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ON THE HOMOLOGY OF SU(n) INSTANTONS

CHARLES P. BOYER, BENJAMIN M. MANN, AND DANIEL WAGGONER

ABSTRACT. In this paper we study the homology of the moduli spaces of instantons associated to principal $\mathbf{SU}(n)$ bundles over the four-sphere. This is accomplished by exploiting an "iterated loop space" structure implicit in the disjoint union of all moduli spaces associated to a fixed $\mathbf{SU}(n)$ with arbitrary instanton number and relating these spaces to the known homology structure of the four-fold loop space on $B\mathbf{SU}(n)$.

Moduli spaces of instantons (self-dual connections with respect to a conformal class of metrics) associated to principal G-bundles $P_k(G) = P$ over S^4 have proven to be basic objects in modern geometry. Here G is any simple compact Lie group and k is the integer that classifies the bundle P and is referred to as the instanton number. We denote these moduli spaces by $\mathscr{M}_k(G)$. There is a natural inclusion

$$i_k(G)\colon \mathscr{M}_k(G) \longrightarrow \Omega^4_k BG \tag{0.1}$$

induced by forgetting the self-duality condition where we have identified the moduli space of based gauge equivalence classes of all connections on P with the kth component of the four-fold loop space $\Omega^4 BG$ [5]. In a fundamental paper, Atiyah and Jones [5] studied the inclusion (0.1) for $G = \mathbf{SU}(2)$ (when S^4 has its standard conformally flat metric) and posed several fundamental questions. In a series of papers Taubes (cf. [24–26]) proved several basic existence theorems, stability theorems in terms of k and provided a basis framework to describe how the topology of $\mathscr{M}_k$ changes as k increases. Taubes' work is much more general than we describe here in that he not only studied general Lie groups G but also replaced S^4 by an arbitrary compact closed Riemannian four-manifold (with arbitrary conformal class of metric).

In [7] it was observed that, over the four-sphere but with $G = \mathbf{Sp}(n)$, the disjoint union of $\mathscr{M}_k$ over all k form a homotopy C_4 space and that iterated loop space techniques may be profitably used to study $H_*(\mathscr{M}_k)$ for individual k. Certain computational results may be obtained immediately from these

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constructions for all $\mathbf{Sp}(n)$ but to obtain more delicate results special facts about $\mathcal{M}_k(\mathbf{Sp}(n))$ for small values of k are required. When $G = \mathbf{Sp}(1) = \mathbf{SU}(2)$ the results of [7] give substantial information about $H_*(\mathcal{M}_k(\mathbf{SU}(2)); \mathbf{Z}/p)$ as well as establish, for k greater than one, that $\mathcal{M}_k(\mathbf{SU}(2))$ is not a Stein manifold. This result should be compared to the theorem of Donaldson [11] that $\mathcal{M}_k(\mathbf{SU}(2))$ is a complex manifold and to an observation of Hitchin (private communication) that $\mathcal{M}_k(\mathbf{SU}(2))$ is hyper-Kähler.

In this paper we combine the constructions of [7] with the computations of [27] to study $H_*(\mathcal{M}_k(\mathbf{SU}(n)); \mathbf{Z}/p)$ for general $n > 2$. This depends on an explicit determination of the inclusion i (0.1) for $k = 1$ which is given in §4.

The first two sections of this paper review previous work on the topology of $\mathcal{M}_k(G)$ which establish the general framework in which we can construct nontrivial classes in $H_*(\mathcal{M}_k(G))$. We then specialize to the case $G = \mathbf{SU}(n)$ and in §3 analyze the natural inclusion i_1. §4 then summarizes results of [27] on $H_*(\Omega^3\mathbf{SU}(n))$ that we will need and in §5 we analyze i_1 on the level of homology. Next, in §6, we are able to state our main results which describe new families of nontrivial homology classes in $H_*(\mathcal{M}_k(\mathbf{SU}(n)))$ (see Theorems 6.4 and 6.6), as well as show that large numbers of the $\mathcal{M}_k(\mathbf{SU}(n))$, which are known to be complex manifolds, cannot admit a Stein manifold structure (see Corollary 6.12). Finally §7 concludes with the proofs of several technical lemmas required in earlier sections.

It is natural to ask if the computations of this paper can be extended to other Lie groups G, in particular for $\mathbf{Sp}(n)$. Basically, in order to use the homology operations constructed in [7] to explicitly study $\mathcal{M}_k(G)$ one must understand not only $H_*(\Omega^3 G)$ but also $\mathcal{M}_k(G)$ and the inclusion i_k for various values of k. Only when $G = \mathbf{SU}(n)$ does it appear that $\mathcal{M}_1(G)$ is sufficiently rich to generate enough new homology in the $\mathcal{M}_k(G)$'s which gives nontrivial homological information not obtainable by other means.

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1. Connections and Yang-Mills instantons

In order for this paper to be relatively self-contained and for us to define the spaces that are our main objects of study we begin with a very brief review of the differential geometric formulation of the Yang-Mills theory over the four-sphere. Good references on the foundational material in this area include [2, 4, 5, and 25].

Let G be a compact, connected, simple Lie group. In later sections we will concentrate on the case $G = \mathbf{SU}(n)$ but the general framework we consider works for arbitrary G. Next let $\pi\colon P(G) \to S^4$ be a principal G-bundle over the four-sphere. As $\pi_3(G) = \mathbf{Z}$, such bundles are classified by the integers and we write $P_k(G)$ or simply P_k for the bundle determined by the integer k. There are two natural spaces associated to P_k, namely $\mathcal{A}_k$, the space of all connections on P_k, and $\mathscr{G}(P_k)$, the gauge group consisting of all bundle automorphisms of P_k which cover the identity map on S^4. Now $\mathscr{A}_k$ is an affine space, and $\mathscr{G}(P_k)$ acts naturally on $\mathscr{A}_k$. However, this action is not in general free. But there is a normal subgroup $\mathscr{G}^b(P_k)$, given by all bundle automorphisms covering the identity that fix the fiber over a base point, say the north pole, and this does act freely on $\mathscr{A}_k$. In the appropriate Sobolev norms $\mathscr{G}^b(P_k)$ is a closed subgroup of $\mathscr{G}(P_k)$, so that the quotient space $\mathscr{A}_k/\mathscr{G}^b(P_k) = \mathscr{C}_k$ has the structure of a smooth infinite-dimensional Banach manifold [14]. Furthermore, since the space $\mathscr{A}_k$ is contractible and

$$\mathscr{G}^b(P_k) \to \mathscr{A}_k \to \mathscr{C}_k \tag{1.1}$$

is a principal fibration, $\mathscr{C}_k$ may be identified with $B\mathscr{G}^b(P_k)$, the classifying space of $\mathscr{G}^b(P_k)$. Moreover, $\mathscr{C}_k$ is homotopy equivalent to $\Omega^3_k G \simeq \Omega^4_k BG$ [5].

Now let $H \subset G$ be any closed Lie subgroup. Then there is a map

$$\psi\colon \mathscr{C}_k(H) \to \mathscr{C}_k(G) \tag{1.2}$$

constructed as follows: Let $\imath\colon P_k(H) \hookrightarrow P_k(G)$ be the inclusion of a subbundle. A connection on $P_k(H)$ is a smooth H-equivariant choice of horizontal subspace $\mathscr{H}_v(H)$ of the tangent space $T_vP_k(H)$ for each $v \in P_k(H)$. We can promote an H-connection to a G-connection as follows: Given $u \in P_k(G)$, choose $v \in P_k(H)$ and $a \in G$ such that $u = \imath(v)a$ and define the horizontal subspace $\mathscr{H}_u(G)$ of $T_uP_k(G)$ by

$$\mathscr{H}_u(G) = R_{a*} \circ \imath_*(\mathscr{H}_v(H)).$$

One easily shows that this is independant of the choice of both v and a and is G-equivariant. Alternatively in terms of the uniquely defined Ad-equivariant 1-forms ω_H and ω_G we define ω_G to be the unique G-connection 1-form which satisfies $\omega_H = \imath^*\omega_G$. Thus, we have a smooth injection

$$\imath_*\colon \mathscr{A}_k(H) \to \mathscr{A}_k(G).$$

Similarly, we define a map $\imath\colon \mathscr{G}^b_k(H) \to \mathscr{G}^b_k(G)$ as follows: Given $f_H \in \mathscr{G}^b_k(H)$, define $f_G \in \mathscr{G}^b_k(G)$ by

$$f_G(u) = R_a \circ \imath \circ f_H(v),$$

where $\imath(v)a = u \in P_k(G)$. Again f_G is independent of this decomposition. Furthermore, as H is closed in G it is easy to check that $\mathscr{G}^b_k(H)$ is a closed subgroup of $\mathscr{G}^b_k(G)$. This gives the smooth map (1.2).

It is well known that principal G-bundles on a manifold X are classified by homotopy classes of maps f from X into the classifying space BG. Furthermore, if X is compact, f factors through a finite skeleton $BG(k)$ of BG for some $k = k(f)$, and in particular, for $G = \mathbf{Sp}(n)$, $BG(k)$ may be taken to be $G_{n,n+k}(\mathbf{H})$, the Grassmannian of quaternionic n-planes in $\mathbf{H}^{n+k}$. There is also the existence of a universal G-connection given by Narasimhan and Ramanan [22] for compact G which we now describe. Given a quaternionic vector bundle E_n on S^4 of quarternionic rank n and an embedding of E_n into H^{n+k}, the trivial $\mathbf{H}^{n+k}$ bundle over S^4, one has the Gauss map $u\colon E_n \to \mathbf{H}^{n+k}$. Here u associates to each $x \in S^4$ a monomorphism of the fiber $E_n|_{\pi^{-1}(x)}$ into $\mathbf{H}^{n+k}$, and thus determines the classifying map $f\colon S^4 \to G_{n,n+k}(\mathbf{H})$. By choosing bases for $\pi^{-1}(x)$ and $\mathbf{H}^{n+k}$, $u(x)$ can be represented as the n by $n+k$ matrix

$$u(x) = \begin{bmatrix} -I_n \\ U \end{bmatrix}, \tag{1.3}$$

where U denotes affine coordinates on $G_{n,n+k}(\mathbf{H})$. Alternatively, the map $U \mapsto u(x)$ gives a local frame for the universal vector bundle over an affine neighborhood in $G_{n,n+k}(\mathbf{H})$. In our case we need the principal $\mathbf{Sp}(n)$ bundle of orthonormal symplectic frames on $G_{n,n+k}(\mathbf{H})$ whose total space is the Stiefel manifold $V_{n,n+k}(\mathbf{H})$. Then the map u satisfies the equation

$$u^* u = I_n, \tag{1.4}$$

where * is the quaternionic adjoint (conjugate transpose). Hence, $u(x)$ in (1.4) above is divided on the right by the n by n quarternionic matrix $\sqrt{I_n + U^* U}$. On $V_{n,n+k}(\mathbf{H})$ there is a universal $\mathbf{Sp}(n)$-connection [22] given by

$$\eta_{\mathbf{Sp}(n)} = u^* du. \tag{1.5}$$

Now it follows from the Peter-Weyl theorem that any compact simple G can be realized as a closed subgroup of $\mathbf{Sp}(n)$ for some n. Thus if $G \hookrightarrow \mathbf{Sp}(n)$ is a closed subgroup, then the action of G on $V_{n,n+k}$ gives a principal G-bundle $G \to V_{n,n+k} \to BG(k)$ which we denote by $\gamma(G)$. Let $\mathbf{g}$ be the Lie algebra of G. Then, since $\mathbf{Sp}(n)$ is compact, the projection to $\mathbf{g}$ of the restriction of $\eta_{\mathbf{Sp}(n)}$ to $\gamma(G)$ defines a connection 1-form η_G on $\gamma(G)$. Furthermore, both the G-bundle $\gamma(G)$ and the connection η_G are universal [22] for quaternionic vector bundles $E_n(G) = P(G) \times_G \mathbf{H}^n$ on S^4. That is, every pair $(P(G), \omega_G)$, where $P(G)$ is a principal G-bundle on S^4 with connection ω, can be obtained by pulling back the pair $(\gamma(G), \eta_G)$ by the classifying map $f\colon S^4 \to G_{n,n+k}(\mathbf{H})$. Furthermore, if H is a closed Lie subgroup of the Lie group G, and $\iota : P(H) \hookrightarrow P(G)$ is a subbundle with connections ω_H and ω_G, its promotion to a G-connection on $P(G)$, then we have

$$\omega_H = \iota^* f_G^* \eta_G = f_H^* \eta_H. \tag{1.6}$$

Associated to every $\omega \in \mathscr{A}_k$ is its curvature $F^\omega = D^\omega \omega$ which may be identified with a section of the adjoint bundle of Lie algebras $(P_k \times_G \mathbf{g}) \otimes \Lambda^2(S^4)$. There is a natural bilinear form on $(P_k \times_G \mathbf{g}) \otimes \Lambda^2(S^4)$ given by the Hodge inner product on $\Lambda^2(S^4)$ and the Killing form on $\mathbf{g}$. The corresponding norm gives the Yang-Mills functional on $\mathscr{A}_k$:

$$\mathscr{Y}\mathscr{M}(\omega) = \int_{S^4} \|F^\omega\|^2 . \tag{1.7}$$

Now one easily sees that $\mathscr{Y}\mathscr{M}$ depends only on the gauge equivalence class of ω, and the conformal class of the metric which we fix to be the standard one. Furthermore, F^ω splits orthogonally into self-dual, F^ω_+, and anti-self-dual, F^ω_-, components (with respect to the Hodge star operator) and it follows from Chern-Weil theory that the self-dual (for $k > 0$) and anti-self-dual (for $k < 0$) connections give absolute minima of $\mathscr{Y}\mathscr{M}$. Since the existence of self-dual or anti-self-dual connections depends on the orientation class of the manifold, we restrict ourselves to the case $k \geq 0$. Such gauge equivalence classes of self-dual connections are known as *instantons* and we denote the set of instantons with respect to the group G by $\mathscr{M}_k(G)$, i.e.,

Definition 1.8. $\mathscr{M}_k(G) = \{\omega \in \mathscr{A}_k(G) : *F^\omega = F^\omega\}/\mathscr{G}^b(P_k(G))$.

