MULTIPICITY OF SOLUTIONS
FOR ELLIPTIC PROBLEMS WITH CRITICAL EXPONENT
OR WITH A NONSYMMETRIC TERM

J. GARCIA AZORERO AND I. PERAL ALONSO

Abstract. We study the existence of solutions for the following nonlinear degenerate elliptic problems in a bounded domain \( \Omega \subset \mathbb{R}^N \):

\[
-\text{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p^*-2}u + \lambda|u|^{q-2}u, \quad \lambda > 0,
\]

where \( p^* \) is the critical Sobolev exponent, and \( u|_{\partial \Omega} = 0 \). By using critical point methods we obtain the existence of solutions in the following cases:

If \( p < q < p^* \), there exists \( \lambda_0 > 0 \) such that for all \( \lambda > \lambda_0 \) there exists a nontrivial solution.

If \( \max(p, p^*-p/(p-1)) < q < p^* \), there exists nontrivial solution for all \( \lambda > 0 \).

If \( 1 < q < p \) there exists \( \lambda_1 \) such that, for \( 0 < \lambda < \lambda_1 \), there exist infinitely many solutions.

Finally, we obtain a multiplicity result in a noncritical problem when the associated functional is not symmetric.

1. Introduction

In this work we will consider the following model problem:

\[
-\text{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p^*-2}u + \lambda|u|^{q-2}u, \quad u|_{\partial \Omega} = 0,
\]

where \( \lambda > 0 \), \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with boundary \( \partial \Omega \), and assume that

(i) \( 1 < p < N \),

(ii) \( p^* = pN/(N-p) \),

(iii) \( 1 < q < p^* \).

Observe that \( p^* \) is the critical exponent in the Sobolev inclusion theorem. The nonlinear differential operator is called \( p\text{-Laplacian}, \Delta_p \). We look for nontrivial
solutions of (1.1), and this question is reduced to show the existence of critical points for the functional

\[ F(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{\lambda}{q} \int_{\Omega} |u|^q - \frac{1}{p^*} \int_{\Omega} |u|^{p^*}. \]

Under hypothesis (1.2), \( F(u) \) is defined on the Sobolev space \( W^{1,p}_0(\Omega) \).

By using the so-called generalized Pohozaev identity, it is possible to prove that, if the domain \( \Omega \) is starshaped, then (1.1) cannot have any nontrivial solution in \( W^{1,p}_0(\Omega) \) if \( \lambda \leq 0 \) (see [P-S], and also [O, G-V, and E]); therefore, we are reduced to consider positive \( \lambda \).

For \( p = q \) the problem is studied in [G-P.1] where the existence of positive solution for the dimensions \( N \) such that \( p^2 \leq N \) is obtained, if \( 0 < \lambda < \lambda_1 \), \( \lambda_1 \) being the first eigenvalue for the \( p \)-Laplacian (\( \lambda_1 \) is isolated and simple, as it is obtained in [Ba]; see also [Bha and A]). The main difficulty in solving problem (1.1) is the lack of compactness in the inclusion of \( W^{1,p}_0(\Omega) \) in \( L^p \), because in general the Palais-Smale condition is not satisfied.

In the case \( p = 2 \), the problem has been solved by Brézis-Nirenberg [B-N]. As in [B-N], we obtain a local Palais-Smale condition for the case \( p \neq 2 \) which is sufficient. This question is handled in §2 by the concentration-compactness principle of P. L. Lions (see [L1 and L2]).

In §3 we analyze the case \( p < q < p^* \) and achieve some new results with respect to those obtained in [G-P.1].

The case \( 1 < q < p \) is managed in §4 by classical critical point theory. See [B-F and G-P.1] for related methods in the subcritical case.

Obviously, more general terms can be handled if their behaviour at 0 and at infinity is the same.

Finally, in §5, we solve some nonsymmetric problems. Following the ideas in [R1] we also obtain multiplicity results in this case.

For the regularity of the solutions, see [T and DiB].

2. The Palais-Smale Condition

A sequence \( \{u_j\} \subset W^{1,p}_0(\Omega) \) is called a Palais-Smale sequence for \( F \), defined by (1.3), if

\[ F(u_j) \to c, \]

\[ F'(u_j) \to 0 \quad \text{in} \quad W^{-1,p'}(\Omega), \quad \text{where} \quad 1/p + 1/p' = 1. \]

If (2.1) implies the existence of a subsequence \( \{u_{j_k}\} \subset \{u_j\} \) which converges in \( W^{1,p}_0(\Omega) \), we say that \( F \) verifies the Palais-Smale condition.

If this strongly convergent subsequence exists only for some \( c \) values, we say that \( F \) verifies a local Palais-Smale condition.

In our case, the main difficulty is the lack of compactness in the inclusion of \( W^{1,p}_0(\Omega) \) in \( L^{p^*} \). Then, we prove a local Palais-Smale condition, which is sufficient although with some restrictions.
The technical results which we must use are based on a measure representation lemma, used by P. L. Lions in the proof of the concentration-compactness principle (see [L1 and L2]).

Let \{u_j\} be a bounded sequence in \(W_{0}^{1, p}(\Omega)\). Then, there is a subsequence, such that \(u_j \rightharpoonup u\), weakly in \(W_{0}^{1, p}(\Omega)\), and
\[
\|\nabla u_j\|^p \rightarrow d\mu \\
\|u_j\|^p \rightarrow d\nu
\]
weakly-* in the sense of measures.

If we take \(\varphi \in C^\infty_0 (\mathbb{R}^N)\), by some calculations with the Sobolev inequality we conclude that
\[
(2.2) \quad \left( \int_\Omega |\varphi|^p \, d\nu \right)^{1/p^*} S^{1/p} \leq \left( \int_\Omega |\varphi|^p \, d\mu \right)^{1/p} + \left( \int_\Omega |\nabla \varphi|^p |u|^p \, dx \right)^{1/p}
\]
where
\[
S = \inf\{\|u\|_{W_{0}^{1, p}(\Omega)} : u \in W_{0}^{1, p}(\Omega), \|u\|_{p^*} = 1\}
\]
is the best constant in the Sobolev inclusion.

If, in (2.2), we suppose \(u \equiv 0\), then we have a reverse Hölder inequality for two different measures. In this situation, we have the following representation of the measures (see P. L. Lions [L1 and L2]):

**Lemma 2.1.** Let \(\mu, \nu\) be two nonnegative and bounded measures on \(\overline{\Omega}\), such that for \(1 < p < r < \infty\) there exists some constant \(C > 0\) such that
\[
\left( \int_\Omega |\varphi|^r \, d\nu \right)^{1/r} \leq C \left( \int_\Omega |\varphi|^p \, d\mu \right)^{1/p} \quad \forall \varphi \in C^\infty_0 (\mathbb{R}^N).
\]
Then, there exist \(\{x_j\}_{j \in J} \subset \overline{\Omega}\) and \(\{\nu_j\}_{j \in J} \subset (0, \infty)\), where \(J\) is at most countable, such that
\[
\nu = \sum_{j \in J} \nu_j \delta_{x_j}, \quad \mu \geq C^{-p} \sum_{j \in J} \nu_j^{p/r} \delta_{x_j},
\]
where \(\delta_{x_j}\) is the Dirac mass at \(x_j\).

If we apply this lemma to \(\nu_j = u_j - u\), we obtain the following result, due to P. L. Lions (see [L1 and L2]):

**Lemma 2.2.** Let \(\{u_j\}\) be a weakly convergent sequence in \(W_{0}^{1, p}(\Omega)\) with weak limit \(u\), and such that
\[
(i) \quad |\nabla u_j|^p \text{ converges in the weak-* sense of measures to a measure } \mu,
(ii) \quad |u_j|^p \text{ converges in the weak-* sense of measures to a measure } \nu.
\]
Then, for some at most countable index set \(J\) we have
\[
(1) \quad \nu = |u|^p + \sum_{j \in J} \nu_j \delta_{x_j}, \quad \nu_j > 0,
(2.3) \quad \mu \geq |\nabla u|^p + \sum_{j \in J} \mu_j \delta_{x_j}, \quad \mu_j > 0,
(3) \quad \nu_j^{p/p^*} \leq \mu_j / S,
\]
where \(x_j \in \overline{\Omega}\).
The relations (2.3) with the hypothesis that the constant $c$ in (2.1) is small enough allow us to prove that the singular part of the measures must be 0, and we have a local Palais-Smale condition.

**Lemma 2.3.** Let $\{v_j\} \subset W^{1,p}_0(\Omega)$ be a Palais-Smale sequence for $F$, defined by (1.3), that is,

(2.4) \[ F(v_j) \to C, \]

(2.5) \[ F'(v_j) \to 0 \text{ in } W^{-1,p'}(\Omega), \quad 1/p + 1/p' = 1. \]

Then, we have

(a) If $p < q < p^*$, and $C < S^{N/p}/N$, there exists a subsequence $\{v_{j_k}\} \subset \{v_j\}$, strongly convergent in $W^{1,p}_0(\Omega)$.

(b) If $1 < q < p$, and $C < S^{N/p}/N - K\lambda^\beta$, where $\beta = p^*/(p^* - q)$ and $K$ depends on $p$, $q$, $N$ and $\Omega$, then there exists a subsequence $\{v_{j_k}\} \subset \{v_j\}$, strongly convergent in $W^{1,p}_0(\Omega)$.

**Proof.** In both cases, by (2.4) and (2.5), it is easy to prove that the sequence $\{v_j\}$ is bounded in $W^{1,p}_0(\Omega)$. Then, if we take the appropriate subsequence, we can assume in both cases (by Lemma 2.2)

\[
 v_j \rightharpoonup v \text{ weakly in } W^{1,p}_0(\Omega),
\]

\[
 v_j \to v \text{ in } L^r, \quad 1 < r < p^*, \text{ and a.e.},
\]

(2.6) \[
 |\nabla v_j|^p \to d\mu \geq |\nabla v|^p + \sum_{k \in J} \mu_k \delta_{x_k},
\]

\[
 |v_j|^{p^*} \to d\nu = |v|^{p^*} + \sum_{k \in J} \nu_k \delta_{x_k}.
\]

Take $x_k \in \overline{\Omega}$ in the support of the singular part of $d\mu, d\nu$. We consider $\phi \in C_0^\infty(\mathbb{R}^N)$, such that

(2.7) \[
 \phi \equiv 1 \text{ on } B(x_k, \epsilon), \quad \phi \equiv 0 \text{ on } B(x_k, 2\epsilon)^c, \quad |\nabla \phi| \leq 2/\epsilon.
\]

It is clear that the sequence $\{\phi v_j\}$ is bounded in $W^{1,p}_0(\Omega)$; then, by using hypothesis (2.5), $\lim (F'(v_j), \phi v_j) = 0$ ($\langle , \rangle$ is the duality product), and

\[
 \int \phi \, d\nu + \lambda \int |v|^q \phi \, dx - \int \phi \, d\mu = \lim \int |\nabla v_j|^{p^* - 2} v_j (\nabla v_j, \nabla \phi) \, dx
\]

($\langle , , \rangle$ is the product in $\mathbb{R}^N$). By (2.6), (2.7), and the Hölder inequality, we obtain

\[
 0 \leq \lim_{\epsilon \to 0} \int |\nabla v_j|^{p^* - 2} v_j (\nabla v_j, \nabla \phi) \, dx \leq C \left( \int_{B(x_k, 2\epsilon)} |v|^{p^*} \right)^{1/p^*} \to 0.
\]
Then,

\[ 0 = \lim_{t \to 0} \left\{ \int \phi \, d\nu + \lambda \int |v|^q \phi \, dx - \int \phi \, d\mu \right\} = \nu_k - \mu_k. \]

By Lemma 2.2, \( \mu_k \geq S_{\nu_k}^{p/p^*} \), i.e. \( \nu_k \geq S_{\nu_k}^{p/p^*} \). That is, \( \nu_k = 0 \), or

(2.8) \[ \nu_k \geq S^{N/p}. \]

(In particular, there are, at most, a finite number of singularities, because \( d\nu \) is a bounded measure.) We will prove that (2.8) is not possible.

Let us assume that there exists a \( k_0 \) with \( \nu_{k_0} \neq 0 \) i.e. \( \nu_{k_0} \geq S^{N/p} \). By (2.4) and (2.6),

\[ C = \lim_{j} F(v_j) \geq F(v) + \left( \frac{1}{p} - \frac{1}{p^*} \right) \sum \nu_k \geq F(v) + \frac{1}{N} S^{N/p}. \]

But, by hypothesis, \( C < S^{N/p}/N \); then, \( F(v) < 0 \). In particular, \( v \neq 0 \), and

\[ 0 < \frac{1}{p} \int |\nabla v|^p < \frac{1}{p^*} \int |v|^{p^*} + \frac{\lambda}{q} \int |v|^q. \]

That is,

(2.9) \[ C = \lim_{j} F(v_j) = \lim_{j} \{ F(v_j) - 1/p \langle F'(v_j), v_j \rangle \} \geq \frac{1}{N} \int |v|^{p^*} + \frac{1}{N} S^{N/p} + \lambda \left( \frac{1}{p} - \frac{1}{q} \right) \int |v|^q. \]

Now we distinguish two cases:

(a) If \( p < q < p^* \), then \( C > S^{N/p}/N \), and this inequality contradicts the hypothesis for this case. Then, \( \nu_k = 0 \) \( \forall k \), and \( \lim_{j} \int |v_j|^p = \int |v|^{p^*} \). By using (2.6), we conclude that \( v_j \to v \) in \( L^{p^*} \), and, finally, because of the continuity of \( \Delta_{-1} \), \( v_j \to v \) in \( W^{1,p}_0 (\Omega) \).

(b) If \( 1 < q < p \), applying the Hölder inequality at (2.8), we have

\[ C \geq \frac{1}{N} S^{N/p} + \frac{1}{N} \int |v|^{p^*} - \lambda \left( \frac{1}{q} - \frac{1}{p} \right) |\Omega|^{(p^*-q)/p^*} \left( \int |v|^{p^*} \right)^{q/p^*}. \]

Let \( f(x) = c_1 x^{p^*} - \lambda c_2 x^q \). This function attains its absolute minimum (for \( x > 0 \)) at the point \( x_0 = (\lambda c_2 q/p c_1)^{1/(p^*-q)} \). That is,

\[ f(x) \geq f(x_0) = -K \lambda^{p^*/(p^*-q)}. \]

But this result contradicts the hypothesis; then, \( \nu_k = 0 \) \( \forall k \), and we conclude. □

Remark 2.4. It is well known that it is impossible to improve this local Palais-Smale condition in case (a); we can construct a Palais-Smale sequence with \( C = S^{N/p}/N \), without any convergent subsequence (see [B]).

In case (b) it is also possible to exhibit a counterexample; we construct this counterexample at the end of §4. □
3. The case $p < q < p^*$

In §2, we have proved that below the level $S^{N/p}/N$, the functional $F$ verifies a local Palais-Smale condition. In this section we will use the Mountain Pass Lemma to prove the existence of a solution for problem (1.1).

We will use the following general version of the Mountain Pass Lemma (see [A-E] for the proof).

**Lemma 3.1.** Let $F$ be a functional on a Banach space $X$, $F \in C^1(X, \mathbb{R})$. Let us assume that there exists $r, R > 0$, such that

(i) $F(u) > r$, $\forall u \in X$ with $\|u\| = R$,

(ii) $F(0) = 0$, and $F(w_0) < r$ for some $w_0 \in X$, with $\|w_0\| > R$.

Let us define $C = \{g \in C([0, 1]; X): g(0) = 0, g(1) = w_0\}$, and

\[
(3.1) \quad c = \inf_{g \in C} \max_{t \in [0,1]} F(g(t)).
\]

Then, there exists a sequence $\{u_j\} \subset X$, such that $F(u_j) \to c$, and $F'(u_j) \to 0$ in $X^*$ (dual of $X$).

In our case, it is easy to see that $F$ verifies (i) and (ii).

If we can prove that

\[
(3.2) \quad c < S^{N/p}/N
\]

then Lemma 3.1 and Lemma 2.3 give the existence of the critical point of $F$.

To obtain (3.2), we choose $v_0 \in W^{1,p}(\Omega)$, with

\[
(3.3) \quad \|v_0\|_{p^*} = 1, \quad \lim_{t \to \infty} F(tv_0) = -\infty;
\]

then, $\sup_{t \geq 0} F(tv_0) = F(t_\lambda v_0)$, for some $t_\lambda > 0$. Thus $t_\lambda$ verifies

\[
(3.4) \quad 0 = t_\lambda^{p-1} \int |\nabla v_0|^p - t_\lambda^{p^*-1} \int |v_0|^{p^*} - \lambda t_\lambda^{q-1} \int |v_0|^q
\]

and we get

\[
0 = t_\lambda^{q-1} \left( t_\lambda^{p^*-q} \left( \int |\nabla v_0|^p \right) - t_\lambda^{p^*-q} - \lambda \int |v_0|^q \right).
\]

Observe that

\[
t_\lambda^{p^*-q} + \lambda \int |v_0|^q \xrightarrow[\lambda \to \infty]{} \infty;
\]

therefore, (3.4) implies $t_\lambda \xrightarrow[\lambda \to \infty]{} 0$. By the continuity of $F$,

\[
\lim_{\lambda \to \infty} \left( \sup_{t \geq 0} F(tv_0) \right) = 0;
\]

then, there exists $\lambda_0$ such that $\forall \lambda \geq \lambda_0$,

\[
\sup_{t \geq 0} F(tv_0) < S^{N/p}/N.
\]
If we take \( w_0 = tv_0 \), with \( t \) large enough to verify \( F(w_0) < 0 \), we get
\[
c \leq \max_{t \in [0,1]} F(g_0(t)) \quad \text{taking} g_0(t) = tw_0.
\]
Therefore, \( c \leq \sup_{t \geq 0} F(tv_0) < S^{N/p}/N \), and we have proved estimate (3.2), for \( \lambda \) large enough. Hence, we can apply Lemma 3.1 and Lemma 2.3, and we have the following result:

**Theorem 3.2.** If \( p < q < p^* \), there exists \( \lambda_0 > 0 \) such that problem (1.1) has a nontrivial solution \( \forall \lambda \geq \lambda_0 \).

By choosing carefully the function \( v_0 \in W^{1,p}_0(\Omega) \) in (3.3), we can prove the following stronger result:

**Theorem 3.3.** If \( \max(p, p^* - p/(p-1)) < q < p^* \), then there exists a nontrivial solution of problem (1.1), \( \forall \lambda > 0 \).

**Proof.** The natural choice is to take an appropriated truncation of
\[
U_\varepsilon(x) = (\varepsilon + c|x - x_0|^{p/(p-1)}(p-N)/p
\]
because they are the functions in \( W^{1,p}(\mathbb{R}^N) \) where the best constant in the Sobolev inclusion is attained. It is well known that they are the unique positives, except for translations and dilations (see [B, L1, L2]).

We can assume that \( 0 \in \Omega \), and consider \( x_0 = 0 \) at (3.5).

Let \( \phi \) be a function \( \phi \in C_0^\infty(\Omega) \), and \( \phi(x) \equiv 1 \) in a neighbourhood of the origin. We define \( u_\varepsilon(x) = \phi(x)U_\varepsilon(x) \). For \( \varepsilon \to 0 \), the behaviour of \( u_\varepsilon \) has to be like \( U_\varepsilon \), and we can estimate the error we get when we take \( u_\varepsilon \) instead of \( U_\varepsilon \).

In this way, taking \( v_\varepsilon = u_\varepsilon/\|u_\varepsilon\|_{p^*} \), we obtain the following estimates (see [B-N, G-P.l] for the details):

1. **Estimate for the gradient:**
   \[
   \|\nabla v_\varepsilon\|_p^p = S + O(\varepsilon^{(N-p)/p}).
   \]

2. **Estimate of** \( \|v_\varepsilon\|_q \):
   - if \( q > p^*(1 - 1/p) \), then
   \[
   C_1 \varepsilon^{(p-1)/p(N-q(N-p)/p)} \leq \|v_\varepsilon\|_q \leq C_2 \varepsilon^{(p-1)/p(N-q(N-p)/p)}.
   \]
   - If \( q = p^*(1 - 1/p) \), then
   \[
   C_1 \varepsilon^{(N-p)q/p^2} |\log \varepsilon| \leq \|v_\varepsilon\|_q \leq C_2 \varepsilon^{(N-p)q/p^2} |\log \varepsilon|.
   \]
   - If \( q < p^*(1 - 1/p) \), then
   \[
   C_1 \varepsilon^{(N-p)q/p^2} \leq \|v_\varepsilon\|_q \leq C_2 \varepsilon^{(N-p)q/p^2}.
   \]

Observe that, if \( p < q < p^* \), then
\[
\|v_\varepsilon\|_q \xrightarrow[\varepsilon \to 0]{} 0.
\]
By using these estimates, we will show that there exists $\varepsilon > 0$, small enough, such that
\[
\sup_{t \geq 0} F(t v_\varepsilon) < S^{N/p}/N.
\]

Then, we conclude as in Theorem 3.2, by using Lemma 3.1 and Lemma 2.3.

Let us consider the functions
\[
g(t) = F(t v_\varepsilon) = \frac{t^p}{p} \int |\nabla v_\varepsilon|^p - \frac{t^{p^*}}{p^*} - \frac{\lambda t^q}{q} \int |v_\varepsilon|^q,
\]
and
\[
\overline{g}(t) = \frac{t^p}{p} \int |\nabla v_\varepsilon|^p - \frac{t^{p^*}}{p^*}.
\]

It is clear that $g(t) \xrightarrow{t \to \infty} -\infty$; then, $\sup_{t \geq 0} F(t v_\varepsilon)$ is attained for some $t_\varepsilon > 0$, and
\[
0 = g'(t_\varepsilon) = t_\varepsilon^{p-1} \left( \int |\nabla v_\varepsilon|^p - t_\varepsilon^{p^* - p} - \lambda t_\varepsilon^{q - p} \int |v_\varepsilon|^q \right).
\]

Therefore,
\[
\int |\nabla v_\varepsilon|^p = t_\varepsilon^{p^* - p} + \lambda t_\varepsilon^{q - p} \int |v_\varepsilon|^q > t_\varepsilon^{p^* - p},
\]
i.e.
\[
(3.11) \quad t_\varepsilon \leq \left( \int |\nabla v_\varepsilon|^p \right)^{1/(p^* - p)}.
\]

This inequality implies
\[
(3.12) \quad \int |\nabla v_\varepsilon|^p \leq t_\varepsilon^{p^* - p} + \lambda \left( \int |\nabla v_\varepsilon|^p \right)^{(q - p)/(p^* - p)} \left( \int |v_\varepsilon|^q \right).
\]
Choosing $\varepsilon$ small enough, by (3.6) and (3.10),
\[
(3.13) \quad t_\varepsilon^{p^* - p} \geq S/2.
\]
That is, we have a lower bound for $t_\varepsilon$, independent of $\varepsilon$. Now, we estimate $g(t_\varepsilon)$.

The function $\overline{g}$ attains its maximum at $t = (\int |\nabla v_\varepsilon|^p)^{1/(p^* - p)}$, and is increasing at the interval $[0, (\int |\nabla v_\varepsilon|^p)^{1/(p^* - p)}]$. Then, by using (3.6), (3.11) and (3.13), we have
\[
g(t_\varepsilon) = \overline{g}(t_\varepsilon) - \frac{\lambda}{q} t_\varepsilon^q \int |v_\varepsilon|^q
\leq \overline{g} \left( \left( \int |\nabla v_\varepsilon|^p \right)^{1/(p^* - p)} \right) - \frac{\lambda}{q} t_\varepsilon^q \int |v_\varepsilon|^q
\leq \frac{1}{N} S^{N/p} + C_3 e^{(N-p)/p} - \frac{\lambda}{q} \left( \frac{S}{2} \right)^{q/(p^* - p)} \int |v_\varepsilon|^q.
\]
Let us suppose \( q > p^*(1 - 1/p) \). Then, we have (3.7), and
\[
g(t_{\varepsilon}) \leq S^{N/p}/N + C^3 e^{(N-p)/p} - \lambda C_1 e^{((p-1)/(p)(N-q(N-p)/p)).}
\]
If
\[
\frac{N-p}{p} > \frac{p-1}{p} \left( N - q \frac{(N-p)}{p} \right),
\]
that is, \( q > p^* - p/(p-1) \), then for \( \varepsilon \) small enough we get \( g(t_{\varepsilon}) < S^{N/p}/N \), and we conclude. \( \Box \)

Remark 3.4. If \( N \geq p^2 \), then \( p^* - \frac{p}{p-1} \leq p^*(1 - \frac{1}{p}) \leq p \), and if \( p < q < p^* \), we have \( q > p^* - \frac{p}{p-1} \). Then \( q \) verifies the estimate (3.7), and we obtain the result of [G-P.1].

If \( N < p^2 \), then \( q > p^* - \frac{p}{p-1} \), and for \( p < q < p^* - \frac{p}{p-1} \) the estimate is insufficient. \( \Box \)

Remark 3.5. It is possible to prove the analogous result for the problem:
\[
\begin{cases}
-\text{div}(|\nabla u|^{p-2}\nabla u) = |u|^{q-2}u + \lambda |u|^q u + f, \quad \lambda > 0 \\
u|_{\partial \Omega} = 0
\end{cases}
\]
if \( f \) is small enough in the norm of \( W^{-1,p'}(\Omega) \). The proof is an adaptation of the above argument. \( \Box \)

4. The case \( 1 < q < p \)

In this section, we will construct a mini-max class of critical points, by using the classical concept and properties of the genus.

Let \( X \) be a Banach space, and \( \Sigma \) the class of the closed and symmetric with respect to the origin subsets of \( X - \{0\} \). For \( A \in \Sigma \), we define the genus \( \gamma(A) \) by
\[
\gamma(A) = \min\{k \in \mathbb{N} : \exists \phi \in C(A; \mathbb{R}^k - \{0\}), \phi(x) = -\phi(-x)\}.
\]
If such a minimum does not exist then we define \( \gamma(A) = +\infty \). The main properties of the genus are the following (see [R1 or R2] for the details):

**Proposition 4.1.** Let \( A, B \in \Sigma \). Then

1. If there exists \( f \in C(A, B) \), odd, then \( \gamma(A) \leq \gamma(B) \).
2. If \( A \subset B \), then \( \gamma(A) \leq \gamma(B) \).
3. If there exists an odd homeomorphism between \( A \) and \( B \), then \( \gamma(A) = \gamma(B) \).
4. If \( S^{N-1} \) is the sphere in \( \mathbb{R}^N \), then \( \gamma(S^{N-1}) = N \).
5. \( \gamma(A \cup B) \leq \gamma(A) + \gamma(B) \).
6. If \( \gamma(B) < +\infty \), then \( \gamma(A - B) \geq \gamma(A) - \gamma(B) \).
7. If \( A \) is compact, then \( \gamma(A) < +\infty \), and there exists \( \delta > 0 \) such that \( \gamma(A) = \gamma(N_\delta(A)) \) where \( N_\delta(A) = \{x \in X : d(x, A) \leq \delta\} \).
If $X_0$ is a subspace of $X$ with codimension $K$, and $\gamma(A) < K$, then $A \cap X_0 \neq \emptyset$.

Given the functional $F$, defined by (1.3), under the hypothesis $q < p$, by Sobolev's inequality we obtain

$$F(u) \geq \frac{1}{p} \int |\nabla u|^p - \frac{1}{p^* S^{p^*/p}} \left( \int |\nabla u|^p \right)^{p^*/p} - \frac{\lambda}{q} C_{p,q} \left( \int |\nabla u|^p \right)^{q/p}.$$

If we define

$$h(x) = \frac{1}{p} x^p - \frac{1}{p^* S^{p^*/p}} x^{p^*} - \frac{\lambda}{q} C_{p,q} x^q$$

then

$$F(u) \geq h(\|\nabla u\|_p).$$

There exists $\lambda_1 > 0$ such that, if $0 < \lambda \leq \lambda_1$, $h$ attains its positive maximum (see Figure 4.1).

Let us assume $0 < \lambda \leq \lambda_1$; choosing $R_0$ and $R_1$ as in Figure 4.1 we make the following truncation of the functional $F$:

Take $\tau: \mathbb{R}^+ \rightarrow [0, 1]$, nonincreasing and $C^\infty$, such that

$$\tau(x) = 1 \quad \text{if} \quad x < R_0,$$

$$\tau(x) = 0 \quad \text{if} \quad x > R_1.$$

Let $\varphi(u) = \tau(\|\nabla u\|_p)$. We consider the truncated functional

$$J(u) = \frac{1}{p} \int |\nabla u|^p - \frac{1}{p^*} \int |u|^{p^*} \varphi(u) - \frac{\lambda}{q} \int |u|^q.$$

As in (4.1), $J(u) \geq \overline{h}(\|\nabla u\|_p)$, with

$$\overline{h}(x) = \frac{1}{p} x^p - \frac{1}{p^* S^{p^*/p}} x^{p^*} \tau(x) - \frac{\lambda}{q} C_{p,q} x^q$$

(see Figure 4.2).
Observe that for \( x \leq R_0 \), \( h = h \), and for \( x \geq R_1 \),
\[
\bar{h}(x) = \frac{1}{p} x^p - \frac{1}{q} C_p q x^q.
\]

The principal properties of \( J \) defined by (4.2) are:

**Lemma 4.2.**  
(1) \( J \in C^1(\mathcal{W}^{1,p}(\Omega), \mathbb{R}) \).

(2) If \( J(u) \leq 0 \), then \( \|\nabla u\|_p < R_0 \), and \( F(v) = J(v) \) for all \( v \) in a small enough neighbourhood of \( u \).

(3) There exists \( \lambda_1 > 0 \), such that, if \( 0 < \lambda < \lambda_1 \), then \( J \) verifies a local Palais-Smale condition for \( c \leq 0 \).

**Proof.** (1) and (2) are immediate. To prove (3), observe that all Palais-Smale sequences for \( J \) with \( c \leq 0 \) must be bounded; then, by Lemma 2.3, if \( \lambda \) verifies \( S^{N/p}/N - K_\lambda^\theta \geq 0 \) there exists a convergent subsequence. \( \Box \)

Observe that, by (2), if we find some negative critical value for \( J \), then we have a negative critical value of \( F \).

Now, we will construct an appropriate mini-max sequence of negative critical values for the functional \( J \).

**Lemma 4.3.** Given \( n \in \mathbb{N} \), there is \( \varepsilon = \varepsilon(n) > 0 \), such that
\[
\gamma(\{u \in \mathcal{W}^{1,p}_0(\Omega) : J(u) \leq -\varepsilon\}) \geq n.
\]

**Proof.** Fix \( n \), let \( E_n \) be an \( n \)-dimensional subspace of \( \mathcal{W}^{1,p}_0(\Omega) \). We take \( u_n \in E_n \), with norm \( \|\nabla u_n\|_p = 1 \). For \( 0 < \rho < R_0 \), we have
\[
J(\rho u_n) = F(\rho u_n) = \frac{1}{p} \rho^p - \frac{1}{p^*} \rho^{p^*} \int |u|^{p^*} - \frac{\lambda}{p} \rho^q \int |u|^q.
\]

\( E_n \) is a space of finite dimension; so, all the norms are equivalent. Then, if we define
\[
\alpha_n = \inf \left\{ \int |u|^{p^*} : u \in E_n, \|\nabla u_n\|_p = 1 \right\} > 0,
\]
\[
\beta_n = \inf \left\{ \int |u|^q : u \in E_n, \|\nabla u_n\|_p = 1 \right\} > 0,
\]
we have
\[
J(\rho u_n) \leq \frac{1}{p} \rho^p - \frac{\alpha_n}{p^*} \rho^{p^*} - \frac{\lambda \beta_n}{q} \rho^q,
\]
and we can choose \( \varepsilon \) (which depends on \( n \)), and \( \eta < R_0 \), such that \( J(\eta u) \leq -\varepsilon \) if \( u \in E_n \), and \( \|\nabla u\|_p = 1 \).

Let \( S_\eta = \{u \in \mathcal{W}^{1,p}_0(\Omega) : \|\nabla u\|_p = \eta \} \). \( S_\eta \cap E_n \subset \{u \in \mathcal{W}^{1,p}_0(\Omega) : J(u) \leq -\varepsilon\} \); therefore, by Proposition 4.1,
\[
\gamma(\{u \in \mathcal{W}^{1,p}_0(\Omega) : J(u) \leq -\varepsilon\}) \geq \gamma(S_\eta \cap E_n) = n. \quad \Box
\]

This lemma allows us to prove the existence of critical points.
Lemma 4.4. Let $\Sigma_k = \{ C \subset W_0^{1,p}(\Omega) - \{0\}, C$ is closed, $C = -C, \gamma(C) \geq k \}$.

Let $c_k = \inf_{C \in \Sigma_k} \sup_{u \in C} J(u), K_c = \{ u \in W_0^{1,p}(\Omega): J'(u) = 0, J(u) = c \}$, and suppose $0 < \lambda < \lambda_1$, where $\lambda_1$ is the constant of Lemma 4.2.

Then, if $c = c_k = c_{k+1} = \cdots = c_{k+r}, \gamma(K_c) \geq r + 1$.

(In particular, the $c_k$'s are critical values of $J$.)

Proof. In the proof, we will use Lemma 4.3, and a classical deformation lemma (see [Be]).

For simplicity, we call $J^{-\varepsilon} = \{ u \in W_0^{1,p}(\Omega): J(u) \leq -\varepsilon \}$. By Lemma 4.3, $\forall k \in \mathbb{N}, \exists \varepsilon(k) > 0$ such that $\gamma(J^{-\varepsilon}) \geq k$.

Because $J$ is continuous and even, $J^{-\varepsilon} \in \Sigma_k$; then, $c_k - \varepsilon(k) < 0, \forall k$. But $J$ is bounded from below; hence, $c_k > -\infty \forall k$.

Let us assume that $c = c_k = \cdots = c_{k+r}$. Let us observe that $c < 0$; therefore, $J$ verifies the Palais-Smale condition in $K_c$, and it is easy to see that $K_c$ is a compact set.

If $\gamma(K_c) \leq r$, there exists a closed and symmetric set $U$, $K_c \subset U$, such that $\gamma(U) \leq r$. (We can choose $U \subset J^0$, because $c < 0$.)

By the deformation lemma, we have an odd homeomorphism

$$\eta: W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega),$$

such that $\eta(J^{c+\delta} - U) \subset J^{c-\delta}$, for some $\delta > 0$. (Again, we must choose $0 < \delta < -c$, because $J$ verifies the Palais-Smale condition on $J^0$, and we need $J^{c+\delta} \subset J^0$.) By definition,

$$c = c_{k+r} = \inf_{C \in \Sigma_{k+r}} \sup_{u \in C} J(u).$$

Then, there exists $A \in \Sigma_{k+r}$, such that $\sup_{u \in A} J(u) < c + \delta$; i.e., $A \subset J^{c+\delta}$, and

$$\eta(A - U) \subset \eta(J^{c+\delta} - U) \subset J^{c-\delta}.$$

But $\gamma(A - U) \geq \gamma(A) - \gamma(U) \geq k$, and $\gamma(\eta(A - U)) \geq \gamma(A - U) \geq k$.

Then, $\eta(A - U) \in \Sigma_k$. And this contradicts (4.4); in fact,

$$\eta(A - U) \in \Sigma_k \implies \sup_{u \in \eta(A - U)} J(u) \geq c_k = c. \quad \Box$$

This lemma proves the following result:

**Theorem 4.5.** Given problem (1.1), with $1 < q < p$, there exists $\lambda_1 > 0$, such that, for $0 < \lambda < \lambda_1$, there exists infinitely many solutions.

**Remark 4.6.** (1) For the truncated functional $J$, a result of Brezis-Oswald [B-O] for $p = 2$, which is extended to a general case for Diaz-Saa [D-S], proves the uniqueness of nontrivial positive solutions.

Then, the solutions that we find change the sign, except for those associated with $c_1$. In fact, $c_1 = \inf J(u)$, and, if $c_1 = J(u_0)$, then $c_1 = J(|u_0|)$. That
is, $|u_0|$ is a nonnegative solution, and, by the maximum principle (see [T]), is strictly positive on $\Omega$.

Observe that there is not a uniqueness result for the nontruncated functional $F$. It remains open the question of the existence of positive solutions with positive energy (solutions as those of §3, for $p < q < p^*$).

(2) It is possible to make another proof of Theorem 4.5, if we replace the truncation of $F$ by a special construction of the deformation function $\eta$. In fact, we can take $\eta$ which acts on $B(0, \rho_0) \subset W^{1,p}_0(\Omega)$, and is the identity otherwise; then we must define

$$\Sigma_k = \{ C \subset B(0, \rho_0) - \{0\}: C \text{ closed, symmetric, } \gamma(C) \geq k \}.$$  

(3) The critical values that we have obtained are negative, and $F$ verifies the Palais-Smale condition for $c < 0$; then, it is easy to see that the set of solutions of Theorem 4.5, is a compact set. \hfill \Box

Now, we can show that it is not possible to extend the Palais-Smale condition that we have proved.

Take $x_0 \in \Omega$, and the balls $B_j = B(x_0, j\delta) \subset \Omega$, and the following $C^\infty(\mathbb{R}^N)$ functions:

$$
\begin{align*}
\phi_\delta &\equiv 1 \quad \text{on } \Omega - B_3, \quad \xi_\delta \equiv 1 \quad \text{on } B_1, \\
\phi_\delta &\equiv 0 \quad \text{on } B_2, \quad \xi_\delta \equiv 0 \quad \text{on } \Omega - B_2, \\
|\nabla \phi_\delta| &< 2/\delta, \quad |\nabla \xi_\delta| < 2/\delta.
\end{align*}
$$

We define $\phi_\delta = \phi_\delta v + \xi_\delta wE$, where $F'(v) = 0$, and $F(v) < 0$, $F(wE) \rightarrow S^{N/p}/N$ and $F'(wE) \rightarrow 0$. (Take $wE = s^{(N-\mu)/\rho^2}vE$, with $vE$ defined in §3.) Later, we shall choose $\varepsilon = \varepsilon(\delta) \rightarrow 0$.

Then, $F'(\phi_\delta) = F'(\phi_\delta v) + F'(\xi_\delta wE)$, and we can show that $F'(\phi_\delta) \rightarrow C < S^{N/p}/N$, with $F'(\phi_\delta) \rightarrow 0$.

But it is not possible to find a convergent subsequence of $\{\phi_\delta\}$, because $\phi_\delta \rightarrow v$ but

$$
\|\phi_\delta - v\|_{W^{1,p}_0(\Omega)} = \|\phi_\delta - 1\|v + \xi_\delta wE\|_{W^{1,p}_0(\Omega)}
\geq \|\xi_\delta wE\|_{W^{1,p}_0(\Omega)} - \|\phi_\delta - 1\|v\|_{W^{1,p}_0(\Omega)} > M > 0
$$

with $M$ independent of $\delta$.

5. A PROBLEM WITHOUT SYMMETRY

We shall consider the following model problem:

$$
-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{q-2}u + f(x), \quad \lambda > 0, \\
|u|_{\partial \Omega} = 0,
$$

where $\Omega$ is a rectangle in $\mathbb{R}^N$, and $p < q < p^*$, $1 < p < N$. When $f \equiv 0$, there are infinitely many solutions $\forall \lambda > 0$. In the proof, we use a mini-max type theory, as in §4, because the associated functional is even.
When \( f \neq 0 \), the associated functional is

\[
I(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{\lambda}{q} \int_{\Omega} |u|^q - \int_{\Omega} fu.
\]

We cannot apply the previous method, because \( I \) is not even; however, it is possible to make use of the method developed by P. Rabinowitz in the case \( p = 2 \) (see [R1 and R2]). For the sake of completeness, we will include here the proofs of the more interesting steps.

The point is the lack of control on the nonsymmetric part of the functional \( I \); that is, \( I(u) - I(-u) \). The idea is to find some appropriated truncation of \( I \), in order to obtain a functional \( J \), in which the nonsymmetric part can be estimated, such that the existence of critical points for \( J \) implies the existence of critical points for \( I \). We start with an a priori estimate, which gives us the idea to make the truncation.

**Lemma 5.1.** There exists a constant \( A = A(\|f\|_{p'}) > 0 \) such that, if \( I'(u) = 0 \), then

\[
\frac{\lambda}{q} \int_{\Omega} |u|^q \leq A(\|I(u)\|^p + 1)^{1/p}.
\]

(The proof is an easy adaptation of those made in [R1].) With this estimate, we make the following truncation: Let \( \chi: \mathbb{R} \rightarrow [0,1] \) such that

\[
\begin{align*}
\chi(x) &= 0, \quad x \geq 2, \\
\chi(x) &= 1, \quad x \leq 1, \\
-2 &\leq \chi'(x) \leq 0
\end{align*}
\]

and

\[
\psi(u) = \chi \left\{ \frac{(\lambda/q) \int |u|^q}{2A(\|I(u)\|^p + 1)^{1/p}} \right\}.
\]

Define

\[
J(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{\lambda}{q} \int_{\Omega} |u|^q - \int_{\Omega} \psi(u)fu.
\]

In particular, Lemma 5.1 implies that, if \( I'(u) = 0 \), then \( J'(u) = 0 \). However, we need just the converse. The main properties of \( J \) are the following (for the proof, see [R1]):

**Lemma 5.2.**

(i) \( J \in C^1(W_0^{1,p}(\Omega), \mathbb{R}) \).

(ii) \( \exists \beta > 0, \beta = \beta(\|f\|_{p'}) \), such that \( |J(u) - J(-u)| \leq \beta(\|J(u)\|^{1/q} + 1) \).

(iii) \( \exists M_0 > 0 \), such that if \( J(u) \geq M_0 \), and \( J'(u) = 0 \), then \( \psi(v) \equiv 1 \) in a neighbourhood of \( u \) (that is, \( J(u) = I(u) \), and \( J'(u) = I'(u) = 0 \)).

(iv) \( \exists M_1 \geq M_0 \) such that \( J \) verifies a local Palais-Smale condition for \( C > M_1 \). That is, if we have a sequence \( \{u_k\} \subset W_0^{1,p}(\Omega) \) such that \( J(u_k) \rightarrow C \) and \( J'(u_k) \rightarrow 0 \), then there exists a convergent subsequence \( \{u_k\} \subset \{u_k\} \).
According to (iii), if we find some critical value for $J$, and it is large enough, then we have a solution of problem (5.1). We will prove a stronger result: we construct a sequence of critical values for $J$, which tends to infinity.

To simplify the notation, we assume $\Omega = (0, 1)^N$. Let $E_k$ be the $k$-dimensional subspace of $W^{1,p}_0(\Omega)$, generated by the first $k$ functions of the basis

$$\{(\sin k_1 \pi x_1 \cdots \sin k_N \pi x_N), \quad k_i \in \mathbb{N}, \quad i = 1, \ldots, N\}$$

(see [G-P.2]).

In this finite dimensional subspace, it is easy to prove that it is possible to construct an increasing sequence of numbers $R_j > 0$ (as big as we wish), such that

$$(5.6) \quad J(u) \leq 0 \quad \text{if} \quad u \in E_j \cap B^C_{R_j}.$$  

Let $D_j = B_{R_j} \cap E_j$, and define

$$(5.7) \quad G_j = \{h \in C(D_j, W^{1,p}_0(\Omega)): h \text{ is odd}, \quad h|_{\delta B_{R_j} \cap E_j} = \text{Id}\},$$

$$(5.8) \quad b_j = \inf_{h \in G_j} \max_{u \in D_j} J(h(u)).$$

First, we prove that the sequence $\{b_j\}$ is well defined, and increasing:

**Proposition 5.3.** Let $b_k$ defined by (5.8). Then, there exists a constant $\beta > 0$, such that

$$(5.9) \quad b_k \geq \beta k^\gamma$$

where $\gamma = pq/N(q - p) - 1$.

**Proof.** Given $h \in G_k$, and $\rho < R_k$, we can prove that $h(D_k) \cap \delta B_\rho \cap E^C_{k-1} \neq \emptyset$. In fact, it suffices to show that $\gamma(h(D_k) \cap \delta B_\rho) \geq k$, and apply property (8) of Proposition 4.1. Let $A = \{x \in D_k: h(x) \in B_\rho\}$. It is clear that $0 \in A$, because $h$ is odd; then, we define $A_0$ the component of $A$ containing $0$. $A_0$ is a bounded and symmetric neighbourhood of $0$ in $E_k$; then, $\gamma(\delta A_0) = k$.

Moreover, $h(\delta A_0) \subset \delta B_\rho$. If not, given $x \in \delta A_0$ such that $h(x) \in B_\rho$, if $x \in D_k$, there exists a neighbourhood of $x$, $U$, such that $h(U) \subset B_\rho$. Then, $x \notin \delta A_0$. Hence, $x \in \delta D_k$; but $h|_{\delta D_k} = \text{Id}$, and this implies that $\|h(x)\| = \|x\| = R_k > \rho$, a contradiction.

Now, if we define $B = \{x \in D_k: h(x) \in B_\rho\}$, we have $\delta A_0 \subset B$, and

$$\gamma(h(D_k) \cap \delta B_\rho) = \gamma(h(B)) \geq \gamma(B) \geq \gamma(\delta A_0) = k.$$  

Note that the condition "$h$ is even" is essential to obtain this result; then, it is an important ingredient in the definition of $G_k$.

Let $u \in \delta B_\rho \cap E^C_{k-1}$; then

$$J(u) \geq \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{\lambda}{q} \int_{\Omega} |u|^q - C_1 \|u\|_p$$
where \( C_1 = C_1(\|f\|_{p'}) \). By using the Gagliardo-Nirenberg inequality,

\[
\left( \int_{\Omega} |u|^q \right)^{1/q} \leq C \left( \int_{\Omega} |\nabla u|^p \right)^{a/p} \left( \int_{\Omega} |u|^p \right)^{(1-a)/p}
\]

with \( a = (N/p)(1 - p/q) \), we get

\[
J(u) \geq \frac{1}{p} \int_{\Omega} |\nabla u|^p - C_1 \left( \int_{\Omega} |\nabla u|^p \right)^{a/p} \left( \int_{\Omega} |u|^p \right)^{(1-a)/p} - C_2 \left( \int_{\Omega} |u|^p \right)^{1/p}.
\]

Moreover, \( u \in E^C_{k-1} \); hence,

\[
\|u\|_p \leq C \|\nabla u\|_p/k^{1/N}
\]

(see [G-P.2] for the proof). Finally, by (5.11) and (5.12), we obtain

\[
J(u) \geq \frac{1}{p} \rho^p - C_1 \left( \frac{C}{k^{q(1-a)/N}} \right) \rho^q - \left( \frac{C_3}{k^{1/N}} \right) \rho
\]

\[
= \rho^p \left( \frac{1}{p} - \frac{C_2}{k^{q(1-a)/N}\rho^{q-p}} \right) - \left( \frac{C_3}{k^{1/N}} \rho \right).
\]

Now, we choose

\[
\rho_k = \left\{ \frac{k^{q(1-a)/N}}{2pC_2} \right\}^{1/(q-p)};
\]

therefore,

\[
J(u) \geq \frac{1}{2p} \rho_k^p - \frac{C_3}{k^{1/N}} \rho_k \geq C k^{pq(1-a)/N(q-p)}
\]

for \( k \) large enough. Then, by equations (5.13) and (5.10), we get \( \forall h \in G_k, \max_{u \in D_k} J(h(u)) \geq C k^\gamma \), where

\[
\gamma = \frac{pq(1-a)}{N(q-p)} = \frac{pq}{N(q-p)} - 1.
\]

And this implies (5.9). \( \Box \)

If we try to prove that \( b_k \) is a critical value for \( J \), we find an important obstruction, unless \( f \equiv 0 \). In fact, in the case \( f \neq 0 \), the associated functional \( J \) is not even, and the deformation lemma give us a homeomorphism \( \eta \) which is not odd. Then, given \( h \in G_k \), in general \( \eta \circ h \notin G_k \), and the classical proof does not work.

However, the sequence \( \{b_k\} \) allows us to prove that other mini-max sequences that we will construct are well defined and verifies the appropriate estimates. Define

\[
U_k = \{ u = te^{k+1} + w, \ t \in [0, R_{k+1}], \ w \in B_{R_{k+1}} \cap E_k, \ ||u|| \leq R_{k+1} \},
\]

\[
\Lambda_k = \{ H \in C(U_k, W_{0,1}) \cap E_k, \ H|_{D_k} \in G_k, \ H|_{(B_{R_{k+1}} \cap E_k) \cup (B_{R_{k+1}} \cap B_{R_{k+1}} \cap E_k)} = Id \},
\]
These mini-max values have the same problem as the $b_k$'s: if $H|_{D_k} \in G_k$, then $H|_{D_k}$ is odd, but $\eta \circ H|_{D_k}$ is not odd, in general. However, it is clear that $c_k \geq b_k$ (compare (5.16) and (5.8)); and if $c_k > b_k$, we can solve our problem, as the following proposition shows:

**Proposition 5.4.** If $c_k > b_k > M_x$ (where $M_1$ is the constant of Lemma 5.2 (iv)), given $\delta \in (0, c_k - b_k)$, we define

$$
\Lambda_k(\delta) = \{ H \in \Lambda_k \text{ such that } J(H(u)) \leq b_k + \delta, \forall u \in D_k \},
$$

$$
c_k(\delta) = \inf_{H \in \Lambda_k(\delta)} \max_{u \in U_k} J(H(u)).
$$

Then, $c_k(\delta)$ is a critical value for $J$.

**Proof.** By definition (5.8) it is clear that $\Lambda_k(\delta) \neq \emptyset$. And, by (5.16) and (5.18), it is also clear that $c_k(\delta) \geq c_k$. Suppose that $c_k(\delta)$ is not a critical value and take $\varepsilon < \frac{1}{2}(c_k - b_k - \delta)$.

By the classical deformation theorem, we obtain the homeomorphism $\eta: W_{1,p}(\Omega) \to W_{1,p}(\Omega)$, with the following properties:

$$
\eta(J^{c_k(\delta) + \varepsilon}) \subset J^{c_k(\delta) - \varepsilon},
$$

$$
\eta(u) = u, \quad \text{if } u \notin J^{-1}([c_k(\delta) - 2\varepsilon, c_k(\delta) + 2\varepsilon]).
$$

Note that if $u \in D_k$ and $H \in \Lambda_k(\delta)$, then

$$
J(H(u)) \leq b_k + \delta < c_k - 2\varepsilon,
$$

that is, if $H \in \Lambda_k(\delta)$, by (5.20) and (5.21), then $\eta \circ H|_{D_k} = H|_{D_k} \in G_k$. Then, we have solved the problem of the lack of symmetry in $\eta$. Now, it is easy to conclude: we prove that $\eta \circ H \in \Lambda_k(\delta)$, and find a contradiction between (5.18) and (5.19). \qed

Finally, it remains to prove that it is impossible to have $c_k = b_k$, $\forall k$.

**Proposition 5.5.** If $c_k = b_k$, $\forall k \geq k^*$, there exist some constants $C > 0$, and $k' \geq k^*$, such that

$$
b_k \leq C k^{\frac{q}{q-1}}, \quad \forall k \geq k'.
$$

**Proof.** Basically, the idea is to use that $D_{k+1} = (U_k) \cup (-U_k)$, and if $H \in \Lambda_k$, it is possible to extend it to a function of $G_k$.

By (5.15) and (5.16), we can choose $H \in \Lambda_k$ such that

$$
\max_{u \in U_k} J(H(u)) \leq c_k + \varepsilon = b_k + \varepsilon,
$$

and, by (5.8), taking the extension of $H$, we have

$$
b_{k+1} \leq \max_{u \in D_{k+1}} J(H(u)).
$$
If the maximum is attained at $U_k$, then
\begin{equation}
(5.24) \quad b_{k+1} \leq \max_{u \in D_{k+1}} J(H(u)) = \max_{u \in U_k} J(H(u)) \leq c_k + \varepsilon \leq b_k + \varepsilon.
\end{equation}

If the maximum is attained on $-U_k$, we can use estimate (iii) of Lemma 5.2, in the following way: Suppose $\max_{u \in D_{k+1}} J(H(u)) = J(H(w))$, for some $w \in -U_k$. Then,
\begin{align*}
J(H(-w)) &\geq J(H(w)) - \beta(|J(H(-w))|^{1/q} + 1) \\
&\geq b_{k+1} - \beta((b_k + \varepsilon)^{1/q} + 1) \geq b_{k+1} - \beta((b_{k+1} + \varepsilon)^{1/q} + 1) > 0,
\end{align*}
for $k$ large. And, if $J(H(-w)) > 0$,
\begin{equation}
(5.25) \quad b_{k+1} \leq J(H(w)) = J(-H(-w)) \\
\leq J(-H(-w)) + \beta(J(H(-w))^{1/q} + 1) \\
\leq (b_k + \varepsilon) + \beta((b_k + \varepsilon)^{1/q} + 1).
\end{equation}

Getting $\varepsilon \to 0$ at (5.24) and (5.25), we obtain $b_{k+1} \leq b_k + \beta(b_k^{1/q} + 1)$. Finally, this inequality implies (5.22); the proof can be made by induction. \(\Box\)

Proposition 5.5, together with Proposition 5.3 and Proposition 5.4, prove the following theorem:

**Theorem 5.6.** Problem (5.1), when $q/(q - 1) < pq/N(q - p) - 1$, has infinitely many solutions, which correspond to a sequence of critical values of the functional (5.2), the sequence tending to infinity.

**Remark 5.7.** Note that, for $p = 2$, Theorem 5.6 is contained in Theorem 10.4 in [R2].

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**References**


SOLUTIONS FOR ELLIPTIC PROBLEMS


Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain