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GENERALIZATIONS OF PICARD'S THEOREM FOR RIEMANN SURFACES

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ABSTRACT. Let D be a plane domain, $E \subset D$ a compact set of capacity zero, and f a holomorphic mapping of $D \backslash E$ into a hyperbolic Riemann surface W. Then there is a Riemann surface W' containing W such that f extends to a holomorphic mapping of D into W'. The same conclusion holds if hyperbolicity is replaced by the assumption that the genus of W be at least two. Furthermore, there is quite a general class of sets of positive capacity which are removable in the above sense for holomorphic mappings into Riemann surfaces of positive genus, except for tori.

1. INTRODUCTION

We owe the following generalization of the big Picard theorem to Ohtsuka [12, 13]:

Let f be a holomorphic mapping of the punctured disc $\{z \in \mathbf{C} \mid 0 < |z| < 1\}$ into a nonexceptional Riemann surface W, i.e., a Riemann surface whose universal covering surface is of hyperbolic type. Then there is a Riemann surface W', containing W, such that f extends to a holomorphic mapping of the disc $\{z \in \mathbf{C} \mid |z| < 1\}$ into W'.

Alternative proofs and simplifications can be found in [4, 6, 16, 17 and 19].

On the other hand, it was shown by Matsumoto [7] that there is no general Picard's theorem for singularities of capacity zero. More precisely: for every compact plane set K of capacity zero there is a compact plane set E of capacity zero and a meromorphic function f in the complement of E such that f omits K and has an essential singularity at each point of E.

Against this background it is quite surprising that holomorphic mappings into compact Riemann surfaces of genus at least 2 do admit a holomorphic extension over a singularity of capacity zero [10, 21].

In this paper we first consider holomorphic mappings into hyperbolic Riemann surfaces. We show that, given a plane domain D, a compact set $E \subset D$ of capacity zero, and a holomorphic mapping f of $D \backslash E$ into a hyperbolic Riemann surface W, there is a supersurface W' of W such that f extends to a holomorphic mapping f^* of D into W'. W' is obtained by adding to W a suitable portion of the Stoïlow ideal boundary of W.

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In the next section, we give a generalization of Nishino's result. It will be shown that the preceding theorem remains valid if hyperbolicity is replaced by the assumption that W be of genus ≥ 2. Our method of proof here is that of Nishino.

The last part of this paper is related to the work of Carleson [3] and Matsumoto [8]. We show that quite a general class of Cantor-type sets, containing sets of positive capacity, is removable for holomorphic mappings into nonexceptional Riemann surfaces of positive genus. Needless to say, the extension of the mappings again presumes an appropriate extension of the surfaces.

2. Mappings into hyperbolic Riemann surfaces

In this section we shall consider holomorphic mappings into hyperbolic Riemann surfaces. Recall that a Riemann surface is said to be hyperbolic provided it carries Green's functions; other Riemann surfaces are termed parabolic.

Theorem 1. *Let $D \subset \mathbf{C}$ be a domain, and let $E \subset D$ be a compact set of capacity zero. Let W be a hyperbolic Riemann surface, and let $f: D \backslash E \to W$ be a holomorphic mapping. Then there exists a Riemann surface $W' \supset W$ such that*

(a) *$W' \backslash W$ is of capacity zero and*

(b) *f extends to a holomorphic mapping $f^*: D \to W'$.*

Proof. We may assume that f is nonconstant. Fix $z_0 \in E$, and choose a Jordan domain $D' \subset D$ containing z_0 such that $\partial D' \subset D \backslash E$. Fix $q \in W$ and let $p \mapsto G(p, q)$ be the Green's function for W with singularity at q. Then $h = G \circ f$ is a positive superharmonic in $\overline{D}' \backslash E$. By a well-known extension theorem in potential theory, h extends to a function h^* superharmonic in $\overline{D}'$. The minimum principle for superharmonic functions implies that $h^*(z) \geq m = \min\{h(z) \mid z \in \partial D'\}$ for each $z \in D'$. It follows that $\mathrm{Cl}(f; z_0)$, the cluster set of f at z_0, is contained in the set $\{p \in W \mid G(p, q) \geq m\} \subsetneq W$. Since E is of capactiy zero, f has the localizable Iversen property [18, pp. 365 and 375; 20, pp. 187–192]. Hence, by Stoïlow's principle on Iversen's property [18, p. 370; 20, p. 189], we have one of the following three alternatives:

(1) $\mathrm{Cl}(f; z_0) = \varnothing$,

(2) $\mathrm{Cl}(f; z_0) = \{p\}$ for some $p \in W$,

(3) $\mathrm{Cl}(f; z_0) = W$.

But the case (3) was just ruled out. Thus, if we denote by W^* the Stoïlow compactification of W and take $\mathrm{Cl}(f; z_0)$ with respect to W^*, $\mathrm{Cl}(f; z_0)$ is always a singleton. In other words, f extends to a continuous mapping $f^*: D \to W^*$. Set $W' = W \cup f^*(E)$. We claim that W' is a manifold, i.e., all ideal boundary points of W which occur as images of some points of E are planar. Thanks to Whyburn's theorem [24, p. 195], it suffices to show that f^* as a mapping of D into W^* is open and light, i.e., f^* preserves open sets and each point-inverse $(f^*)^{-1}(p)$, $p \in W'$, is totally disconnected (observe that the foldings described in Whyburn's theorem cannot occur in this situation).

