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MULTIPLICITY OF SOLUTIONS FOR ELLIPTIC PROBLEMS WITH CRITICAL EXPONENT OR WITH A NONSYMMETRIC TERM

J. GARCIA AZORERO AND I. PERAL ALONSO

ABSTRACT. We study the existence of solutions for the following nonlinear degenerate elliptic problems in a bounded domain $\Omega \subset \mathbf{R}^N$

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p^*-2}u + \lambda|u|^{q-2}u, \qquad \lambda > 0,$$

where p^* is the critical Sobolev exponent, and $u|_{\delta\Omega} \equiv 0$. By using critical point methods we obtain the existence of solutions in the following cases:

If $p < q < p^*$, there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ there exists a nontrivial solution.

If $\max(p, p^* - p/(p-1)) < q < p^*$, there exists nontrivial solution for all $\lambda > 0$.

If $1 < q < p$ there exists λ_1 such that, for $0 < \lambda < \lambda_1$, there exist infinitely many solutions.

Finally, we obtain a multiplicity result in a noncritical problem when the associated functional is not symmetric.

1. INTRODUCTION

In this work we will consider the following model problem:

$$(1.1) \qquad \begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p^*-2}u + \lambda|u|^{q-2}u, \\ u|_{\delta\Omega} = 0, \end{gathered}$$

where $\lambda > 0$, Ω is a bounded domain in $\mathbf{R}^N$ with boundary $\delta\Omega$, and assume that

$$(1.2) \qquad \begin{aligned} &\text{(i)} && 1 < p < N, \\ &\text{(ii)} && p^* = pN/(N-p), \\ &\text{(iii)} && 1 < q < p^*. \end{aligned}$$

Observe that p^* is the critical exponent in the Sobolev inclusion theorem. The nonlinear differential operator is called *p-Laplacian*, Δ_p. We look for nontrivial

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solutions of (1.1), and this question is reduced to show the existence of critical points for the functional

$$F(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{q}\int_\Omega |u|^q - \frac{1}{p^*}\int_\Omega |u|^{p^*}. \tag{1.3}$$

Under hypothesis (1.2), $F(u)$ is defined on the Sobolev space $W_0^{1,p}(\Omega)$.

By using the so-called generalized Pohozaev identity, it is possible to prove that, if the domain Ω is starshaped, then (1.1) cannot have any nontrivial solution in $W_0^{1,p}(\Omega)$ if $\lambda \le 0$ (see [P-S], and also [O, G-V, and E]); therefore, we are reduced to consider positive λ.

For $p = q$ the problem is studied in [G-P.1] where the existence of positive solution for the dimensions N such that $p^2 \le N$ is obtained, if $0 < \lambda < \lambda_1$, λ_1 being the first eigenvalue for the p-Laplacian (λ_1 is isolated and simple, as it is obtained in [Ba]; see also [Bha and A]). The main difficulty in solving problem (1.1) is the lack of compactness in the inclusion of $W_0^{1,p}(\Omega)$ in L^{p^*}, because in general the Palais-Smale condition is not satisfied.

In the case $p = 2$, the problem has been solved by Brézis-Nirenberg [B-N]. As in [B-N], we obtain a local Palais-Smale condition for the case $p \ne 2$ which is sufficient. This question is handled in §2 by the concentration-compactness principle of P. L. Lions (see [L1 and L2]).

In §3 we analyze the case $p < q < p^*$ and achieve some new results with respect to those obtained in [G-P.1].

The case $1 < q < p$ is managed in §4 by classical critical point theory. See [B-F and G-P.1] for related methods in the subcritical case.

Obviously, more general terms can be handled if their behaviour at 0 and at infinity is the same.

Finally, in §5, we solve some nonsymmetric problems. Following the ideas in [R1] we also obtain multiplicity results in this case.

For the regularity of the solutions, see [T and DiB].

2. The Palais-Smale condition

A sequence $\{u_j\} \subset W_0^{1,p}(\Omega)$ is called a Palais-Smale sequence for F, defined by (1.3), if

$$\begin{aligned} &F(u_j) \to c, \\ &F'(u_j) \to 0 \quad \text{in } W^{-1,p'}(\Omega), \quad \text{where } 1/p + 1/p' = 1. \end{aligned} \tag{2.1}$$

If (2.1) implies the existence of a subsequence $\{u_{j_k}\} \subset \{u_j\}$ which converges in $W_0^{1,p}(\Omega)$, we say that F verifies the Palais-Smale condition.

If this strongly convergent subsequence exists only for some c values, we say that F verifies a *local* Palais-Smale condition.

In our case, the main difficulty is the lack of compactness in the inclusion of $W_0^{1,p}(\Omega)$ in L^{p^*}. Then, we prove a local Palais-Smale condition, which is sufficient although with some restrictions.

The technical results which we must use are based on a measure representation lemma, used by P. L. Lions in the proof of the concentration-compactness principle (see [L1 and L2]).

Let $\{u_j\}$ be a bounded sequence in $W_0^{1,p}(\Omega)$. Then, there is a subsequence, such that $u_j \rightharpoonup u$, weakly in $W_0^{1,p}(\Omega)$, and

$$\begin{matrix} |\nabla u_j|^p \rightharpoonup d\mu \\ |u_j|^{p^*} \rightharpoonup d\nu \end{matrix} \quad \text{weakly-}^* \text{ in the sense of measures.}$$

If we take $\varphi \in C_0^\infty(\mathbf{R}^N)$, by some calculations with the Sobolev inequality we conclude that

$$\left(\int_\Omega |\varphi|^{p^*} d\nu\right)^{1/p^*} S^{1/p} \le \left(\int_\Omega |\varphi|^p d\mu\right)^{1/p} + \left(\int_\Omega |\nabla\varphi|^p |u|^p dx\right)^{1/p} \tag{2.2}$$

where

$$S = \inf\{\|u\|_{W_0^{1,p}(\Omega)} : u \in W_0^{1,p}(\Omega),\ \|u\|_{p^*} = 1\}$$

is the best constant in the Sobolev inclusion.

If, in (2.2), we suppose $u \equiv 0$, then we have a reverse Hölder inequality for two differents measures. In this situation, we have the following representation of the measures (see P. L. Lions [L1 and L2]):

Lemma 2.1. *Let μ, ν be two nonnegative and bounded measures on $\overline{\Omega}$, such that for $1 \le p < r < \infty$ there exists some constant $C > 0$ such that*

$$\left(\int_\Omega |\varphi|^r d\nu\right)^{1/r} \le C \left(\int_\Omega |\varphi|^p d\mu\right)^{1/p} \quad \forall \varphi \in C_0^\infty(\mathbf{R}^N).$$

Then, there exist $\{x_j\}_{j\in J} \subset \overline{\Omega}$ and $\{\nu_j\}_{j\in J} \subset (0, \infty)$, where J is at most countable, such that

$$\nu = \sum_{j\in J} \nu_j \delta_{x_j}, \qquad \mu \ge C^{-p} \sum_{j\in J} \nu_j^{p/r} \delta_{x_j},$$

where δ_{x_j} is the Dirac mass at x_j.

