Y] that whenever the dimension of M is greater than or equal to 7, the developing map is not surjective. Furthermore if one assumes Schoen's positive mass theorem (so far not published) then the condition to their main theorem will be satisfied and consequently the developing map is not surjective for all dimensions of M. (The author learned this while he was visiting Keio University.)

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