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COMPLEX REPRESENTATIONS OF MATRIX SEMIGROUPS

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ABSTRACT. Let M be a finite monoid of Lie type (these are the finite analogues of linear algebraic monoids) with group of units G. The multiplicative semigroup $\mathscr{M}_n(F)$, where F is a finite field, is a particular example. Using Harish-Chandra's theory of cuspidal representations of finite groups of Lie type, we show that every complex representation of M is completely reducible. Using this we characterize the representations of G extending to irreducible representations of M as being those induced from the irreducible representations of certain parabolic subgroups of G. We go on to show that if F is any field and S any multiplicative subsemigroup of $\mathscr{M}_n(F)$, then the semigroup algebra of S over any field of characteristic zero has nilpotent Jacobson radical. If $S = \mathscr{M}_n(F)$, then this algebra is Jacobson semisimple. Finally we show that the semigroup algebra of $\mathscr{M}_n(F)$ over a field of characteristic zero is regular if and only if $\mathrm{ch}(F) = p > 0$ and F is algebraic over its prime field.

INTRODUCTION

Complex representations of the finite general linear groups were determined by Green [7]. A description of the complex characters of finite groups of Lie type was obtained by Deligne and Lusztig [4] using the étale cohomology of Grothendieck and Artin. This is a particularly significant accomplishment in light of the classification of finite simple groups.

Going from finite groups to finite semigroups, one runs into problems at the very first step since a complex representation of a finite semigroup need not be completely reducible. Munn [14] and Ponizovskii [21] characterized the semisimplicity of complex semigroup algebras in terms of the invertibility of the sandwich matrices over appropriate complex group algebras. However, showing invertibility of a matrix over a group algebra can be a formidable task. Faddeev [6] gives an outline of an incomplete proof that the semigroup algebra $\mathbb{C}[\mathscr{M}_n(\mathbb{F}_q)]$ is semisimple. In his approach he needs to show that the eigenvalues of certain complex matrices derived from the sandwich matrices are all powers of q. However he only proves that the traces of these matrices are sums of certain powers of q. Since no further details have ever been published, we can only view the main 'theorem' of the paper as a conjecture. W. D. Munn has informed us that he too worked on the problem for several years and in partic-

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ular proved the semisimplicity of $\mathbb{C}[\mathscr{M}_n(\mathbb{F}_q)]$ for $n \leq 4$. Irreducible modular representations of $\mathscr{M}_n(\mathbb{F}_q)$ were recently studied by Harris and Kuhn [11] in connection with some problems in algebraic topology. In this paper, we are interested in the much more general situation of finite monoids of Lie type. These were called regular split monoids in [24]. They are the finite analogues of linear algebraic monoids [23]–[28], in the same way that finite groups of Lie type or finite groups with split BN-pairs are the finite analogues of linear algebraic groups. Any finite group of Lie type has associated with it an infinite family of finite monoids of Lie type. The main result of this paper is that if M is any finite monoid of Lie type, then the semigroup algebra $\mathbb{C}[M]$ is semisimple. This is accomplished in several steps. First the problem is reduced to considering the universal three $\mathscr{J}$-class monoids of Lie type. Second the sandwich matrices are shown to have a triangular form in a very weak sense. Third the sandwich matrices are greatly simplified with respect to irreducible cuspidal representations of the maximal subgroup of the $\mathscr{J}$-class. Having handled the cuspidal situation, we go to arbitrary irreducible representations according to the philosophy of Harish-Chandra [9, 10]. This is accomplished by considering certain four $\mathscr{J}$-class monoids of Lie type.

In the final section we study the semigroup algebras of infinite linear semigroups. First, we prove that $J(K[S])$ is nilpotent for all semigroups $S \subseteq \mathscr{M}_n(F)$, where K, F are arbitrary fields with $\mathrm{ch}(K) = 0$. It is then shown that $J(K[\mathscr{M}_n(F)]) = 0$ in this case. Finally, $K[\mathscr{M}_n(F)]$ is a von Neumann regular algebra if and only if F is algebraic over its prime subfield and of prime characteristic. In fact the algebra then is a union of an ascending chain of finite dimensional semisimple algebras.

1. Preliminaries

We begin by briefly reviewing some basics of semigroup theory [3]. Let S be a semigroup. If S is a monoid (i.e. has an identity element), then $S^1 = S$. Otherwise $S^1 = S \cup \{1\}$. $\mathscr{J}, \mathscr{R}, \mathscr{L}, \mathscr{H}$ will denote the usual Green's relations on S : $a\mathscr{J}b$ if $S^1aS^1 = S^1bS^1$, $a\mathscr{R}b$ if $aS^1 = bS^1$, $a\mathscr{L}b$ if $S^1a = S^1b$, $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$. We let $E(S)$ denote the set of idempotents of S. If $X \subseteq S$, then $E(X) = X \cap E(S)$ and $\langle X \rangle$ is the subsemigroup of S generated by X. If J is a $\mathscr{J}$-class of S, then $J^0 = J \cup \{0\}$ with

$$a \circ b = \begin{cases} ab & \text{if } ab \in J, \\ 0 & \text{otherwise.} \end{cases}$$

J is regular if $E(J) \neq \varnothing$. If some power of each element of S lies in a subgroup (for example when S is finite), then J^0 is either a null semigroup or else a completely 0-simple semigroup.

Let S be a completely 0-simple semigroup with a maximal subgroup H. Then by [3, Theorem 3.5], S has a Rees representation $S \simeq \mathfrak{M}(H, I, M; \mathscr{P})$ where $\mathscr{P}$ is an $M \times I$ sandwich matrix. Now, for every field K, the contracted semigroup ring $K_0[S]$ may be identified with the so-called Munn ring

$\mathfrak{M}(K[H], I, M; \mathscr{P})$ consisting of all generalized $(I \times M)$-matrices over the group ring $K[H]$ with finitely many nonzero entries, and multiplication extending that in S. That is, $AB = A \circ \mathscr{P} \circ B$ for $A, B \in \mathfrak{M}(K[H], I, M; \mathscr{P})$, where $\circ$ stands for the usual multiplication of matrices. We refer to [3, §5.2, and 13] for basic results on Munn rings. Note that, if S has a zero element z, then $K[S] \simeq K_0[S] \oplus Kz \simeq K_0[S] \oplus K$. Thus, the irreducible representations of $K[S]$ and $K_0[S]$ are strongly related. In fact, the representations of S (over K) mapping z onto zero correspond exactly to the $K_0[S]$-modules. Let $a \in K_0[S], a \neq 0$. Then $a = \sum \lambda_s s$, $0 \neq \lambda_s \in K$. The *support*, $\operatorname{supp}(a) = \{s \in S | \lambda_s \neq 0\}$.

One of the basic facts on the structure of $K_0[S]$ is the description of its Jacobson radical $J(K_0[S])$. Here, we say that a matrix A lies over a subset X of $K[H]$ if all entries of A are in X.

Proposition 1.1. *Let $S = \mathfrak{M}(H, I, M; \mathscr{P})$ be a completely 0-simple semigroup. Then, for every field K we have*

$$J(K_0[S]) = \{A \,|\, \mathscr{P} \circ A \circ \mathscr{P} \text{ lies over } J(K[H])\}.$$

Moreover, if $\operatorname{ch}(K) = 0$, *and H is a linear group, then*

$$J(K_0[S]) = \{A \,|\, \mathscr{P} \circ A \circ \mathscr{P} = 0\}.$$

Proof. The former assertion is established in [13]. The latter then follows from the semisimplicity of group rings of linear groups in characteristic zero, cf. [20, Lemma 7.4.1, Lemma 7.4.4, and Exercise 15 to §6].

If S is a regular semigroup with finitely many $\mathscr{J}$-classes, then every irreducible representation of it is in fact an irreducible representation of a principal factor of S [3, Theorem 5.33]. For any chain $S = S_1 \supset S_2 \supset \cdots \supset S_t$ such that S_i is an ideal in S_{i-1} for $i = 2, \ldots, t$, we get the corresponding subideal chain in $K[S]$ such that $K[S_{i-1}]/K[S_i] \simeq K_0[S_{i-1}/S_i]$. Hence the semisimplicity problem for $K[S]$ is reduced to the corresponding problems for $K[S_1]$ and all $K_0[S_{i-1}/S_i]$. This may be restated in a more general form as follows.