If we apply this lemma to $v_j = u_j - u$, we obtain the following result, due to P. L. Lions (see [L1 and L2]):

Lemma 2.2. *Let $\{u_j\}$ be a weakly convergent sequence in $W_0^{1,p}(\Omega)$ with weak limit u, and such that*

(i) *$|\nabla u_j|^p$ converges in the weak-* sense of measures to a measure μ,*

(ii) *$|u_j|^{p^*}$ converges in the weak-* sense of measures to a measure ν.*

Then, for some at most countable index set J we have

$$\begin{aligned} &(1)\quad \nu = |u|^{p^*} + \sum_{j\in J} \nu_j \delta_{x_j}, \qquad \nu_j > 0, \\ &(2)\quad \mu \ge |\nabla u|^p + \sum_{j\in J} \mu_j \delta_{x_j}, \qquad \mu_j > 0, \\ &(3)\quad \nu_j^{p/p^*} \le \mu_j / S, \end{aligned} \tag{2.3}$$

where $x_j \in \overline{\Omega}$.

The relations (2.3) with the hypothesis that the constant c in (2.1) is small enough allow us to prove that the singular part of the measures must be 0, and we have a local Palais-Smale condition.

Lemma 2.3. *Let $\{v_j\} \subset W_0^{1,p}(\Omega)$ be a Palais-Smale sequence for F, defined by (1.3), that is,*

$$F(v_j) \to C, \tag{2.4}$$

$$F'(v_j) \to 0 \quad in\ W^{-1,p'}(\Omega), \qquad 1/p + 1/p' = 1. \tag{2.5}$$

Then, we have

(a) *If $p < q < p^*$, and $C < S^{N/p}/N$, there exists a subsequence $\{v_{j_k}\} \subset \{v_j\}$, strongly convergent in $W_0^{1,p}(\Omega)$.*

(b) *If $1 < q < p$, and $C < S^{N/p}/N - K\lambda^{\beta}$, where $\beta = p^*/(p^* - q)$ and K depends on p, q, N and Ω, then there exists a subsequence $\{v_{j_k}\} \subset \{v_j\}$, strongly convergent in $W_0^{1,p}(\Omega)$.*

Proof. In both cases, by (2.4) and (2.5), it is easy to prove that the sequence $\{v_j\}$ is bounded in $W_0^{1,p}(\Omega)$. Then, if we take the appropriate subsequence, we can assume in both cases (by Lemma 2.2)

$$\begin{gathered} v_j \rightharpoonup v \text{ weakly in } W_0^{1,p}(\Omega), \\ v_j \to v \text{ in } L^r,\ 1 < r < p^*, \text{ and a.e.}, \\ |\nabla v_j|^p \rightharpoonup d\mu \geq |\nabla v|^p + \sum_{k\in J} \mu_k \delta_{x_k}, \\ |v_j|^{p^*} \rightharpoonup d\nu = |v|^{p^*} + \sum_{k\in J} \nu_k \delta_{x_k}. \end{gathered} \tag{2.6}$$

Take $x_k \in \overline{\Omega}$ in the support of the singular part of $d\mu$, $d\nu$. We consider $\varphi \in C_0^\infty(\mathbf{R}^N)$, such that

$$\varphi \equiv 1 \text{ on } B(x_k, \varepsilon), \quad \varphi \equiv 0 \text{ on } B(x_k, 2\varepsilon)^c, \quad |\nabla\varphi| \leq 2/\varepsilon. \tag{2.7}$$

It is clear that the sequence $\{\varphi v_j\}$ is bounded in $W_0^{1,p}(\Omega)$; then, by using hypothesis (2.5), $\lim\langle F'(v_j), \varphi v_j\rangle = 0$ ($\langle\ ,\ \rangle$ is the duality product), and

$$\int \varphi\, d\nu + \lambda \int |v|^q \varphi\, dx - \int \varphi\, d\mu = \lim_j \int |\nabla v_j|^{p-2} v_j (\nabla v_j, \nabla\varphi)\, dx$$

($(\cdot, \cdot)$ is the product in $\mathbf{R}^N$). By (2.6), (2.7), and the Hölder inequality, we obtain

$$0 \leq \lim_j \left| \int |\nabla v_j|^{p-2} v_j (\nabla v_j, \nabla\varphi)\, dx \right| \leq C \left(\int_{B(x_k, 2\varepsilon)} |v|^{p^*} \right)^{1/p^*} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Then,

$$0 = \lim_{\varepsilon \to 0} \left\{ \int \varphi \, d\nu + \lambda \int |v|^q \varphi \, dx - \int \varphi \, d\mu \right\} = \nu_k - \mu_k.$$

By Lemma 2.2, $\mu_k \geq S\nu_k^{p/p^*}$, i.e. $\nu_k \geq S\nu_k^{p/p^*}$. That is, $\nu_k = 0$, or

$$\nu_k \geq S^{N/p}. \tag{2.8}$$

(In particular, there are, at most, a finite number of singularities, because $d\nu$ is a bounded measure.) We will prove that (2.8) is not possible.

Let us assume that there exists a k_0 with $\nu_{k_0} \neq 0$ i.e. $\nu_{k_0} \geq S^{N/p}$. By (2.4) and (2.6),

$$C = \lim_j F(v_j) \geq F(v) + \left(\frac{1}{p} - \frac{1}{p^*}\right) \sum \nu_k \geq F(v) + \frac{1}{N} S^{N/p}.$$

But, by hypothesis, $C < S^{N/p}/N$; then, $F(v) < 0$. In particular, $v \not\equiv 0$, and

$$0 < \frac{1}{p} \int |\nabla v|^p < \frac{1}{p^*} \int |v|^{p^*} + \frac{\lambda}{q} \int |v|^q.$$

That is,

$$\begin{aligned} C &= \lim_j F(v_j) = \lim_j \{F(v_j) - 1/p\langle F'(v_j), v_j\rangle\} \\ &\geq \frac{1}{N} \int |v|^{p^*} + \frac{1}{N} S^{N/p} + \lambda \left(\frac{1}{p} - \frac{1}{q}\right) \int |v|^q. \end{aligned} \tag{2.9}$$

Now we distinguish two cases:

(a) If $p < q < p^*$, then $C > S^{N/p}/N$, and this inequality contradicts the hypothesis for this case. Then, $\nu_k = 0 \; \forall k$, and $\lim_j \int |v_j|^{p^*} = \int |v|^{p^*}$. By using (2.6), we conclude that $v_j \to v$ in L^{p^*}, and, finally, because of the continuity of Δ_p^{-1}, $v_j \to v$ in $W_0^{1,p}(\Omega)$.