Suppose $U \subset D$ is a nonempty open set such that $f^*(U)$ contains a boundary point, say p. Pick out $z \in U$ such that $f^*(z) = p$. Since $f^* \mid D\backslash E$ is holomorphic and nonconstant, we have $z \in E$, and sets of capacity zero being removable singularities for bounded holomorphic functions, $p \in f^*(E) \cap \beta$, where $\beta = W^*\backslash W$, the ideal boundary of W. Choose a Jordan domain U' containing z such that $\overline{U}' \subset U$ and $\partial U' \subset U\backslash E$, and let V stand for a noncompact subregion of W with compact relative boundary ∂V such that $p \in \beta_V$, the relative ideal boundary of V, and $(V \cup \partial V) \cap f(\partial U') = \varnothing$. Since p is a boundary point of $f^*(U')$ and $f^*(U') \cap (V \cap \partial V \cup \beta_V)$ is compact (note that $f^*(U') \cap (V \cup \partial V \cup \beta_V) = f^*(\overline{U}') \cap (V \cup \partial V \cup \beta_V))$, $(V \cup \partial V \cup \beta_V)\backslash f^*(U')$ is a nonempty open set. Hence $f^*(U')$ must have boundary points in V also. But this contradicts the openness of f^* at the preimages of such points. It follows that f^* defines an open mapping of D into W^* (or W'). The lightness of f^* is immediate by the total disconnectedness of E.

Let $p \in f^*(E) \cap \beta$. Since p is planar, we can find a planar subregion V of W with compact relative boundary such that $p \in \beta_V$ and $V \cup \beta_V \subset W'$. Thus we may realize p as a compact connected set in $\mathbf{C}$. That this realization is a singleton follows readily from the fact that sets of capacity zero are removable for bounded holomorphic functions. Actually, the realization of β_V is a compact set of capacity zero. This follows from the corresponding property of E and the fact that a nonconstant holomorphic function is a local homeomorphism off a discrete set (recall that a countable union of sets of capacity zero is again of capacity zero [18, p. 371]). Finally, since sets of capacity zero are removable for conformal mappings, we see that W' can indeed be given a (unique) conformal structure, compatible with that of W, such that f^* becomes a holomorphic mapping of D into W' and $W'\backslash W$ is of capacity zero. □

Remarks. (1) The planar character of the boundary elements in $\beta \cap f^*(E)$ can also be shown by means of the Riemann-Hurwitz relation in the manner indicated by Heins [4] in the case of an isolated singularity.

(2) In the special case that W is planar, the result is well known. On the other hand, in the next section it will be shown that the hypothesis that W be hyperbolic is superfluous in case the genus of W is at least 2. However, the proof there makes use of Theorem 1.

(3) Theorem 1 also holds true for quasiconformal mappings. Indeed, by the Stoïlow factorization, f admits a representation $f = g \circ \varphi$, where $\varphi: D\backslash E \to D'\backslash E'$ $(D' \subset \mathbf{C})$ is a quasiconformal homeomorphism and $g: D'\backslash E' \to W$ is holomorphic. Furthermore, sets of capacity zero are removable for quasiconformal homeomorphisms, so that φ extends to a quasiconformal homeomorphism $\varphi^*: D \to D'$. Finally, since sets of capacity zero are preserved under φ^*, the assertion follows from Theorem 1.

Suppose now that V is a subregion of a Riemann surface such that ∂V is compact and β_V, the relative ideal boundary of V, is of harmonic measure zero. Consider a holomorphic mapping of V into a hyperbolic Riemann surface W. This situation can be regarded as a generalization of the preceding one with β_V in the role of the exceptional set E. Making use of the minimum principle valid for superharmonic functions on Riemann surfaces whose ideal boundary is of harmonic measure zero [2] and of Stoïlow's principle on Iversen's property as in the proof of the preceding theorem, one readily establishes the following result, which constitutes a part of Théorème 3 in [14, p. 141]. We omit the details.

Theorem 2. *Let V and β_V be as above, and let f be a holomorphic mapping of V into a hyperbolic Riemann surface W. Then f extends to a continuous map $f^*: V \cup \beta_V \to W \cup \beta$, where β denotes the Stoïlow boundary of W. Furthermore, $W \cap f^*(\beta_V)$ is of capacity zero, and $\beta \cap f^*(\beta_V)$ is of harmonic measure zero.*

Owing to the Stoïlow factorization, Theorem 2 can be formulated in terms of quasiconformal mappings also, as is the case in [14]. We conclude this section with an open question. Let D be a plane domain and let $E \subset D$ be a compact set of class N_B, i.e., a null-set for bounded holomorphic functions. Let f be a holomorphic mapping of $D \backslash E$ into a Riemann surface W and suppose that $\mathrm{Cl}(f; z_0)$, $z_0 \in E$, is neither empty nor a singleton. Must then $\mathrm{Cl}(f; z_0)$ coincide with W?

3. Mappings into Riemann surfaces of genus ≥ 2

We begin by recalling some basic notions of Ahlfors' theory of covering surfaces [1] specialized to the needs of the present paper. Let W be a Riemann surface of genus $g \geq 2$ (the case $g = \infty$ is not excluded), and let W_0 be a relatively compact subregion of W, also of genus ≥ 2, whose boundary ∂W_0 consists of a finite number of analytic Jordan curves (if W is compact, we may take $W_0 = W$). As a nonexceptional Riemann surface, W carries the Poincaré metric inherited from the open unit disc. Let $\rho(z)|dz|$ be the restriction of this metric to $\overline{W}_0$. Let V be a finite Riemann surface with border $\overline{V} \backslash V$ (we may assume that V is a subregion of an unspecified Riemann surface with respect to which the closure is taken), and let $f: \overline{V} \to \overline{W}_0$ be a nonconstant holomorphic mapping. We equip $\overline{V}$ with the pullback of $\rho(z)|dz|$ with respect to f. The part of $\overline{V} \backslash V$ mapped into the interior of $\overline{W}_0$ is said to be the relative boundary of $\overline{V}$ with respect to $\overline{W}_0$ and denoted by $\partial_0 \overline{V}$. Let $A(\cdot)$ and $L(\cdot)$ stand for the area and the length, respectively. The so-called "Metrisch-topologischer Hauptsatz" of Ahlfors [1, p. 168] asserts that

$$(1) \qquad A(\overline{W}_0) \cdot \max\{\chi, 0\} \geq \chi_0 A(\overline{V}) - kL(\partial_0 \overline{V}),$$

where χ (resp. χ_0) stands for the characteristic of $\overline{V}$ (resp. $\overline{W}_0$), and $k > 0$ is a constant depending only on $\overline{W}_0$ (and its metric). Relying on (1), Nishino [10, p. 104] derived an isoperimetric inequality which is the key ingredient in his work (and in what follows).