Lemma 1.2. *Let A be a ring and let $A = I_n \supset I_{n-1} \supset \cdots \supset I_1 \supset I_0 = 0$ be a chain such that I_{j-1} is an ideal in I_j for $j = 1, \ldots, n$. If $J(I_j/I_{j-1}) = 0$ ($J(I_j/I_{j-1})$ is nil, $J(I_j/I_{j-1})^m = 0$ for some m, respectively), then $J(A) = 0$ ($J(A)$ is nil, $J(A)^{m^n} = 0$, respectively).*

Proof. We proceed by induction on n. If $n = 1$, then the assertion is obvious. Assume that $n > 1$, and the assertion holds for I_{n-1}. We know that $J(A) \cap I_{n-1} = J(I_{n-1})$. Moreover, $J(A) \subseteq I_{n-1}$ ($J(A)$ is nil modulo I_{n-1}, $J(A)^m \subseteq I_{n-1}$, respectively) by the hypothesis on I_n/I_{n-1}. The result follows.

We note that the problem of describing $J(K[S])$ for finite semigroups was studied by many authors [5, 8, 12]. In particular, Munn [14], and Ponizovskii [21] showed that, for $\operatorname{ch}(K) = 0$, $J(K[S]) = 0$ if and only if S is regular, and the sandwich matrices arising from (arbitrary) Rees presentations of the principal factors of S are invertible in the corresponding matrix rings. Clearly, this is a consequence of Proposition 1.1 and Lemma 1.2.

The following observation is useful when changing the base field within the same characteristic, when studying the radical.

Proposition 1.3. *Let S be a finite semigroup. Then, for any fields $K \subseteq L$, we have $J(L[S]) = L \cdot J(K[S])$.*

Proof. Let F be the prime subfield of K. Clearly $J(F[S]) \subseteq J(K[S])$ because it is nilpotent. Now $K[S] \simeq K \otimes_F F[S]$ and hence $J(K[S]) = K \cdot J(F[S])$ by a general result on tensor product extensions [20, Theorem 7.3.8]. Similarly $J(L[S]) = L \cdot J(F[S])$, and the result follows.

In view of the above, throughout the rest of this section we restrict our attention to the case where $K = \mathbb{C}$. In some special cases, checking that the sandwich matrix $\mathscr{P}$ is invertible requires dealing with an arbitrary $\mathscr{R}$-class of S only. This is illustrated below.

Proposition 1.4. *Let $S = \mathfrak{M}(H, m, m; \mathscr{P}) = J^0$ be a finite completely 0-simple semigroup. Assume that for any e, f in $E(J)$ there exists $\sigma \in \mathrm{Aut}(S)$ such that $\sigma(e) = f$. If $x_1, \ldots, x_m \in K[H]$, for a field K, are such that $(x_1, \ldots, x_m)\mathscr{P} = (1, 0, \ldots, 0)$, then $\mathscr{P}$ is invertible in $\mathscr{M}_m(K[H])$.*

Proof. Let e be an idempotent of S lying in the $\mathscr{R}$-class corresponding to the first row of the matrix semigroup $\mathfrak{M}(H, m, m; \mathscr{P})$. The hypothesis on $\mathscr{P}$ implies that there exists an element x in $K_0[S]$ such that $\mathrm{supp}(x) \subseteq eS$, and

$$xy = \begin{cases} y & \text{if } \mathrm{supp}(y) \subseteq eS, \\ 0 & \text{if } \mathrm{supp}(y) \cap eS = \varnothing \end{cases}$$

(x corresponds to the matrix of which the first row is $(x_1, \ldots, x_m)$, the other rows zero, under the natural isomorphism $K_0[S] \simeq \mathfrak{M}(K[H], m, m; \mathscr{P})$). In other words, x is a left identity of the algebra $K_0[eS]$ such that it annihilates on the left the remaining $\mathscr{R}$-classes of S. Choose nonzero idempotents $e_1 = e, e_2, \ldots, e_m$ from distinct nonzero $\mathscr{R}$-classes of S. By the hypothesis, there exist automorphisms σ_i of S such that $\sigma_i(e) = e_i$, $i = 1, \ldots, m$. Since every σ_i permutes the nonzero $\mathscr{R}$-classes of S, it is therefore clear that every $\sigma_i(x)$ is a left identity of $K_0[e_iS]$ which annihilates all e_jS, $j \neq i$ (here σ_i are treated as the induced automorphisms of $K_0[S]$). It follows easily that $\sum_{i=1}^m \sigma_i(x)$ is a left identity of $K_0[S]$. Then $\mathscr{P}$ must be left invertible, and since S is finite, we see that $\mathscr{P}$ is invertible in $\mathscr{M}_m(K[H])$.

If φ is a representation of the maximal subgroup H of $\mathfrak{M}(H, I, M; \mathscr{P})$, then by $\varphi(\mathscr{P})$ we denote the $(M \times I)$-matrix with entries $\varphi(p_{mi})$, $m \in M$, $i \in I$, where $\mathscr{P} = (p_{mi})$.

Corollary 1.5. *Suppose that for every irreducible representation φ of the maximal subgroup H of a finite completely 0-simple semigroup $S = \mathfrak{M}(H, m, m; \mathscr{P}) = J^0$ there exist $A_1, \ldots, A_m$ in $\varphi(\mathbb{C}[H])$ such that $(A_1, \ldots, A_m)\varphi(\mathscr{P}) = (1, 0, \ldots, 0)$. Assume also that for any $e, f \in E(J)$ there is $\sigma \in \mathrm{Aut}(S)$ such that*

$\sigma(e) = f$. *Then* $\mathscr{P}$ *is invertible in* $\mathscr{M}_m(\mathbb{C}[H])$, *and* $\mathbb{C}[S]$ *is a semisimple algebra.*

Proof. In view of the semisimplicity of $\mathbb{C}[H]$ we see that the hypothesis of Proposition 1.4 is satisfied. Hence the result follows.

We will further need some basic observations on irreducible modules over semigroup rings of finite regular semigroups.

Proposition 1.6. *Let* S *be a finite regular semigroup with zero, and* I *an ideal of* S. *Let* J *be a* $\mathscr{J}$*-class of* S *such that* $JI = IJ = I$ *and* $J^2 \subset J^0 = J \cup \{0\}$. *Let* V *be an irreducible left* $\mathbb{C}_0[S]$*-module such that* $IV \neq 0$. *Then, as a* $\mathbb{C}_0[J^0]$*-module,* V *is completely reducible with no null constituents.*

Proof. Let $\varphi\colon S \to \operatorname{End} V$ be the associated representation. So $\operatorname{End} V$ is spanned by $\varphi(I)$. Since $J^2 \subseteq J^0$, it follows that the span A of $\varphi(J)$ is a subalgebra. Let N denote the radical of A. Then we have the natural onto mappings $K_0[J^0] \to K_0[\varphi(J^0)] \to A$. Since $R = K_0[J^0]$ is finite dimensional, we see that $J(R)$ maps onto $J(A)$. By Proposition 1.1, it follows that $ANA = 0$. Since $JI = IJ = I$, we see that $\varphi(I)N\varphi(I) = 0$. So $N = 0$, and V is completely reducible as a $\mathbb{C}_0[J^0]$-module. If $u \in V$ is such that $Ju = 0$, then the fact that $IJ = I$ implies $Iu = 0$. From the hypothesis on I we derive that $u = 0$. The result follows.

Let $S = \mathfrak{M}(H, m, m; \mathscr{P})$ be a completely 0-simple semigroup, and let V be a nonnull irreducible $\mathbb{C}_0[S]$-module. Then $\sum_{i=1}^m e_iV = V$ where $e_1, \dots, e_m$ are chosen from all distinct nonzero $\mathscr{R}$-classes of S. Moreover, $\dim e_iV = \dim e_jV$ for all i, j, because $u_{ij}e_jS = e_iS$ for some $u_{ij} \in S$. Hence, we always have $\dim V \leq m \dim e_iV$. On the other hand, it may be derived from [3, Theorems 5.46, 5.51] that the equality holds if and only if the matrix $\varphi(\mathscr{P})$ is invertible in $\mathscr{M}_m(\varphi(\mathbb{C}[H]))$ where $\varphi\colon \mathbb{C}_0[S] \to \mathscr{M}_r(\mathbb{C})$ is the corresponding ring homomorphism. To indicate a simple reason why this is true, let us factor φ by

$$\mathbb{C}_0[S] \to \mathfrak{M}(\varphi(\mathbb{C}[H]), m, m;\ \varphi(\mathscr{P})) \to \mathscr{M}_r(\mathbb{C}).$$

Then e_iV is an irreducible $(e_i\mathbb{C}_0[S]e_i \simeq \mathbb{C}[H])$-module. The equality $\dim V = m \dim e_iV$ is then equivalent to $r = mn$ where $\varphi(\mathbb{C}[H]) \simeq \mathscr{M}_n(\mathbb{C})$. But this is equivalent to the fact that the latter homomorphism in the above factorization is an isomorphism. This can happen if and only if $\varphi(\mathscr{P})$ is an invertible matrix. Otherwise $\mathbb{C}_0[\varphi(S)]$ has a nonzero annihilator.