(b) If $1 < q < p$, applying the Hölder inequality at (2.8), we have

$$C \geq \frac{1}{N} S^{N/p} + \frac{1}{N} \int |v|^{p^*} - \lambda \left(\frac{1}{q} - \frac{1}{p}\right) |\Omega|^{(p^*-q)/p^*} \left(\int |v|^{p^*}\right)^{q/p^*}.$$

Let $f(x) = c_1 x^{p^*} - \lambda c_2 x^q$. This function attains its absolute minimum (for $x > 0$) at the point $x_0 = (\lambda c_2 q / p^* c_1)^{1/(p^*-q)}$. That is,

$$f(x) \geq f(x_0) = -K\lambda^{p^*/(p^*-q)}.$$

But this result contradicts the hypothesis; then, $\nu_k = 0 \; \forall k$, and we conclude. □

Remark 2.4. It is well known that it is impossible to improve this local Palais-Smale condition in case (a); we can construct a Palais-Smale sequence with $C = S^{N/p}/N$, without any convergent subsequence (see [B]).

In case (b) it is also possible to exhibit a counterexample; we construct this counterexample at the end of §4. □

3. The case $p < q < p^*$

In §2, we have proved that below the level $S^{N/p}/N$, the functional F verifies a local Palais-Smale condition. In this section we will use the Mountain Pass Lemma to prove the existence of a solution for problem (1.1).

We will use the following general version of the Mountain Pass Lemma (see [A-E] for the proof).

Lemma 3.1. *Let F be a functional on a Banach space X, $F \in C^1(X, \mathbf{R})$. Let us assume that there exists $r, R > 0$, such that*

(i) *$F(u) > r$, $\forall u \in X$ with $\|u\| = R$,*

(ii) *$F(0) = 0$, and $F(w_0) < r$ for some $w_0 \in X$, with $\|w_0\| > R$.*

Let us define $\mathbf{C} = \{g \in C([0, 1]; X) : g(0) = 0,\ g(1) = w_0\}$, and

$$c = \inf_{g \in \mathbf{C}} \max_{t \in [0,1]} F(g(t)). \tag{3.1}$$

Then, there exists a sequence $\{u_j\} \subset X$, such that $F(u_j) \to c$, and $F'(u_j) \to 0$ in X^* (dual of X).

In our case, it is easy to see that F verifies (i) and (ii).

If we can prove that

$$c < S^{N/p}/N \tag{3.2}$$

then Lemma 3.1 and Lemma 2.3 give the existence of the critical point of F.

To obtain (3.2), we choose $v_0 \in W_0^{1,p}(\Omega)$, with

$$\|v_0\|_{p^*} = 1, \qquad \lim_{t \to \infty} F(tv_0) = -\infty; \tag{3.3}$$

then, $\sup_{t \geq 0} F(tv_0) = F(t_\lambda v_0)$, for some $t_\lambda > 0$. Thus t_λ verifies

$$0 = t_\lambda^{p-1} \int |\nabla v_0|^p - t_\lambda^{p^*-1} \int |v_0|^{p^*} - \lambda t_\lambda^{q-1} \int |v_0|^q \tag{3.4}$$

and we get

$$0 = t_\lambda^{q-1} \left(t_\lambda^{p-q} \left(\int |\nabla v_0|^p \right) - t_\lambda^{p^*-q} - \lambda \int |v_0|^q \right).$$

Observe that

$$t_\lambda^{p^*-q} + \lambda \int |v_0|^q \underset{\lambda \to \infty}{\longrightarrow} \infty;$$

therefore, (3.4) implies $t_\lambda \underset{\lambda \to \infty}{\longrightarrow} 0$. By the continuity of F,

$$\lim_{\lambda \to \infty} \left(\sup_{t \geq 0} F(tv_0) \right) = 0;$$

then, there exists λ_0 such that $\forall \lambda \geq \lambda_0$,

$$\sup_{t \geq 0} F(tv_0) < S^{N/p}/N.$$

If we take $w_0 = tv_0$, with t large enough to verify $F(w_0) < 0$, we get

$$c \le \max_{t\in[0,1]} F(g_0(t)) \quad \text{taking } g_0(t) = tw_0.$$

Therefore, $c \le \sup_{t\ge 0} F(tv_0) < S^{N/p}/N$, and we have proved estimate (3.2), for λ large enough. Hence, we can apply Lemma 3.1 and Lemma 2.3, and we have the following result:

Theorem 3.2. *If* $p < q < p^*$, *there exists* $\lambda_0 > 0$ *such that problem* (1.1) *has a nontrivial solution* $\forall\lambda \ge \lambda_0$.

By choosing carefully the function $v_0 \in W_0^{1,p}(\Omega)$ in (3.3), we can prove the following stronger result:

Theorem 3.3. *If* $\max(p, p^* - p/(p-1)) < q < p^*$, *then there exists a nontrivial solution of problem* (1.1), $\forall\lambda > 0$.

Proof. The natural choice is to take an appropriated truncation of

$$U_\varepsilon(x) = (\varepsilon + c|x - x_0|^{p/(p-1)})^{(p-N)/p} \tag{3.5}$$

because they are the functions in $W^{1,p}(\mathbf{R}^N)$ where the best constant in the Sobolev inclusion is attained. It is well known that they are the unique positives, except for translations and dilations (see [B, L1, L2]).

We can assume that $0 \in \Omega$, and consider $x_0 = 0$ at (3.5).

Let ϕ be a function $\phi \in C_0^\infty(\Omega)$, and $\phi(x) \equiv 1$ in a neighbourhood of the origin. We define $u_\varepsilon(x) = \phi(x)U_\varepsilon(x)$. For $\varepsilon \to 0$, the behaviour of u_ε has to be like U_ε, and we can estimate the error we get when we take u_ε instead of U_ε.

In this way, taking $v_\varepsilon = u_\varepsilon/\|u_\varepsilon\|_{p^*}$, we obtain the following estimates (see [B-N, G-P.1] for the details):

(1) Estimate for the gradient:

$$\|\nabla v_\varepsilon\|_p^p = S + O(\varepsilon^{(N-p)/p}). \tag{3.6}$$

(2) Estimate of $\|v_\varepsilon\|_q$:
if $q > p^*(1 - 1/p)$, then

$$C_1\varepsilon^{((p-1)/p)(N-q(N-p)/p)} \le \|v_\varepsilon\|_q^q \le C_2\varepsilon^{((p-1)/p)(N-q(N-p)/p)}. \tag{3.7}$$

If $q = p^*(1 - 1/p)$, then

$$C_1\varepsilon^{(N-p)q/p^2}|\log\varepsilon| \le \|v_\varepsilon\|_q^q \le C_2\varepsilon^{(N-p)q/p^2}|\log\varepsilon|. \tag{3.8}$$

If $q < p^*(1 - 1/p)$, then

$$C_1\varepsilon^{(N-p)q/p^2} \le \|v_\varepsilon\|_q^q \le C_2\varepsilon^{(N-p)q/p^2}. \tag{3.9}$$

Observe that, if $p < q < p^*$, then

$$\|v_\varepsilon\|_q^q \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{3.10}$$

By using these estimates, we will show that there exists $\varepsilon > 0$, small enough, such that

$$\sup_{t\geq 0} F(tv_\varepsilon) < S^{N/p}/N.$$

Then, we conclude as in Theorem 3.2, by using Lemma 3.1 and Lemma 2.3.