It follows from work of Atiyah, Hitchin, and Singer [4], Taubes [25], and Uhlenbeck [13] that $\mathscr{M}_k(G)$ is a smooth manifold of dimension $p_1(\mathbf{g})$, the first Pontrjagin number associated to the adjoint bundle $P_k \times_G \mathbf{g}$. These spaces, known as the (based) moduli spaces of instantons, are our fundamental objects of interest. Atiyah, Hitchin, and Singer [4] gives the dimension of $\mathscr{M}_k(G)$ for compact, simple, simply connected G.

Next we review the construction of the moduli space of instantons and recall the well-known description of instantons on S^4 in terms of linear algebra given by Atiyah, Drinfeld, Hitchin, and Manin [3]. For full details see Atiyah [2]. This description was used in [7] to construct a homotopy C_4 structure on $\mathscr{M}(\mathbf{Sp}(n))$. Using Lemma 1.19 below the construction extends to a homotopy C_4 structure on $\mathscr{M}(G)$ for arbitrary G. Notice that on S^4 the Gauss map gives the short exact sequence

$$0 \to E_n \to H^{n+k} \to kL \to 0,$$

where kL denotes k copies of the quaternionic Hopf bundle and H^{n+k} is the trivial bundle of quaternionic rank $n+k$. The orthogonal splitting of this sequence, with respect to the flat metric on H^{n+k}, determines the classifying map f in terms of the embedding of the normal bundle kL. This embedding can be represented by the matrix

$$v(x) = \begin{bmatrix} \Lambda \\ B - xI \end{bmatrix} \tag{1.9}$$

satisfying the condition that

$$\operatorname{rank} v(x) = k \quad \text{for all } x \in \mathbf{HP}(1), \tag{1.10}$$

where Λ and B are certain n by k and k by k quaternionic matrices. The orthogonality condition is given by the equation $u^* v(x) = 0$.

In the case of $G = \mathbf{Sp}(n)$ the classifying map in affine coordinates on $G_{n,n+k}(\mathbf{H})$ takes the form

$$f(x) = (\Lambda(B - xI)^{-1})^* . \tag{1.11}$$

In the case of general G there are constraints on (Λ, B), but, more importantly, the corresponding connection 1-form is still given by $\omega_G = f^*\eta_G$.

Returning to the case $G = \mathbf{Sp}(n)$, let $M_{n,k}(\mathbf{H})$ denote the normed linear space of n by k quaternionic matrices. We shall consider various subspaces of $M_{n,k}(\mathbf{H}) \times M_{k,k}(\mathbf{H})$ which give rise to connections through the ADHM construction. First we define the space

$$\widehat{\mathscr{B}_k} = \{(\Lambda, B) \in M_{n,k}(\mathbf{H}) \times M_{k,k}(\mathbf{H}) : \text{condition (1.10) is satisfied}\}. \tag{1.12}$$

Clearly $\widehat{\mathscr{B}_k}$ is an open set of the normed linear space $M_{n+k,k}(\mathbf{H})$. The discussion above thus describes a continuous map $\hat{\alpha}_k : \widehat{\mathscr{B}_k} \to \mathscr{A}_k$. Now the real orthogonal group $O(k)$ acts naturally on $M_{n+k,k}(\mathbf{H}) \simeq M_{n,k}(\mathbf{H}) \times M_{k,k}(\mathbf{H})$ by sending

$$(\Lambda, B) \mapsto (\Lambda T, T^{-1}BT), \tag{1.13}$$

where $T \in O(k)$. This clearly leaves $\widehat{\mathscr{B}_k}$ invariant. Thus, by restriction, we obtain the quotient space $\mathscr{B}_k = \widehat{\mathscr{B}_k}/O(k)$. Moreover, one easily sees that changing (Λ, B) to $(\Lambda T, T^{-1}BT)$ does not affect $\omega_{\mathbf{Sp}(n)}$, so we get a continuous map $\mathscr{B}_k \to \mathscr{A}_k$. Thus composing this map with the natural projection $\mathscr{A}_k \to \mathscr{C}_k$ gives a continuous map α_k from $\mathscr{B}_k$ to $\mathscr{C}_k$, associating to equivalance classes of pairs (Λ, B) satisfying (1.12) gauge equivalence classes of connections in P_k.

So far we have just constructed connections from quaternionic "rational maps" given by the Λ's and the B's without considering their associated Yang-Mills energy. However, Atiyah [2, II-3.11] gives the following explicit formula for the curvature of these connections:

$$F = N dx \rho^{-2} d\bar{x} N^*.$$

Here N is a projection operator and

$$\rho^2 = v^*v = (B^* - \bar{x}I)(B - xI) + \Lambda^*\Lambda.$$

Therefore, the matrix $\rho^2(x)$ can be used to compute the Yang-Mills energy of the associated connection. But as $dx \wedge d\bar{x}$ spans the self-dual two-forms on an affine open set of S^4, the connection will be self-dual if and only if ρ^2 is real. The reality of ρ^2 is equivalent to the following two conditions [2]:

$$\begin{aligned} &\text{(i) } B \text{ is symmetric,} \\ &\text{(ii) } B^*B + \Lambda^*\Lambda \text{ is real.} \end{aligned} \tag{1.14}$$

Thus, we define subspaces $\widehat{\mathscr{M}_k} \subset \widehat{\mathscr{V}_k} \subset \widehat{\mathscr{B}_k}$ by

$$\widehat{\mathscr{V}_k} = \{(\Lambda, B) \in \widehat{\mathscr{B}_k} : B \text{ is symmetric}\},$$

and

$$\widehat{\mathscr{M}_k} = \{(\Lambda, B) \in \widehat{\mathscr{V}_k} : B^*B + \Lambda^*\Lambda \text{ is real}\}.$$

As conditions (1.14) are unaltered by the $O(k)$-action on $\widehat{\mathscr{B}_k}$, there are well defined quotients spaces, and it is a remarkable theorem of Atiyah, Drinfeld, Hitchin, and Manin [3] that the above construction gives *all* instantons of charge k, namely

Theorem 1.15 [3]. $\mathscr{M}_k = \mathscr{M}_k(\mathbf{Sp}(n)) \simeq \widehat{\mathscr{M}_k}/O(k)$.

In the case of a G-connection, where G is a proper closed subgroup of $\mathbf{Sp}(n)$, the ADHM Theorem 1.15 [3] still holds and the above discussion goes through equally as well except that the classifying map f lands in $BG(k)$. The composition of f with the natural projection of $BG(k)$ onto $G_{n,n+k}(\mathbf{H})$ does have the explicit form given in (1.11) with additional constraints on the Λ and B matrices, but the important point is that the G-connection ω_G is still obtained by pulling back the universal G-connection η_G on $\gamma(G)$ by this classifying map. So the moduli space of G-instantons for any compact simple Lie group G is given by

(1.16) $$\mathscr{M}_k(G) = \{[\Lambda, B] \in \mathscr{M}_k : \omega^{(\Lambda, B)} \text{ is a } G\text{-instanton}\}.$$

Similarly, we have

(1.17) $$\mathscr{V}_k(G) = \{[\Lambda, B] \in \mathscr{V}_k : \omega^{(\Lambda, B)} \text{ is a } G\text{-connection}\}.$$

The following simple observation will allow us to pass easily from the case where $G = \mathbf{Sp}(n)$ to the more general situation where G is any compact simple Lie group. Let $\omega_t = f_t^*(\eta_{\mathbf{Sp}(n)})$ be a path of connections induced from a path of classifying maps $f_t\colon S^4 \to B\mathbf{Sp}(n)$. Since $\mathbf{Sp}(n)/G \to BG \to B\mathbf{Sp}(n)$ is a fibration [28], if ω_0 is reducible to a G-connection then the entire path f_t lifts to BG and ω_t is reducible to a G-connection for all t. Furthermore, it is easy to check that if

(1.18) $$\iota_k(G)\colon \widehat{\mathscr{V}_k}(G) \to \widehat{\mathscr{V}_k}(\mathbf{Sp}(n))$$

is the natural inclusion then any path through classifying maps given by elements of $\widehat{\mathscr{V}_k}(\mathbf{Sp}(n))$ that starts in $\widehat{\mathscr{V}_k}(G)$ lifts to a path through classifying maps given by elements in $\widehat{\mathscr{V}_k}(G)$. Summarizing we have

Lemma 1.19. *Let $\omega_t = f_t^*(\eta_{\mathbf{Sp}(n)})$ be a path of $\mathbf{Sp}(n)$ connections induced from a path of classifying maps $f_t\colon S^4 \to B\mathbf{Sp}(n)$ for $t \in [0, 1]$. Then*

(1) *If ω_0 reduces to a G-connection then ω_t reduces to a G-connection for all $t \in [0, 1]$.*

(2) *If ω_0 reduces to a G-connection and for all $t \in [0, 1]$ f_t is represented by an element of $\widehat{\mathscr{V}_k}(\mathbf{Sp}(n))$ then for all $t \in [0, 1]$ f_t is represented by an element of $\widehat{\mathscr{V}_k}(G)$.*

2. On the topology of instantons

We wish to study the global topology of the spaces $\mathscr{M}_k(G)$ defined in §1. In this section we review some results of Atiyah and Jones [5], Taubes [25, 26], as well as [7], on the topology of $\mathscr{M}_k(G)$.

We begin with a theorem of Atiyah and Jones:

Theorem 2.1 [5]. *$(i_k)_q\colon H_q(\mathscr{M}_k(\mathbf{SU}(2))) \to H_q(\mathscr{C}_k(\mathbf{SU}(2)))$ is a surjection for $q << k$.*

Atiyah and Jones then posed the following questions:

(1) Is $(i_k)_q$ actually a homology isomorphism through a range?
(2) Can the range of the surjection (isomorphism) $q = q(k)$ be explicitly determined as a function of k?
(3) Is $(i_k)_q$ a surjection on homotopy groups through a range?
(4) Is i_k a homotopy equivalence through a range?
(5) Are similar results true if $\mathbf{SU}(2) = \mathbf{Sp}(1)$ is replaced by more general compact simple G?

Question (4), which is now commonly known as the Atiyah-Jones conjecture, still remains open at this time. However, Taubes [26] has shown that a stable version of the Atiyah-Jones conjecture is true. In addition, the following theorem of Taubes [26] answers question (5) above.

Theorem 2.2 [26]. *Let G be any simple compact Lie group. Then $(i_k)_q\colon \pi_q(\mathscr{M}_k(G)) \to \pi_q(\mathscr{C}_k(G))$ and $(i_k)_q\colon H_q(\mathscr{M}_k(G)) \to H_q(\mathscr{C}_k(G))$ are surjections for $q << k$.*

While Theorem 2.2 is an extremely powerful existence theorem it does not give explicit computational information. For $G = \mathbf{SU}(2)$ [7] gave an explicit bound for the surjection dimension $q(k)$ in terms of k, showed there are nonzero classes in $H_{6k-3}(\mathscr{M}_k(\mathbf{SU}(2)))$ for infinitely many k, and, in general, characterized large families of nontrivial homology classes in $H_q(\mathscr{M}_k(\mathbf{SU}(2)))$ in terms of homology operations on iterated loop spaces. The program used in [7] consisted of considering the union, over all positive values of k, of the natural inclusions $\imath_k(\mathbf{SU}(2))\colon \mathscr{M}_k(\mathbf{SU}(2)) \to \mathscr{C}_k(\mathbf{SU}(2))$ to obtain the natural inclusion

$$\imath(\mathbf{SU}(2))\colon \mathscr{M}(\mathbf{SU}(2)) \to \mathscr{C}(\mathbf{SU}(2)) \to \Omega^4 B\mathbf{SU}(2), \tag{2.3}$$

where $\mathscr{M}(\mathbf{SU}(2)) = \coprod_{k>0}\mathscr{M}_k(\mathbf{SU}(2))$, $\mathscr{C}(\mathbf{SU}(2)) = \coprod_{k>0}\mathscr{C}_k(\mathbf{SU}(2))$, and $\imath(\mathbf{SU}(2))$ is the natural inclusion of the positively indexed path components into the total iterated loop space. As iterated loop spaces have a very rich topological structure, which has been successfully studied by many people, it is quite natural to try to use (2.3) to impose a similiar structure on instantons.

We now summarize the main technical result of [7] which, combined with Lemma 1.19 of §1, shows that $\mathscr{M}(G)$ is a well-behaved homotopy C_4 space, for all compact simple G, and, that up to homotopy, $\imath(G)$ behaves like a C_4 map.

Alternatively, one could use the method of Taubes' patching [26] to construct such structure maps. While these constructions are valid for general groups G, explicit calculations were given in [7] only when $G = \mathbf{SU}(2)$. Later we will combine these structure maps with the computations of [27] to make the homology calculations for $\mathbf{SU}(n)$ which occupy the rest of the paper.

More precisely, it was shown in [7], for $G = \mathbf{Sp}(n)$, that there are maps

$$\phi_j(\mathbf{Sp}(n)) \colon C_4(j) \times_{\Sigma_j} (\mathscr{M}(\mathbf{Sp}(n)))^j \longrightarrow \mathscr{M}(\mathbf{Sp}(n)) \tag{2.4}$$

such that (2.4) gives $\mathscr{M}(\mathbf{Sp}(n))$ a homotopy C_4 operad structure. Here C_4 is the little cubes operad [6, 18]. To be more precise about the terminology we should say that a homotopy C_n space is simply a space admitting C_n like structure maps where one requires the usual diagrams to commute only up to homotopy rather than to strictly commute.

The structure maps $\phi_j(\mathbf{Sp}(n))$ of (2.4) constructed in [7] were given explicitly in terms of the ADHM construction as follows. Think of little cubes in I^4 as big cubes in R^4 in the obvious way. Fix a homeomorphism of I^4 with $H^1 = R^4$. Then a point in $C_4(p)$ is equivalent under this fixed homeomorphism with p disjoint open cubes in H^1 (with sides parallel to the axes). Let $q_1, \ldots, q_p$ denote the centers (the points whose coordinates are given by the midpoints of each side) of the p disjoint cubes and let $e_{i,j} = d_H(q_i, q_j)$ for $i \neq j$ be the distance between the distinct centers. Thus $\hat{e} = \min_{i \neq j}(e_{i,j}) > 0$ and we let $e = \max(1, 1/\hat{e})$.

Definition 2.5. Let $b_i = (\Lambda_i, B_i) \in \mathscr{M}_k(G)$ for $1 \leq i \leq p$. Then

$$\psi_\delta(G)(c_{12\cdots p}, b_1, \ldots, b_p) = b_\delta = (\Lambda_\delta, B_\delta) \in \mathscr{B}_{pk}(G),$$

where

(1) $\Lambda_\delta = \frac{\delta}{e}(\frac{1}{\|b_1\|}\Lambda_1, \ldots, \frac{1}{\|b_p\|}(\Lambda_p))$,

(2) $B_\delta = \operatorname{diag}(q_1 I + \frac{\delta}{e\|b_1\|}B_1, \ldots, q_p I + \frac{\delta}{e\|b_p\|}B_p)$, the pk by pk block diagonal matrix with the k by k matrix $q_i I + \frac{\delta}{e\|b_i\|}B_i$ in the ith diagonal block, and

(3) the q_i's and e are uniquely determined from $c_{12\cdots p} \in C_4(p)$ in the manner described in the paragraph above.

Of course, one must check that $\psi_\delta(G)$ is well defined. But this follows exactly as in [7] for the $\mathbf{Sp}(n)$ case. Furthermore, it follows from Theorem 6.10 of [7] and Lemma 1.19 that when $\delta = 1$, $\psi_1(G)$ has its image in $\mathscr{V}_k(G)$, and thus, by applying Lemma 1.19 again, that $\psi_\delta(G)$ has its image in $\mathscr{V}_k(G)$ for all $\delta > 0$. Next, it follows by exactly the same arguments as in [7] that there exists $\delta_0 > 0$ (depending continuously on ε and on the entries in the b_i) such that for all $0 < t \leq 1$, $\psi_{t\delta_0}(G) : C_4(p) \times_{\Sigma_p} (\mathscr{M}_k(G))^p \to \mathscr{B}_{pk}(G)$ maps into $\mathscr{B}^\varepsilon_{pk}(G)$ and for appropriate choice of ε we may compose with the Taubes strong deformation retraction [25] from $\mathscr{B}^\varepsilon_{pk}(G)$ to $\mathscr{M}_{pk}(G)$ to obtain the maps $\phi_j(G)$ in the following two theorems.

Theorem 2.6. *For all compact, simple Lie groups G there are maps*

$$\phi_j(G)\colon C_4(j) \times_{\Sigma_j} (\mathcal{M}(G))^j \longrightarrow \mathcal{M}(G) \tag{2.7}$$

which make

$$\mathcal{M}(G) = \mathcal{M} = \coprod_{k\geq 0} \mathcal{M}_k(G) \tag{2.8}$$

a homotopy C_4 operad space.

Theorem 2.9. *The following diagram commutes up to homotopy*:

$$\begin{tikzcd} C_4(j) \times_{\Sigma_j} (\Omega^4 BG)^j \arrow[r, "\vartheta_j"] & \Omega^4 BG \\ C_4(j) \times_{\Sigma_j} (\mathcal{M}(G))^j \arrow[u] \arrow[r, "\phi_j(G)"] & \mathcal{M}(G) \arrow[u] \end{tikzcd} \tag{2.10}$$

Here ϑ_j is the standard C_4 operad structure corresponding to the standard four-fold loop sum structure on $\Omega^4 BG$.

Proof. Again, this follows from the proof when $G = \mathbf{Sp}(n)$ [7] and Lemma 1.19.

In §5 we summarize results of [27] on the C_4 homology operations on $H_*(\Omega^4 B\mathbf{SU}(n), \mathbf{Z}/p)$. The remainder of the paper then uses those results and the naturality of diagram (2.10) to obtain nontrivial information about $H_*(\mathcal{M}_k(\mathbf{SU}(n)), \mathbf{Z}/p)$.

3. Instanton number one

In this section we examine the natural inclusion $i_1\colon \mathcal{M}_1 \longrightarrow \mathcal{C}_1$. It was shown in [7] that, when $G = \mathbf{Sp}(1)$, i_1 is homotopic to the well-known J-homomorphism $J\colon SO(3) \longrightarrow \Omega_1^3 S^3$. This observation was a vital ingredient in the homological calculations given in [7]. For $G = \mathbf{SU}(n)$ the analysis is more complicated, but it is still possible to obtain nontrivial homological information about i_1. In later sections we will use these results to obtain interesting homological information about $H_*(\mathcal{M}_k(G))$. In the following we denote the centralizer of $\mathbf{SU}(2)$ in G by $C(G)$ or simply C.

Proposition 3.1. *Let G be a compact simple simply connected Lie group. The based moduli space $\mathcal{M}_1(G)$ fibers trivially over $\mathcal{M}_1'(G)$ with fiber $C(G)\backslash G$. Furthermore, the composition of maps*

$$C\backslash G \xrightarrow{j'} \mathcal{M}_1(G) \xrightarrow{i_1} \mathcal{C}_1(G) \xrightarrow{\theta} \Omega_1^3 G \tag{3.2}$$

is given by the map

$$J'(G)(Cg) = [x \mapsto g^{-1}\iota(x) g], \tag{3.3}$$

where j' *and* i_1 *are natural inclusions,* $\imath$ *is a fixed embedding of* $\mathbf{SU}(2)$ *into* G, *and* θ *is the Atiyah-Jones equivalence* [5].

Proof. By a result of Atiyah, Hitchin, and Singer [4] (Theorem 8.4) every self-dual G-connection on P_1 is gauge equivalent to a reducible $\mathbf{SU}(2) = \mathbf{Sp}(1)$-connection. Hence, $\mathscr{M}_1'(G)$ is homeomorphic to $\mathscr{M}_1'(\mathbf{SU}(2))$ which is homeomorphic to the five ball B^5. Fix a self-dual connection $A_0 \in \mathscr{A}_1(\mathbf{SU}(2)) \subset \mathscr{A}_1(G)$ representing a class $\bar{A}_0 \in \mathscr{M}_1'$. Consider a family of unbased gauge transformations $\{f_g\}$ that are constant on a neighborhood of some 3-sphere S_R^3 near infinity, and are parametrized by $g \in G$. This gives a continuous map $\hat{j}'\colon G \longrightarrow \mathscr{M}_1(G)$ sending $g \in G$ to the orbit $f_g\mathscr{G}^b(G)$ in $\mathscr{M}_1(G)$. This orbit can be represented by the connection $g^{-1}A_0g$. Furthermore, since A_0 lies in an $\mathbf{SU}(2)$ subgroup of G, the map $\hat{j}'$ passes to the quotient $C\backslash G$ to give a well-defined map $j'\colon C\backslash G \longrightarrow \mathscr{M}_1(G)$. Since the choice of $\bar{A}_0$ was arbitrary, this describes the fiber at every point and gives a trivial bundle since $\mathscr{M}_1'(G)$ is homeomorphic to the five ball.

To prove the last statement, we recall the description of the Atiyah-Jones map θ given by Atiyah [2]. Given a connection A we can choose an asymptotic gauge so that as $x \longrightarrow \infty$, $A \longrightarrow h^{-1}(x)dh(x)$. Near infinity this gives the continuous map $h\colon S_R^3 \longrightarrow G$. Normalizing the sphere S_R^3 we get an element of $\Omega_1^3 G$, and, recalling that for instantons $k = 1$ connections are reducible to $\mathbf{SU}(2)$-connections [4], gives a map from S^3 to itself. So

$$\theta(A) = [x \mapsto h(x)].$$

Since $k = 1$ there must be a self-dual connection, say A_0, that corresponds to the identity map $x \longrightarrow x$. Composing the maps just described gives precisely the map $_{\imath}J'(G)$ in (3.3) above. □

Remark. We have used left cosets in Proposition 3.1 because of the standard convention that principal G-bundles have a free right G-action. However, in the fibration sequences that follow, the standard notational convention is to use right cosets. To stay with standard conventions we will use right coset notation for the quotient space for the rest of the paper. That is, we will consider the homotopy equivalent quotient space and composition

$$G/C \xrightarrow{j} \mathscr{M}_1(G) \xrightarrow{i_1} \mathscr{C}_1(G) \xrightarrow{\theta} \Omega_1^3 G$$

given by

$$J(G)(gC) = [x \mapsto g\imath(x)g^{-1}] \tag{3.4}$$

which is obviously homotopy equivalent to (3.3).

For future reference we note the following two elementary facts:

Lemma 3.5. *$C(\mathbf{SU}(n))$ is homeomorphic to* $\mathbf{U}(n-2)$.

Proof. It is easy to check that elements of $C(\mathbf{SU}(n))$ are those of the form

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & A \end{bmatrix},$$

where $A \in \mathbf{U}(n-2)$ and $\det(A) = a^{-2}$. $\square$

It is not generally true that the homeomorphism given in Lemma 3.5, or any such homeomorphism, is a group homomorphism.

Lemma 3.6. *$C(\mathbf{Sp}(n))$ is homeomorphic to* $\mathbf{Z}/2 \times \mathbf{Sp}(n-1)$.

Proof. Again it is easy to check that elements of $C(\mathbf{Sp}(n))$ are precisely those of the form

$$\begin{bmatrix} a & 0 \\ 0 & aA \end{bmatrix},$$

where $a \in \mathbf{Z}/2$ is the center of $\mathbf{Sp}(1)$ and $A \in \mathbf{Sp}(n-1)$. $\square$

The homeomorphism given in Lemma 3.6 actually is a group homomorphism.

For $n > 2$, there exists a fibration

$$S^{2n-3} = \mathbf{U}(n-1)/C \longrightarrow \mathbf{SU}(n)/C \longrightarrow \mathbf{SU}(n)/\mathbf{U}(n-1) = \mathbf{CP}^{n-1},$$

where the inclusion of C into $\mathbf{U}(n-1)$ is given by

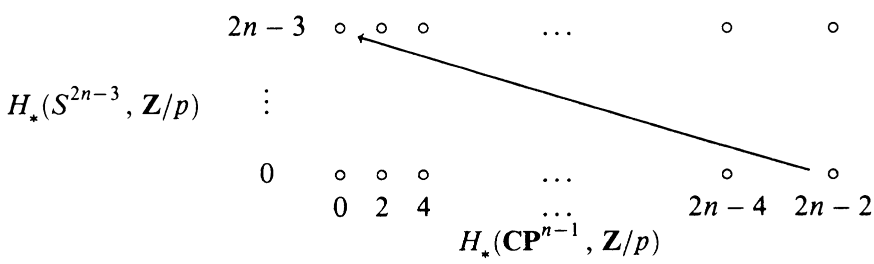
$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & A \end{bmatrix} \mapsto \begin{bmatrix} a & 0 \\ 0 & A \end{bmatrix}, \tag{3.7}$$

where $a \in S^1$, $A \in \mathbf{U}(n-2)$, $\det(A) = a^{-2}$ and the inclusion of $\mathbf{U}(n-1)$ into $\mathbf{SU}(n)$ is given by

$$B \mapsto \begin{bmatrix} \det(B)^{-1} & 0 \\ 0 & B \end{bmatrix} \tag{3.8}$$

for $B \in \mathbf{U}(n-1)$.

The mod p Serre spectral sequence for this fibration has only one possible nontrivial differential as shown below.



Thus, for $n > 2$, $H_*(\mathbf{SU}(n)/C, \mathbf{Z}/p)$ is isomorphic to one of the following groups.

(1) $H_*(\mathbf{CP}^{n-1}, \mathbf{Z}/p) \otimes H_*(S^{2n-3}, \mathbf{Z}/p)$,

(2) $H_*(\mathbf{CP}^{n-2}, \mathbf{Z}/p) \otimes H_*(S^{2n-1}, \mathbf{Z}/p)$.

In either case, $H_*(\mathbf{SU}(n)/C, \mathbf{Z}/p)$ is isomorphic to $H_*(\mathbf{CP}^{n-2}, \mathbf{Z}/p)$, as a coalgebra, through dimension $2n-4$. Since the diagram

$$\begin{array}{ccc} \mathbf{SU}(n)/C_n & \rightarrow & \mathbf{CP}^{n-1} \\ \downarrow j_n & & \downarrow \\ \mathbf{SU}(n+1)/C_{n+1} & \rightarrow & \mathbf{CP}^n \end{array}$$

commutes, where j_n is defined by

(3.9) $$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$$

for $A \in \mathbf{SU}(n)/C_n$, $(j_n)_*$ is an isomorphism on $H_i(-; \mathbf{Z}/p)$ for $i \leq 2n-4$. We record these facts in the following lemma.

Lemma 3.10. *Let* $n > 2$. *Through dimension* $2n-4$, $H_*(\mathbf{SU}(n)/C; \mathbf{Z}/p)$ *is isomorphic as a coalgebra to* $H_*(\mathbf{CP}^{n-2}; \mathbf{Z}/p)$. *Furthermore,* $(j_n)_*$ *is an isomorphism through dimension* $2n-4$.

We can completely determine the group structure of $H_*(\mathbf{SU}(n)/C; \mathbf{Z}/p)$ when $p = 2$.

Proposition 3.11. *The following are isomorphic as groups.*

(1) $H_*(\mathbf{SU}(2)/C; \mathbf{Z}/2) \cong H_*(\mathbf{RP}^3; \mathbf{Z}/2)$.

(2) $H_*(\mathbf{SU}(n)/C; \mathbf{Z}/2) \cong H_*(\mathbf{CP}^{n-1}; \mathbf{Z}/2) \otimes H_*(S^{2n-3}; \mathbf{Z}/2)$ *for* $n > 2$ *and even.*

(3) $H_*(\mathbf{SU}(n)/C; \mathbf{Z}/2) \cong H_*(\mathbf{CP}^{n-2}; \mathbf{Z}/2) \otimes H_*(S^{2n-1}; \mathbf{Z}/2)$ *for* $n > 2$ *and odd.*

Proof. $\mathbf{SU}(2)/C$ is homeomorphic to $\mathbf{RP}^3$ which implies the first statement. To prove the second statement it suffices to show that there is a nonzero element in $H_{2n-3}(\mathbf{SU}(n)/C; \mathbf{Z}/2)$. Consider the following commutative diagram

(3.12)
$$\begin{array}{ccccc} S^1 \times S^{2n-3} & & & & \\ \| & & & & \\ \mathbf{U}(n-1)/(\mathbf{Z}/2 \times \mathbf{SU}(n-2)) & \longrightarrow & \mathbf{SU}(n)/(\mathbf{Z}/2 \times \mathbf{SU}(n-2)) & \longrightarrow & \mathbf{CP}^{n-1} \\ \downarrow & & \downarrow f & & \downarrow \mathrm{id} \\ S^1 & \longrightarrow & \mathbf{RP}^{2n-1} & \longrightarrow & \mathbf{CP}^{n-1} \end{array}$$

where each row is a fibration. The inclusion of $\mathbf{Z}/2 \times \mathbf{SU}(n-2)$ in $\mathbf{U}(n-1)$ is given by

$$a \times A \mapsto \begin{bmatrix} a & 0 \\ 0 & A \end{bmatrix},$$

where $a \in \mathbf{Z}/2$ and $A \in \mathbf{SU}(n-2)$. The inclusion of $\mathbf{U}(n-1)$ into $\mathbf{SU}(n)$ is given by (3.8) and, finally, f is defined by mapping the equivalence class of $A \in \mathbf{SU}(n)$ to the first column of A. By comparing the integral Serre spectral sequences for the two rows of (3.12), we see that in the spectral sequence for the middle row there must be differentials which we indicate below.

$$\begin{array}{ll}
 & \begin{array}{r|cccccc}
2n-2 & \circ & \circ & \circ & \cdots & \circ & \circ \\
2n-3 & \circ & \circ & \circ & \cdots & \circ & \circ \\
H_*(S^1 \times S^{2n-3}; \mathbf{Z}) \quad \vdots & & & & & & \\
1 & \circ & \circ & \circ & \cdots & \circ & \circ \\
0 & \circ & \circ & \circ & \cdots & \circ & \circ \\
\hline
 & 0 & 2 & 4 & \cdots & 2n-4 & 2n-2
\end{array} \\
 & \qquad\qquad H_*(\mathbf{CP}^{n-1}, \mathbf{Z})
\end{array}$$

Here the bottom differentials are all multiplication by 2. Thus the group $H_{2n-3}(\mathbf{SU}(n)/(\mathbf{Z}/2 \times \mathbf{SU}(n-2)); \mathbf{Z})$ is either $\mathbf{Z}$ or $\mathbf{Z} \times \mathbf{Z}/2$. When n is even there exists a map $g\colon \mathbf{RP}^{2n-1} \to \mathbf{SU}(n)/(\mathbf{Z}/2 \times \mathbf{SU}(n-2))$ such that $f \circ g$ is the identity. g is given explicitly by the formula

$$g \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 & -\bar{x}_2 \\ x_2 & \bar{x}_1 \\ \vdots & \vdots \\ x_{n-1} & -\bar{x}_n \\ x_n & \bar{x}_{n-1} \end{bmatrix}.$$

So there is an element of order two in $H_{2n-3}(\mathbf{SU}(n)/(\mathbf{Z}/2 \times \mathbf{SU}(n-2)); \mathbf{Z})$, which implies that $H_{2n-3}(\mathbf{SU}(n)/(\mathbf{Z}/2 \times \mathbf{SU}(n-2)); \mathbf{Z}/2) = \mathbf{Z}/2 \times \mathbf{Z}/2$. Now consider the mod 2 Serre spectral sequence for the following fibration:

$$S^1 = C/(\mathbf{Z}/2 \times \mathbf{SU}(n-2)) \to \mathbf{SU}(n)/(\mathbf{Z}/2 \times \mathbf{SU}(n-2)) \to \mathbf{SU}(n)/C.$$

Since

$$H_{2n-3}(\mathbf{SU}(n)/(\mathbf{Z}/2 \times \mathbf{SU}(n-2)); \mathbf{Z}/2) = \mathbf{Z}/2 \times \mathbf{Z}/2$$

and

$$H_{2n-4}(\mathbf{SU}(n)/C; \mathbf{Z}/2) = \mathbf{Z}/2,$$

there must be a nonzero element in $H_{2n-3}(\mathbf{SU}(n)/C; \mathbf{Z}/2)$. This completes the proof of the second statement.

To prove the third statement, consider the fibration

$$S^1 = C_n/\mathbf{SU}(n-2) \to \mathbf{SU}(n)/\mathbf{SU}(n-2) \xrightarrow{s} \mathbf{SU}(n)/C_n,$$

where the inclusion of $\mathbf{SU}(n-2)$ into C_n is given by

$$A \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A \end{bmatrix},$$

where $A \in \mathbf{SU}(n-2)$. By analyzing the Serre spectral sequence for this fibration we see that if $H_*(\mathbf{SU}(n)/C; \mathbf{Z}/2)$ was given by 3.11(2), then s_* would be an isomorphism on $H_{2n-3}(-; \mathbf{Z}/2)$ and zero on $H_{2n-1}(-; \mathbf{Z}/2)$. If n is odd, then

$$Sq_*^2 : H_{2n-1}(\mathbf{SU}(n)/\mathbf{SU}(n-2); \mathbf{Z}/2) \to H_{2n-3}(\mathbf{SU}(n)/\mathbf{SU}(n-2); \mathbf{Z}/2)$$

is an isomorphism. So if s_* were an isomorphism on $H_{2n-3}(-; \mathbf{Z}/2)$, then s_* would be nonzero on $H_{2n-1}(-; \mathbf{Z}/2)$, which is a contradiction. This implies the third statement. □

Remark. It will follow from our work in §5 that, for $n > 2$ and $n \not\equiv 0 \mod p$, $H_*(\mathbf{SU}(n)/C; \mathbf{Z}/p)$ is given by the mod p analog of 3.11 but we will not need this fact in the sequel.

Theorem 3.13. *$\mathbf{Sp}(n)/C$ is homeomorphic to $\mathbf{RP}^{4n-1}$.*

Proof. The homeomorphism is given by sending the equivalence class of $A \in \mathbf{Sp}(n)$ to the first column of A. □

Although we have included Lemma 3.6 and Theorem 3.13, $H_*(\mathbf{Sp}(n)/C)$ is too small to support enough homology on the image of $i_1(\mathbf{Sp}(n))$ to take advantage of the operad maps given in §2. For this reason we concentrate on $G = \mathbf{SU}(n)$ for the rest of the paper.

4. The homology of $\Omega^3 G$

In this section we will summarize what is known about the homology of the triple loop spaces of S^n and $\mathbf{SU}(n)$. From the homotopy theory point of view it is worthwhile to notice that the triple loops on a Lie group are always a four-fold loop space. We first recall the definitions of the classic homology operations on iterated loop spaces (cf. [10, §1, pp. 213–219]). Given a C_{n+1} space X with structure map $\vartheta : C_{n+1} \times_{\Sigma_j} X^j \longrightarrow X$ one can use ϑ to push equivariant cells of $C_{n+1} \times X^j$ into X in mod p homology. More precisely, one has

Definition 4.1 [10]. Let X be a C_{n+1} space with $x \in H_q(X, \mathbf{Z}/p)$ and $y \in H_r(X, \mathbf{Z}/p)$. Then define

(1) For $i < n$

$$Q_{i(p-1)}(x) = \vartheta_{p*}(e_{i(p-1)} \otimes x^p) \in H_{pq+i(p-1)}(X, \mathbf{Z}/p)$$

and, for p odd,

$$Q_{i(p-1)-1}(x) = \vartheta_{p*}(e_{i(p-1)-1} \otimes x^p) \in H_{pq+i(p-1)-1}(X, \mathbf{Z}/p).$$

(2) For $p = 2$ and $s < q$

$$Q^s(x) = 0$$

while if $s \geq q$ then

$$Q^s(x) = Q_{s-q}(x).$$

(3) For $p > 2$ and $2s < q$

$$Q^s(x) = 0$$

while if $2s \geq q$ then

$$Q^s(x) = (-1)^s \nu(q) Q_{(2s-q)(p-1)}(x),$$

where $\nu(q) = (-1)^{q(q-1)(p-1)/4}(((p-1)/2)!)^q$.

(4) For $2s \leq q$

$$\beta Q^s(x) = 0$$

while if $2s > q$

$$\beta Q^s(x) = (-1)^s \nu(q) Q_{(2s-q)(p-1)-1}(x).$$

(5) For $p = 2$

$$\xi_n(x) = \vartheta_2(e_n \otimes x \otimes x) \in H_{2q+n}(X, \mathbf{Z}/p).$$

(6) For p odd

$$\xi_n = (-1)^{(n+q)/2} \nu(q) \vartheta_{p*}(e_{n(p-1)} \otimes x^p) \in H_{pq+n(p-1)}(X, \mathbf{Z}/p).$$

(7) For p odd and $n + q$ even

$$\zeta_n = (-1)^{(n+q)/2} \nu(q) \vartheta_{p*}(e_{n(p-1)-1}) \otimes x^p \in H_{pq+n(p-1)-1}(X, \mathbf{Z}/p).$$

(8)

$$\lambda_n(x, y) = (-1)^{nq+1} \psi_*(\iota \otimes x \otimes y) \in H_{n+q+r}(X, \mathbf{Z}/p).$$

Here $\psi\colon C_{n+1}(2) \times X \times X \to X$ is the Σ_2 equivariant map without the $\mathbf{Z}/2$ quotient action on the domain and $\iota \in H_n(C_{n+1}(2), \mathbf{Z}/p) \cong H_n(S^n, \mathbf{Z}/p)$ is the fundamental class.

Here:

(1) The cells $e_i \in H_i(C_n(p)/\Sigma_p, \mathbf{Z}/p(q))$ are dual to the i-dimensional generator in the image of $H^i(B\Sigma_p, \mathbf{Z}/p(q)) \to H^i(C_n(p)/\Sigma_p, \mathbf{Z}/p(q))$, see [10].

(2) 4.1(8) defines the Browder operation [9].

(3) $Q_0(x) = x^p$, the p-fold Pontrjagin product of x with itself.

(4) In general the top operation ξ_n behaves very much like a Dyer-Lashof operation ($\xi_n = Q_{n(p-1)}$ if X is a C_{n+2} operad space). Theorem 1.3 of [10] catalogs the precise differences.

(5) We are interested in C_4 spaces, thus $n = 3$ in our calculations.

These operations have many nice algebraic properties. For example, the following is true.

Theorem 4.2. *Let X be a C_{n+1} space with $x \in H_q(X, \mathbf{Z}/2)$, $y \in H_r(X, \mathbf{Z}/2)$. Then the following relations are true:*

(1) *The Cartan Relations:*

$$Q_a(x * y) = \sum_{i=0}^{a} Q_i(x) * Q_{a-i}(y). \tag{4.3}$$

(2) *The Nishida Relations:*

$$Sq_*^r Q_s x = \sum_i (r-2i, s-2r+2i+q) Q_{s-r+2i} Sq_*^i x \tag{4.4}$$

for $x \in H_q(X; \mathbf{Z}/2)$.

(3) *The Adem Relations: If $r > s$*

$$Q_r Q_s = \sum (2i-r-s, r-i-1) Q_{r+2s-2i} Q_i, \tag{4.5}$$

where (i, j) is the binomial coefficient $(i+j)!/i!j!$.

There are analogous Adem, Cartan, and Nishida relations for odd primes (see [10, pp. 213–214]).

The map $\phi(G)$ described in Theorem 2.6 can be used to construct analogous operations in $\mathscr{M}(G)$ and one can use diagram (2.10) to compare the results with the better-known structure of iterated loop operations in $\Omega^4 BG$. Furthermore, since homology is compactly supported it is routine to use homotopies such as in Theorem 6.12 of [7] along with Lemma 1.19 to verify that the diagrams used to derive (4.3), (4.4) and (4.5) (cf. [18]) all homotopy commute on the $(\mathscr{M}(G), \phi(G))$ level. Thus, we have

Corollary 4.6. *The Cartan, Nishida, and Adem relations all hold in the group $H_*(\mathscr{M}(G), \mathbf{Z}/p)$.*

These homology operations are crucial in describing the homology of iterated loop spaces. We first recall the homology of $\Omega^n S^j$, as these are the building blocks for the homology of $\Omega^3 \mathbf{SU}(n)$. Let ι be a generator of $H_{j-n}(\Omega^n S^j, \mathbf{Z}/p) \cong \mathbf{Z}/p$ for $j > n$.

Theorem 4.7 [12, 19]. *For $j > n$, $H_*(\Omega^n S^j, \mathbf{Z}/2) \cong \mathbf{Z}/2(\iota, Q_I(\iota))$, a polynomial algebra over $\mathbf{Z}/2$, under the loop sum Pontrjagin product, on generators ι and $Q_I(\iota) = Q_{i_1} Q_{i_2} \cdots Q_{i_k}(\iota)$, where $I = (i_1, \dots, i_k)$ satisfies $0 < i_1 \leq i_2 \leq \cdots \leq i_k < n$.*

Notice that $H_*(\Omega^3 S^3, \mathbf{Z}/2)$ can be described solely in terms of iterated operations Q_1 and Q_2. However, as $\Omega^3 S^3 \simeq \Omega^4 BS^3$, $Q_3 = \xi_3$ exists in $H_*(\Omega^4 BS^3, \mathbf{Z}/2)$ and in $H_*(\mathscr{M}, \mathbf{Z}/2)$ where it is highly nontrivial as was shown in [7]. To state the analog of Theorem 4.7 for odd primes we need a bit more notation. Let

$$\beta^{\varepsilon_1} Q_{s_1} \cdots \beta^{\varepsilon_k} Q_{s_k}(1) = Q_I(1) \tag{4.8}$$

be an iterated mod p operation on ι.

Definition 4.9. Let $I = (\varepsilon_1, s_1, \dots, \varepsilon_k, s_k)$ and p be an odd prime. I is n-admissible if:

(1) $0 < s_1 \leq s_2 \leq \cdots \leq s_k < n$.
(2) $\varepsilon_i = 0$ or 1.
(3) $\varepsilon_i \equiv s_i - s_{i-1} \bmod 2$ for $1 < i \leq k$.

Recall that the symmetric algebra, $S(-)$, is the product over $\mathbf{Z}/p$ of polynomial algebras on even dimensional generators and of exterior algebras on odd dimensional generators.

Theorem 4.10 [19, 10]. *Let p be an odd prime and $2j+1 > n$. As algebras, under the loop sum Pontrjagin product,*

$$H_*(\Omega^n S^{2j+1}, \mathbf{Z}/p) \cong S(\iota, Q_I(\iota)), \tag{4.11}$$

where $I = (\varepsilon_1, s_1, \dots, \varepsilon_k, s_k)$ is n-admissible and $s_k \equiv 2j+1-n \bmod 2$.

To describe $H_*(\Omega^n S^n, \mathbf{Z}/p)$ recall the identity map $S^n \to S^n$ represents the base point in the 1-component $\Omega^n_1 S^n$, and thus, a distinguished homology class $[1] \in H_0(\Omega^n_1 S^n, \mathbf{Z}/p)$. Furthermore, if x and y are homology classes carried by the k and l components of $\Omega^n S^n$, then $x * y$ and $Q_{i(p-1)}(x)$ are carried by the $k+l$ and pk components respectively.

Theorem 4.12 [12, 19]. *$H_*(\Omega^n_0 S^n, \mathbf{Z}/2) \cong \mathbf{Z}/2(Q_I(1) * [-2^k])$, a polynomial algebra over $\mathbf{Z}/2$, under the loop sum Pontrjagin product, on generators $Q_I(1) * [-2^k] = Q_{i_1} Q_{i_2} \cdots Q_{i_k}(1) * [-2^k]$, where $I = (i_1, \dots, i_k)$ satisfies $0 < i_1 \leq i_2 \leq \cdots \leq i_k < n$.*

Theorem 4.13 [19, 10]. *Let p be an odd prime. As algebras, under the loop sum Pontrjagin product,*

$$H_*(\Omega^{2n+1}_0 S^{2n+1}, \mathbf{Z}/p) \cong S(Q_I(1) * [-p^k]), \tag{4.14}$$

where $I = (\varepsilon_1, s_1, \dots, \varepsilon_k, s_k)$ is $2n+1$-admissible and $s_k \equiv 0 \bmod 2$.

Next we recall that, if $n \leq p$, then localized at p, $\mathbf{SU}(n)$ is homotopy equivalent to the product of odd spheres $S^3 \times \cdots \times S^{2n-1}$. Thus, for $n \leq p$

$$H_*(\Omega^3 \mathbf{SU}(n), \mathbf{Z}/p) \cong \bigotimes_{i=1}^{n-1} H_*(\Omega^3 S^{2i+1}, \mathbf{Z}/p). \tag{4.15}$$

To state the analog when $n > p$ we need to define

$$Q_s^a = \overbrace{Q_s \cdots Q_s}^{a \text{ times}}. \tag{4.16}$$

Theorem 4.17 [27]. *Let $n > p$. There are choices of elements*

(1) $[1] \in H_0(\Omega^3_1 \mathbf{SU}(n), \mathbf{Z}/p) \subset H_0(\Omega^3 \mathbf{SU}(n), \mathbf{Z}/p)$,
(2) $u_i \in H_{2i-2}(\Omega^3_0 \mathbf{SU}(n), \mathbf{Z}/p)$,
(3) $v_i \in H_{2pi-3}(\Omega^3_0 \mathbf{SU}(n), \mathbf{Z}/p)$,

such that $H_*(\Omega_0^3\mathbf{SU}(n), \mathbf{Z}/p)$ *is isomorphic to one of the following algebras*:

I. *Let* $p = 2$.

$$\begin{aligned}&\mathbf{Z}/2[Q_2^a(Q_2[1]*[-2]) \mid a \geq 0]\\ &\quad \otimes \mathbf{Z}/2\left[Q_2^a(u_i) \mid a \geq 0,\ 1 < i \leq \left[\frac{n-1}{2}\right],\ i \not\equiv 0 \bmod 2\right]\\ &\quad \otimes \mathbf{Z}/2\left[Q_1^aQ_2^b(u_i) \mid a, b \geq 0,\ \left[\frac{n-1}{2}\right] < i \leq n-1,\ i \not\equiv 0 \bmod 2\right]\\ &\quad \otimes \mathbf{Z}/2\left[Q_1^aQ_3^b(v_i) \mid a, b \geq 0,\ \left[\frac{n-1}{2}\right] < i \leq n-1,\ i \equiv 0 \bmod 2\right].\end{aligned}$$

II. *Let* $p > 2$.

$$\begin{aligned}&\mathbf{Z}/p[Q_{2(p-1)}^a(Q_{2(p-1)}[1]*[-p]) \mid a \geq 0]\\ &\quad \otimes \mathbf{Z}/p[Q_{2(p-1)}^a(u_i) \mid a \geq 0,\ 1 < i \leq n-1,\ i \not\equiv 0 \bmod p]\\ &\quad \otimes E\left[Q_{p-1}^a\beta Q_{2(p-1)}^b(u_i) \mid a \geq 0,\ b > 0,\ \left[\frac{n-1}{p}\right] < i \leq n-1,\ i \not\equiv 0 \bmod p\right]\\ &\quad \otimes \mathbf{Z}/p\left[\beta Q_{p-1}^a\beta Q_{2(p-1)}^b(u_i) \mid a, b > 0,\ \left[\frac{n-1}{p}\right] < i \leq n-1,\ i \not\equiv 0 \bmod p\right]\\ &\quad \otimes E\left[Q_{p-1}^aQ_{3(p-1)}^b(v_i) \mid a, b \geq 0,\ \left[\frac{n-1}{p}\right] < i \leq n-1,\ i \equiv 0 \bmod p\right]\\ &\quad \otimes \mathbf{Z}/p\left[\beta Q_{p-1}^aQ_{3(p-1)}^b(v_i) \mid a > 0,\ b \geq 0,\ \left[\frac{n-1}{p}\right] < i \leq n-1,\ i \equiv 0 \bmod p\right].\end{aligned}$$

It is useful to know how the following maps act on homology

$$\Omega^3\mathbf{SU}(n) \xrightarrow{\Omega^3 i} \Omega^3\mathbf{SU}(n+1) \xrightarrow{\Omega^3\pi} \Omega^3 S^{2n+1}, \tag{4.18}$$

where i is the inclusion

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$$

and π maps $B \in \mathbf{SU}(n+1)$ to the last column of B. The elements [1], u_k, and v_k can be chosen so that $(\Omega^3 i)_*([1]) = [1]$, $(\Omega^3 i)_*(u_k) = u_k$, and $(\Omega^3 i)_*(v_k) = v_k$, provided u_k and v_k exist in $H_*(\Omega^3\mathbf{SU}(n+1); \mathbf{Z}/p)$. The following theorem explicitly gives the kernel of $(\Omega^3 i)_*$ and the image of $(\Omega^3\pi)_*$.

Theorem 4.19 [27]. *The isomorphisms in Theorem* 4.17 *can be chosen so that*:

(1) *For* $n \not\equiv 0 \bmod p$, $(\Omega^3\pi)_*$ *is a surjection sending the element* $u_n \in H_{2n-2}(\Omega^3\mathbf{SU}(n+1), \mathbf{Z}/p)$ *to* $\iota \in H_{2n-2}(\Omega^3 S^{2n+1}, \mathbf{Z}/p)$ *and* $(\Omega^3 i)_*$ *is an injection.*

(2) *For* $n \equiv 0 \bmod p$, *the image of* $(\Omega^3\pi)_*$ *is generated as an algebra by elements of the form*

$$(\Omega^3\pi)_*(Q_1^aQ_3^b(v_n)) = Q_1^aQ_2^{b+1}(\iota)$$

when $p = 2$, *while for* $p > 2$

$$(\Omega^3\pi)_*(Q^a_{p-1}Q^b_{3(p-1)}(v_n)) = Q^a_{p-1}\beta Q^{b+1}_{2(p-1)}(\iota)$$

and

$$(\Omega^3\pi)_*(\beta Q^{a+1}_{p-1}Q^b_{3(p-1)}(v_n)) = \beta Q^{a+1}_{p-1}\beta Q^{b+1}_{2(p-1)}(\iota).$$

(3) *For* $n \equiv 0 \bmod p$, *but* $n \not\equiv 0 \bmod p^2$ *the kernel of* $(\Omega^3 i)_*$ *is generated as an algebra by elements of the form* $Q_1^{a+1}Q_2^b(u_{n/2})$ *for* $p = 2$ *and by elements of the form* $Q^a_{p-1}Q^b_{2(p-1)}(u_{n/p})$ *and* $\beta Q^{a+1}_{p-1}Q^{b+1}_{2(p-1)}(u_{n/p})$ *for* $p > 2$.

(4) *For* $n \equiv 0 \bmod p^2$ *the kernel of* $(\Omega^3 i)_*$ *is generated as an algebra by elements of the form*: $Q_1^a Q_3^b(v_{n/2})$ *for* $p = 2$ *and by elements of the form* $Q^a_{p-1}\beta Q^{b+1}_{3(p-1)}(v_{n/p})$ *and* $\beta Q^{a+1}_{p-1}\beta Q^b_{3(p-1)}(v_{n/p})$ *for* $p > 2$.

Here the relations described in (2)–(4) *above hold for all* $a, b \geq 0$.

Remark. The element $u_{n-1} \in H_{2n-4}(\Omega_0^3\mathbf{SU}(n); \mathbf{Z}/p)$, $(n-1) \not\equiv 0 \bmod p$, can be chosen to be any element whose image under $(\Omega^3 p)_*$ is the generator of $H_{2n-4}(\Omega^3 S^{2n-1}; \mathbf{Z}/p)$. In §5 we will make an explicit choice for the u_i's.

5. Homological calculations for $k = 1$

In this section we compute the information we will require in the following sections about the mod p homology of the natural inclusion $i_1 : \mathscr{M}_1(\mathbf{SU}(n)) \longrightarrow \mathscr{C}_1(\mathbf{SU}(n))$.

We begin with $p = 2$ and $n = 3$.

Proposition 5.1. *The map* $J(\mathbf{SU}(3)) \colon \mathbf{SU}(3)/C \longrightarrow \Omega_1^3\mathbf{SU}(3)$ *is nontrivial on* $H_5(-, \mathbf{Z}/2)$.

Proof. There is a commutative diagram

$$\begin{array}{ccccc} \Sigma(\mathbf{CP}^2) & \xrightarrow{g} & \mathbf{SU}(3) & \xrightarrow{J(\mathbf{SU}(3))} & \Omega^3\mathbf{SU}(3) \\ c\downarrow & & & & \downarrow \Omega^3\pi \\ S^5 & & \xrightarrow{H} & & \Omega^3 S^5 \end{array} \tag{5.2}$$

where g is the inclusion of the bottom two cells and c is the collapse map given by pinching the bottom cell to the base point.

The following sequence of lemmas will show the adjoint of H is homotopic to the suspension of the Hopf map, which we denote by $\Sigma(h)$. Since $\Sigma(h)$ generates $\pi_8(S^5)$ and there is a spherical class in $H_5(\Omega^3 S^5, \mathbf{Z})$ whose mod 2 reduction is nonzero, $\Sigma(h)$ must be nonzero on $H_5(-, \mathbf{Z}/2)$. Hence, $J(\mathbf{SU}(3))$ must be nonzero on $H_5(-, \mathbf{Z}/2)$. □

Remark. Proposition 5.1 was the starting point of our homological calculations needed to bring the results of [7] and [27] together. We would like to thank

Fred Cohen for providing us, at the beginning of our study, with an alternative proof to the one presented here.

Now define $F_{\alpha,x}\colon \mathbf{C}^n \to \mathbf{C}^n$ for $\alpha \in S^1$ and $x \in S^{2n-1} \subset \mathbf{C}^n$ by the formula

$$F_{\alpha,x}(z) = z - (1-\alpha)\langle z, x\rangle x, \tag{5.3}$$

where $\langle -, -\rangle$ is the usual inner product on $\mathbf{C}^n$. It is routine to verify

Lemma 5.4. $F_{\alpha,x}$ *satisfies the following properties:*

(1) $F_{\alpha,x}$ *fixes the subspace perpendicular to* x *and multiplies* x *by* α.
(2) $F_{\alpha,x}$ *is linear.*
(3) $\langle F_{\alpha,x}(z_1), F_{\alpha,x}(z_2)\rangle = \langle z_1, z_2\rangle$.
(4) $\det(F_{\alpha,x}) = \alpha$.
(5) *The inverse of* $F_{\alpha,x}$ *is* $F_{\bar\alpha,x}$.
(6) *For* $y \in U(n)$, $y \cdot F_{\alpha,x} = F_{\alpha,y(x)} \cdot y$.
(7) $F_{\alpha,x} = F_{\alpha,\beta x}$ *for all* $\beta \in S^1$.
(8) $F_{1,x} = \mathrm{id}$.

Now define

$$g\colon S^1 \wedge \mathbf{CP}^{n-1} \longrightarrow \mathbf{SU}(n) \tag{5.5}$$

by $g(\alpha, x) = F_{\bar\alpha,x_0} \cdot F_{\alpha,x}$, where $x_0 = (1, 0, \ldots, 0)$ is the base point of $\mathbf{CP}^{n-1}$. It is easy to check, using the properties of $F_{\alpha,x}$, that g is well defined. Recall, from remark (3.4), that $J(\mathbf{SU}(n))\colon \mathbf{SU}(n) \to \Omega^3\mathbf{SU}(n)$ is given by

$$J(\mathbf{SU}(n))(x)(y) = x \cdot y \cdot x^{-1}, \tag{5.6}$$

where $x \in \mathbf{SU}(n)$ and $y \in S^3 = \mathbf{SU}(2) \subset \mathbf{SU}(n)$. It is convenient to use the identification of S^3 and $\mathbf{SU}(2)$ given by

$$y = (y_1, y_2) \leftrightarrow \begin{bmatrix} y_2 & -y_1 \\ \bar y_1 & \bar y_2 \end{bmatrix} \tag{5.7}$$

in what follows.

Diagram (5.2) is actually the lower half of the following commutative diagram:

$$\begin{array}{ccccc}
S^1 \wedge \mathbf{CP}^{n-1} & \xrightarrow{g} & \mathbf{SU}(n) & \xrightarrow{J(\mathbf{SU}(n))} & \Omega^3\mathbf{SU}(n) \\
\downarrow i & & & & \downarrow \Omega^3 i \\
S^1 \wedge \mathbf{CP}^{n} & \xrightarrow{g} & \mathbf{SU}(n+1) & \xrightarrow{J(\mathbf{SU}(n+1))} & \Omega^3\mathbf{SU}(n+1) \\
\downarrow c & & & & \downarrow \Omega^3 \pi \\
S^1 \wedge (\mathbf{CP}^{n}/\mathbf{CP}^{n-1}) & & \xrightarrow{H} & & \Omega^3 S^{2n+1}
\end{array} \tag{5.8}$$

where π is the evaluation map on $y_0 = (0, \dots, 0, 1) \in \mathbf{C}^n$. Lemma 5.4 can be used to show that the adjoint of H, which we denote by K, is explicitly given by

$$K(\alpha \wedge x \wedge y) = F_{\bar{\alpha}, x_0} \cdot F_{\alpha, x} \cdot y \cdot F_{\bar{\alpha}, x} \cdot F_{\alpha, x_0}(y_0)$$
$$= F_{\bar{\alpha}, x_0} \cdot F_{\alpha, x} \cdot F_{\bar{\alpha}, y(x)}(y_0) = F_{\bar{\alpha}, x_0}(v),$$

where

$$v = (y_0 - (1 - \bar{\alpha})\langle y_0, x\rangle(y(x) - x) - (1 - \alpha)(1 - \bar{\alpha})\langle y_0, x\rangle(1 - \langle y(x), x\rangle)x).$$

Furthermore, we have the following explicit formula for the suspension of the Hopf map $\Sigma(h): S^8 \to S^5$.

Lemma 5.9. $\Sigma(h): S^1 \wedge (\mathbf{CP}^2/\mathbf{CP}^1) \wedge S^3 \to S^5$ *is given by*

$$\Sigma(h)(\alpha \wedge (x_1, x_2, x_3) \wedge (y_1, y_2)) = F_{\bar{\alpha}, x_0}(\gamma z_1, z_2),$$

where

$$\gamma = -2\bar{x}_3\sqrt{(1 - \Re(\alpha))(1 - \Re(y_2))},$$

$$z_1 = \left[x_3\bar{x}_3 \begin{bmatrix} y_2 & -y_1 \\ \bar{y}_1 & \bar{y}_2 \end{bmatrix} + (1 - x_3\bar{x}_3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

and

$$z_2 = 1 - (1 - \alpha)2(1 - \Re(y_2))(1 - x_3\bar{x}_3)x_3\bar{x}_3.$$

Proof. This is a direct but long computation and is given in §7. □

Finally, the next lemma finishes the proof of Proposition 5.1.

Lemma 5.10. *K and $\Sigma(h)$ are homotopic.*

Proof. A direct, but lengthy, calculation shows that $\Sigma(h)(v)$ is never equal to $-K(v)$ and thus the straight line homotopy

$$\frac{t\Sigma(h) + (1 - t)K}{\| t\Sigma(h) + (1 - t)K \|}$$

is well defined. Details are given in §7. □

We now consider general values for n and p. Recall that one can choose $u_{n-1} \in H_{2n-4}(\Omega_0^3\mathbf{SU}(n); \mathbf{Z}/p)$ to be any element such that the image of u_{n-1} under $(\Omega^3\pi)_*$ is the generator of $H_{2n-4}(\Omega^3 S^{2n-1}; \mathbf{Z}/p)$.

Proposition 5.11. *Let p be a prime and $k \not\equiv 0 \bmod p$, $1 < k < n$. Then $u_k * [1] \in H_{2k-2}(\Omega_1^3\mathbf{SU}(n); \mathbf{Z}/p)$ is in the image of*

$$J(\mathbf{SU}(n)): \mathbf{SU}(n)/C \longrightarrow \Omega_1^3\mathbf{SU}(n).$$

Proof. We will show that there is a homotopy commutative diagram

$$(5.12)\qquad \begin{array}{ccccc} \mathbf{CP}^{k-1} & \xrightarrow{f} & \mathbf{SU}(k+1)/C & \xrightarrow{J(\mathbf{SU}(k+1))} & \Omega_1^3\mathbf{SU}(k+1) \\ c\downarrow & & & & \downarrow \Omega^3\pi \\ S^{2k-2} & & \xrightarrow{K} & & \Omega^3 S^{2k+1} \end{array}$$

where f is the inclusion onto the $2k-2$ skeleton of $\mathbf{SU}(k+1)/C$, c is the collapse map given by pinching $\mathbf{CP}^{k-2} \subset \mathbf{CP}^{k-1}$ to the base point and the adjoint of K is a map of degree k. Thus, if $k \not\equiv 0 \bmod p$, then the generator $\iota \in H_{2k-2}(\Omega^3 S^{2k+1}, \mathbf{Z}/p)$ is in the image of $(f \circ J(\mathbf{SU}(k+1)) \circ \Omega^3\pi)_*$. So we can choose $u_k \in H_{2k-2}(\Omega_0^3\mathbf{SU}(k+1); \mathbf{Z}/p)$ so that $u_k * [1]$ is the image of the generator of $H_{2k-2}(\mathbf{CP}^{n-1}; \mathbf{Z}/p)$ under the map $(f \circ J(\mathbf{SU}(k+1)))_*$. Since the diagram

$$(5.13)\qquad \begin{array}{ccc} \mathbf{SU}(k+1)/C & \xrightarrow{J(\mathbf{SU}(k+1))} & \Omega_1^3\mathbf{SU}(k+1) \\ \downarrow & & \downarrow \Omega^3 i \\ \mathbf{SU}(n)/C & \xrightarrow{J(\mathbf{SU}(n))} & \Omega_1^3\mathbf{SU}(n) \end{array}$$

commutes, $u_k * [1]$ is in the image of $J(\mathbf{SU}(n))_*$ for $k \not\equiv 0 \bmod p$, $1 < k < n$. □

The following sequences of lemmas will show the existence of diagram (5.12). We begin by defining $g\colon S^{2k-1} \subset \mathbf{C}^k \to \mathbf{C}^{k+1}$ by the formula

$$g\begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} -\bar{x}_2 x_1^2 \\ \bar{x}_1 x_1^2 - \bar{x}_3 x_2^2 \\ \vdots \\ \bar{x}_{k-2} x_{k-2}^2 - \bar{x}_k x_{k-1}^2 \\ \bar{x}_{k-1} x_{k-1}^2 \\ \bar{x}_k x_k^2 \end{bmatrix}.$$

It is easy to verify

Lemma 5.14. g *satisfies the following*:

(1) $g(\alpha x) = \alpha g(x)$ *for* $\alpha \in S^1 \subset \mathbf{C}$.
(2) $g(x) \neq 0$.
(3) $g(x)$ *is perpendicular to* x.

Choose $y_1, \dots, y_{k-1} \in S^{2k+1}$ so that the matrix

$$\left[x, \frac{g(x)}{\|g(x)\|}, y_1, \dots, y_{k-1}\right]$$

is an element of $\mathbf{SU}(k+1)$. Define $f\colon \mathbf{CP}^{k-1} \longrightarrow \mathbf{SU}(k+1)/C$ by sending $x \in \mathbf{CP}^{k-1}$ to the matrix above. Lemma 5.14 guarantees that f is well defined. Recall, from remark (3.4), that $J(\mathbf{SU}(k+1))\colon \mathbf{SU}(k+1) \to \Omega^3\mathbf{SU}(k+1)$ is given by

$$J(\mathbf{SU}(k+1))(x)(y) = x \cdot y \cdot x^{-1},$$

where $x \in \mathbf{SU}(k+1)$ and $y \in S^3 = \mathbf{SU}(2) \subset \mathbf{SU}(k+1)$. It is convenient to use the identification of S^3 and $\mathbf{SU}(2)$ given by

$$y = (a, b) \leftrightarrow \begin{bmatrix} \bar{a} & b \\ -\bar{b} & a \end{bmatrix}$$

in what follows. The composite $(\Omega^3\pi \circ J(\mathbf{SU}(k+1)) \circ f)$ is explicitly given by the formula

$$(\Omega^3\pi \circ J(\mathbf{SU}(k+1)) \circ f)(x)(y)$$
$$= \begin{bmatrix} \alpha b x_1 \bar{x}_k - \beta(a-1)x_1^2\bar{x}_2\bar{x}_k \\ \alpha b x_2\bar{x}_k - \beta(a-1)x_2^2\bar{x}_3\bar{x}_k + \gamma_2(a-1)x_1\bar{x}_k \\ \vdots \\ \alpha b x_i\bar{x}_k - \beta(a-1)x_i^2\bar{x}_{i+1}\bar{x}_k + \gamma_i(a-1)x_{i-1}\bar{x}_k \\ \vdots \\ \alpha b x_{k-1}\bar{x}_k - \beta(a-1)x_{k-1}^2\bar{x}_k^2 + \gamma_{k-1}(a-1)x_{k-2}\bar{x}_k \\ \alpha b x_k\bar{x}_k + \gamma_k(a-1)x_{k-1}\bar{x}_k \\ 1 + (a-1)\|x_k\|^6/\|g(x)\|^2 \end{bmatrix}$$

where

(1) $\alpha = \|x_k\|^2/\|g(x)\|$,
(2) $\beta = \|x_k\|^2/\|g(x)\|^2$,
(3) $\gamma_i = \|x_{i-1}\|\|x_k\|^2/\|g(x)\|$, and
(4) we note that α, β and γ_i are all nonnegative constants.

Lemma 5.15. *A choice for the composite $K \circ c$ is given by*

$$(K \circ c)(x)(y) = \begin{bmatrix} \lambda x_{k-1}^k x_k^k/\|x_{k-1}\|^{k-1}\|x_k\|^{k-1} \\ -\lambda x_1\bar{x}_k \\ \vdots \\ -\lambda x_{k-2}\bar{x}_k \\ b\|x_k\|^2 \\ 1 + (a-1)\|x_k\|^2 \end{bmatrix},$$

where $\lambda = \sqrt{2(1 - \Re(a))} \geq 0$.

Proof. See §7. □

The next lemma finishes the proof of Proposition 5.11.

Lemma 5.16. *$\Omega^3\pi \circ J(\mathbf{SU}(k+1)) \circ f$ and $K \circ c$ are homotopic.*

Proof. Direct calculation shows that $\Omega^3\pi \circ J(\mathbf{SU}(k+1)) \circ f$ is never equal to $-K \circ c$ and thus the two maps are homotopic via the standard straight line homotopy. Again, these details are given in §7. □

We can now give an alternate proof of part of 8.9 and 8.10 of [7].

Corollary 5.17. *Both*

$$Q_1([1]) * [-1] \in H_1(\Omega_1^3\mathbf{SU}(2); \mathbf{Z}/2)$$

and

$$Q_2([1]) * [-1] \in H_2(\Omega_1^3\mathbf{SU}(2); \mathbf{Z}/2)$$

are in the image of $J(\mathbf{SU}(2))_*$.

Remark. [7, 8.9] shows that

$$Q_3([1]) * [-1] \in H_3(\Omega_1^3\mathbf{SU}(2); \mathbf{Z}/2)$$

is in the image of $J(\mathbf{SU}(2))_*$.

Proof. By Theorem 4.17, $H_4(\Omega_1^3\mathbf{SU}(4); \mathbf{Z}/2)$ is generated by $u_3 * [1]$ and $(Q_2([1])*[-1])^2$ and the image of these two elements generate $H_4(\Omega_1^3\mathbf{SU}; \mathbf{Z}/2)$. Since there is a nonprimitive element in $H_4(\Omega_1^3\mathbf{SU}; \mathbf{Z}/2)$, the reduced diagonal of $u_3 * [1]$ must be $(Q_2([1]) * [-1]) \otimes (Q_2([1]) * [-1])$. By Proposition 5.11, $u_3 * [1]$ is in the image of $J(\mathbf{SU}(4))_*$, so that $Q_2([1]) * [-1]$ must also be in the image of $J(\mathbf{SU}(4))_*$.

Consider the following homotopy commutative diagram:

$$\begin{array}{ccccc} \Omega S^1 & \to & C_2 & \to & C_3 \\ \downarrow & & \downarrow & & \downarrow \\ \Omega S^5 & \to & \mathbf{SU}(2) & \to & \mathbf{SU}(3) \\ \downarrow & & \downarrow & & \downarrow \\ F & \to & \mathbf{SU}(2)/C_2 & \xrightarrow{j_2} & \mathbf{SU}(3)/C_3 \end{array} \tag{5.18}$$

in which both the rows and columns are fibrations. It is easy to see that $H_1(F; \mathbf{Z}/2) = \mathbf{Z}/2$ and by Proposition 3.11:

$$H_1(\mathbf{SU}(3)/C_3; \mathbf{Z}/2) = 0,$$
$$H_2(\mathbf{SU}(3)/C_3; \mathbf{Z}/2) = \mathbf{Z}/2,$$
$$H_2(\mathbf{SU}(2)/C_2; \mathbf{Z}/2) = \mathbf{Z}/2.$$

Thus in the Serre spectral sequence for (5.18), the class in $H_2(\mathbf{SU}(3)/C_3; \mathbf{Z}/2)$ cannot support a differential. This implies that $(j_2)_*$ is an isomorphism on $H_2(-; \mathbf{Z}/2)$. By Lemma 3.10, the map $(j_n)_*$ is an isomorphism on $H_i(-; \mathbf{Z}/2)$

for $i \leq 2n-4$, thus the composite map $\mathbf{SU}(2)/C_2 \to \mathbf{SU}(4)/C_4$ is an isomorphism on $H_2(-;\mathbf{Z}/2)$. Since the diagram

$$\begin{array}{ccc} \mathbf{SU}(2)/C_2 & \xrightarrow{J(\mathbf{SU}(2))} & \Omega_1^3\mathbf{SU}(2) \\ \downarrow & & \downarrow \\ \mathbf{SU}(4)/C_4 & \xrightarrow{J(\mathbf{SU}(4))} & \Omega_1^3\mathbf{SU}(4) \end{array}$$

commutes, $Q_2([1]) * [-1]$ is in the image of $J(\mathbf{SU}(2))_*$.

Since $Sq_*^1(Q_2([1])*[-1]) = Q_1([1])*[-1]$, the image of $J(\mathbf{SU}(2))_*$ contains $Q_1([1]) * [-1]$.

Remark. Since the map $\mathbf{SU}(2)/C_2 \to \mathbf{SU}(n)/C_n$ induces an isomorphism on $H_2(-;\mathbf{Z}/2)$, $J(\mathbf{SU}(n))_*$ is a monomorphism on $H_2(-;\mathbf{Z}/2)$.

Corollary 5.19. *For $1 < k < n$ and $k \equiv 0 \bmod p$, there is an element $x_k \in H_{2k-2}(\Omega_1^3\mathbf{SU}(n);\mathbf{Z}/p)$, which is in the image of $H_{2k-2}(\Omega_1^3\mathbf{SU}(k-1);\mathbf{Z}/p)$, such that $u_2 * u_{k-1} * [1] + x_k$ is in the image of $J(\mathbf{SU}(n))$. In the case $p = 2$, u_2 is understood to be $Q_2[1] * [-2]$.*

Proof. For $1 \leq j \leq n-2$, denote the generator of $H_{2j}(\mathbf{SU}(n)/C;\mathbf{Z}/p)$ by w_j. By Lemma 3.10 the reduced diagonal of w_j is given by

$$\bar{\Delta} w_j = \sum_{i=1}^{j-1} w_i \otimes w_{j-i}.$$

By Proposition 5.11 and Corollary 5.17,

$$J(\mathbf{SU}(n))_* w_{k-2} = u_{k-1} * [1] \quad \text{and} \quad J(\mathbf{SU}(n))_* w_1 = u_2 * [1].$$

Thus $(u_2*[1])\otimes(u_{k-1}*[1])$ is a term in the reduced diagonal of $J(\mathbf{SU}(n))_* w_{k-1}$. This implies that $J(\mathbf{SU}(n))_* w_{k-1}$ is nonzero and contains the term $u_2 * u_{k-1} * [1]$. Since all other generators in $H_{2k-2}(\Omega_1^3\mathbf{SU}(n);\mathbf{Z}/p)$ are in the image of $H_{2k-2}(\Omega_1^3\mathbf{SU}(k-1);\mathbf{Z}/p)$, this proves the corollary. □

Corollary 5.20. *For n even, there is an element $x_n \in H_{2n-2}(\Omega_1^3\mathbf{SU}(n);\mathbf{Z}/2)$, which is in the image of $H_{2n-2}(\Omega_1^3\mathbf{SU}(n-1);\mathbf{Z}/2)$, such that*

$$u_2 * u_{n-1} * [1] + x_n$$

is in the image of $J(\mathbf{SU}(n))_*$.

Proof. The proof is similar to that of Corollary 5.19.

Remark. The results of this section, together with the description of $H_*(\mathbf{SU}(n)/C;\mathbf{Z}/p)$ given in Lemma 3.11, imply that $J(\mathbf{SU}(n))_*$ is a monomorphism through dimension $2n-4$ for all n and p. Furthermore, when $p = 2$ and n is even, $J(\mathbf{SU}(n))_*$ is also a monomorphism in dimension $2n-2$.

6. SU(n) INSTANTONS

In this section we study the case $G = \mathbf{SU}(n)$. As G is fixed in this section we will suppress the $\mathbf{SU}(n)$ in the notation. We are now ready to use the results of the previous sections to construct many new nontrivial classes in $H_*(\mathcal{M}_k, \mathbf{Z}/p)$ for all primes p. Theorem 2.9 implies the following diagrams commute:

$$(6.1)\quad \begin{array}{ccc} H_s(\Omega^4_k B\mathbf{SU}(n), \mathbf{Z}/p) \otimes H_t(\Omega^4_l B\mathbf{SU}(n), \mathbf{Z}/p) & \xrightarrow{*} & H_{s+t}(\Omega^4_{k+l} B\mathbf{SU}(n), \mathbf{Z}/p) \\ i_{k_*} \otimes i_{l_*} \uparrow & & \uparrow i_{k+l_*} \\ H_s(\mathcal{M}_k, \mathbf{Z}/p) \otimes H_t(\mathcal{M}_l, \mathbf{Z}/p) & \xrightarrow{*} & H_{s+t}(\mathcal{M}_{k+l}, \mathbf{Z}/p) \end{array}$$

$$(6.2)\quad \begin{array}{ccc} H_r(\Omega^4_k B\mathbf{SU}(n), \mathbf{Z}/p) & \xrightarrow{Q_s} & H_{2s(p-1)+r}(\Omega^4_{pk} B\mathbf{SU}(n), \mathbf{Z}/p) \\ i_{k_*} \uparrow & & \uparrow i_{pk_*} \\ H_r(\mathcal{M}_k, \mathbf{Z}/p) & \xrightarrow{Q_s} & H_{2s(p-1)+r}(\mathcal{M}_{pk}, \mathbf{Z}/p) \end{array}$$

and the Browder operation

$$(6.3)\quad \begin{array}{ccc} H_r(\Omega^4_k B\mathbf{SU}(n), \mathbf{Z}/p) \otimes H_t(\Omega^4_l B\mathbf{SU}(n), \mathbf{Z}/p) & \xrightarrow{\lambda_3} & H_{r+t+3}(\Omega^4_{k+l} B\mathbf{SU}(n), \mathbf{Z}/p) \\ i_{k_*} \otimes i_{l_*} \uparrow & & \uparrow i_{k+l_*} \\ H_r(\mathcal{M}_k, \mathbf{Z}/p) \otimes H_t(\mathcal{M}_l, \mathbf{Z}/p) & \xrightarrow{\lambda_3} & H_{r+t+3}(\mathcal{M}_{k+l}, \mathbf{Z}/p) \end{array}$$

First let $n = 3$ and $p = 2$. Starting with generators $z_0 = [1] \in H_0(\mathcal{M}_1, \mathbf{Z}/2)$ and $z_5 \in H_5(\mathcal{M}_1, \mathbf{Z}/2)$ one computes iterated Dyer-Lashof operations on these generators and loop sums of such elements, and by using the commutativity of diagrams (6.1) and (6.2) and the known loop sum Dyer-Lashof structure of $H_*(\Omega^4 B\mathbf{SU}(3), \mathbf{Z}/2)$ one recovers the $\mathbf{Z}/2$ version of the generalized Atiyah-Jones theorem for $\mathbf{SU}(3)$, first shown by Taubes, that $H_q(\mathcal{M}_k) \longrightarrow H_q(\mathcal{C}_k)$ is a surjection for $k >> q$. For example we immediately find $Q_2(1)$ is in image $i_*: H_*(\mathcal{M}_2, \mathbf{Z}/2) \longrightarrow H_*(\mathcal{C}_2, \mathbf{Z}/2)$. We note in passing that $Q_2 Q_2 \cdots Q_2(1) \in H_{2^{l+1}-2}(\mathcal{C}_{2^l}, \mathbf{Z}/2)$ is in image (i_*) and thus we obtain a nonzero class in $H_q(\mathcal{M}_k, \mathbf{Z}/2)$ for q approximately $\frac{1}{4}\dim(\mathcal{M}_k)$. In addition,

$$Q_{i_1} Q_{i_2} \cdots Q_{i_l}(z_5) \in H_d(\mathcal{C}_{2^l}, \mathbf{Z}/2)$$

is in image (i_*). Furthermore, using these elements it is possible to obtain a better bound for the range $q = q(k)$ through which i_* is a surjection. However, computing iterated Dyer-Lashof operations on $u_k * [1]$ and on $u_2 * u_k * [1] + x_{k+1}$ in $H_*(\Omega^3 \mathbf{SU}(n); \mathbf{Z}/p)$ is delicate in general because of the presence of possibly nontrivial Browder operations there. Our main computational results are as follows:

Theorem 6.4. *$H_*(\mathcal{M}_k(\mathbf{SU}(n)); \mathbf{Z}/2)$ contains elements of the form*

$$(6.5)\qquad z = z(I_1, \ldots, I_m, j_1, \ldots, j_m) = Q_{I_1}(z_{j_1}) * \cdots * Q_{I_m}(z_{j_m}) * [k - l]$$

for all sequences $(I_1, \ldots, I_m, j_1, \ldots, j_m)$ such that:

(1) $l = \sum_{i=1}^m 2^{l(I_i)} \leq k$.

(2) *Each* j_i *runs through the following possibilities*:

$$j_i = \begin{cases} 2k, & \text{for } 0 \le k \le n-2, \\ \text{also } 5, & \text{if } n = 3 \text{ or } 4, \\ \text{also } 2n-2, & \text{if } n \text{ is even.} \end{cases}$$

(3) *Each* Q_{I_i} *is of the form*

$$Q_{I_i} = \begin{cases} Q_2^a, & \text{for } j_i \le n-3 \text{ and } j_i \text{ even}, \\ Q_1^c Q_2^b Q_3^a, & \text{for } a \cdot b \cdot c = 0,\ n-3 \le j_i \text{ and } j_i \text{ even}, \\ Q_1^b Q_3^a, & \text{for } a, b \ge 0 \text{ and } j_i = 5. \end{cases}$$

(4) *The image of* z *in* $H_*(\Omega^3\mathbf{SU}(n); \mathbf{Z}/2)$ *is given by replacing* z_0 *by* $[1]$, z_2 *by* $Q_2[1]*[1]$, z_{2k} *by* $u_{k+1}*[1]$ *when* k *is even*, z_{2k} *by* $u_2 * u_k * [1] + x_{k+1}$ *when* k *is odd, and* z_5 *by* v_2.

For odd primes,

Theorem 6.6. $H_*(\mathcal{M}_k(\mathbf{SU}(n)); \mathbf{Z}/p)$ *contains elements of the form*

$$z = z(I_1, \ldots, I_m, j_1, \ldots, j_m) = Q_{I_1}(z_{j_1}) * \cdots * Q_{I_m}(z_{j_m}) * [k-l] \tag{6.7}$$

for all sequences $(I_1, \ldots, I_m, j_1, \ldots, j_m)$ *such that*:

(1) $l = \sum_{i=1}^m p^{l(I_i)} \le k$.
(2) $j_i = 2k$ *for* $0 \le k \le n-2$.
(3) *Each* Q_{I_i} *is of the form*

$$Q_{I_i} = \begin{cases} Q_{2p-2}^a, & \text{for } a \ge 0 \text{ and all } j_i, \\ Q_{p-1}^b \beta Q_{2p-2}^a, & \text{for } a > 0,\ b \ge 0 \text{ and } j_i > \frac{2(n-p-1)}{p}, \\ \beta Q_{p-1}^b \beta Q_{2p-2}^a, & \text{for } a > 0,\ b > 0 \text{ and } j_i > \frac{2(n-p-1)}{p}. \end{cases}$$

(4) *The image of* z *in* $H_*(\Omega_1^3\mathbf{SU}(n); \mathbf{Z}/p)$ *is given by replacing* z_0 *by* $[1]$, z_{2k} *by* $u_{k+1} * [1]$ *for* $1 \le k \le n-2$ *when* $k \not\equiv -1 \bmod p$, *and* z_{2k} *by* $u_2 * u_k * [1] + x_{k+1}$ *when* $k \equiv -1 \bmod p$.

Remark. By a long but direct computation, using the properties of the Dyer-Lashof algebra, the image of the z's given in Theorems 6.4 and 6.6 are nonzero in $H_*(\Omega^3\mathbf{SU}(n); \mathbf{Z}/p)$.

For n even, this gives an element in $H_{2^a(2n+1)-3}(\mathcal{M}_{2^a}(\mathbf{SU}(n)); \mathbf{Z}/2)$ which maps to

$$Q_3^a(u_2 * u_{n-1} * [1] + x_n) \in H_{2^a(2n+1)-3}(\Omega^3_{2^a}(\mathbf{SU}(n)); \mathbf{Z}/2)$$

and is hence nonzero. For $a \ge 2$ this class occurs above the middle dimension of $\mathcal{M}_{2^a}$. Furthermore, by taking appropriate products we obtain

Corollary 6.9. *For* $k = 4$ *or* $k > 8$, *the moduli space* $\mathcal{M}_k(\mathbf{SU}(2n))$ *has nontrivial homology above the middle dimension.*

(Also, we have the special case $n = 3$ due to the class v_2 there.)

Corollary 6.10. *For $k > 1$, the moduli space $\mathcal{M}_k(\mathbf{SU}(3))$ has nontrivial homology above the middle dimension.*

Similarly, the results of [7] imply

Corollary 6.11. *For $k > 1$, the moduli space $\mathcal{M}_k(\mathbf{SU}(2))$ has nontrivial homology above the middle dimension.*

Corollaries 6.9, 6.10, and 6.11 have interesting analytical consequences. We first recall that Donaldson has shown that, for all k, $\mathcal{M}_k(G)$ is a complex manifold. Furthermore, Hitchin observed that $\mathcal{M}_k(G)$ is hyper-Kähler; that is, $\mathcal{M}_k(G)$ has an entire S^2 worth of complex structures. In another direction however, the Lefschetz theorem (cf. [21]) states that a complex analytic manifold of complex dimension r which bianalytically embeds as a closed subset of a complex Euclidean space has the homotopy type of an r-dimensional CW complex. Of course, Stein manifolds are particularly nice examples of such embeddable complex manifolds. Hence, we have

Corollary 6.12. *For $k > 1$, $\mathcal{M}_k(\mathbf{SU}(2))$ and $\mathcal{M}_k(\mathbf{SU}(3))$ cannot be given the structure of a Stein manifold. Furthermore, for all $n > 1$, if $k = 4$ or $k > 8$, $\mathcal{M}_k(\mathbf{SU}(2n))$ cannot be given the structure of a Stein manifold.*

7. The proofs of the technical lemmas

Proof of Lemma 5.9. $\Sigma(h)$ can be explicitly constructed from the following five maps.

(1) The collapse map $c\colon \mathbf{CP}^2/\mathbf{CP}^1 \to S^4$ given by

$$(x_1, x_2, x_3) \mapsto (2x_1\bar{x}_3, 2x_2\bar{x}_3, 1 - 2x_3\bar{x}_3).$$

(2) The smash map $s_{4,3}\colon S^4 \wedge S^3 \to S^7$ given by

$$(z_1, z_2, z_3) \wedge (y_1, y_2) \mapsto (az_1, az_2, by_1, by_2 + (1-b)),$$

where $a = \sqrt{(1 - \Re(y_2))/2}$ and $b = (1 - z_3)/2$.

(3) The classical Hopf map $h\colon S^7 \to S^4$ given by

$$\begin{bmatrix} z_1 & w_1 \\ z_2 & w_2 \end{bmatrix} \mapsto \left[2 \begin{bmatrix} w_2 & -w_1 \\ \bar{w}_1 & \bar{w}_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, 2(w_1\bar{w}_1 + w_2\bar{w}_2) - 1 \right].$$

(4) The homeomorphism $\theta\colon S^1 \wedge S^4 \to S^1 \wedge S^4$ given by

$$\alpha \wedge (z_1, z_2, z_3) \mapsto \alpha \wedge (-\bar{\alpha} z_1, -z_2, z_3).$$

(5) Finally, the smash map $s_{1,4}\colon S^1 \wedge S^4 \to S^5$ given by

$$\alpha \wedge (z_1, z_2, z_3) \mapsto \left(cz_1, cz_2, 1 - \frac{(1-\alpha)(1-z_3)}{2} \right).$$

where $c = \sqrt{(1 - \Re(\alpha))/2}$.

The proof of Lemma 5.9 now follows directly from the fact that $\Sigma(h)$ is explicitly given by the evident composition induced, in sequence, by the maps given in items (1)–(5) above.

$$\begin{gathered} S^1 \wedge (\mathbf{CP}^2/\mathbf{CP}^1) \wedge S^3 \\ \downarrow \\ S^1 \wedge S^4 \wedge S^3 \\ \downarrow \\ S^1 \wedge S^4 \wedge S^3 \\ \downarrow \\ S^1 \wedge S^7 \\ \downarrow \\ S^1 \wedge S^4 \\ \downarrow \\ S^5 \quad \square \end{gathered}$$

Proof of Lemma 5.10. We need to show that $\Sigma(h)$ is never equal to $-K$. Let $x = (x_1, x_2, x_3) \in S^5$, $\hat{x} = (x_1, x_2)$, and $y = (y_1, y_2) \in S^3 = \mathbf{SU}(2)$. If $y(\hat{x}) = \gamma\hat{x} + \hat{z}$, where $\hat{z}$ is perpendicular to $\hat{x}$, then $1 - \langle y(x), x\rangle = (1 - x_3\bar{x}_3)(1-\gamma)$ and $\Re(y_2) = \Re(\gamma)$. Thus, $K(\alpha \wedge x \wedge y)$ is $F_{\bar{\alpha}, x_0}$ applied to

$$\begin{bmatrix} -(1-\bar{\alpha})\bar{x}_3(y(\hat{x}) - \hat{x}) - (1-\alpha)(1-\bar{\alpha})\bar{x}_3(1 - x_3\bar{x}_3)(1-\gamma)\hat{x} \\ 1 - (1-\alpha)(1-\bar{\alpha})x_3\bar{x}_3(1-x_3\bar{x}_3)(1-\gamma) \end{bmatrix}$$

and $\Sigma(h)(\alpha \wedge x \wedge y)$ is $F_{\bar{\alpha}, x_0}$ applied to

$$\begin{bmatrix} -2\bar{x}_3\sqrt{(1-\Re(\alpha))(1-\Re(\gamma))}(x_3\bar{x}_3(y(\hat{x}) - \hat{x}) + \hat{x}) \\ 1 - 2(1-\alpha)(1-\Re(\gamma))x_3\bar{x}_3(1-x_3\bar{x}_3) \end{bmatrix}.$$

When $\alpha = 1$, $x_3\bar{x}_3 = 0$, $x_3\bar{x}_3 = 1$ or $\gamma = 1$ both K and $\Sigma(h)$ map $v = (\alpha \wedge x \wedge y)$ to the base point. If $y(\hat{x}) = \gamma\hat{x} + \hat{z}$, where $\hat{z}$ is both perpendicular to $\hat{x}$ and nonzero, then $\hat{x}$ and $y(\hat{x}) - \hat{x}$ are linearly independent. But, in this case, $(1-\alpha) \neq -2x_3\bar{x}_3\sqrt{(1-\Re(\alpha))(1-\Re(\gamma))}$ as the right-hand side is real and negative. Finally, if $\hat{z} = 0$ and $\Sigma(h)(v) = -K(v)$ then we would have

$$(7.1)\quad (1-\bar{\alpha}) - (1-\alpha)(1-\bar{\alpha})(1-x_3\bar{x}_3) = \frac{2\sqrt{(1-\Re(\alpha))(1-\Re(\gamma))}}{(1-\gamma) - 2x_3\bar{x}_3\sqrt{(1-\Re(\alpha))(1-\Re(\gamma))}}$$

and

$$(7.2)\quad 2 - (1-\alpha)(1-\bar{\alpha})x_3\bar{x}_3(1-x_3\bar{x}_3)(1-\gamma) = (1-\gamma)(1-\bar{\gamma})x_3\bar{x}_3(1-x_3\bar{x}_3)(1-\alpha).$$

However, (7.1) implies that

$$\Im(1-\bar{\alpha}) = \Im\left(\frac{\sqrt{(1-\alpha)(1-\bar{\alpha})(1-\gamma)(1-\bar{\gamma})}}{1-\gamma}\right)$$

which implies that $\alpha = \gamma$. But then (7.1) also implies that $x_3\bar{x}_3 = \frac{1}{2}$ and (7.2) would then imply that $1 - (1-\alpha)^2(1-\bar{\alpha})/4 = 0$ which is impossible as α is real. □

Proof of Lemma 5.15. The adjoint of $K \circ c$ is the composite $S^3 \wedge \mathbf{CP}^{k-1} \xrightarrow{\mathrm{id}\wedge c} S^3 \wedge \mathbf{CP}^{k-1}/\mathbf{CP}^{k-2} \xrightarrow{\phi} S^{2k+1} \xrightarrow{\mathbf{k}} S^{2k+1}$, where ϕ is any homeomorphism and $\mathbf{k}$ is any degree k map. It is easy to check that we may use the following choice of ϕ:

$$\begin{aligned}\phi((a, b) \wedge (x_1, \dots, x_k)) \\ = (\lambda x_{k-1}\bar{x}_k, -\lambda x_1\bar{x}_k, \dots, -\lambda x_{k-2}\bar{x}_k, b x_k\bar{x}_k, 1 + (a-1)x_k\bar{x}_k),\end{aligned}$$

where $\lambda = \sqrt{2(1 - \Re(a))}$. Now define $\mathbf{k}: S^{2k+1} \longrightarrow S^{2k+1}$ by

$$\mathbf{k}(y_1, \dots, y_{k+1}) = \left(\frac{y_1^k}{\|y_1\|^{k-1}}, y_2, \dots, y_{k+1}\right).$$

Clearly, $\mathbf{k}$ is of degree k. The adjoint of the composite $\mathbf{k} \circ \phi \circ (\mathrm{id} \wedge c)$ is the map given by Lemma 4.15. □

We conclude with

Proof of Lemma 5.16. We will assume that

$$(7.3) \qquad (\Omega^3\pi \circ J(\mathbf{SU}(k+1)) \circ f)((a, b) \wedge x) = -(K \circ c)((a, b) \wedge x)$$

and show that $a = b = 0$, which is a contradiction. Equation (7.3) implies the following four equalities:

$$(7.4) \qquad 1 + (a-1)\frac{\|x_k\|^6}{\|g(x)\|^2} = -1 - (a-1)\|x_k\|^2,$$

$$(7.5) \qquad \alpha b\|x_k\|^2 + (a-1)\gamma_k x_{k-1}\bar{x}_k = -b\|x_k\|^2,$$

$$(7.6) \qquad \alpha b x_i\bar{x}_k - \beta(a-1)x_i^2\bar{x}_{i+1}\bar{x}_k + \gamma_i(a-1)x_{i-1}\bar{x}_k = \lambda x_{i-1}\bar{x}_k,$$

$$(7.7) \qquad \alpha b x_1\bar{x}_k - \beta(a-1)x_1^2\bar{x}_2\bar{x}_k = -\lambda\frac{x_{k-1}^k\bar{x}_k^k}{\|x_{k-1}\|^{k-1}\|x_k\|^{k-1}},$$

where α, β, γ_i, λ are all nonnegative, $1 < i < k$, and $\alpha = \beta = \gamma_i = 0$ only if $(a, b) \wedge x$ is the base point.

Equation (7.4) implies that $x_k \neq 0$ and that $a-1$ is negative. Thus, $(a, b) \wedge x$ is not the base point and α, β, γ_i are all strictly positive. This together with equation (7.5) implies that $x_{k-1}\bar{x}_k = \delta_{k-1}b$ where $\delta_{k-1} > 0$. Now assume that $x_j\bar{x}_k = \delta_j b^{k-j}$ for $1 < i \le j \le k-1$. Using equation (7.6) and the hypothesis above we obtain $x_{i-1}\bar{x}_k = \delta_{i-1}b^{k-i+1}$ where $\delta_{i-1} > 0$. Thus, $x_i\bar{x}_k = \delta_i b^{k-i}$ for $1 \leq i \leq k-1$. This together with equation (7.7) implies that $b^k = 0$. Hence, $x_i = 0$ for $1 \leq i \leq k-1$ and $\|x_k\| = 1 = \|g(x)\|$. Along with equation (7.4) this implies that $a = 0$. □

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW MEXICO, ALBUQUERQUE, NEW MEXICO 87131.

E-mail address: CBOYER@GAUSS.UNM.EDU

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW MEXICO, ALBUQUERQUE, NEW MEXICO 87131

E-mail address: MANN@GAUSS.UNM.EDU

DEPARTMENT OF MATHEMATICS, AGNES SCOTT COLLEGE, DECATUR, GEORGIA 30030