Proposition 1.7. *Let* $S = \mathfrak{M}(H, m, m;\ \mathscr{P}) = J^0$ *be a finite completely* 0*-simple semigroup,* $e = e^2 \in J$. *Let* V *be an* $\mathbb{C}_0[S]$*-module which is completely reducible with no null constituents. If* $\varphi\colon H \to GL(eV)$ *is the associated representation, then* $\varphi(\mathscr{P})$ *is invertible in* $\mathscr{M}_m(\varphi(\mathbb{C}[H]))$ *if and only if* $\dim V = m \cdot \dim eV$.

Proof. Let $V = V_1 \oplus \cdots \oplus V_k$ where each V_i is a nonnull irreducible $\mathbb{C}_0[S]$-module. Then $eV = eV_1 \oplus \cdots \oplus eV_k$. Let $\varphi_i : H \to GL(eV_i)$ be the associated representations. So $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_k$, and $\varphi(\mathscr{P})$ is the direct sum of the matrices $\varphi_1(\mathscr{P}), \ldots, \varphi_k(\mathscr{P})$. It follows that $\varphi(\mathscr{P})$ is invertible if and only if each $\varphi_i(\mathscr{P})$ is invertible. Since for all i $\dim V_i \leq m \dim eV_i$, we see that the assertion follows from the remarks preceding the proposition.

Our final result in this section establishes a link between induced group representations, and representations of finite monoids in the case of invertible sandwich matrices.

Proposition 1.8. *Let M be a finite monoid with zero and group of units G, and let $e \in E(M)$. Suppose $J = GeG$ is a $\mathscr{J}$-class of M such that $J^0 = J \cup \{0\}$ is an ideal of M. Let V be an irreducible $\mathbb{C}_0[M]$-module such that $eV \neq 0$. Let $P = \{a \in G \mid ae = eae\}$, and let $\varphi : P \to GL(eV)$, $\psi : G \to GL(V)$ denote the associated representations. Suppose the sandwich matrix $\mathscr{P}$ of J^0 (under a Rees presentation of J^0) is $m \times m$ and invertible. Then ψ is equivalent to the induced representation φ_P^G. In particular $\dim V = \frac{|G|}{|P|} \dim eV$.*

Proof. We know that the invertibility of $\mathscr{P}$ implies that $\dim V = m \cdot \dim eV$. Let $1 = g_1, \ldots, g_k$ be coset representatives of P in G. Suppose that $g_ie\mathscr{R}g_je$. Then $g_i^{-1}g_je\mathscr{R}e$, so $g_i^{-1}g_j \in P$ and $g_iP = g_jP$. Hence $i = j$. It follows that $k \leq m$. Let $x \in G$. Then $xP = g_iP$ for some i. Since $Pe = ePe$, we get $xeV = xPeV = g_iPeV = g_ieV$. So $W = \sum g_ieV$ is invariant under J^0. Since J^0 is an ideal in M, and V is an irreducible $\mathbb{C}_0[M]$-module, $W = V$. Hence $\dim V \leq k \cdot \dim eV$, $k = m$, and $V = g_1eV \oplus \cdots \oplus g_meV$.

Let λ, χ denote the characters of φ, ψ respectively. Let $g \in G$. Now $gg_ieV = g_jeV$ where $gg_iP = g_jP$. Clearly g_ieV contributes to the trace of $\varphi(g)$ exactly when $i = j$. But then $g_i^{-1}gg_i \in P$, and what is contributed is exactly $\lambda(g_i^{-1}gg_i)$. Hence

$$\chi(g) = \sum_{g_i^{-1}gg_i \in P} \lambda(g_i^{-1}gg_i) = \frac{1}{|P|} \sum_{\substack{x \in G \\ x^{-1}gx \in P}} \lambda(x^{-1}gx).$$

Hence $\chi = \lambda_P^G$ and the result follows.

2. Monoids of Lie type

Let G be a finite group. Then G admits a BN-pair (or has a *Tits system*) if there are subgroups B, N of G which generate G such that $T = B \cap N \lhd N$ and the *Weyl group* $W = N/T$ has a generating set Γ of order two elements such that:

(T1) $\sigma B\theta \subseteq B\sigma B \cup B\sigma\theta B$ for all $\sigma \in W$, $\theta \in \Gamma$.

(T2) $\theta B\theta \neq B$ for all $\theta \in \Gamma$.

Rather amazingly, the above conditions impose severe restrictions on W so that W is in fact a Coxeter group, i.e. the orders of $\theta\theta'$ $(\theta, \theta' \in \Gamma)$ determine a presentation of W. The possibilities for these orders are $1, 2, 3, 4, 6, 8$.

If $\sigma \in W$, then the length $\ell(\sigma)$ is defined to be the smallest positive integer k such that σ is a product of k elements of Γ. The length of the identity element 1 is defined to be zero. It turns out W has a unique element σ_0 of maximum length. Then σ_0 is of order 2 and $\sigma_0\Gamma\sigma_0 = \Gamma$. Let $B^- = \sigma_0 B\sigma_0$. If $I \subseteq \Gamma$, then let W_I denote the subgroup of W generated by I. It turns out that $P_I = BW_IB$ and $P_I^- = B^- W_I B^-$ are subgroups of G which are their own normalizers . Moreover, these are the only subgroups of G containing B and B^- respectively. In particular G is the disjoint union of $B\sigma B$ $(\sigma \in W)$. This is called the *Bruhat decomposition.* The conjugates of B are called *Borel subgroups.* The conjugates of P_I, $I \subseteq \Gamma$, are called *parabolic subgroups.* If $x \in G$, then the parabolic subgroup $P = x^{-1}P_I x$ is said to be *opposite* to $P^- = x^{-1}P_I^- x$. Thus if P, P^- are opposite parabolic subgroups, then $a^{-1}P^- a$ $(a \in P)$ is the set of all opposites of P. We refer to [1, 2, 29] for details.

We will assume as in [2, Chapter 2] that G admits a split BN-pair satisfying some commutator relations. Then there is a normal subgroup U of B such that $B = UT$, $U \cap T = \{1\}$. So $B^- = U^- T$ where $U^- = \sigma_0 U\sigma_0$. Moreover, if $I \subseteq \Gamma$, $L_I = P_I \cap P_I^-$, then there exist $U_I \lhd P_I$, $U_I^- \lhd P_I^-$ such that $U_I \cap P_I = U_I^- \cap P_I^- = \{1\}$ and $P_I = L_I U_I$, $P_I^- = L_I U_I^-$. Then U_I, U_I^- are called the *unipotent radicals* of P_I, P_I^- respectively. L_I is called a *Levi factor* of P_I. If P, P^- are two opposite parabolic subgroups of G, $L = P \cap P^-$, then $P = LR_u(P)$, $P^- = LR_u(P^-)$ where $R_u(P)$ and $R_u(P^-)$ are the unipotent radicals of P and P^-, respectively. The Levi factor L of P depends on the choice of P^-. However the unipotent radical $R_u(P)$ is independent of the choice of P^-. Any two Levi factors of P are conjugate by some element of $R_u(P)$.

U is generated by certain *root subgroups* X_α $(\alpha \in \Phi^+)$. In fact, U is the product of the X_α's in *any order.* Similarly U^- is the product of certain *root subgroups* X_α $(\alpha \in \Phi^- = -\Phi^+)$. Moreover W permutes the root subgroups X_α $(\alpha \in \Phi = \Phi^+ \cup \Phi^-)$ and T normalizes the root subgroups X_α $(\alpha \in \Phi)$. If $I \subseteq \Gamma$, then L_I is generated by T and the X_α's such that X_α, $X_{-\alpha}$ are both contained in P_I. U_I is the product in any order of the X_α $(\alpha \in \Phi^+)$ such that $X_{-\alpha} \not\subseteq P_I$. U_I^- is the product in any order of the $X_{-\alpha}$ with $X_\alpha \subseteq U_I$. See [2, Chapter 2] for details.

Example 2.1. Let $G = GL(n, F)$ where F is a finite field. Then B, B^- consist of upper and lower triangular matrices, respectively. T consists of the diagonal matrices. W consists of the permutation matrices,

$$\sigma_0 = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}.$$

U, U^- consist of unipotent (1 is the only eigenvalue) upper and lower triangular matrices, respectively. If $I \subseteq \Gamma$, then P_I, P_I^- consist of block upper and lower triangular matrices, respectively. L_I consists of block diagonal matrices. W_I consists of block diagonal permutation matrices. Φ^+ is in 1-1 correspondence with ordered pairs (i, j) with $1 \leq i < j \leq n$. Φ^- is in 1-1 correspondence with ordered pairs (i, j) with $1 \leq j < i \leq n$. The root subgroup

$$X_{i,j} = \{I + \alpha E_{ij} | \alpha \in F\}$$

where E_{ij} is the $n \times n$ matrix with 1 in the (i, j) component and zero elsewhere.

We will need the following later.

Lemma 2.2. *Let $I \subseteq \Gamma$, $\sigma_1, \sigma_2 \in W$ be such that σ_1 is of minimum length in $W_I\sigma_1$. If $\ell(\sigma_1) \geq \ell(\sigma_2)$ and $\sigma_1 \neq \sigma_2$, then $\sigma_1 B \cap P_I^- P_I \sigma_2 = \emptyset$.*

Proof. Let $P = P_I$, $P^- = P_I^-$, $L = L_I$. Then by the Bruhat decomposition for L,

$$P^- P = U_I^- L U_I = U_I^- (B^- \cap L) W_I (B \cap L) U_I = B^- W_I B.$$

By (T1), for any $\sigma \in W$, $\theta \in \Gamma$,

$$B^- \sigma B \theta = \sigma_0 B \sigma_0 \sigma B \theta \subseteq \sigma_0 B \sigma_0 \sigma B \cup \sigma_0 B \sigma_0 \sigma \theta B = B^- \sigma B \cup B^- \sigma \theta B.$$

Let $\sigma_2 = \theta_1 \cdots \theta_t$ where $\theta_1, \ldots, \theta_t \in \Gamma$, $\ell(\sigma_2) = t$. Then

$$P^- P \sigma_2 = B^- W_I B \theta_1 \cdots \theta_t \subseteq \bigcup_{i_1 < \cdots < i_r} B^- W_I \theta_{i_1} \cdots \theta_{i_r} B.$$

Suppose $\sigma_1 B \cap P_I^- P_I \sigma_2 \neq \emptyset$. By the Bruhat decomposition, the union $G = \bigcup B^- \sigma B$ $(\sigma \in W)$ is disjoint. Hence $\sigma_1 \in W_I \theta_{i_1} \cdots \theta_{i_r}$ for some $i_1 < \cdots < i_r$. Since σ_1 is of minimum length in $W_I \sigma_1$, $r = t = \ell(\sigma_1)$ and $\sigma_1 = \sigma_2$. This contradiction completes the proof.

By a *monoid of Lie type* (on G) we mean a regular monoid M having zero 0 and group of units G, with M generated by G and the idempotent set E such that:

(1) If $e \in E$ then

$$P = P(e) = \{a \in G | ae = eae\},$$
$$P^- = P^-(e) = \{a \in G | ea = eae\}$$

are opposite parabolic subgroups of G with $xe = e$ for $x \in R_u(P)$ and $ey = e$ for all $y \in R_u(P^-)$.

(2) If $e, f \in E$ with $eM = fM$ or $Me = Mf$, then $x^{-1}ex = f$ for some $x \in G$.

In [24], these monoids are called regular split monoids on G. They are the finite analogues of linear algebraic monoids [23]–[28]. Monoids of Lie type are classified in [24]. With every group of Lie type, there is associated an infinite family of monoids of Lie type. Of course, the multiplicative matrix semigroup $\mathscr{M}_n(F)$ is the first example. As other examples, consider:

Example 2.3. Let F be a finite field,

$$M = \{A \otimes B \mid A, B \in \mathscr{M}_n(F), \ A^tB = BA^t = \alpha I \text{ for some } \alpha \in F\}.$$

Example 2.4 (Symplectic monoid). Let F be a finite field of characteristic $\neq 2$. Let m be a positive integer, $n = 2m$. Let P_0 denote the $m \times m$ matrix

$$\begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}, \quad P = \begin{bmatrix} 0 & P_0 \\ -P_0 & 0 \end{bmatrix}.$$

Let

$$M' = \{A \in \mathscr{M}_n(F) \mid A^tPA = \alpha P \text{ for some } \alpha \in F\}.$$

Let G, E denote the group of units and the diagonal idempotent set of M', respectively. Then the monoid M generated by E and G is a monoid of Lie type.

The simplest example of a monoid of Lie type on G is of course $M = G \cup \{0\}$. We now give the description of the universal three $\mathscr{J}$-class monoids of Lie type on G. Let $I \subseteq \Gamma$. If $x \in G$, let $\overline{x} \in L_I^0 = L_I \cup \{0\}$ be given by

$$\overline{x} = \begin{cases} 0 & \text{if } x \notin P_I^- P_I, \\ c & \text{if } x \in U_I^- c U_I, \ c \in L_I. \end{cases} \tag{1}$$

Since $U_I \lhd P_I$, $U_I^- \lhd P_I^-$, we have for all $x \in G$,

$$\overline{ax} = \overline{a}\,\overline{x}, \quad \overline{xb} = \overline{x}\,\overline{b} \quad \text{for all } a \in P_I^-, \ b \in P_I. \tag{2}$$

In particular, since $L_I = P_I \cap P_I^-$,

$$\overline{ax} = a\overline{x}, \quad \overline{xa} = \overline{x}a \quad \text{for all } a \in L_I. \tag{3}$$

Let

$$M_I = M_I(G) = G \cup GeG \cup \{0\}.$$

If $x, y, s, t \in G$, define $xey = set$ if

$$s^{-1}x \in P_I, \qquad ty^{-1} \in P_I^-, \qquad \overline{s^{-1}x} = \overline{ty^{-1}}.$$

If $a = xey$, $b = set \in GeG$, define

$$ab = \begin{cases} 0 & \text{if } ys \notin P_I^- P_I, \\ xe\overline{ys}t & \text{if } ys \in P_I^- P_I. \end{cases}$$

For the details that then M_I is a well-defined monoid of Lie type, see the proof of [24, Theorem 3.8]. The $\mathscr{H}$-class of e is $eL_I \cong L_I$. Moreover, every three $\mathscr{J}$-class monoid of Lie type is a homomorphic image of some M_I (L_I is replaced by a homomorphic image). Hence any principal factor of a monoid of Lie type is a homomorphic image of a principal factor of some M_I. Hence (see §1), to show that every complex representation of a finite monoid of Lie type is completely reducible, it suffices to show that each $\mathbb{C}[M_I]$ is semisimple.

There is only one nontrivial sandwich matrix associated with M_I, namely that coming from the $\mathscr{J}$-class GeG. This is easily described as follows. Let $P = P_I$, $P^- = P_I^-$, $L = L_I$. Let

$$G/P^- = \{P^- a_i \mid i = 1, \ldots, t\},$$
$$G/P = \{b_i P \mid i = 1, \ldots, t\}.$$

Then the sandwich matrix $\mathscr{P}$ is the $t \times t$ matrix $(\overline{a_i b_j})$.

Our goal is to show that $\mathscr{P}$ is invertible over $\mathbb{C}[L]$. By Corollary 1.5 this is equivalent to showing that for every irreducible representation $\varphi\colon L \to GL(n, \mathbb{C})$, we can find $A_1, \ldots, A_t \in \mathscr{M}_n(\mathbb{C})$ such that

$$[A_1, \ldots, A_t]\varphi(\mathscr{P}) = [I_n, 0, \ldots, 0].$$

We begin by considering cuspidal representations [9, 10, 2, Chapter 9]. φ is *cuspidal* if for every proper parabolic subgroup Q of L with unipotent radical V,

$$\sum_{v \in V} \varphi(v) = 0.$$

Following is the key lemma in proving our main theorem.

Lemma 2.5. *Let $\varphi\colon L \to GL(n, \mathbb{C})$ be an irreducible cuspidal representation. Then there exist $A_1, \ldots, A_t \in \mathscr{M}_n(\mathbb{C})$ such that*

$$[A_1, \ldots, A_t]\varphi(\mathscr{P}) = [I_n, 0, \ldots, 0].$$

Proof. We begin by carefully choosing the coset representatives of G/P and G/P^-. Any coset of W_I in W has a unique element of minimum length [2, Proposition 2.3.3]. Let these (right coset) representatives be $1 = \sigma_1, \ldots, \sigma_s$. Then by the Bruhat decomposition,

$$G = B^- W B = \bigcup B^- W_I \sigma_i B = \bigcup P^- \sigma_i B.$$

Thus $G = \bigcup P^- \sigma_i B$. We claim that this union is disjoint. For suppose $P^- \sigma_i B = P^- \sigma_j B$. Suppose $\ell(\sigma_i) \geq \ell(\sigma_j)$. Since $\sigma_i B$ intersects $P^- \sigma_j$, we see by Lemma 2.2 that $\sigma_i = \sigma_j$. Also by the Bruhat decomposition,

$$G = BWB = \bigcup B\sigma_i^{-1} W_I B = \bigcup B\sigma_i^{-1} P.$$

Thus $G = \bigcup B\sigma_i^{-1} P$. We claim that this union is also disjoint. For suppose $B\sigma_i^{-1} P = B\sigma_j^{-1} P$. Then $P\sigma_i B = P\sigma_j B$. Suppose $\ell(\sigma_i) \geq \ell(\sigma_j)$. Since $\sigma_i B$ intersects $P\sigma_j$, we see by Lemma 2.2 that $\sigma_i = \sigma_j$.

For $i = 1, \ldots, s$, let

$$V_i = U \cap \sigma_i^{-1} U_I \sigma_i, \quad Y_i = U \cap \sigma_i^{-1} U_I^- \sigma_i, \quad Z_i = U \cap \sigma_i^{-1} L \sigma_i.$$

Since U is the product of the positive root subgroups in any order, we see that $U = Y_i Z_i V_i$. Hence

$$P^- \sigma_i B = P^- \sigma_i U = P^- \sigma_i V_i, \qquad i = 1, \ldots, s,$$
$$B\sigma_i^{-1} P = U\sigma_i^{-1} P = Y_i \sigma_i^{-1} P, \qquad i = 1, \ldots, s.$$

Let $\sigma_i = Tn_i$, $i = 1, \dots, s$. Suppose $v_1, v_2 \in V_i$ such that $P^- n_i v_1 = P^- n_i v_2$. So $n_i v_1 v_2^{-1} n_i^{-1} \in P^-$. But $n_i v_1 v_2^{-1} n_i^{-1} \in U_I$. So $v_1 v_2^{-1} = 1$ and $v_1 = v_2$. Hence $n_i v, v \in V_i$, $i = 1, \dots, s$, is a set of distinct (right) coset representatives for G/P^-. Suppose $y_1, y_2 \in Y_i$ such that $y_1 n_i^{-1} P = y_2 n_i^{-1} P$. Then $n_i y_1^{-1} y_2 n_i^{-1} \in P$. But $n_i y_1^{-1} y_2 n_i^{-1} \in U_I^-$. Hence $y_1^{-1} y_2 = 1$ and $y_1 = y_2$. Thus yn_i^{-1}, $y \in Y_i$, $i = 1, \dots, s$, is a set of distinct (left) coset representatives for G/P.

We next arrange the σ's so that $1 = \sigma_1, \dots, \sigma_r \in N_W(L)$ and $\sigma_k \notin N_W(L)$ for $r+1 \le k \le s$. Further let $\ell(\sigma_1) \le \ell(\sigma_2) \le \cdots \le \ell(\sigma_r)$. Then note that

$$Y_1 = U \cap U_I^- = \{1\}.$$

If $x \in G$, let $\overline{x} \in L \cup \{0\}$ be as in (1). Now the entry in the $(n_i v,\ yn_j^{-1})$ position of $\varphi(\mathscr{P})$ is $\varphi(\overline{n_i v y n_j^{-1}})$. Thus we are trying to find $A_{n_i v} \in \mathscr{M}_n(\mathbb{C})$ for $v \in V_i$, $i = 1, \dots, s$, such that

$$\sum_{n_i v} A_{n_i v} \varphi(\overline{n_i v}) = I_n, \tag{4}$$

$$\sum_{n_i v} A_{n_i v} \varphi(\overline{n_i v y n_j^{-1}}) = 0 \quad \text{for } y \in Y_j,\ j = 2, \dots, s. \tag{5}$$

Let $1 \le i \le r$. Then $\sigma_i \in N_W(L)$. Thus for any root subgroup X_α of U_I, $\sigma_i X_\alpha \sigma_i^{-1}$ is either contained in U_I or U_I^-. Since U_I is a product of its root subgroups in any order, we see that

$$U_I = Y_i V_i, \qquad i = 1, \dots, r. \tag{6}$$

Now for $y \in Y_i$, $v \in V_i$, $x \in G$, we have by (2),

$$\overline{n_i y v x} = \overline{(n_i y n_i^{-1})(n_i v x)} = \overline{(n_i y n_i^{-1})}\,\overline{(n_i v x)} = \overline{n_i v x}. \tag{7}$$

It follows from (6), (7) that for all $x \in G$,

$$\sum_{u \in U_I} \varphi(\overline{n_i u x}) = |Y_i| \sum_{v \in V_i} \varphi(\overline{n_i v x}), \qquad i = 1, \dots, r. \tag{8}$$

Now let $j \ge r+1$. By [2, Proposition 2.8.9], $Q = \sigma_j P \sigma_j^{-1} \cap L$ is a parabolic subgroup of L with unipotent radical $V = \sigma_j U_I \sigma_j^{-1} \cap L$. Suppose $Q = L$. Then $\sigma_j^{-1} L \sigma_j \subseteq P$. Since $j \ge r+1$, $\sigma_j \notin N_W(L)$. Since L is generated by T and root subgroups, there is a root subgroup $X_\alpha \subseteq L$ such that $\sigma_j^{-1} X_\alpha \sigma_j \not\subseteq L$. So $\sigma_j^{-1} X_\alpha \sigma_j \subseteq U_I$. So $X_{-\alpha} \subseteq L$, $\sigma_j^{-1} X_{-\alpha} \sigma_j \subseteq U_I^-$, contradicting the assumption that $\sigma_j^{-1} L \sigma_j \subseteq P$. Hence $Q \ne L$. Since φ is cuspidal, we have

$$\sum_{a \in V} \varphi(a) = 0. \tag{9}$$

Let $a \in V$, $i \le r$. Since $n_j^{-1} a n_j \in U_I$, we have by (3),

$$\begin{aligned}\sum_{u\in U_I} \varphi(\overline{n_i u n_j^{-1}})\varphi(a) &= \sum_{u\in U_I} \varphi(\overline{n_i u n_j^{-1} a}) \\ &= \sum_{u\in U_I} \varphi(\overline{n_i (u n_j^{-1} a n_j) n_j^{-1}}) = \sum_{u\in U_I} \varphi(\overline{n_i u n_j^{-1}}).\end{aligned}$$

Hence by (9),

$$\sum_{u\in U_I} \varphi(\overline{n_i u n_j^{-1}}) = 0. \tag{10}$$

Now let $w \in U$. Then $w = \ell v$ for some $\ell \in L$, $v \in U_I$. Since $n_i \in N_G(L)$, we have by (10), (3),

$$\begin{aligned}\sum_{u\in U_I} \varphi(\overline{n_i u w n_j^{-1}}) &= \sum_{u\in U_I} \varphi(\overline{n_i u \ell v n_j^{-1}}) \\ &= \varphi(n_i \ell n_i^{-1}) \sum_{u\in U_I} \varphi(\overline{n_i (\ell^{-1} u \ell) v n_j^{-1}}) \\ &= \varphi(n_i \ell n_i^{-1}) \sum_{u\in U_I} \varphi(\overline{n_i u n_j^{-1}}) = 0.\end{aligned}$$

Thus for $w \in U$,

$$\sum_{u\in U_I} \varphi(\overline{n_i u w n_j^{-1}}) = 0, \qquad i \le r,\ j \ge r+1. \tag{11}$$

From general considerations, we know that if a solution to (4), (5) exists, it must be such that $A_{n_i v} = A_{n_i}$ for $v \in V_i$, $i = 1, \dots, r$. So at this time let $A_{n_i v} = A_{n_i}$ for $v \in V_i$, $i = 1, \dots, r$. We further let $A_{n_i v} = 0$ for $i \ge r+1$, $v \in V_i$. Then we see by (8), (11) that (5) is automatically valid for $j \ge r+1$, $y \in Y_j$. Hence (4), (5) now become by (8)

$$\sum_{i=1}^{r} \frac{1}{|Y_i|} A_{n_i} \sum_{u\in U_I} \varphi(\overline{n_i u}) = I_n, \tag{12}$$

$$\sum_{i=1}^{r} \frac{1}{|Y_i|} A_{n_i} \sum_{u\in U_I} \varphi(\overline{n_i u y n_j^{-1}}) = 0, \qquad y \in Y_j,\ j = 2, \dots, r. \tag{13}$$

Now for $j \le r$, $\sigma_j \in N_G(L)$. Hence $\sigma_j^{-1} U_I^- \sigma_j \cap L = \{1\}$. Hence by [2, Proposition 2.8.6]

$$\begin{aligned}Y_j = U \cap \sigma_j^{-1} U_I^- \sigma_j &\subseteq P \cap \sigma_j^{-1} U_I^- \sigma_j \\ &= [L \cap \sigma_j^{-1} U_I^- \sigma_j][U_I \cap \sigma_j^{-1} U_I^- \sigma_j] \subseteq U_I.\end{aligned}$$

Thus (13) simplifies to

$$\sum_{i=1}^{r} \frac{1}{|Y_i|} A_{n_i} \sum_{u\in U_I} \varphi(\overline{n_i u n_j^{-1}}) = 0, \qquad j = 2, \dots, r. \tag{14}$$

For $i \le r$, $\sigma_i U_I \sigma_i^{-1} \cap L = \{1\}$ and hence

$$\sigma_i U_I \sigma_i^{-1} \subseteq U_I^- U_I.$$

Thus

$$\overline{n_i u n_i^{-1}} = 1 \quad \text{for } u \in U_I,\ i = 1, \dots, r. \tag{15}$$

Furthermore, by Lemma 2.2,

$$\overline{n_i u n_j^{-1}} = 0 \quad \text{for } 1 \le j < i \le r,\ u \in U_I. \tag{16}$$

By (15), (16) we see that (12), (14) finally become:

$$|U_I| A_{n_1} = I_n, \tag{17}$$

$$\frac{|U_I|}{|Y_j|} A_{n_j} + \sum_{i=1}^{j-1} \frac{1}{|Y_i|} A_{n_i} \sum_{u \in U_I} \varphi(\overline{n_i u n_j^{-1}}) = 0 \quad \text{for } j = 2, \dots, r. \tag{18}$$

We can now solve for $A_{n_1}, \dots, A_{n_r}$, inductively. This completes the proof.

We can now finally prove our main theorem.

Theorem 2.6. *Every complex representation of a finite monoid of Lie type is completely reducible.*

Proof. As noted before, it suffices to show that $\mathbb{C}[M_I(G)]$ is semisimple for any $I \subseteq \Gamma$. We prove this by induction on $|I|$. If $|I| = 0$, every irreducible representation of L_I is cuspidal and hence the result follows from Lemma 2.5 and Corollary 1.5. So let $|I| > 0$. Let $M_I = G \cup GeG \cup \{0\}$, $\mathscr{P}$ be the sandwich matrix of GeG, and $L = L_I \cong eL$. If $x \in G$, let $\overline{x}$ be as in (1). By Corollary 1.5 and Lemma 2.5, it suffices to show that $\varphi(\mathscr{P})$ is invertible for any noncuspidal irreducible representation φ of L. Let φ be such a representation. By [2, Theorem 9.2.3] there exist $K \subseteq I$, $K \neq I$ and an irreducible cuspidal representation θ of L_K such that if $\overline{\theta}$ is the trivial extension of θ to $Q = P_K \cap L$ (i.e., $\overline{\theta}(u) = 1$ for $u \in U_K \cap L$), then φ is a component of $\psi = \overline{\theta}_Q^L$. It therefore suffices to prove that $\psi(\mathscr{P})$ is invertible. Let $P_I = P$, $P_I^- = P^-$. Since $K \subseteq I$, we have,

$$P_K \subseteq P, \quad P_K^- \subseteq P^-, \quad L_K \subseteq L, \quad U_I^- \subseteq U_K^-, \quad U_I \subseteq U_K.$$

Let $M_K = G \cup GfG \cup \{0\}$ with $C_G(f) = L_K$. Let $M = M_I \cup M_K$ with the G's identified and the zeros identified. So

$$M = G \cup GeG \cup GfG \cup \{0\}.$$

For $a = xey \in GeG$, $b = sft \in GfG$ define

$$ab = \begin{cases} x\overline{ys}ft & \text{if } ys \in P^- P, \\ 0 & \text{otherwise}, \end{cases}$$

$$ba = \begin{cases} sf\overline{tx}y & \text{if } tx \in P^- P, \\ 0 & \text{otherwise}. \end{cases}$$

The multiplication on M_I, M_K is as before. That M is now a well-defined monoid of Lie type follows from [24, Theorem 3.8]. Moreover $e > f$ and eMe is a monoid of Lie type on L with $eMe \cong M_K(L)$. By the induction hypothesis $\mathbb{C}[eMe]$ is semisimple. Now $GfG \cup \{0\}$ is the unique nonzero minimal ideal of M. Hence by [3, Theorems 5.33, 5.51] there is an irreducible $\mathbb{C}_0[M]$-module V such that the $\mathbb{C}[L_K]$-module fV has representation θ. Now eV is an irreducible $\mathbb{C}_0[eMe]$-module. By Proposition 1.8, the $\mathbb{C}[L]$-module eV has the representation ψ and,

$$\dim eV = \dim fV \cdot \frac{|L|}{|P_K \cap L|}. \tag{19}$$

Also by the induction hypothesis, the sandwich matrix of GfG is invertible. Hence again by Proposition 1.8,

$$\dim V = \dim fV \cdot \frac{|G|}{|P_K|}. \tag{20}$$

Since $P_K \subseteq P_I = P$, we have by [2, Theorem 2.8.7] that

$$P_K = (P_K \cap L)(P_K \cap U_I) = (P_K \cap L)U_I.$$

Hence by (19), (20),

$$\dim V = \dim eV \cdot \frac{|G|}{|L| \cdot |U_I|} = \dim eV \cdot \frac{|G|}{|P|}.$$

If $t = |G|/|P|$, then the sandwich matrix $\mathscr{P}$ of GeG is $t \times t$. Hence by Propositions 1.6 and 1.7, $\psi(\mathscr{P})$ is invertible. This completes the proof of the theorem.

By Proposition 1.8 and Theorem 2.6, we have

Corollary 2.7. *Let M be a finite monoid of Lie type with group of units G and idempotent set E. For $e \in E$, let $P(e) = \{x \in G \mid xe = exe\}$, $L(e) = \{x \in G \mid xe = ex\}$, $N(e) = \{x \in G \mid xe = ex = e\}$. Let θ be an irreducible complex representation of $L(e)$ containing $N(e)$ in its kernel. Let $\overline{\theta}$ be the trivial extension of θ to $P(e)$, φ the representation induced from $P(e)$ to G. Then φ extends to an irreducible representation of M. Moreover, any complex representation of G, extending to an irreducible representation of M, is obtained in this manner.*

Remark 2.8. The work of Munn [14] and Ponizovskii [21] classifies the irreducible representations of M according to which $\mathscr{J}$-class they come from. In [2, Chapter 9], the irreducible representations of G are classified according to which parabolic subgroup they come from. Corollary 2.7 provides a connection.

By Proposition 1.3 and Theorem 2.6 we have

Corollary 2.9. *Every representation of a finite monoid of Lie type, over a field of characteristic zero, is completely reducible.*

Corollary 2.10. *Let F be a finite field. Then every representation of $\mathcal{M}_n(F)$, over a field of characteristic zero, is completely reducible.*

We conjecture that if M is any finite monoid of Lie type, then $K[M]$ is semisimple for any field K with characteristic not dividing the order of the group of units G. At this point we can show that $K[M]$ is semisimple for fields K of all but finitely many characteristics. This is a consequence of Corollary 2.9 and the following observation.

Proposition 2.11. *Let S be a finite semigroup such that $J(K[S]) = 0$ for a field K of characteristic zero. Then there exists a finite set of primes Z such that $J(L[S]) = 0$ for every field L with $\mathrm{ch}(L) \notin Z$.*

Proof. Since $J(K[S]) = 0$, it is known that S has no null principal factors. Moreover, the sandwich matrices $\mathscr{P}_1, \dots, \mathscr{P}_k$ arising from Rees representations of all principal factors $T_j = \mathfrak{M}(G_j, I_j, M_j; \mathscr{P}_j)$ of S are invertible in the corresponding matrix rings $\mathcal{M}_{|M_j|}(K[G_j])$. In view of Proposition 1.3 we may assume that $K = \mathbb{Q}$. Define Z as the set of primes not dividing any of $|G_1|, \dots, |G_k|$, and not dividing the denominators of the coefficients of $\mathscr{P}_1^{-1}, \dots, \mathscr{P}_k^{-1}$. Then $J(L[G_j]) = 0$ by Maschke's theorem for L with $\mathrm{ch}(L) \notin Z$. Moreover, the matrices $\mathscr{P}_j^{(p)}$ are invertible in $\mathcal{M}_{|M_j|}(L[G_j])$, $p = \mathrm{ch}\, L$, where $(\mathscr{P}_j^{(p)})^{-1}$ is defined by

$$(\mathscr{P}_j^{(p)})^{-1} = \left(\frac{a_{mi}^{(j)} \bmod p}{b_{mi}^{(j)} \bmod p}\right) \quad \text{if}\, \mathscr{P}_j^{-1} = \left(\frac{a_{mi}^{(j)}}{b_{mi}^{(j)}}\right)$$

with $a_{mi}^{(j)}$, $b_{mi}^{(j)}$ relatively prime for all fixed $m \in M$, $i \in I$. Therefore, again by Munn's and Ponizovskii's result, $L[S]$ is semisimple.

Our main result shows that a large family of naturally arising finite linear semigroups have complex semisimple algebras. However it should be noted that there are many naturally arising finite regular semigroups, whose complex semigroup algebras are far from being semisimple. For instance, let T_n be the full transformation semigroup on the set $\{1, \dots, n\}$. Clearly, T_n embeds into every linear semigroup $\mathcal{M}_n(F)$. It is easy to see that $K[T_n]$ is not semisimple for every field K if $n \geq 2$. For example, if $n = 2$, and $T_2 = \{1, \sigma, a, b\}$ with $\sigma^2 = 1$, then $ax = a$, $bx = b$ for every x in T_2. Thus, $a - b$ annihilates on the left the augmentation ideal $\omega(K[T_2])$. It follows that $J(K[T_2]) = K(a-b)$ if $\mathrm{ch}(K) \neq 2$, and $J(K[T_2]) = K(a-b) + K(1-\sigma)$ if $\mathrm{ch}(K) = 2$. More generally, if $n \geq 2$, then the set I of transformations of rank 1 is a completely simple ideal of T_n, the sandwich matrix of which is not square. Therefore $K[T_n]$ cannot be semisimple. However, it was conjectured in [22] that $K[T_n]$ is always an algebra of finite representation type whenever it is nonmodular (that is, $\mathrm{ch}(K)$ does not divide orders of the subgroups of T_n). This was established for $n \leq 3$, and also for a family of submonoids of T_n for arbitrary n.

3. Infinite linear semigroups

In this section we discuss possible extensions of the semisimplicity of $K[\mathscr{M}_n(F)]$ for finite fields F and K of characteristic zero, to the case of arbitrary fields F, K. To this end, we first study the radical of the semigroup algebra of an arbitrary linear semigroup.

Lemma 3.1. *Let S be a semigroup with a chain of ideals $S = S_k \supset S_{k-1} \supset \cdots \supset S_1$ such that S_1 and all S_j/S_{j-1} are completely 0-simple or nilpotent. Denote by ℓ the maximum of the nilpotency indices of the nilpotent factors S_j/S_{j-1}, or S_1. Assume further that $J(K[G]) = 0$ for every subgroup G of S, and some field K. Then there exists $r = r(k, \ell)$ such that $J(K[S])^r = 0$.*

Proof. Consider the chain of ideals $K[S] = K[S_k] \supset \cdots \supset K[S_1]$ of $K[S]$. From Lemma 1.2 it follows that it is enough to show that the radicals of the algebras $K[S_j]/K[S_{j-1}]$, $j = 2, \ldots, k$, and $K[S_1]$ are nilpotent, and there exists a bound (dependent on ℓ only) on their nilpotency indices. We know that $K[S_j]/K[S_{j-1}]$ is isomorphic to the contracted semigroup algebra $K_0[S_j/S_{j-1}]$. If $(S_j/S_{j-1})^m = 0$, then $K_0[S_j/S_{j-1}]^m = 0$. Similarly, if S_1 is nilpotent and $(S_1)^m = 0$, then $K_0[S_1]^m = 0$. Since $K[S_1] \simeq K_0[S_1] \oplus K$, it follows that $J(K[S_1])^m \simeq K_0[S_1]^m = 0$ in this case, too. On the other hand, if $T = S_j/S_{j-1}$ or $T = S_1$ is completely 0-simple, then from Proposition 1.1 it follows that $J(K_0[T])^3 = 0$ because of the hypothesis on the subgroups of S. (Note that the maximal subgroup of T is a subgroup of S.) The result follows.

Recall that S is called strongly π-regular if a power of any element of S lies in a subgroup of S.

Corollary 3.2. *Let $S \subseteq \mathscr{M}_n(F)$ be a strongly π-regular linear semigroup over a field F. Then there exists $r = r(n)$ such that $J(K[S])^r = 0$ for every field K of characteristic zero.*

Proof. We know that $J(K[G]) = 0$ for every linear group G. Moreover, it was shown in [17] that S admits a chain of ideals $S = S_k \supset \cdots \supset S_1$ such that S_1 and all factors S_j/S_{j-1}, $j = 2, \ldots, k$, are completely 0-simple or nilpotent, and there exists a number $N = N(n)$ such that k and the nilpotency indices of the nilpotent S_j/S_{j-1}, S_1, are bounded by N. Therefore, the assertion follows from Lemma 3.1.

Lemma 3.3. *Let $S \subseteq \mathscr{M}_n(F)$ be a linear semigroup. Then for every field K of characteristic zero $J(K[S])$ is a nil ideal.*

Proof. Let $S_j = \{a \in S \mid \mathrm{rk}(a) \le j\}$, $j = 0, 1, \ldots, n$. Then $K[S] = K[S_n] \supseteq \cdots \supseteq K[S_t]$ is a chain of ideals of $K[S]$, where t is minimal with $S_t \neq \varnothing$. Since $K[S_j]/K[S_{j-1}] \simeq K_0[S_j/S_{j-1}]$, then by Lemma 1.2 it is enough to show that all $J(K_0[S_j/S_{j-1}])$, $J(K_0[S_t])$ are nil ideals. Let T denote any of the semigroups S_j/S_{j-1}, S_t. As in [18, Lemma 3], it may be shown that for every

$s, t \in T$, the supports of nonzero elements in $sJ(K_0[T])t$ lie in a null $\mathscr{H}$-class of the corresponding principal factor U of $\mathscr{M}_n(F)$. Therefore, since $t \in T$ is arbitrary,

$$(sJ(K_0[T])t\,K_0[T^1])^2 = 0$$

because $sJ(K_0[T])tT^1$ must annihilate in $K_0[T]$ the $\mathscr{R}$-class of s in U. It follows that $J(K_0[T])^3 \subseteq \sum_{s, t \in T} sJ(K_0[T])t$ lies in a sum of nilpotent ideals of $K_0[T]$. Thus, $J(K_0[T])$ must be a nil ideal.

Lemma 3.4. *Let R be a prime ring with a family of ideals I_α, $\alpha \in A$, such that $\bigcap_{\alpha \in A} I_\alpha = 0$. Then, for every field K, $K_0[R]$ is a subdirect product of all $K_0[R/I_\alpha]$, $\alpha \in A$, where R and the quotient rings R/I_α are viewed as semigroups under multiplication.*

Proof. Let $\varphi_\alpha : K_0[R] \to K_0[R/I_\alpha]$ be the natural homomorphisms, and let ρ_α be the corresponding congruences on R. That is $(s, t) \in \rho_\alpha$ if $\varphi_\alpha(s) = \varphi_\alpha(t)$, for $s, t \in R$. Suppose that $a \in K_0[R]$ is such that $\varphi_\alpha(a) = 0$ for all $\alpha \in A$. Then A is a disjoint union of its subsets $A_1, \dots, A_m$, $m \geq 1$, such that for every $i = 1, \dots, m$, and every $\alpha, \beta \in A_i$, the congruences ρ_α, ρ_β coincide when restricted to $\operatorname{supp}(a)$. Let $I_i = \bigcap_{\alpha \in A_i} I_\alpha$. Since $I_1 \cap \cdots \cap I_m = 0$, we see that $I_k = 0$ for some k because R is prime. Thus, we can assume that $A = A_k$.

If $\alpha \in A$, then $a = a_1 + \cdots + a_m$ where every $\operatorname{supp}(a_j)$ is one of the ρ_α-classes restricted to supp(a). Moreover, this presentation does not depend on the choice of $\alpha \in A$. The fact that $t - u \in I_\alpha$ for every $u, t \in \operatorname{supp}(a_j)$ implies that $t - u \in \bigcap I_\alpha = 0$. It follows that $a_j = \lambda t$ for some $\lambda \in K$, $t \in R$. The fact that $\varphi_\alpha(a) = 0$ implies that $\varphi_\alpha(a_j) = 0$. Therefore $\lambda = 0$ or $\varphi_\alpha(t) = 0$ for all $\alpha \in A$. It follows that $\lambda = 0$ or $t \in \bigcap I_\alpha = 0$, so that $a_j = 0$. Consequently $a = 0$, which completes the proof.

We are now ready for the first main result of this section. Recall that the contracted semigroup ring of a semigroup S with no zero element is defined by $K_0[S] = K[S]$.

Theorem 3.5. *Let $n \geq 1$. Then there exists $r = r(n)$ such that for every field F and every linear semigroup $S \subseteq \mathscr{M}_n(F)$ we have $J(K[S])^r = 0$ if K is a field of characteristic zero.*

Proof. It is enough to show that $J(K_0[S])^r = 0$ where r is chosen as in Corollary 3.2. Let $a_1, \dots, a_r \in J(K_0[S])$. Consider the subring R of F generated by all entries of the matrices in $\bigcup_{i=1}^r \operatorname{supp}(a_i) = T$. Then $a_1, \dots, a_r \in R$, and since $J(K_0[S])$ is a nil ideal by Lemma 3.3 then $a_1, \dots, a_r \in J(K_0[S]) \cap K_0[T] \subseteq J(K_0[T])$. Moreover $K_0[T] \subseteq K_0[\mathscr{M}_n(R)]$. By the Hilbert Nullstellensatz $\mathscr{M}_n(R)$ is a subdirect product of some matrix algebras $\mathscr{M}_n(F_\alpha)$ for a family of finite fields F_α, $\alpha \in \mathscr{A}$. It then follows from Lemma 3.4 that $K_0[\mathscr{M}_n(R)]$ is a subdirect product of all algebras $K_0[\mathscr{M}_n(F_\alpha)]$, $\alpha \in \mathscr{A}$. Therefore, $K_0[T]$ is a subdirect product of some $K_0[T_\alpha]$, $\alpha \in \mathscr{A}$, where T_α is the image of

T under the corresponding homomorphism $\varphi_\alpha\colon K_0[\mathscr{M}_n(R)] \to K_0[\mathscr{M}_n(F_\alpha)]$. If we show that $\varphi_\alpha(a_1)\cdots\varphi_\alpha(a_r) = 0$ for every $\alpha \in \mathscr{A}$, then $a_1\cdots a_r = 0$, which will establish the assertion. Now, T_α is a finite linear semigroup, so it satisfies the hypothesis of Corollary 3.2. Therefore $J(K_0[T_\alpha])^r = 0$. Since $\varphi_\alpha(a_j) \in J(\varphi_\alpha(K_0[T])) = J(K_0[T_\alpha])$, then $\varphi_\alpha(a_1)\cdots\varphi_\alpha(a_r) \in J(K_0[T_\alpha])^r = 0$ as desired.

Since, for any semigroup algebra $K[S]$ satisfying a polynomial identity, the algebra $K[S]/B(K[S])$, B denoting the prime radical, embeds into a subdirect product of matrix algebras of bounded size over fields, then the above theorem may be viewed as an extension of the main result of [19] asserting that $J(K[S]) = B(K[S])$.

The reasoning of the proof of the above result may be refined if $S = \mathscr{M}_n(F)$. In this case every T_α coincides with $\mathscr{M}_n(F_\alpha)$, $\alpha \in \mathscr{A}$. By Corollary 2.10 we know that $J(K[\mathscr{M}_n(F_\alpha)]) = 0$, so that $K[S]$ is a subdirect product of semisimple rings. We therefore have the following result.

Theorem 3.6. *Let F be an arbitrary field. Then $J(K[\mathscr{M}_n(F)]) = 0$ for every field K of characteristic zero.*

The above result may be strengthened in the case of algebraic extensions of finite fields.

Theorem 3.7. *Let F be a field algebraic over its prime subfield,* $\mathrm{ch}(K) > 0$. *Then $K[\mathscr{M}_n(F)]$ is a union of semisimple artinian rings, so it is a regular ring.*

Proof. Clearly, every element $a \in K[\mathscr{M}_n(F)]$ is contained in $K[\mathscr{M}_n(R)]$ where R is the (finite) subfield of F generated by the entries of all matrices in $\mathrm{supp}(a)$. From Corollary 2.10 it follows that $K[R]$ is a semisimple artinian ring. The result follows.

We note that the periodicity of the semigroup S is a necessary condition for $K[S]$ to be regular [16]. Therefore, $K[\mathscr{M}_n(F)]$ cannot be regular whenever F does not satisfy the hypothesis of Theorem 3.7. Let us also observe that if F is algebraic over its prime subfield, then by Theorem 3.7, every principal factor semigroup ring $K_0[T]$ also is regular as a homomorphic image of an ideal of $K[\mathscr{M}_n(F)]$. It then follows that the sandwich matrix arising from any Rees presentation of T is locally invertible in the sense of [15].

References

1. A. Borel and J. Tits, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. **27** (1965), 55–151.
2. R. W. Carter, *Finite groups of Lie type: Conjugacy classes and complex characters*, Wiley, New York, 1985.
3. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys, no. 7, Amer. Math. Soc., Providence, R.I., 1961.
4. P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), 103–161.

5. M. P. Drazin, *Maschke's theorem for semigroups*, J. Algebra **72** (1981), 269–278.

6. D. K. Faddeev, *On representations of a full matrix semigroup over a finite field*, Dokl. Akad. Nauk SSSR **230** (1976), 1290–1293.

7. J. A. Green, *The characters of the finite general linear groups*, Trans. Amer. Math. Soc. **80** (1955), 402–447.

8. T. E. Hall, *The radical of the algebra of any finite semigroup over any field*, J. Austral. Math. Soc. **11** (1970), 350–352.

9. Harish-Chandra, *Automorphic forms on semisimple Lie groups*, Lecture Notes in Math., vol. 62, Springer-Verlag, New York, 1968.

10. ——, *Eisenstein series over finite fields*, Functional Analysis and Related Fields, Springer-verlag, New York, 1970, pp. 76–88.

11. J. C. Harris and N. J. Kuhn, *Stable decompositions of classifying spaces of finite abelian p-groups*, Math. Proc. Cambridge Philos. Soc. **103** (1988), 427–449.

12. G. Lallement and M. Petrich, *Irreducible matrix representations of finite semigroups*, Trans. Amer. Math. Soc. **139** (1969), 393–412.

13. D. B. McAlister, *Rings related to completely 0-simple semigroups*, J. Austral. Math. Soc. **12** (1971), 257–274.

14. W. D. Munn, *On semigroup algebras*, Proc. Cambridge Philos. Soc. **51** (1955), 1–15.

15. ——, *On the regularity of certain semigroup algebras*, Semigroups, Academic Press, London, 1980, pp. 207–224.

16. J. Okniński, *On regular semigroup rings*, Proc. Roy. Soc. Edinburgh **99** (1984), 145–151.

17. ——, *Strongly π-regular matrix semigroups*, Proc. Amer. Math. Soc. **93** (1985), 215–217.

18. ——, *On semigroup algebras satisfying polynomial identities*, Semigroup Forum **33** (1986), 87–102.

19. ——, *On the radical of semigroup rings satisfying polynomial identities*, Math. Proc. Cambridge Philos. Soc. **99** (1986), 435–445.

20. D. S. Passman, *The algebraic structure of group rings*, Wiley-Interscience, New York, 1977.

21. I. S. Ponizovskii, *On matrix representations of associative systems*, Mat. Sb. **38** (1956), 241–260.

22. ——, *Some examples of semigroup algebras of finite representation type*, Analytic Number Theory and Theory of Functions, vol. 8, Zap. Nauchn. Sem. LOMI 160, "Nauka", Leningrad, 1987.

23. M. S. Putcha, *A semigroup approach to linear algebraic groups*, J. Algebra **80** (1983), 164–185.

24. ——, *Monoids on groups with BN-pairs*, J. Algebra **120** (1989), 139–169.

25. ——, *Linear algebraic monoids*, London Math. Soc. Lecture Note Series, vol. 133, Cambridge Univ. Press, 1988.

26. M. S. Putcha and L. E. Renner, *The system of idempotents and the lattice of $\mathscr{J}$-classes of reductive algebraic monoids*, J. Algebra **116** (1988), 385–399.

27. L. E. Renner, *Classification of semisimple algebraic monoids*, Trans. Amer. Math. Soc. **292** (1985), 192–223.

28. ——, *Analogue of the Bruhat decomposition for algebraic monoids*, J. Algebra **101** (1986), 303–338.

29. J. Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Math., vol. 384, Springer-Verlag, New York, 1974.

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