Let us consider the functions

$$g(t) = F(tv_\varepsilon) = \frac{t^p}{p}\int |\nabla v_\varepsilon|^p - \frac{t^{p^*}}{p^*} - \frac{\lambda t^q}{q}\int |v_\varepsilon|^q,$$

and

$$\overline{g}(t) = \frac{t^p}{p}\int |\nabla v_\varepsilon|^p - \frac{t^{p^*}}{p^*}.$$

It is clear that $g(t) \underset{t\to\infty}{\longrightarrow} -\infty$; then, $\sup_{t\geq 0} F(tv_\varepsilon)$ is attained for some $t_\varepsilon > 0$, and

$$0 = g'(t_\varepsilon) = t_\varepsilon^{p-1}\left(\int |\nabla v_\varepsilon|^p - t_\varepsilon^{p^*-p} - \lambda t_\varepsilon^{q-p}\int |v_\varepsilon|^q\right)..$$

Therefore,

$$\int |\nabla v_\varepsilon|^p = t_\varepsilon^{p^*-p} + \lambda t_\varepsilon^{q-p}\int |v_\varepsilon|^q > t_\varepsilon^{p^*-p},$$

i.e.

$$t_\varepsilon \leq \left(\int |\nabla v_\varepsilon|^p\right)^{1/(p^*-p)}. \tag{3.11}$$

This inequality implies

$$\int |\nabla v_\varepsilon|^p \leq t_\varepsilon^{p^*-p} + \lambda\left(\int |\nabla v_\varepsilon|^p\right)^{(q-p)/(p^*-p)}\left(\int |v_\varepsilon|^q\right). \tag{3.12}$$

Choosing ε small enough, by (3.6) and (3.10),

$$t_\varepsilon^{p^*-p} \geq S/2. \tag{3.13}$$

That is, we have a lower bound for t_ε, independent of ε. Now, we estimate $g(t_\varepsilon)$.

The function $\overline{g}$ attains its maximum at $t = (\int |\nabla v_\varepsilon|^p)^{1/(p^*-p)}$, and is increasing at the interval $[0, (\int |\nabla v_\varepsilon|^p)^{1/(p^*-p)}]$. Then, by using (3.6), (3.11) and (3.13), we have

$$\begin{aligned} g(t_\varepsilon) &= \overline{g}(t_\varepsilon) - \frac{\lambda}{q}t_\varepsilon^q \int |v_\varepsilon|^q \\ &\leq \overline{g}\left(\left(\int |\nabla v_\varepsilon|^p\right)^{1/(p^*-p)}\right) - \frac{\lambda}{q}t_\varepsilon^q\int |v_\varepsilon|^q \\ &\leq \frac{1}{N}S^{N/p} + C_3\varepsilon^{(N-p)/p} - \frac{\lambda}{q}\left(\frac{S}{2}\right)^{q/(p^*-p)}\int |v_\varepsilon|^q. \end{aligned}$$

Let us suppose $q > p^*(1 - 1/p)$. Then, we have (3.7), and

$$g(t_\varepsilon) \leq S^{N/p}/N + C_3\varepsilon^{(N-p)/p} - \lambda C_1\varepsilon^{\{((p-1)/p)(N-q(N-p)/p)\}}. \tag{3.14}$$

If

$$\frac{N-p}{p} > \frac{p-1}{p}\left(N - q\frac{(N-p)}{p}\right),$$

that is, $q > p^* - p/(p-1)$, then for ε small enough we get $g(t_\varepsilon) < S^{N/p}/N$, and we conclude. □

Remark 3.4. If $N \geq p^2$, then $p^* - \frac{p}{p-1} \leq p^*(1 - \frac{1}{p}) \leq p$, and if $p < q < p^*$, we have $q > p^* - \frac{p}{p-1}$. Then q verifies the estimate (3.7), and we obtain the result of [G-P.1].

If $N < p^2$, then $p < p^*(1 - \frac{1}{p}) < p^* - \frac{p}{p-1}$, and for $p < q \leq p^* - \frac{p}{p-1}$ the estimate is insufficient. □

Remark 3.5. It is possible to prove the analogous result for the problem:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p^*-2}u + \lambda|u|^{q-2}u + f, \ \lambda > 0 \\ u|_{\delta\Omega} = 0 \end{cases}$$

if f is small enough in the norm of $W^{-1,p'}(\Omega)$. The proof is an adaptation of the above argument. □

4. The case $1 < q < p$

In this section, we will construct a mini-max class of critical points, by using the classical concept and properties of the *genus.*

Let X be a Banach space, and Σ the class of the closed and symmetric with respect to the origin subsets of $X - \{0\}$. For $A \in \Sigma$, we define the genus $\gamma(A)$ by

$$\gamma(A) = \min\{k \in \mathbf{N} \colon \exists\phi \in \mathbf{C}(A; \mathbf{R}^k - \{0\}), \phi(x) = -\phi(-x)\}.$$

If such a minimum does not exist then we define $\gamma(A) = +\infty$. The main properties of the genus are the following (see [R1 or R2] for the details):

Proposition 4.1. *Let* $A, B \in \Sigma$. *Then*

(1) *If there exists* $f \in \mathbf{C}(A, B)$, *odd, then* $\gamma(A) \leq \gamma(B)$.

(2) *If* $A \subset B$, *then* $\gamma(A) \leq \gamma(B)$.

(3) *If there exists an odd homeomorphism between* A *and* B, *then* $\gamma(A) = \gamma(B)$.

(4) *If* S^{N-1} *is the sphere in* $\mathbf{R}^N$, *then* $\gamma(S^{N-1}) = N$.

(5) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

(6) *If* $\gamma(B) < +\infty$, *then* $\gamma(\overline{A - B}) \geq \gamma(A) - \gamma(B)$.

(7) *If* A *is compact, then* $\gamma(A) < +\infty$, *and there exists* $\delta > 0$ *such that* $\gamma(A) = \gamma(N_\delta(A))$ *where* $N_\delta(A) = \{x \in X \colon d(x, A) \leq \delta\}$.

(8) *If X_0 is a subspace of X with codimension K, and $\gamma(A) < K$, then* $A \cap X_0 \neq \emptyset$.

Given the functional F, defined by (1.3), under the hypothesis $q < p$, by Sobolev's inequality we obtain

$$F(u) \geq \frac{1}{p}\int |\nabla u|^p - \frac{1}{p^* S^{p^*/p}}\left(\int |\nabla u|^p\right)^{p^*/p} - \frac{\lambda}{q}C_{p,q}\left(\int |\nabla u|^p\right)^{q/p}.$$

If we define

$$h(x) = \frac{1}{p}x^p - \frac{1}{p^* S^{p^*/p}}x^{p^*} - \frac{\lambda}{q}C_{p,q}x^q$$

then

$$F(u) \geq h(\|\nabla u\|_p). \tag{4.1}$$

There exists $\lambda_1 > 0$ such that, if $0 < \lambda \leq \lambda_1$, h attains its positive maximum (see Figure 4.1).

Let us assume $0 < \lambda \leq \lambda_1$; choosing R_0 and R_1 as in Figure 4.1 we make the following truncation of the functional F:

Take $\tau: \mathbf{R}^+ \to [0, 1]$, nonincreasing and $\mathbf{C}^\infty$, such that

$$\tau(x) = 1 \quad \text{if } x \leq R_0,$$
$$\tau(x) = 0 \quad \text{if } x \geq R_1.$$

Let $\varphi(u) = \tau(\|\nabla u\|_p)$. We consider the truncated functional

$$J(u) = \frac{1}{p}\int |\nabla u|^p - \frac{1}{p^*}\int |u|^{p^*}\varphi(u) - \frac{\lambda}{q}\int |u|^q. \tag{4.2}$$

As in (4.1), $J(u) \geq \overline{h}(\|\nabla u\|_p)$, with

$$\overline{h}(x) = \frac{1}{p}x^p - \frac{1}{p^* S^{p^*/p}}x^{p^*}\tau(x) - \frac{\lambda}{q}C_{p,q}x^q \tag{4.3}$$

(see Figure 4.2).

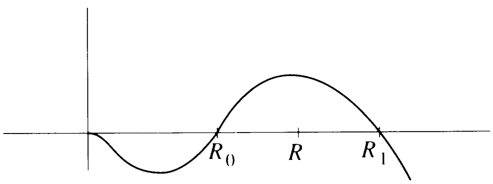

FIGURE 4.1

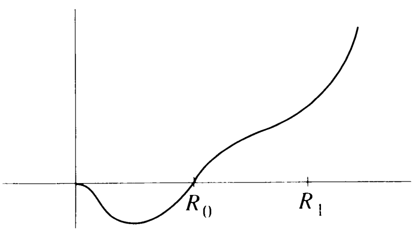

FIGURE 4.2

Observe that for $x \le R_0$, $\overline{h} = h$, and for $x \ge R_1$,

$$\overline{h}(x) = \tfrac{1}{p}x^p - \tfrac{\lambda}{q}C_{p,q}x^q.$$

The principal properties of J defined by (4.2) are:

Lemma 4.2. (1) $J \in \mathbf{C}^1(W_0^{1,p}(\Omega), \mathbf{R})$.

(2) *If* $J(u) \le 0$, *then* $\|\nabla u\|_p < R_0$, *and* $F(v) = J(v)$ *for all* v *in a small enough neighbourhood of* u.

(3) *There exists* $\lambda_1 > 0$, *such that, if* $0 < \lambda < \lambda_1$, *then* J *verifies a local Palais-Smale condition for* $c \le 0$.

Proof. (1) and (2) are immediate. To prove (3), observe that all Palais-Smale sequences for J with $c \le 0$ must be bounded; then, by Lemma 2.3, if λ verifies $S^{N/p}/N - K\lambda^\beta \ge 0$ there exists a convergent subsequence. □

Observe that, by (2), if we find some negative critical value for J, then we have a negative critical value of F.

Now, we will construct an appropriate mini-max sequence of negative critical values for the functional J.

Lemma 4.3. *Given* $n \in \mathbf{N}$, *there is* $\varepsilon = \varepsilon(n) > 0$, *such that*

$$\gamma(\{u \in W_0^{1,p}(\Omega) : J(u) \le -\varepsilon\}) \ge n.$$

Proof. Fix n, let E_n be an n-dimensional subspace of $W_0^{1,p}(\Omega)$. We take $u_n \in E_n$, with norm $\|\nabla u_n\|_p = 1$. For $0 < \rho < R_0$, we have

$$J(\rho u_n) = F(\rho u_n) = \frac{1}{p}\rho^p - \frac{1}{p^*}\rho^{p^*}\int |u|^{p^*} - \frac{\lambda}{\rho}\rho^q \int |u|^q.$$

E_n is a space of finite dimension; so, all the norms are equivalent. Then, if we define

$$\alpha_n = \inf\left\{\int |u|^{p^*} : u \in E_n,\ \|\nabla u_n\|_p = 1\right\} > 0,$$
$$\beta_n = \inf\left\{\int |u|^q : u \in E_n,\ \|\nabla u_n\|_p = 1\right\} > 0,$$

we have

$$J(\rho u_n) \le \frac{1}{p}\rho^p - \frac{\alpha_n}{p^*}\rho^{p^*} - \frac{\lambda\beta_n}{q}\rho^q,$$

and we can choose ε (which depends on n), and $\eta < R_0$, such that $J(\eta u) \le -\varepsilon$ if $u \in E_n$, and $\|\nabla u\|_p = 1$.

Let $S_\eta = \{u \in W_0^{1,p}(\Omega) : \|\nabla u\|_p = \eta\}$. $S_\eta \cap E_n \subset \{u \in W_0^{1,p}(\Omega) : J(u) \le -\varepsilon\}$; therefore, by Proposition 4.1,

$$\gamma(\{u \in W_0^{1,p}(\Omega) : J(u) \le -\varepsilon\}) \ge \gamma(S_\eta \cap E_n) = n. \quad \square$$

This lemma allows us to prove the existence of critical points.

Lemma 4.4. *Let* $\Sigma_k = \{C \subset W_0^{1,p}(\Omega)-\{0\},\ C$ *is closed,* $C=-C,\ \gamma(C)\geq k\}$.

Let $c_k = \inf_{C\in\Sigma_k}\sup_{u\in C} J(u)$, $K_c = \{u \in W_0^{1,p}(\Omega): J'(u)=0,\ J(u)=c\}$, *and suppose* $0<\lambda<\lambda_1$, *where* λ_1 *is the constant of Lemma* 4.2.

Then, if $c = c_k = c_{k+1} = \cdots = c_{k+r}$, $\gamma(K_c)\geq r+1$.

(*In particular, the* c_k *'s are critical values of* J.)

Proof. In the proof, we will use Lemma 4.3, and a classical deformation lemma (see [Be]).

For simplicity, we call $J^{-\varepsilon} = \{u \in W_0^{1,p}(\Omega): J(u)\leq -\varepsilon\}$. By Lemma 4.3, $\forall k\in \mathbf{N}$, $\exists \varepsilon(k)>0$ such that $\gamma(J^{-\varepsilon})\geq k$.

Because J is continuous and even, $J^{-\varepsilon}\in\Sigma_k$; then, $c_k\leq -\varepsilon(k)<0$, $\forall k$. But J is bounded from below; hence, $c_k>-\infty\ \forall k$.

Let us assume that $c=c_k=\cdots=c_{k+r}$. Let us observe that $c<0$; therefore, J verifies the Palais-Smale condition in K_c, and it is easy to see that K_c is a compact set.

If $\gamma(K_c)\leq r$, there exists a closed and symmetric set U, $K_c\subset U$, such that $\gamma(U)\leq r$. (We can choose $U\subset J^0$, because $c<0$.)

By the deformation lemma, we have an odd homeomorphism

$$\eta: W_0^{1,p}(\Omega)\to W_0^{1,p}(\Omega),$$

such that $\eta(J^{c+\delta}-U)\subset J^{c-\delta}$, for some $\delta>0$. (Again, we must choose $0<\delta<-c$, because J verifies the Palais-Smale condition on J^0, and we need $J^{c+\delta}\subset J^0$.) By definition,

$$c = c_{k+r} = \inf_{C\in\Sigma_{k+r}}\sup_{u\in C} J(u).$$

Then, there exists $A\in\Sigma_{k+r}$, such that $\sup_{u\in A} J(u) < c+\delta$; i.e., $A\subset J^{c+\delta}$, and

$$\eta(A-U)\subset \eta(J^{c+\delta}-U)\subset J^{c-\delta}. \tag{4.4}$$

But $\gamma(\overline{A-U})\geq\gamma(A)-\gamma(U)\geq k$, and $\gamma(\eta(\overline{A-U}))\geq\gamma(\overline{A-U})\geq k$.

Then, $\eta(\overline{A-U})\in\Sigma_k$. And this contradicts (4.4); in fact,

$$\eta(\overline{A-U})\in\Sigma_k \quad \text{implies} \quad \sup_{u\in\eta(\overline{A-U})} J(u)\geq c_k = c. \quad \square$$

This lemma proves the following result:

Theorem 4.5. *Given problem* (1.1), *with* $1<q<p$, *there exists* $\lambda_1>0$, *such that, for* $0<\lambda<\lambda_1$, *there exists infinitely many solutions.*

Remark 4.6. (1) For the truncated functional J, a result of Brezis-Oswald [B-O] for $p=2$, which is extended to a general case for Diaz-Saa [D-S], proves the uniqueness of nontrivial positive solutions.

Then, the solutions that we find change the sign, except for those associated with c_1. In fact, $c_1=\inf J(u)$, and, if $c_1=J(u_0)$, then $c_1=J(|u_0|)$. That is, $|u_0|$ is a nonnegative solution, and, by the maximum principle (see [T]), is strictly positive on Ω.

Observe that there is not a uniqueness result for the nontruncated functional F. It remains open the question of the existence of positive solutions with positive energy (solutions as those of §3, for $p < q < p^*$).

(2) It is possible to make another proof of Theorem 4.5, if we replace the truncation of F by a special construction of the deformation function η. In fact, we can take η which acts on $B(0, R_0) \subset W_0^{1,p}(\Omega)$, and is the identity otherwise; then we must define

$$\overline{\Sigma}_k = \{C \subset B(0, R_0) - \{0\}: C \text{ closed}, \text{ symmetric}, \ \gamma(C) \geq k\}.$$

(3) The critical values that we have obtained are negative, and F verifies the Palais-Smale condition for $c < 0$; then, it is easy to see that the set of solutions of Theorem 4.5, is a compact set. □

Now, we can show that it is not possible to extend the Palais-Smale condition that we have proved.

Take $x_0 \in \Omega$, and the balls $B_j = B(x_0, j\delta) \subset \Omega$, and the following $C_0^\infty(\mathbf{R}^N)$ functions:

$$\begin{aligned} &\varphi_\delta \equiv 1 \quad on\ \Omega - B_3, && \xi_\delta \equiv 1 \quad on\ B_1, \\ &\varphi_\delta \equiv 0 \quad on\ B_2, && \xi_\delta \equiv 0 \quad on\ \Omega - B_2, \\ &|\nabla\varphi_\delta| < 2/\delta, && |\nabla\xi_\delta| < 2/\delta. \end{aligned}$$

We define $\phi_\delta = \varphi_\delta v + \xi_\delta w_\varepsilon$, where $F'(v) = 0$, and $F(v) < 0$, $F(w_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} S^{N/p}/N$ and $F'(w_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0$. (Take $w_\varepsilon = S^{(N-p)/p^2} v_\varepsilon$, with v_ε defined in §3.) Later, we shall choose $\varepsilon = \varepsilon(\delta) \underset{\delta\to 0}{\longrightarrow} 0$.

Then, $F(\phi_\delta) = F(\varphi_\delta v) + F(\xi_\delta w_\varepsilon)$, and we can show that $F(\phi_\delta) \underset{\delta\to 0}{\longrightarrow} C < S^{N/p}/N$, with $F'(\phi_\delta) \underset{\delta\to 0}{\longrightarrow} 0$.

But it is not possible to find a convergent subsequence of $\{\phi_\delta\}$, because $\phi_\delta \rightharpoonup v$ but

$$\begin{aligned} \|\phi_\delta - v\|_{W_0^{1,p}(\Omega)} &= \|(\varphi_\delta - 1)v + \xi_\delta w_\varepsilon\|_{W_0^{1,p}(\Omega)} \\ &\geq \|\xi_\delta w_\varepsilon\|_{W_0^{1,p}(\Omega)} - \|(\varphi_\delta - 1)v\|_{W_0^{1,p}(\Omega)} > M > 0 \end{aligned}$$

with M independent of δ.

5. A problem without symmetry

We shall consider the following model problem:

$$(5.1) \qquad \begin{aligned} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &= \lambda|u|^{q-2}u + f(x), \qquad \lambda > 0, \\ u|_{\delta\Omega} &= 0, \end{aligned}$$

where Ω is a rectangle in $\mathbf{R}^N$, and $p < q < p^*$, $1 < p < N$. When $f \equiv 0$, there are infinitely many solutions $\forall\lambda > 0$. In the proof, we use a mini-max type theory, as in §4, because the associated functional is even.

When $f \not\equiv 0$, the associated functional is

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{q}\int_\Omega |u|^q - \int_\Omega fu. \tag{5.2}$$

We cannot apply the previous method, because I is not even; however, it is possible to make use of the method developed by P. Rabinowitz in the case $p = 2$ (see [R1 and R2]). For the sake of completeness, we will include here the proofs of the more interesting steps.

The point is the lack of control on the nonsymmetric part of the functional I; that is, $I(u) - I(-u)$. The idea is to find some appropriated truncation of I, in order to obtain a functional J, in which the nonsymmetric part can be estimated, such that the existence of critical points for J implies the existence of critical points for I. We start with an a priori estimate, which gives us the idea to make the truncation.

Lemma 5.1. *There exists a constant* $A = A(\|f\|_{p'}) > 0$ *such that, if* $I'(u) = 0$, *then*

$$\frac{\lambda}{q}\int_\Omega |u|^q \le A(|I(u)|^p + 1)^{1/p}.$$

(The proof is an easy adaptation of those made in [R1].) With this estimate, we make the following truncation: Let $\chi : \mathbf{R} \to [0, 1]$ such that

$$\begin{aligned} &\chi(x) = 0, \qquad x \ge 2, \\ &\chi(x) = 1, \qquad x \le 1, \\ &-2 \le \chi'(x) \le 0 \end{aligned} \tag{5.3}$$

and

$$\psi(u) = \chi\left\{\frac{(\lambda/q)\int |u|^q}{2A(|I(u)|^p + 1)^{1/p}}\right\}. \tag{5.4}$$

Define

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{q}\int_\Omega |u|^q - \int_\Omega \psi(u) fu. \tag{5.5}$$

In particular, Lemma 5.1 implies that, if $I'(u) = 0$, then $J'(u) = 0$. However, we need just the converse. The main properties of J are the following (for the proof, see [R1]):

Lemma 5.2.

(i) $J \in \mathbf{C}^1(W_0^{1,p}(\Omega), \mathbf{R})$.

(ii) $\exists \beta > 0$, $\beta = \beta(\|f\|_{p'})$, *such that* $|J(u) - J(-u)| \le \beta(|J(u)|^{1/q} + 1)$.

(iii) $\exists M_0 > 0$, *such that if* $J(u) \ge M_0$, *and* $J'(u) = 0$, *then* $\psi(v) \equiv 1$ *in a neighbourhood of* u (*that is,* $J(u) = I(u)$, *and* $J'(u) = I'(u) = 0$).

(iv) $\exists M_1 \ge M_0$ *such that* J *verifies a local Palais-Smale condition for* $C > M_1$. *That is, if we have a sequence* $\{u_k\} \subset W_0^{1,p}(\Omega)$ *such that* $J(u_k) \to C$ *and* $J'(u_k) \to 0$, *then there exists a convergent subsequence* $\{u_{k_j}\} \subset \{u_k\}$.

According to (iii), if we find some critical value for J, and it is large enough, then we have a solution of problem (5.1). We will prove a stronger result: we construct a sequence of critical values for J, which tends to infinity.

To simplify the notation, we assume $\Omega = (0, 1)^N$. Let E_k be the k-dimensional subspace of $W_0^{1,p}(\Omega)$, generated by the first k functions of the basis

$$\{(\sin k_1 \pi x_1 \cdots \sin k_N \pi x_N),\ k_i \in \mathbf{N},\ i = 1, \dots, N\}$$

(see [G-P.2]).

In this finite dimensional subspace, it is easy to prove that it is possible to construct an increasing sequence of numbers $R_j > 0$ (as big as we wish), such that

$$J(u) \le 0 \quad \text{if } u \in E_j \cap B_{R_j}^C. \tag{5.6}$$

Let $D_j = B_{R_j} \cap E_j$, and define

$$G_j = \{h \in \mathbf{C}(D_j, W_0^{1,p}(\Omega)): h \text{ is odd},\ h|_{\delta B_{R_j} \cap E_j} = \mathrm{Id}\}, \tag{5.7}$$

$$b_j = \inf_{h \in G_j} \max_{u \in D_j} J(h(u)). \tag{5.8}$$

First, we prove that the sequence $\{b_j\}$ is well defined, and increasing:

Proposition 5.3. *Let b_k defined by (5.8). Then, there exists a constant $\beta > 0$, such that*

$$b_k \ge \beta k^\gamma \tag{5.9}$$

where $\gamma = pq/N(q-p) - 1$.

Proof. Given $h \in G_k$, and $\rho < R_k$, we can prove that $h(D_k) \cap \delta B_\rho \cap E_{k-1}^C \ne \varnothing$. In fact, it suffices to show that $\gamma(h(D_k) \cap \delta B_\rho) \ge k$, and apply property (8) of Proposition 4.1. Let $A = \{x \in D_k : h(x) \in B_\rho\}$. It is clear that $0 \in A$, because h is odd; then, we define A_0 the component of A containing 0. A_0 is a bounded and symmetric neighbourhood of 0 in E_k; then, $\gamma(\delta A_0) = k$.

Moreover, $h(\delta A_0) \subset \delta B_\rho$. If not, given $x \in \delta A_0$ such that $h(x) \in B_\rho$, if $x \in D_k$, there exists a neighbourhood of x, U, such that $h(U) \subset B_\rho$. Then, $x \notin \delta A_0$. Hence, $x \in \delta D_k$; but $h|_{\delta D_k} = \mathrm{Id}$, and this implies that $\|h(x)\| = \|x\| = R_k > \rho$, a contradiction.

Now, if we define $B = \{x \in D_k : h(x) \in B_\rho\}$, we have $\delta A_0 \subset B$, and

$$\gamma(h(D_k) \cap \delta B_\rho) = \gamma(h(B)) \ge \gamma(B) \ge \gamma(\delta A_0) = k.$$

Note that the condition "h is even" is essential to obtain this result; then, it is an important ingredient in the definition of G_k.

Let $u \in \delta B_\rho \cap E_{k-1}^C$; then

$$J(u) \ge \frac{1}{p} \int_\Omega |\nabla u|^p - \frac{\lambda}{q} \int_\Omega |u|^q - C_1 \|u\|_p$$

where $C_1 = C_1(\|f\|_{p'})$. By using the Gagliardo-Nirenberg inequality,

$$\left(\int_\Omega |u|^q\right)^{1/q} \le C\left(\int_\Omega |\nabla u|^p\right)^{a/p}\left(\int_\Omega |u|^p\right)^{(1-a)/p} \tag{5.10}$$

with $a = (N/p)(1 - p/q)$, we get

(5.11)

$$J(u) \ge \frac{1}{p}\int_\Omega |\nabla u|^p - C_1\left(\int_\Omega |\nabla u|^p\right)^{qa/p}\left(\int_\Omega |u|^p\right)^{q(1-a)/p} - C_2\left(\int_\Omega |u|^p\right)^{1/p}.$$

Moreover, $u \in E_{k-1}^C$; hence,

$$\|u\|_p \le C\|\nabla u\|_p / k^{1/N} \tag{5.12}$$

(see [G-P.2] for the proof). Finally, by (5.11) and (5.12), we obtain

$$\begin{aligned} J(u) &\ge \frac{1}{p}\rho^p - C_1\left(\frac{C}{k^{q(1-a)/N}}\right)\rho^q - \left(\frac{C_3}{k^{1/N}}\right)\rho \\ &= \rho^p\left(\frac{1}{p} - \frac{C_2}{k^{q(1-a)/N}}\rho^{q-p}\right) - \frac{C_3}{k^{1/N}}\rho. \end{aligned}$$

Now, we choose

$$\rho_k = \left\{\frac{k^{q(1-a)/N}}{2pC_2}\right\}^{1/(q-p)};$$

therefore,

$$J(u) \ge \frac{1}{2p}\rho_k^p - \frac{C_3}{k^{1/N}}\rho_k \ge Ck^{pq(1-a)/N(q-p)} \tag{5.13}$$

for k large enough. Then, by equations (5.13) and (5.10), we get $\forall h \in G_k$, $\max_{u\in D_k} J(h(u)) \ge Ck^\gamma$, where

$$\gamma = \frac{pq(1-a)}{N(q-p)} = \frac{pq}{N(q-p)} - 1.$$

And this implies (5.9). □

If we try to prove that b_k is a critical value for J, we find an important obstruction, unless $f \equiv 0$. In fact, in the case $f \not\equiv 0$, the associated functional J is not even, and the deformation lemma give us a homeomorphism η which is not odd. Then, given $h \in G_k$, in general $\eta \circ h \notin G_k$, and the classical proof does not work.