Fundamental Lemma. *Let* $\overline{W}_0$, $\overline{V}$, *and* f *be as above, and assume that* $\overline{V}$ *is planar. Then* $A(\overline{V}) \leq \alpha L(\partial_0 V)$, *where* $\alpha > 0$ *is a constant depending only on* $\overline{W}_0$.

We are now ready to establish our main result in this section.

Theorem 3. *Let* $D \subset \mathbf{C}$ *be a domain, and let* $E \subset D$ *be a compact set of capacity zero. Let* W *be a Riemann surface of genus* ≥ 2, *and let* $f: D\backslash E \to W$ *be a holomorphic mapping. Then there exists a Riemann surface* $W' \supset W$ *such that*

(a) $W'\backslash W$ *is of capacity zero and*

(b) f *extends to a holomorphic mapping* $f^*: D \to W'$.

Proof. Without loss of generality, we may assume that D is the open unit disc and f is defined and holomorphic on ∂D. Thus $f(\partial D)$ is a piecewise analytic closed curve in W. Hence the number of the components of $W\backslash f(\partial D)$ is finite. We have two alternatives: each component of $W\backslash f(\partial D)$ has finite genus, or some of these components are of infinite genus.

We first assume that the components of $W\backslash f(\partial D)$ are of finite genus. Then the same is true of W itself. Thus W can be regarded as a subregion of a compact Riemann surface W^* of genus ≥ 2. We choose $W_0 = W^*$ and let $\rho(z)|dz|$ denote the Poincaré metric of W_0. Since E is of capacity zero, there is a harmonic function u defined in $\overline{D}\backslash E$ such that $u(z) = 0$ for $z \in \partial D$, $u(z) \to +\infty$ as $z \to z_0$ for every $z_0 \in E$, and $\int_{\partial D} *du = 2\pi$ [11, p. 9]. Consider the mapping

$$z \mapsto e^{u(z)+iv(z)}, \qquad z \in D\backslash E, \tag{2}$$

where v stands for a conjugate of u. It is not single-valued. However, making suitable slits along the v-lines $\{z \mid v(z) = \text{constant}\}$, we obtain a doubly connected domain which is mapped by (2) conformally onto the exterior of the closed unit disc $\{w \in \mathbf{C} \mid |w| \leq 1\}$ with a countable number of slits of the form $\{w \mid \arg w = c,\ |w| \geq c'\}$ removed [11, p. 13]. Let ψ stand for this conformal homeomorphism.

Let $r > 1$ and set $D_r = \{z \in D \mid e^{u(z)} < r\}$. It is not hard to show that D_r is connected. Thus $f \mid \overline{D}_r: \overline{D}_r \to W_0$ defines a finite covering in the sense of Ahlfors. Let F_r stand for the set $\partial D_r \cap D = \{z \in D \mid e^{u(z)} = r\}$. We claim that $\underline{\lim}_{r\to\infty} L(F_r)/A(\overline{D}_r) = 0$. To prove this we argue in the manner that has been standard since the work of Ahlfors. Suppose there is $k > 0$ such that

$$L(F_r)/A(\overline{D}_r) \geq k \quad \text{for each } r > 1. \tag{3}$$

To evaluate $A(\overline{D}_r)$ and $L(F_r)$ we pass to the w-plane via the mapping ψ. Denoting by (r, φ) the polar coordinates in the w-plane we have

$$A(\overline{D}_r) = \int_1^r \int_0^{2\pi} (\widehat{\rho}(w))^2 r\,dr\,d\varphi, \qquad L(F_r) = \int_0^{2\pi} \widehat{\rho}(w) r\,d\varphi,$$

where $\widehat{\rho}(w)|dw|$ is the pullback of the metric $\rho(z)|dz|$ with respect to $f \circ \psi^{-1}$.

Observing that

$$\frac{dA(\overline{D}_r)}{dr} = \int_0^{2\pi} (\widehat{\rho})^2 r \, d\varphi$$

and applying the Schwarz inequality to obtain

$$\left(\int_0^{2\pi} \widehat{\rho} r \, d\varphi \right)^2 \le 2\pi r \int_0^{2\pi} (\widehat{\rho})^2 r \, d\varphi ,$$

we get

$$(L(F_r))^2 \le 2\pi r \frac{dA(\overline{D}_r)}{dr}.$$

Combining this with (3) gives

$$k^2 \frac{dr}{2\pi r} \le \frac{dA(\overline{D}_r)}{A(\overline{D}_r)^2} \quad \text{for } r > 1.$$

Integration and letting $r \to \infty$ leads to a contradiction. Hence there is a sequence (r_n) such that

$$r_n \to \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{L(F_{r_n})}{A(\overline{D}_{r_n})} = 0. \tag{4}$$

We next show that $A(\overline{D}_r)$ remains bounded as $r \to \infty$. It suffices to prove that the sequence $(A(\overline{D}_{r_n}))$ is bounded. Invoking the Fundamental Lemma, we find a positive constant α such that $A(\overline{D}_{r_n}) \le \alpha(L(F_{r_n}) + L(\partial D))$ for every $n \in N$. By (4) we have $L(F_{r_n}) \le \frac{1}{2\alpha} A(\overline{D}_{r_n})$ for large n. It follows that $A(\overline{D}_{r_n}) \le 2\alpha \cdot L(\partial D)$ for large n.

We are now in a position to show that f admits a holomorphic extension to the whole of D. Fix $z_0 \in E$. Of course, it is enough to show that the cluster set of f at z_0 (taken with respect to W_0) is a singleton. Since $r \mapsto A(D_r)$ is bounded, we find an $r > 0$ such that f omits a closed set, say F, of positive area measure in $\overline{D} \backslash \overline{D}_r$. By Theorem II15A in [18, p. 137], $W_0 \backslash F$ carries a nonconstant bounded holomorphic function, say g. Since E is removable for bounded holomorphic functions, $\mathrm{Cl}(g \circ f; z_0)$ is a singleton. It follows that $\mathrm{Cl}(f; z_0)$ is a singleton in W_0 or a proper continuum in $\partial(W_0 \backslash F)$. In either case, $\mathrm{Cl}(f; z_0)$ is nowhere dense in W_0. Fix $p_0 \in W_0 \backslash \mathrm{Cl}(f; z_0)$ and let V be a neighborhood of $\mathrm{Cl}(f; z_0)$ such that $p_0 \notin \overline{V}$. Then we can find a neighborhood U of z_0 such that $f(U \backslash E) \subset V$. Now there is a rational function, say h, in W_0 with a single pole at p_0. Considering $h \circ f$, we realize that $\mathrm{Cl}(f; z_0)$ is indeed a singleton. Let f^* stand for the extended mapping. We set $W' = W \cup f^*(D) = W \cup f^*(E)$. That $W' \backslash W$ is of capacity zero follows as in the proof of Theorem 1.

It remains to consider the case that at least one of the components of $W \backslash f(\partial D)$, say V, is of infinite genus. Let W_0 be a relatively compact subregion of V whose boundary ∂W_0 consists of a finite number of analytic Jordan curves and whose genus ≥ 2. Let $\rho(z)|dz|$ stand for the Poincaré metric of V

restricted to $\overline{W}_0$. We will show that actually no point of $D\backslash E$ is mapped into $\overline{W}$ by f. The proof is by contradiction. So assume that $f^{-1}(W_0) \cap D \neq \varnothing$, and let D' stand for a component of $f^{-1}(W_0) \cap D$. Of course, $\partial D' \subset D$. Set $E' = E \cap \overline{D'}$, and let u, v, and ψ be as in the first part of the proof but this time defined with respect to the set E' instead of E. We notice that E' is nonempty, for otherwise f would give a proper mapping of D' onto W_0. But this is impossible by the Riemann-Hurwitz relation (or by the Fundamental Lemma). Assume $r > 1$ and set $D'_r = D' \cap \{z \in D \mid e^{u(z)} < r\}$, $F_r = \partial D'_r \cap D'$. Then $f \mid \overline{D'_r}$ defines a finite covering $\overline{D'_r} \to \overline{W}_0$ with F_r as the relative boundary of $\overline{D'_r}$ with respect to $\overline{W}_0$. Note that $\overline{D'_r}$ is not necessarily connected, but this is inessential. Passing to the w-plane we have

$$A(\overline{D'_r}) = \int_1^r \int_{\phi_r} (\widehat{\rho}(w))^2 r\,dr\,d\varphi, \qquad L(F_r) = \int_{\phi_r} \widehat{\rho}(w) r\,d\varphi,$$

where $\widehat{\rho}(w)|dw|$ is the pullback of the metric $\rho(z)|dz|$ with respect to $f \circ \psi^{-1}$ and $\phi_r = \{\varphi \in [0, 2\pi) \mid \psi^{-1}(re^{i\varphi}) \in D'\}$. Arguing as before and denoting the measure of ϕ_r by $m(\phi_r)$, we obtain

$$(L(F_r))^2 \le rm(\phi_r)\frac{dA(\overline{D'_r})}{dr} \le 2\pi r \frac{dA(\overline{D'_r})}{dr}. \tag{5}$$

By the Fundamental Lemma, there is an $\alpha > 0$ such that

$$A(\overline{D'_r}) \le \alpha L(F_r) \tag{6}$$

(since an inequality of this type is valid for each component of $\overline{D'_r}$, (6) is obtained simply by adding). Combining (5) and (6) gives

$$\frac{dr}{r} \le 2\pi\alpha^2 \frac{dA(\overline{D'_r})}{(A(\overline{D'_r}))^2}.$$

The promised contradiction is obtained by integration and sending r to ∞. It follows that $f^{-1}(W_0) \cap D = \varnothing$, as was asserted. Hence f can be regarded as a holomorphic mapping of $D\backslash E$ into the hyperbolic Riemann surface $W\backslash\overline{U}$, where $\overline{U}$ is a small disc in W_0. Thus Theorem 1 applies. The proof is complete. □

Remarks. (1) We want to stress once more that the preceding argument is essentially the same as Nishino's [10], whose work includes the theorem in the case that W is compact [10, pp. 106–109]. A simpler proof of this special case was given by Suzuki [21]. An interesting generalization to higher dimensions can be found in [22].

(2) Simple examples show that Theorem 3 does not hold if the genus of W is 0 or 1 even if W is nonexceptional.

(3) Theorem 3 holds true for quasiconformal mappings also; cf. Remark 3 following Theorem 1.

A Riemann surface W is said to be maximal provided there does not exist a Riemann surface W' such that $W \subsetneq W'$. Of course, all compact surfaces are maximal.

Corollary 1. *Let $D \subset \mathbf{C}$ be a domain, and let $E \subset D$ be a compact set of capacity zero. Let W be a maximal Riemann surface of genus ≥ 2, and let $f: D\backslash E \to W$ be a holomorphic mapping. Then f extends to a holomorphic mapping $f^*: D \to W$.*

Corollary 2 is Théorème 1 in [10, p. 109].

Corollary 2. *Let W_1 be a parabolic Riemann surface of genus g_1, $0 \leq g_1 < \infty$, and let W_2 be a Riemann surface of genus g_2, $g_2 \geq 2$. Let f be a holomorphic mapping of W_1 into W_2. Then there exist compact Riemann surfaces W_i^* with $W_i \subset W_i^*$, $i = 1, 2$, such that f extends to a holomorphic mapping $f^*: W_1^* \to W_2^*$. Furthermore, $g_1 \geq g_2$.*

Proof. By assumption, there is a compact Riemann surface W_1^* of genus g_1 such that $W_1 \subset W_1^*$ and $E = W_1^* \backslash W_1$ is of capacity zero. Let $D \subset W_1^*$ be a planar domain containing E. By the preceding theorem, we can find a Riemann surface W_2^* with $W_2 \subset W_2^*$ such that $f \mid D\backslash E$ extends to a holomorphic mapping of D into W_2^*. Thus we have extended f to a holomorphic map $f^*: W_1^* \to W_2^*$. Since W_1^* is compact, the same is true of W_2^*. The last assertion follows from the Riemann-Hurwitz relation (observe that the genus of W_2 equals that of W_2^*). □

The next corollary provides an affirmative answer to a question of M. Suzuki communicated by Ohtsuka [15, p. 380].

Corollary 3. *Let $D \subset \mathbf{C}$ be a domain and let $E \subset D$ be a compact set of capacity zero. Suppose that W is a Riemann surface and f is a holomorphic mapping of $D\backslash E$ into W. If $\mathrm{Cl}(f; z_0)$ coincides with W for some $z_0 \in E$, then W is parabolic and of genus at most* 1.

4. Mappings into nonexceptional Riemann surfaces of positive genus

In this section we show that certain sets of positive capacity are nonessential singularities for mappings into Riemann surfaces of positive genus, except for tori. This result is not very surprising in view of the work of Carleson [3]. It is perhaps of some interest to notice that the only analytic tool we shall need, apart from a standard result on removable singularities, is the principle of hyperbolic metric. The basic ideas, however, are from the work of Carleson and Matsumoto [8].

The examples we shall produce are Cantor type sets. Therefore, we briefly describe their construction. Let I_0 stand for the interval $[-2^{-1}, 2^{-1}]$ of length 1 on the real axis. We first remove from I_0 centrally a segment of length $1-\xi_1$, $0 < \xi_1 < 1$. The remaining set I_1 consists of equal segments $I_{1,1}$ and $I_{1,2}$

of total length ξ_1. Next we remove from $I_{1,1}$ and $I_{1,2}$ centrally a segment of length $2^{-1}(1-\xi_2)\xi_1$, $0 < \xi_2 < 1$. The result is a set of total length $\xi_1\xi_2$. Continuing in this manner we obtain the Cantor set $\bigcap_{i=1}^{\infty} I_i$ of length $\prod_{i=1}^{\infty}\xi_i$. It is known [9, p. 153] that the capacity of $\bigcap_{i=1}^{\infty} I_i$ vanishes if and only if

$$\sum_{i=1}^{\infty}\frac{\log \xi_i^{-1}}{2^i} = \infty. \tag{7}$$

Theorem 4. *Let $D \subset \mathbf{C}$ be the open unit disc. Let (ξ_n) be a sequence such that $0 < \xi_n < 1$ for each n and $\lim_{n\to\infty}\xi_n = 0$, and let E stand for the Cantor set corresponding to (ξ_n). Suppose W is a nonexceptional Riemann surface of positive genus and f is a holomorphic mapping of $D\backslash E$ into W. Then there is a Riemann surface $W' \supset W$ such that*

(a) *$W'\backslash W$ is of linear measure zero and*

(b) *f extends to a holomorphic mapping $f^*: D \to W'$.*

Proof. It is perhaps in order to begin by observing that a set of linear measure zero, being invariant under conformal mappings, is well defined on any Riemann surface.

Let $z \in E$ be arbitrary. Unless $\mathrm{Cl}(f; z)$ is empty or a singleton, it contains a nondegenerate continuum. Suppose for the moment that we have ruled out the last possibility. Then f extends to a continuous mapping f^* of D into W^*, the Stoïlow compactification of W. The set E, being of linear measure zero, is removable for bounded holomorphic functions. Therefore, the very argument given in the proof of Theorem 1 shows that, adding the set $f^*(E)$ to W, we obtain a Riemann surface W' such that f^* is holomorphic as a mapping of D into W'. Finally, it is a simple matter to verify that $W'\backslash W$, a subset of $f^*(E)$, is of linear measure zero. Accordingly, all that remains is to rule out the case that $\mathrm{Cl}(f; z)$ contains a proper continuum.

Let $\rho(z)|dz|$ be the Poincaré metric on W, and let $\delta(\cdot, \cdot)$ be the distance function induced by this metric. Let $d(S)$ stand for the (hyperbolic) diameter of a set $S \subset W$. Set $U(p_0, r) = \{p \in W \mid \delta(p, p_0) < r\}$. Assume now that there is $z_0 \in E$ such that $\mathrm{Cl}(f; z_0)$ contains a proper continuum, say K. Fix $p_0 \in K$ and choose an a, $0 < a < 1$, such that $K \cap (W\backslash\overline{U(p_0, 4a)}) \neq \varnothing$. Assume also, as we may, that $\overline{U(p_0, 4a)}$ is contained in a parametric disc of W. Let W_0 be a relatively compact subregion of W, with nice boundary and of positive genus, such that $K \cup \overline{U(p_0, 4a)} \subset W_0$. Fix $b > 0$ such that each set of the form $\overline{U(p, r)}$, $p \in W$, $r \leq b$, which meets $\overline{W}_0$, is contained in a parametric disc of W (hence such a $U(p, r)$ may be called a hyperbolic disc).

Next choose $n_0 \in N$ so large that

$$\xi_n < \min\{\tfrac{1}{2}e^{-(24/a)\pi^2}, \tfrac{1}{2}e^{-(3/b)\pi^2}\} \quad \text{for } n \geq n_0. \tag{8}$$

Of the 2^{n_0+1} intervals (in the construction of E) let $J_{1,1}$ be the one which contains z_0, and let J_0 be the superinterval of $J_{1,1}$. Let l_0 and l_1 stand for the length of J_0 and $J_{1,1}$, respectively. Let $C_{1,1}$ be the circle of radius $\frac{1}{2}(l_0 l_1)^{1/2}$ with center at the midpoint of $J_{1,1}$, and let $C'_{1,1}$ and $C''_{1,1}$ be the circles of radii l_1 and $\frac{1}{4}l_0$, respectively, and concentric with $C_{1,1}$. We then define the corresponding objects with respect to the subintervals $J_{2,1}$ and $J_{2,2}$ of $J_{1,1}$ arising in the next step of the construction of E. We obtain the circles $C_{2,i}$, $C'_{2,i}$, and $C''_{2,i}$ of radii $\frac{1}{2}(l_1 l_2)^{1/2}$, l_2, and $\frac{1}{4}l_1$, respectively, $i = 1, 2$. Continuing in this way, we get three sequences of circles: (C_{ij}), (C'_{ij}), and (C''_{ij}), $i = 1, 2, \ldots$, $j = 1, \ldots, 2^{i-1}$. Let G_{ij} be the triply connected domain bounded by the circles C_{ij}, $C_{i+1,2j-1}$, and $C_{i+1,2j}$, $i = 1, \ldots, j = 1, \ldots, 2^{i-1}$. Let D_{ij} stand for the disc bounded by C_{ij}. Then

$$\overline{D}_{1,1} \backslash E = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} \overline{G}_{ij}. \tag{9}$$

Because $p_0 \in \mathrm{Cl}(f; z_0)$, we can find $z' \in D_{1,1} \backslash E$ such that $p' = f(z') \in U(p_0, a/2)$. By (9), $z' \in \overline{G}_{kj_k}$ for some $k \in N$, $j_k \in \{1, \ldots, 2^{k-1}\}$. Obviously, there is a unique $j_{k-1} \in \{1, \ldots, 2^{k-2}\}$ such that $\overline{G}_{kj_k} \cap \overline{G}_{k-1, j_{k-1}} \neq \varnothing$. Similarly, there is a unique $j_{k-2} \in \{1, \ldots, 2^{k-3}\}$ such that $\overline{G}_{k-1, j_{k-1}} \cap \overline{G}_{k-2, j_{k-2}} \neq \varnothing$. Continuing in this way we obtain a chain of domains $(G_{kj_k}, \ldots, G_{1,1})$ such that $\bigcup_{i=1}^{k} \overline{G}_{ij_i}$ is connected. Consider the set $f^{-1}(\partial U(p_0, 2a)) \cap (\bigcup_{i=1}^{k} \overline{G}_{ij_i})$. If this set is empty, then

$$f(\overline{G}_{1,1}) \subset U(p_0, 2a) \tag{A}$$

by the connectedness of $\bigcup_{i=1}^{k} \overline{G}_{ij_i}$. Otherwise we set

$$h = \max\left\{ i \in \{1, \ldots, k\} | f^{-1}(\partial U(p_0, 2a)) \cap \overline{G}_{ij_i} \neq \varnothing \right.$$
$$\left. \text{for some } j_i \in \{1, \ldots, 2^{i-1}\} \right\}.$$

Thus

$$f^{-1}(\partial U(p_0, 2a)) \cap \overline{G}_{hj_h} \neq \varnothing. \tag{B}$$

We are going to show that both cases lead to a contradiction: (A) implying that $f(\overline{D}_{1,1} \backslash E) \subset U(p_0, 3a)$ and (B) that $f(\overline{D}_{hj_h} \backslash E) \cap U(p_0, a/2) = \varnothing$.

We first assume that $f(\overline{G}_{1,1}) \subset U(p_0, 2a)$. Consider the concentric circles $C_{1,1}$, $C'_{1,1}$, and $C''_{1,1}$. Let $A_{1,1}$ be the annulus bounded by $C'_{1,1}$ and $C''_{1,1}$. It is well known (and easy to verify) that the hyperbolic length of the circle $\{z \in \mathbf{C} \mid |z| = 1\}$ with respect to the annulus $\{z \in \mathbf{C} \mid 1/R < |z| < R\}$ is $\pi^2/2 \log R$. This, combined with (8), implies that the hyperbolic length of $C_{1,1}$

(with respect to $A_{1,1}$) is

$$\frac{\pi^2}{\log(l_0/4l_1)} = \frac{\pi^2}{\log(2\xi_{n_0+1})^{-1}} < \frac{a}{24}.$$

By the principle of hyperbolic metric, this is also true if the hyperbolic metric is taken with respect to $D\backslash E$. By the same principle

$$s(f(C_{1,1})) < a/24, \tag{10}$$

where $s(\cdot)$ denotes the hyperbolic length (with respect to W). By (8), (10) also holds for $C_{2,1}$ and $C_{2,2}$. Let U_1, U_2, and U_3 be hyperbolic discs of radius $a/24$ containing $f(C_{1,1})$, $f(C_{2,1})$, and $f(C_{2,2})$, respectively. We claim that the set $\overline{U}_1 \cup \overline{U}_2 \cup \overline{U}_3$ is connected. If this is not the case, then $W_0' = W_0\backslash(\overline{U}_1 \cup \overline{U}_2 \cup \overline{U}_3)$ is connected and the valence function of $f \mid G_{1,1}$ is positive and constant in W_0' (if it were identically zero, $f(G_{1,1})$ could not be connected). Hence, letting $G_{1,1}'$ denote a component of $f^{-1}(W_0')\cap G_{1,1}$, f defines a proper mapping of $G_{1,1}'$ onto W_0'. But this is impossible by virtue of the Riemann-Hurwitz formula, since $G_{1,1}'$ is planar and W_0' is of positive genus. The assertion follows. Hence there is a hyperbolic disc U (concentric with one of the U_i's) of radius $a/8$ such that $\bigcup_{i=1}^3 U_i \subset U$. Clearly $f(G_{1,1}) \subset U$, whence $d(f(G_{1,1})) < a/4$.

We next claim that, given any $i \in N$, $j \in \{1, \dots, 2^{i-1}\}$,

$$d(f(G_{ij})) < a/4 \quad \text{provided that } f(C_{ij}) \cap U(p_0, 3a) \neq \varnothing. \tag{11}$$

By (8), $s(f(C_{ij})) < a/24$ and

$$s(f(C_{i+1,2j-1})),\ s(f(C_{i+1,2j})) < c = \min\{a/24, b/3\}.$$

Thus $f(C_{ij})$ is contained in a hyperbolic disc, say U_1 again, of radius $a/24$. If $f(C_{i+1,2j-1})$ and $f(C_{i+1,2j})$ meet $\overline{W}_0$, we can find hyperbolic discs U_2 and U_3 of radius c containing $f(C_{i+1,2j-1})$ and $f(C_{i+1,2j})$, respectively (recall the choice of b). Modifying the above argument slightly, we see that $\bigcup_{i=1}^3 \overline{U}_i$ must be connected and $f(G_{ij})$ is contained in a hyperbolic disc of radius $a/8$. Thus $d(f(G_{ij})) < a/4$. Similarly, it can be shown that the case $f(C_{i+1,2j-1}) \cap \overline{W}_0 = \varnothing$ or $f(C_{i+1,2j}) \cap \overline{W}_0 = \varnothing$ leads to a disagreement with the Riemann-Hurwitz formula.

We are going to complete the proof of the claim that $f(\overline{D}_{1,1}\backslash E) \subset U(p_0, 3a)$ (thereby showing that $f(C_{ij}) \subset U(p_0, 3a)$ for all i, j). The proof is again by contradiction. If $f(D_{1,1}\backslash E) \not\subset U(p_0, 3a)$, we may set $m = \min\{i \in N \mid f(\overline{G}_{ij})\cap\partial U(p_0, 3a) \neq \varnothing$ for some $j \in \{1, \dots, 2^{i-1}\}\}$. Fix $j_m \in \{1, \dots, 2^{m-1}\}$ such that $f(\overline{G}_{mj_m}) \cap \partial U(p_0, 3a) \neq \varnothing$. Observe that $f(C_{mj}) \cap U(p_0, 3a) \neq \varnothing$ for every $j \in \{1, \dots, 2^{m-1}\}$.

Consider the circle $C_{2,1}$. It is contained in $\overline{G}_{1,1} \cup \overline{G}_{2,1}$. By (11),

$$d(f(\overline{G}_{1,1} \cup \overline{G}_{2,1})) < a/2.$$

Fix $p_1 \in f(\overline{G}_{1,1} \cup \overline{G}_{2,1})$. Then $f \mid G_{1,1} \cup C_{2,1} \cup G_{2,1}$ can be regarded as a holomorphic mapping into the hyperbolic disc $U(p_1, a/2) \subset U(p_0, 4a)$. Since the annulus bounded by $C'_{2,1}$ and $C''_{2,1}$ is contained in $G_{1,1} \cup C_{2,1} \cup G_{2,1}$, the principle of hyperbolic metric yields

$$s_1(f(C_{2,1})) < \frac{\pi^2}{\log(2\xi_{n_0+2})^{-1}} < \frac{a}{24},$$

where $s_1(\cdot)$ denotes the hyperbolic length with respect to the disc $U(p_1, a/2)$. Now it is a simple matter to verify that $\rho(z) < \frac{a}{2}\rho_1(z) < \frac{1}{2}\rho_1(z)$, where $\rho_1(z)$ denotes the density of the Poincaré metric of $U(p_1, a/2)$. Hence $s(f(C_{2,1})) < a/48$. The same argument gives $s(f(C_{ij})) < a/48$ for all $i = 2, \ldots, m$, $j = 1, \ldots, 2^{i-1}$. We infer that $d(f(\overline{G}_{ij})) < a/8$ for all $i = 2, \ldots, m-1$, $j = 1, \ldots, 2^{i-1}$. Repeating the same reasoning we obtain $d(f(\overline{G}_{ij})) < a/16$ for $i = 3, \ldots, m-2$, $j = 1, \ldots, 2^{i-1}$, and generally

$$\begin{aligned} &d(f(\overline{G}_{ij})) < a \cdot 2^{-i-1} \quad \text{for } i = 1, \ldots, \left[\frac{m+1}{2}\right], \qquad j = 1, \ldots, 2^{i-1}, \\ (12)\quad &d(f(\overline{G}_{m-i,j})) < a \cdot 2^{-i-2} \quad \text{for } i = 0, \ldots, \left[\frac{m}{2}\right] - 1, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, \ldots, 2^{m-i-1}. \end{aligned}$$

Let $(\overline{G}_{m,j_m}, \overline{G}_{m-1,j_{m-1}}, \ldots, \overline{G}_{1,1})$ be the (uniquely determined) chain such that $\overline{G}_{ij_i} \cap \overline{G}_{i-1,j_{i-1}} \neq \varnothing$ for all $i = m, \ldots, 2$. By (12), $d(\bigcup_{i=1}^m f(\overline{G}_{ij_i})) < a$. But this is impossible, since $\bigcup_{i=1}^m f(\overline{G}_{ij_i})$ meets both $U(p_0, 2a)$ and $\partial U(p_0, 3a)$. It follows that $f(\overline{D}_{1,1} \backslash E) \subset U(p_0, 3a)$ as was asserted. Recalling that $\mathrm{Cl}(f; z_0) \cap (W \backslash \overline{U(p_0, 4a)}) \neq \varnothing$, we see that (A) is indeed contradictory.

The case (B) is now readily settled. Consider the circles C_{hj_h}, $C_{h+1,2j_h-1}$, and $C_{h+1,2j_h}$ bounding G_{hj_h}. By (8), $s(f(C_{hj_h}))$, $s(f(C_{h+1,2j_h-1}))$, and $s(f(C_{h+1,2j_h}))$ are all $< c = \min\{a/24, b/3\}$. Suppose first that each of the image curves meets $\overline{W}_0$. By the choice of b, we find the hyperbolic discs U_1, U_2, and U_3 of radius c containing $f(C_{h,j_h})$, $f(C_{h+1,2j_h-1})$, and $f(C_{h+1,2j_h})$, respectively. Assume $\bigcup_{i=1}^3 \overline{U}_i$ is not connected. Then $W \backslash \bigcup_{i=1}^3 \overline{U}_i$ is connected, and the valence function of $f \mid G_{hj_h}$ is positive and constant in $W \backslash \bigcup_{i=1}^3 \overline{U}_i$. As before, this leads to a contradiction to the Riemann-Hurwitz relation. Thus $\bigcup_{i=1}^3 \overline{U}_i$ is connected, and there is a hyperbolic disc U of radius $3c$ which contains $\bigcup_{i=1}^3 \overline{U}_i$. Since $W \backslash \overline{U}$ is connected and of positive genus, we have $f(G_{hj_h}) \subset U$, whence $d(f(G_{hj_h})) < 6c \leq a/4$. The cases where at least one of the image curves fails to meet $\overline{W}_0$ are readily seen to be incompatible with the Riemann-Hurwitz relation. Since $d(f(G_{hj_h})) < a/4$ and $f(\overline{G}_{hj_h}) \cap \partial U(p_0, 2a) \neq \varnothing$, we have $f(\overline{G}_{hj_h}) \subset U(p_0, \frac{5}{2}a) \backslash \overline{U(p_0, \frac{3}{2}a)}$. From now on we proceed in exactly the same manner as in case (A). It turns out that $f(\overline{D}_{hj_h} \backslash E) \subset U(p_0, \frac{7}{2}a) \backslash \overline{U(p_0, \frac{1}{2}a)}$, contradicting the facts that $z' \in \overline{D}_{hj_h} \backslash E$ and $p' = f(z') \in U(p_0, \frac{1}{2}a)$. It follows that for every $z \in E$, $\mathrm{Cl}(f; z)$ is either empty or a singleton. This conclusion completes the proof. □

Remarks. (1) It follows from (7) that the capacity of E may be positive. For example, we may take $\xi_n = 1/n$, $n \in N$ (indeed, even $\xi_n = 1/n^n$ would do).

(2) Let E be as in the preceding theorem. Then the theorem remains valid if E is replaced by $E \times E$. First, $E \times E$ is of linear measure zero. Secondly, $D \backslash E \times E$ can be exhausted by domains bounded by circles sharing the essential properties of C_{ij}, C'_{ij}, and C''_{ij} above. Even more general formulations could be obtained in the manner indicated by Matsumoto [8, pp. 189–191]. Observe that the branching number of the exhaustion, related to the number of the omitted values in the work of Matsumoto, plays an inessential role in the case of holomorphic mappings into Riemann surfaces of positive genus.

(3) The preceding proof applies with minor modifications to meromorphic functions omitting four values in the Riemann sphere. Thus we have a parallel proof for Theorem 3 in [8, p. 197]. Instead, Theorem 4 does not hold for meromorphic functions omitting three values. This follows readily from (the proof of) Theorem 1 in [23, p. 9]. Hence the assumption that W be of positive genus cannot be dropped.

(4) In [19], it is shown that there are linear sets of linear measure zero which are not removable for holomorphic mappings into a Riemann surface of genus ≥ 2.

We conclude this paper by briefly discussing countable exceptional sets for holomorphic mappings into tori.

Theorem 5. *Let $D \subset \mathbf{C}$ be the open unit disc and let (a_n), $a_n \in D$, be a sequence such that $|a_{n+1}| \leq |a_n|$ for each $n \in N$ and $\underline{\lim}_{n\to\infty} |a_{n+1}|/|a_n| = 0$. Set $E = \{0\} \cup \{a_n \mid n \in N\}$. Suppose W is a compact Riemann surface of genus 1, $p_0 \in W$, and f is a holomorphic mapping of $D \backslash E$ into $W \backslash \{p_0\}$. Then f extends to a holomorphic mapping $f^*: D \to W$.*

Proof. By the theorem of Ohtsuka, mentioned at the outset of this paper, f admits a holomorphic extension to every a_n, $n \in N$. We retain the notation f for this extension. It remains to extend f to 0.

Remove from W a small disc U containing p_0 in its interior, and set $W_0 = W \backslash \overline{U}$. Let $\delta(\cdot, \cdot)$ denote the hyperbolic distance in $W \backslash \{p_0\}$, and set $U(p, r) = \{q \in W \backslash \{p_0\} \mid \delta(q, p) < r\}$. Pick out $a > 0$ such that each set of the form $U(p, r)$, $r < a$, which meets $\overline{W}_0$ is contained in a parametric disc of $W \backslash \{p_0\}$. Then choose $n_0 \in N$ such that $|a_{n_0+1}|/|a_{n_0}| < e^{-(4/a)\pi^2}$ and set

$C_n = \{z \in D \mid |z| = (|a_n||a_{n+1}|)^{1/2}\}$, $n \in N$. By the principle of hyperbolic metric, $s(f(C_{n_0})) < a/4$, where $s(\cdot)$ again denotes the hyperbolic length. Now suppose that $n > n_0$ and $|a_{n+1}|/|a_n| < e^{-(4/a)\pi^2}$, and let A_n stand for the annulus bounded by C_{n_0} and C_n. Observing that $s(f(C_n)) < a/4$ and arguing as in the proof of the preceding theorem, we see that $f(A_n) \cap W_0$ is contained in a hyperbolic disc of radius $\frac{1}{2}a$. It follows that $\mathrm{Cl}(f; 0)$ cannot be the whole of W. Hence $\mathrm{Cl}(f; 0)$ is a singleton (see the proof of Theorem 1). This observation completes the proof. □

Remarks. (1) It is easy to construct a torus W, an exceptional set E of the form $\{a^n \mid n \in N\} \cup \{0\}$, and a holomorphic mapping $f: D\backslash E \to W\backslash\{p\}$ for some $p \in W$ such that f has an essential singularity at 0.

(2) Following Lehto, sets described in the preceding theorem could be termed 0-Picard sets for holomorphic mappings into tori [5, 23]. By Theorem 4, there are also perfect 0-Picard sets for such mappings.

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