However, the sequence $\{b_k\}$ allows us to prove that other mini-max sequences that we will construct are well defined and verifies the appropriate estimates. Define

$$U_k = \{u = te_{k+1} + w,\ t \in [0, R_{k+1}],\ w \in B_{R_{k+1}} \cap E_k,\ \|u\| \le R_{k+1}\}, \tag{5.14}$$

$$\Lambda_k = \{H \in C(U_k, W_0^{1,p}(\Omega)),\ H|_{D_k} \in G_k,\ H|_{(\delta B_{R_{k+1}} \cap E_{k+1}) \cup (B_{R_{k+1}} \cap B_{R_k}^C \cap E_k)} = \mathrm{Id}\}, \tag{5.15}$$

$$c_k = \inf_{H\in\Lambda_k} \max_{u\in U_k} J(H(u)). \tag{5.16}$$

These mini-max values have the same problem as the b_k's: if $H|_{D_k} \in G_k$, then $H|_{D_k}$ is odd, but $\eta \circ H|_{D_k}$ is not odd, in general. However, it is clear that $c_k \geq b_k$ (compare (5.16) and (5.8)); and if $c_k > b_k$, we can solve our problem, as the following proposition shows:

Proposition 5.4. *If* $c_k > b_k > M_1$ *(where* M_1 *is the constant of Lemma* 5.2 (iv)*), given* $\delta \in (0, c_k - b_k)$, *we define*

$$\Lambda_k(\delta) = \{H \in \Lambda_k \text{ such that } J(H(u)) \leq b_k + \delta, \forall u \in D_k\}, \tag{5.17}$$

$$c_k(\delta) = \inf_{H\in\Lambda_k(\delta)} \max_{u\in U_k} J(H(u)). \tag{5.18}$$

Then, $c_k(\delta)$ *is a critical value for* J.

Proof. By definition (5.8) it is clear that $\Lambda_k(\delta) \neq \varnothing$. And, by (5.16) and (5.18), it is also clear that $c_k(\delta) \geq c_k$. Suppose that $c_k(\delta)$ is not a critical value and take $\varepsilon < \frac{1}{2}(c_k - b_k - \delta)$.

By the classical deformation theorem, we obtain the homeomorphism $\eta: W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$, with the following properties:

$$\eta(J^{c_k(\delta)+\varepsilon}) \subset J^{c_k(\delta)-\varepsilon}, \tag{5.19}$$

$$\eta(u) = u, \quad \text{if } u \notin J^{-1}([c_k(\delta) - 2\varepsilon, c_k(\delta) + 2\varepsilon]). \tag{5.20}$$

Note that if $u \in D_k$ and $H \in \Lambda_k(\delta)$, then

$$J(H(u)) \leq b_k + \delta < c_k - 2\varepsilon, \tag{5.21}$$

that is, if $H \in \Lambda_k(\delta)$, by (5.20) and (5.21), then $\eta \circ H|_{D_k} = H|_{D_k} \in G_k$. Then, we have solved the problem of the lack of symmetry in η. Now, it is easy to conclude: we prove that $\eta \circ H \in \Lambda_k(\delta)$, and find a contradiction between (5.18) and (5.19). □

Finally, it remains to prove that it is impossible to have $c_k = b_k$, $\forall k$.

Proposition 5.5. *If* $c_k = b_k$, $\forall k \geq k^*$, *there exist some constants* $C > 0$, *and* $k' \geq k^*$, *such that*

$$b_k \leq C k^{q/(q-1)}, \quad \forall k \geq k'. \tag{5.22}$$

Proof. Basically, the idea is to use that $D_{k+1} = (U_k) \cup (-U_k)$, and if $H \in \Lambda_k$, it is possible to extend it to a function of G_k.

By (5.15) and (5.16), we can choose $H \in \Lambda_k$ such that

$$\max_{u\in U_k} J(H(u)) \leq c_k + \varepsilon = b_k + \varepsilon,$$

and, by (5.8), taking the extension of H, we have

$$b_{k+1} \leq \max_{u\in D_{k+1}} J(H(u)). \tag{5.23}$$

If the maximum is attained at U_k, then

$$b_{k+1} \leq \max_{u \in D_{k+1}} J(H(u)) = \max_{u \in U_k} J(H(u)) \leq c_k + \varepsilon \leq b_k + \varepsilon. \tag{5.24}$$

If the maximum is attained on $-U_k$, we can use estimate (iii) of Lemma 5.2, in the following way: Suppose $\max_{u \in D_{k+1}} J(H(u)) = J(H(w))$, for some $w \in -U_k$. Then,

$$\begin{aligned} J(H(-w)) &\geq J(H(w)) - \beta(|J(H(-w))|^{1/q} + 1) \\ &\geq b_{k+1} - \beta((b_k + \varepsilon)^{1/q} + 1) \geq b_{k+1} - \beta((b_{k+1} + \varepsilon)^{1/q} + 1) > 0, \end{aligned}$$

for k large. And, if $J(H(-w)) > 0$,

$$\begin{aligned} b_{k+1} &\leq J(H(w)) = J(-H(-w)) \\ &\leq J(H(-w)) + \beta(J(H(-w))^{1/q} + 1) \\ &\leq (b_k + \varepsilon) + \beta((b_k + \varepsilon)^{1/q} + 1). \end{aligned} \tag{5.25}$$

Getting $\varepsilon \to 0$ at (5.24) and (5.25), we obtain $b_{k+1} \leq b_k + \beta(b_k^{1/q} + 1)$. Finally, this inequality implies (5.22); the proof can be made by induction. □

Proposition 5.5, together with Proposition 5.3 and Proposition 5.4, prove the following theorem:

Theorem 5.6. *Problem* (5.1), *when* $q/(q-1) < pq/N(q-p) - 1$, *has infinitely many solutions, which correspond to a sequence of critical values of the functional* (5.2), *the sequence tending to infinity.*

Remark 5.7. Note that, for $p = 2$, Theorem 5.6 is contained in Theorem 10.4 in [R2].

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Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain