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NONSINGULAR ALGEBRAIC CURVES IN $\mathbf{R}P^1 \times \mathbf{R}P^1$

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ABSTRACT. We give some restrictions for the mutual position of the connected components of a nonsingular algebraic curve in the product space $\mathbf{R}P^1 \times \mathbf{R}P^1$ of two real projective lines. We obtain our main theorem by calculating the Brown invariant of a certain quadratic form determined by the algebraic curve. Moreover, we consider a double covering of $\mathbf{C}P^1 \times \mathbf{C}P^1$ branched along the complexification of our curve and antiholomorphic involutions that are the lifts of the complex conjugation.

0. INTRODUCTION

The first half of Hilbert's famous 16th problem poses the question of the mutual position of the connected components of a nonsingular algebraic curve in the real projective plane $\mathbf{R}P^2$. In [5] it was proved that a curve of degree d cannot have more than $\frac{1}{2}(d-1)(d-2)+1$ components (Harnack's inequality) and that curves with this maximum number of components (M-curves) exist. For M-curves and $(M-1)$-curves (i.e., curves with $\frac{1}{2}(d-1)(d-2)$ components), Gudkov [3] made the following conjectures: If $F = 0$ is the equation of an M- (resp. $(M-1)$-) curve of even degree d and the set $B^+ = \{F \geq 0\}$ is orientable, then $\chi(B^+) \equiv (\frac{d}{2})^2$ (resp. $(\frac{d}{2})^2 \pm 1$) (mod 8), where $\chi(B^+)$ means the Euler characteristic of B^+. His conjecture for M-curves was first proved by Arnol′d [1] in the following weakened form: $\chi(B^+) \equiv (\frac{d}{2})^2$ (mod 4) (Arnol′d's congruence), and completely proved by Rokhlin (Rokhlin's congruence). They both discovered deep connections between the topology of real plane algebraic curves and the topology of 4-dimensional manifolds. By a similar method, Kharlamov, Gudkov, and Krakhnov proved Gudkov's conjecture for $(M-1)$-curves. These congruences are collected, for example, in [10]. Furthermore, Marin [9] proved all these congruences simultaneously by applying Guillou and Marin's generalization [4] of Rokhlin's signature theorem and calculating the Brown invariant [2] of a certain quadratic form.

It is natural to attempt to extend results on plane curves to space curves. Actually, for example, Hilbert [6], Gudkov [3], and Zvonilov [11] treat curves on a quadric in $\mathbf{R}P^3$. It is well known that $\mathbf{R}P^1 \times \mathbf{R}P^1$ can be embedded in

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$\mathbf{R}P^3$ as the quadric $\{Z_0Z_3 - Z_1Z_2 = 0\}$, when each curve of bidegree (d, d) in $\mathbf{R}P^1 \times \mathbf{R}P^1$ corresponds to a curve that is the intersection of the quadric and a surface of degree d in $\mathbf{R}P^3$.

In this paper we study nonsingular algebraic curves of bidegree (d, r) in $\mathbf{R}P^1 \times \mathbf{R}P^1$. Now we formulate our situation.

Let $F(X_0, X_1; Y_0, Y_1)$ $(\neq 0)$ be a real bihomogeneous polynomial of (bi-) degree (d, r) with $dr \neq 0$. We assume that both d and r are even. Our main object is the zero locus of the polynomial F in $\mathbf{R}P^1 \times \mathbf{R}P^1$, which is denoted by $\mathbf{R}A$. We also consider the zero locus of F in $\mathbf{C}P^1 \times \mathbf{C}P^1$, which is denoted by A, and assume that A is nonsingular. Then it is a compact connected Riemann surface of genus $(d-1)(r-1)$. Hence, our "Harnack-type" inequality is as follows (for the proof, see [10, (3.1)]): $l_A \leq (d-1)(r-1)+1$, where l_A denotes the number of connected components of $\mathbf{R}A$. Remark that the sharpness of this inequality is proved in [7].

We say $\mathbf{R}A$ is an *M-curve* if $l_A = (d-1)(r-1)+1$, and an *$(M-\iota)$-curve* if $l_A = (d-1)(r-1)+1-\iota$ $(\iota = 1, 2, \dots)$.

Let E_i $(i = 1, \dots, l_A)$ denote the components of $\mathbf{R}A$, which are circles embedded in $\mathbf{R}P^1 \times \mathbf{R}P^1$. While all the components of a curve of even degree in $\mathbf{R}P^2$ are homologous to zero, it is possible for a curve of even degree in $\mathbf{R}P^1 \times \mathbf{R}P^1$ to have some "nontrivial" components. Therefore, we have to devise the treatment of such components. This is the main purpose of this paper. We call a component E_i that is homologous to zero in $\mathbf{R}P^1 \times \mathbf{R}P^1$ an *oval*, and any other component a *nontrivial component*. Let l' (resp. l'') denote the number of ovals (resp. nontrivial components): $l_A = l' + l''$.

For an oval E_i, $\mathbf{R}P^1 \times \mathbf{R}P^1 \backslash E_i$ consists of two connected components, one of which is diffeomorphic to an open disk and called the *interior* of the oval. The other component is called the *exterior* of the oval. In the case $l'' = 0$, we assume that $F < 0$ in the intersection of the exteriors of all the ovals of $\mathbf{R}A$.

Since d and r are even, we can define B^+ (resp. B^-) to be the set $\{F \geq 0$ (resp. $\leq 0)\}$ in $\mathbf{R}P^1 \times \mathbf{R}P^1$. The topology of B^+ or B^- describes the mutual position of the components of $\mathbf{R}A$ in $\mathbf{R}P^1 \times \mathbf{R}P^1$.

We also consider the pair $(A, \mathbf{R}A)$. We say $\mathbf{R}A$ is a *dividing* curve if $A \backslash \mathbf{R}A$ is disconnected. M-curves are always dividing curves (see [10, proof of (6.3)]).

We apply Marin's [9] arguments as mentioned earlier to our situation. Namely, we consider the quotient space $\mathbf{C}P^1 \times \mathbf{C}P^1/\tau$ and its subspace $W = A/\tau \cup B^+$, where $\tau: \mathbf{C}P^1 \times \mathbf{C}P^1 \to \mathbf{C}P^1 \times \mathbf{C}P^1$ is the complex conjugation. The following fact is known.

Proposition (Letizia [8, p. 311]). *$\mathbf{C}P^1 \times \mathbf{C}P^1/\tau$ is diffeomorphic to the 4-sphere S^4.*

On the other hand, $W = A/\tau \cup B^+$ is a (possibly nonorientable) PL closed 2-dimensional submanifold of $\mathbf{C}P^1 \times \mathbf{C}P^1/\tau$. By the proposition just made, we can define the *Rokhlin form* (see [4]) as $q: H_1(W; \mathbf{Z}_2) \to \mathbf{Z}_4$ for the pair

$(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W)$. We define $\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W)$ $(\in \mathbf{Z}_8)$ to be the *Brown invariant* (see [2]) of q. Then by using Guillou and Marin's [4] generalization of Rokhlin's signature theorem, we get the following formula (for the proof, see §2):

$$(*) \qquad (\chi(B^+) - dr/2)_{\mathrm{mod}\,8} = \alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W) \quad \text{in } \mathbf{Z}_8.$$

Hence, we want to calculate $\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W)$. We get the following results.

Theorem. *In the case $l'' = 0$, we have the following.*

(1) *If* $\mathbf{R}A$ *is an* M*-curve, then* $\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W) = 0$.

(2) *If* $\mathbf{R}A$ *is an* $(M-1)$*-curve, then* $\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W) = \pm 1$.

(3) *If* $\mathbf{R}A$ *is a not-dividing* $(M-2)$*-curve, then* $\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W) = 0$, ± 2.

(4) *If* $\mathbf{R}A$ *is a dividing curve, then* $\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W) = 0,\ 4$.

In the case $l'' > 0$, we have the following. (For the definition of $\hat{l}$, see (3) in the following section.)

(1′) *If* $\mathbf{R}A$ *is an* M*-curve and* $\hat{l}$ *is odd, then* $\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W) = 0,\ \pm 2$.

(2′) *If* $\mathbf{R}A$ *is a dividing curve and* $\hat{l}$ *is even, then* $\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W) = 0$, 4.

We see that (1) and (2) of our theorem correspond to Rokhlin's congruence and Kharlamov-Gudkov-Krakhnov's congruence, respectively, while (4) and (2′) correspond to Arnol′d's congruence.

In the following section we apply our general results to curves of degree (4, 4). This degree seems to be the lowest one that is interesting. In §2 we prove our theorem. In the last section we consider a double covering of $\mathbf{C}P^1 \times \mathbf{C}P^1$ branched along A as in Arnol′d [1] or Wilson [10] and obtain some other results.

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1. Preliminaries and application to curves of degree (4, 4)

We use the notation given in the preceding section.

We say an oval E_i *surrounds* an oval E_j if the interior (recall §0) of E_i contains E_j. An oval surrounded by an even (resp. odd) number of ovals is called an *even* (resp. *odd*) oval. The number of even (resp. odd) ovals is denoted by P (resp. N).

For each component E_i of $\mathbf{R}A$, we have $[E_i] = s_i[\infty \times \mathbf{R}P^1] + t_i[\mathbf{R}P^1 \times \infty]$ in $H_1(\mathbf{R}P^1 \times \mathbf{R}P^1; \mathbf{Z})$ for some integers s_i and t_i. Since $\mathbf{R}A$ is nonsingular, the following are concluded. (i) If $s_i t_i = 0$, then $|s_i| + |t_i| \leq 1$. If $s_i t_i \neq 0$, then s_i and t_i are relatively prime. (ii) For any i and j, we have $s_i t_j - t_i s_j = 0$. By (i) and (ii), $[E_i] = \pm[E_j]$ for any nontrivial components E_i and E_j. Hence, we may assume that there exist integers s and t such that

$$(1) \qquad [E_i] = s[\infty \times \mathbf{R}P^1] + t[\mathbf{R}P^1 \times \infty] \quad \text{in } H_1(\mathbf{R}P^1 \times \mathbf{R}P^1; \mathbf{Z})$$

for every nontrivial component E_i.

In the case $l'' > 0$, we write E_i $(i = 1, \dots, l')$ and E_i $(i = l'+1, \dots, l_A)$ for the ovals and the nontrivial components of $\mathbf{R}A$, respectively. Then $\mathbf{R}P^1 \times \mathbf{R}P^1 \backslash (E_{l'+1} \cup \cdots \cup E_{l_A})$ consists of l'' components. Each of them is diffeomorphic to $\mathrm{Int}(S^1 \times I)$. Let R_i $(i = 1, \dots, l'')$ denote their closures provided that $F < 0$ in the intersection of $\mathrm{Int}R_1$ and the exteriors of all the ovals in R_1, and $\partial R_1 = E_{l_A} \cup E_{l'+1}$, $\partial R_2 = E_{l'+1} \cup E_{l'+2}, \dots, \partial R_{l''} = E_{l_A - 1} \cup E_{l_A}$. Then it turns out that l'' is even.

Let P^i (resp. N^i) denote the number of even (resp. odd) ovals in R_i.

Remark 1.1. We have

$$\chi(B^+) = P - N \quad \text{if } l'' = 0$$

and

$$\chi(B^+) = \sum_i (-1)^{i+1}(P^i - N^i) \quad \text{if } l'' > 0.$$

An oval is called *positive* (resp. *zero*, *negative*) if the Euler characteristic of the intersection of its interior and the exteriors of all the ovals surrounded by it is 1 (resp. zero, negative). The numbers of positive, zero, and negative even (resp. odd) ovals are denoted by P_+, P_0, and P_- (resp. N_+, N_0, and N_-).

We call a set of ovals of $\mathbf{R}A$ totally ordered by their surroundings a *nest*. The number of ovals in a nest is called the *depth* of the nest.

For a dividing curve (recall §0) $\mathbf{R}A$, it is easily seen [10, §6] that $A \backslash \mathbf{R}A$ consists of two connected components, which are interchanged by the complex conjugation τ and

$$l_A \equiv (d-1)(r-1) + 1 \pmod 2. \tag{2}$$

Let A^+ and A^- denote the closures of these components. The natural orientation of A^+ and A^- determine on $\mathbf{R}A$ two opposite orientations. These are called *complex orientations* of $\mathbf{R}A$. In the case $l'' > 0$, we set

$$\hat{l} = \#\{i\colon \text{even} \mid \text{any orientation of } \partial R_i \text{ induced by an orientation of } R_i \text{ does not coincide with any complex orientation of } \partial R_i\}. \tag{3}$$

Now we restrict ourselves to the case $(d, r) = (4, 4)$.

Remark 1.2. For a curve $\mathbf{R}A$ of degree $(4, 4)$, by considering intersection numbers, we get the following:

(i) In the case $l'' = 0$, the depth of a nest of $\mathbf{R}A$ is at most 2, hence $N_- = N_0 = 0$.

(ii) If $l'' > 0$, then $l'' = 2$ or 4. In the case $l'' = 2$; $(|s|, |t|) = (1, 0)$, $(0, 1)$, $(1, 1)$, $(2, 1)$, or $(1, 2)$, and the depth of a nest of $\mathbf{R}A$ is at most 1, hence $N = P_- = P_0 = 0$. In the case $l'' = 4$, $(|s|, |t|) = (1, 0)$, $(0, 1)$, or $(1, 1)$. In the case $l'' = 4$ or $(|s|, |t|) = (2, 1)$ or $(1, 2)$; $\mathbf{R}A$ has no oval.

By the "Harnack-type" inequality stated in §0, for a curve of degree $(4, 4)$, we have

(4) $l_A \leq 10$.

TABLE 1.1. Classification of curves of degree (4, 4)
(possible cases and their notations)

$l'' = 0$ notation	P_+	P_0	P_-	N_+	N_0	N_-	
ϕ	0	0	0	0	0	0	
n	$1 \leq n \leq 9$	0	0	0	0	0	
$\frac{1}{1}$	0	1	0	1	0	0	
$\frac{1}{1}n$	$1 \leq n \leq 8$	1	0	1	0	0	
$\frac{1}{1}\frac{1}{1}$	0	2	0	2	0	0	
$\frac{m}{1}$	0	0	1	$2 \leq m \leq 9$	0	0	
$\frac{m}{1}n$	$1 \leq n \leq 7$	0	1	$2 \leq m \leq 8$	0	0	$m+n \leq 9$ $m+n = 9 \Rightarrow m = 5$ $m+n = 8 \Rightarrow m = 4, 5$

$l'' > 0$ notation	l''	P^1	P^2	$(\lvert s\rvert, \lvert t\rvert)$	
$4(p, q)$	4	0	0	$(p, q) = (1, 0), (0, 1), (1, 1)$	
$2(p, q)$	2	0	0	$(p, q) = (2, 1), (1, 2)$	
$2(p, q; m, n)$	2	$0 \leq m \leq 8$	$0 \leq n \leq 8$	$(p, q) = (1, 0), (0, 1), (1, 1)$	$m+n \leq 8$

And moreover, by our theorem, with the formula $(*)$ (recall §0) and Remark 1.1, we get the following restrictions:

(5) If $l_A = 10$ and $l'' = 0$, then $(P, N) = (9, 1)$, $(5, 5)$, or $(1, 9)$.

(6) If $l_A = 9$ and $l'' = 0$, then $(P, N) = (9, 0)$, $(8, 1)$, $(5, 4)$, $(4, 5)$, or $(1, 8)$.

(7) If $\mathbf{R}A$ is not dividing and $l_A = 8$ and $l'' = 0$, then $P - N \equiv 0, \pm 2 \pmod 8$.

(8) If $\mathbf{R}A$ is dividing and $l'' = 0$, then $P - N \equiv 0 \pmod 4$.

(9) If $l_A = 10$ and $l'' = 2$ and $\hat{l} = 1$, then $P^1 - P^2 \equiv 0, \pm 2 \pmod 8$.

(10) If $\mathbf{R}A$ is dividing and $l'' = 2$ and $\hat{l} = 0$, then $P^1 - P^2 \equiv 0 \pmod 4$.

By (2) above, we have the following:

(11) If $\mathbf{R}A$ is dividing and $l'' = 2$, then $P^1 - P^2 \equiv 0 \pmod 2$.

In addition, by Proposition 3.1, which will be stated in §3, we have the following:

(12) If $l'' = 0$, then $P_- + P_0 \leq 2$.

(13) If $l'' = 0$ and $\mathbf{R}A$ is not dividing, then $P_- + P_0 \leq 1$.

(14) If $l'' = 0$ and $\mathbf{R}A$ is dividing and B^+ has a component whose Euler characteristic is not zero, then $P_- + P_0 \leq 1$.

(15) If $l'' = 4$, then $\mathbf{R}A$ is dividing.

Remark 1.3. (i) If $l'' = 0$ and $P_- \geq 1$, then $P_- = 1$ and $P_0 = 0$ by (13) and (14). (ii) If $l'' = 0$ and $P_0 = 2$, then $\mathbf{R}A$ is dividing by (13), and $P_+ = 0$ by (14).

Thus the curves of degree $(4, 4)$ are classified as Table 1.1.

At present, we know the existence of the following cases.

(16) n $(1 \leq n \leq 8)$, $\frac{1}{1}$, $\frac{1}{1}n$ $(1 \leq n \leq 4, 6 \leq n \leq 8)$, $\frac{1}{1}\frac{1}{1}$, $\frac{m}{1}$ $(2 \leq m \leq 4)$, $\frac{2}{1}1$, $\frac{3}{1}1$, $\frac{2}{1}2$, $4(1, 0)$, $4(0, 1)$, $4(1, 1)$, $2(2, 1)$, $2(1, 2)$, $2(1, 0; m, n)$ $(m + n \leq 4, (m, n) = (4, 1), (1, 4), (5, 1), (1, 5), (3, 3))$, $2(0, 1; m, n)$ (the same), $2(1, 1 : m, n)$ $(m \leq 4$ and $n = 0$, or $m = 0$ and $n \leq 4)$

(17) $\frac{4}{1}4$, 9, $\frac{3}{1}3$, $\frac{2}{1}3$, $2(1, 0; 6, 2)$, $2(1, 0; 2, 6)$, $2(1, 0; 4, 4)$, $2(0, 1; 6, 2)$, $2(0, 1; 2, 6)$, $2(0, 1; 4, 4)$

We can show the existence of the cases listed in (16) (respectively, (17)) by a method similar to Harnack [5] (respectively, Gudkov [3, p. 42]). We will devote the details of our construction to another paper. But the existence problem of the cases listed in Table 1.1 is not yet solved.

2. Proof of Theorem

We first prove the formula $(*)$ in §0. By Guillou and Marin's generalization [4] of Rokhlin's signature theorem, we have

$$(\mathrm{Sign}(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau) - (W \circ W)_{\mathbf{C}P^1 \times \mathbf{C}P^1/\tau})_{\mathrm{mod}\,16} = 2\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W)$$

in $\mathbf{Z}_{16}$, where $\mathrm{Sign}(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau)$ means the signature of the manifold $\mathbf{C}P^1 \times \mathbf{C}P^1/\tau$ and $(W \circ W)_{\mathbf{C}P^1 \times \mathbf{C}P^1/\tau}$ means the self-intersection number of W in $\mathbf{C}P^1 \times \mathbf{C}P^1/\tau$. By the proposition stated in §0, $\mathrm{Sign}(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau) = \mathrm{Sign}(S^4) = 0$. On the other hand, since τ is written as the complex conjugation $(x, y) \mapsto (\overline{x}, \overline{y})$ of $\mathbf{C}^2$ in a local coordinate neighborhood of each point of $\mathbf{R}P^1 \times \mathbf{R}P^1$, by the same arguments as in [9, p. 53], we have

$$\begin{aligned}(W \circ W)_{\mathbf{C}P^1 \times \mathbf{C}P^1/\tau} &= (A/\tau \circ A/\tau)_{\mathbf{C}P^1 \times \mathbf{C}P^1/\tau} + (B^+ \circ B^+)_{\mathbf{C}P^1 \times \mathbf{C}P^1/\tau} \\ &= \tfrac{1}{2}(A \circ A)_{\mathbf{C}P^1 \times \mathbf{C}P^1} + 2(-\chi(B^+)) = dr - 2\chi(B^+).\end{aligned}$$

Thus $(*)$ follows.

After the following arguments, we will prove our theorem.

We consider a handlebody decomposition of W. We first decompose A/τ. A/τ is a 2-dimensional manifold whose boundary is $\mathbf{R}A$. Since $\tau|A$ is written as the complex conjugation: $x \mapsto \overline{x}$ of $\mathbf{C}$ in a local coordinate neighborhood of each point of $\mathbf{R}A$, as in [9, p. 52], we see that A/τ is orientable if and only if $\mathbf{R}A$ is dividing. Suppose that $\mathbf{R}A$ is an $(M - \imath)$-curve $(\imath = 0, 1, 2, \dots)$, where an $(M - 0)$-curve means an M-curve. Then

$$\chi(A/\tau) = \tfrac{1}{2}(2 - 2g(A)) = 2 - (g(A) + 1) = 2 - \imath - l_A,$$

where we set $g(A) = (d - 1)(r - 1)$. Hence by classical arguments we get a handlebody representation

$$h^0 \cup (h_1 \cup \cdots \cup h_{g(A)+1}) \cup h^2 \tag{1}$$

of the triad $(A/\tau;\, \phi, \mathbf{R}A)$, where h^0, h_j, and h^2 represent a 0-handle D^2, a 1-handle $D^1_j \times D^1_j$, and a 2-handle D^2_1, respectively. The way of attaching these handles is as follows; we say a 1-handle h_j is attached *orientably* (resp. *nonorientably*) if $h^0 \cup h_j$ is orientable (resp. nonorientable). (i) In the case A/τ is orientable, first:

(2) we attach h_1 to ∂D^2 orientably, h_2 to the boundary component that contains $D^1_1 \times \{1\}$ orientably so that $D^1_1 \times \{1\}$ and $D^1_2 \times \{-1\}$ will lie in the same boundary component, and $h_3, \dots, h_{\iota/2}$ in the same way.

We next attach $h_{\iota/2+j}$ $(j = 1, \dots, \iota/2)$ to $D^1_j \times \{\pm 1\}$ orientably so that $\{\pm 1\} \times D^1_{\iota/2+j}$ will lie in $D^1_j \times \{\pm 1\}$. We attach $h_{\iota+1}$ to the boundary component that contains $D^1_{\iota/2} \times \{1\}$ orientably so that $D^1_{\iota/2} \times \{1\}$ and $D^1_{\iota+1} \times \{-1\}$ will lie in the same boundary component, and $h_{\iota+2}, h_{\iota+3}, \dots, h_{g(A)+1}$ in the same way (Figure 2.1). We last attach h^2 to the boundary component that contains $D^1_{\iota/2+1} \times \{-1\}$. (ii) In the case A/τ is nonorientable, we first attach $h_1, \dots, h_\iota$ (resp. $h_{\iota+1}, \dots, h_{g(A)+1}$) nonorientably (resp. orientably) to the same place as (2) (Figure 2.2). We last attach h^2 to the boundary component that contains $D^1_1 \times \{-1\}$.

In what follows, we often omit the statements in the case $l'' = 0$.

We now introduce new notations for the ovals $E_1, \dots, E_{l'}$. We call an oval surrounded by j $(= 0, 1, \dots)$ ovals a *j-oval*. Let ${}_jE^i_k$ $(k = 1, \dots, s_{ij})$ denote the j-ovals in R_i provided that $i - j \equiv 1 \pmod 2$. Then the closure of the intersection of the interior of ${}_jE^i_k$ and the exteriors of all the ovals surrounded by it is a component of B^+ (Figure 2.3). Let B^+_{ijk} denote this component, and ${}_{jk}E^i_a$ $(a = 1, \dots, r_{ijk})$ denote the $j+1$-ovals surrounded by

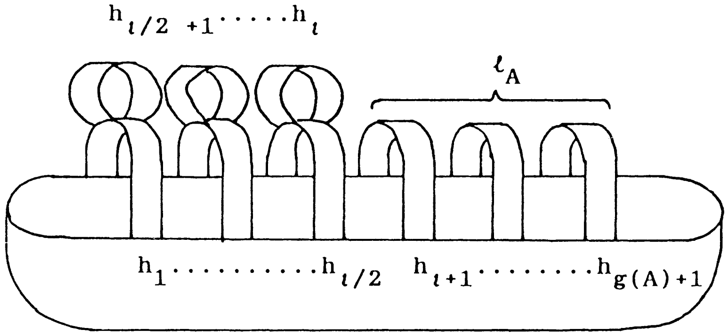

FIGURE 2.1

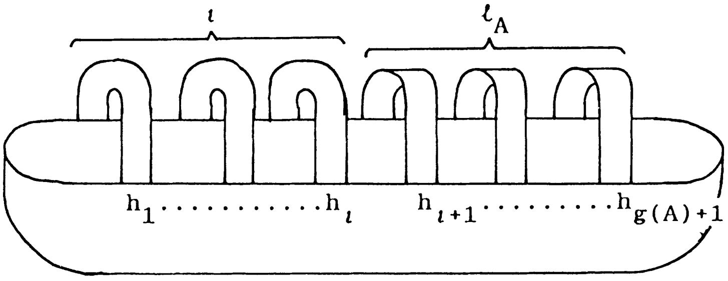

FIGURE 2.2.

${}_jE^i_k$. The closure of the intersection of R_{2j} and the exteriors of all the ovals contained in it is also a component of B^+ (see Figure 2.3). Let B^+_j denote this component, and E^{2j}_b $(b = 1, \dots, p_j)$ denote the 0-ovals in R_{2j}.

We order all the components of $\mathbf{R}A$ as follows:

$$
\begin{aligned}
(3)\qquad & {}_{01}E^1_1, \dots, {}_{01}E^1_{r_{101}}; {}_0E^1_1; {}_{02}E^1_1, \dots, {}_{02}E^1_{r_{102}}; {}_0E^1_2; \dots\dots; {}_0E^1_{s_{10}};\\
& {}_{21}E^1_1, \dots, {}_{21}E^1_{r_{121}}; {}_2E^1_1; {}_{22}E^1_1, \dots, {}_{22}E^1_{r_{122}}; {}_2E^1_2; \dots\dots; {}_2E^1_{s_{12}};\\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots ;\\
& {}_{11}E^2_1, \dots, {}_{11}E^2_{r_{211}}; {}_1E^2_1; {}_{12}E^2_1, \dots, {}_{12}E^2_{r_{212}}; {}_1E^2_2; \dots\dots; {}_1E^2_{s_{21}};\\
& {}_{31}E^2_1, \dots, {}_{31}E^2_{r_{231}}; {}_3E^2_1; {}_{32}E^2_1, \dots, {}_{32}E^2_{r_{232}}; {}_3E^2_2; \dots\dots; {}_3E^2_{s_{23}};\\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots ;\\
& {}_{11}E^{l''}_1, \dots, {}_{11}E^{l''}_{r_{l''11}}; {}_1E^{l''}_1; \dots\dots\dots; \qquad {}_1E^{l''}_{s_{l''1}};\\
& {}_{31}E^{l''}_1, \dots, {}_{31}E^{l''}_{r_{l''31}}; {}_3E^{l''}_1; \dots\dots\dots; \qquad {}_3E^{l''}_{s_{l''3}};\\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots ;\\
& E_{l'+1}; E^2_1, \dots, E^2_{p_1}; E_{l'+2};\\
& E_{l'+3}; E^4_1, \dots, E^4_{p_2}; E_{l'+4};\\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots ;\\
& E_{l_A-1}; E^{l''}_1, \dots, E^{l''}_{p_{l''/2}}; E_{l_A}.
\end{aligned}
$$

Here, only in the case A/τ is orientable (i.e., $\mathbf{R}A$ is dividing), do we make the following convention for the order (3).

(4) We fix an orientation of B^+ and a complex orientation of $\mathbf{R}A$, and divide the components of $\mathbf{R}A$ into two classes: $C = \{$components on which the orientation of B^+ determines the complex orientation$\}$ and $C' = \{$all other components$\}$. Let C_{ijk} (resp. C_j) denote the class that contains ${}_jE^i_k$ (resp. $E_{l'+2j}$). We gather up the components ${}_{jk}E^i_a$ (resp. E^{2j}_b) that belong to C_{ijk}

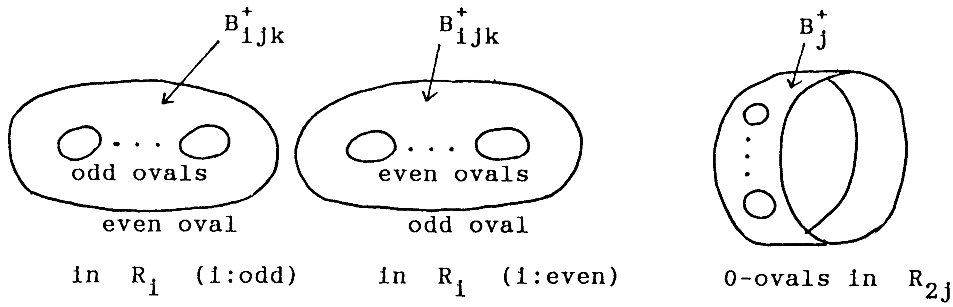

FIGURE 2.3. THE COMPONENTS OF B^+

(resp. C_j), and push them backward. Let ${}_{jk}E^i_a$ $(a = 1, \dots, a_{ijk})$ (resp. E^{2j}_b $(b = 1, \dots, b_j)$) be the forward components.

For handlebody representation (1), we may assume the boundary component which contains $D^1_{\iota+1} \times \{1\}$, that which contains $D^1_{\iota+2} \times \{1\}$, $\dots$, and that which contains $D^1_{g(A)+1} \times \{1\}$ correspond to the components of $\mathbf{R}A$, just as in order (3).

On the other hand, if we prepare a handlebody decomposition of the triad $(B^+_{ijk}; {}_{jk}E^i_1 \amalg \cdots \amalg {}_{jk}E^i_{r_{ijk}} \amalg {}_jE^i_k, \varnothing)$ (respectively, $(B^+_j; E_{l'+2j-1} \amalg E^{2j}_1 \amalg \cdots \amalg E^{2j}_{p_j} \amalg E_{l'+2j}, \varnothing)$) for each (i, j, k) (resp. j), then we obtain a handlebody decomposition of W by attaching the handlebodies to handlebody (1). In other words, we attach some new handles to (1) as follows. Set $M = \{m | \iota + 1 \leq m \leq g(A)+1$ and $D^1_m \times \{-1\}$ lies in some ${}_{jk}E^i_a$, $E_{l'+2j-1}$ or $E^{2j}_b\}$. Only in the case A/τ is orientable, set $M' = \{m \in M | D^1_m \times \{-1\}$ lies in $E_{l'+2j-1}$ (provided that $E_{l'+2j-1}$ belongs to C_j and $b_j \geq 1$), $E^{2j}_{b_j}$ or ${}_{jk}E^i_{a_{ijk}}\}$. For any m $(\in M)$, we attach a new 1-handle $\overline{h}_m = \overline{D}^1_m \times \overline{D}^1_m$ to $D^1_m \times \{\pm 1\}$ so that $\{\pm 1\} \times \overline{D}^1_m$ will lie in $D^1_m \times \{\pm 1\}$. Here, only in the case A/τ is orientable, from convention (4), we attach $\overline{h}_m = \overline{D}^1_m \times \overline{D}^1_m$ nonorientably (resp. orientably) for any m $(\in M'$ (resp. $M \backslash M'$)) (Figure 2.4). Finally, we attach some 2-handles in the trivial way. Thus we get a handlebody decomposition of W.

Now let $\circ\colon H_1(W; \mathbf{Z}_2) \otimes H_1(W; \mathbf{Z}_2) \to \mathbf{Z}_2$ be the intersection form. Then $(H_1(W; \mathbf{Z}_2), \circ)$ is a $\mathbf{Z}_2$-vector space with an inner product. To consider the structure of $(H_1(W; \mathbf{Z}_2); \circ)$, we choose some embedded circles in W as follows. In the case A/τ is orientable, for any m $(= 1, \dots, \iota/2; \iota+1, \dots, g(A)+1)$, we choose an embedding $f_m\colon I = [-1, 1] \to D^2$ (the 0-handle) such that $f_m(\pm 1) = (\pm 1, 0)$ in $D^1_m \times D^1_m$ and the images $f_m(I)$ are disjoint, and we set $S_m = f_m(I) \cup D^1_m \times \{0\}$. Moreover, for any m $(= \iota/2 + 1, \dots, \iota)$, we choose an embedding $f_m\colon I \to D^1_{m-\iota/2} \times D^1_{m-\iota/2}$ such that $f_m(\pm 1) = (\pm 1, 0)$

in $D^1_m \times D^1_m$ and $f_m(I)$ intersects $S_{m-\imath/2}$ at only one point transversely, and we set $S_m = f_m(I) \cup D^1_m \times \{0\}$. Finally, for any $m (\in M)$, we choose an embedding $\overline{f}_m \colon I \to D^1_m \times D^1_m$ such that $\overline{f}_m(\pm 1) = (\pm 1, 0)$ in $\overline{D}^1_m \times \overline{D}^1_m$ and $\overline{f}_m(I)$ intersects S_m at only one point transversely, and we set $\overline{S}_m = \overline{f}_m(I) \cup \overline{D}_m \times \{0\}$. In the case A/τ is nonorientable, for any m $(= 1, \dots, g(A)+1)$, we choose a circle S_m as in the case A/τ is orientable and $m = 1, \dots, \imath/2$; $\imath+1, \dots, g(A)+1$. Moreover, for any m $(\in M)$, we choose a circle $\overline{S}_m$ as in the case A/τ is orientable.

Remark 2.1. The following elements form a basis of $H_1(W; \mathbf{Z}_2)$: $[S_m]$ $(m = 1, \dots, \imath)$; $[S_m]$ and $[\overline{S}_m]$ $(m \in M)$. Hence we have

$$\dim H_1(W; \mathbf{Z}_2) = \imath + 2\#(M)$$
$$= \begin{cases} \imath + 2N & (\text{if } l'' = 0), \\ \imath + 2\left(\sum\limits_{i\,:\,\text{even}} P^i + \sum\limits_{i\,:\,\text{odd}} N^i + l''/2 \right) & (\text{if } l'' > 0). \end{cases}$$

Remark 2.2. (i) In the case A/τ is orientable, we have

$$H_1(W; \mathbf{Z}_2) = \bigoplus_{m=1}^{\imath/2} \langle [S_m], [S_{m+\imath/2}] \rangle_{\mathbf{Z}_2} \bigoplus_{m \in M} \langle [S_m], [\overline{S}_m] \rangle_{\mathbf{Z}_2},$$

$[S_m] \circ [S_m] = 0$ $(m = 1, \dots, \imath)$, $[S_m] \circ [S_{m+\imath/2}] = 1$ $(m = 1, \dots, \imath/2)$, $[S_m] \circ [S_m] = 0$ $(m \in M)$, $[S_m] \circ [\overline{S}_m] = 1$ $(m \in M)$, $[\overline{S}_m] \circ [\overline{S}_m] = 0$ $(m \in M \backslash M')$, and $[\overline{S}_m] \circ [\overline{S}_m] = 1$ $(m \in M')$. (ii) In the case A/τ is nonorientable, we have

$$H_1(W; \mathbf{Z}_2) = \bigoplus_{m=1}^{\imath} \langle [S_m] \rangle_{\mathbf{Z}_2} \bigoplus_{m \in M} \langle [S_m], [\overline{S}_m] \rangle_{\mathbf{Z}_2},$$

$[S_m] \circ [S_m] = 1$ $(m = 1, \dots, \imath)$, $[S_m] \circ [S_m] = 0$ $(m \in M)$, and $[S_m] \circ [\overline{S}_m] = 1$ $(m \in M)$.

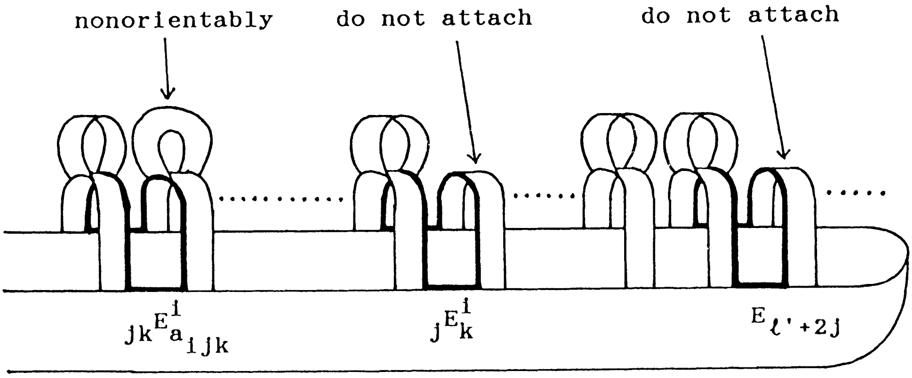

FIGURE 2.4

Now we consider the following subspace L of $H_1(W; \mathbf{Z}_2)$:

$$L = \begin{cases} \langle [E] | E \text{ is an odd oval of } \mathbf{R}A \rangle_{\mathbf{Z}_2} & (\text{if } l'' = 0), \\ \langle [{}_{jk}E_a^i] \rangle_{\mathbf{Z}_2} \oplus (\langle [E_b^{2j}] \rangle_{\mathbf{Z}_2} + \langle [\partial R_i] | i\colon \text{odd} \rangle_{\mathbf{Z}_2}) & (\text{if } l'' > 0). \end{cases}$$

Remark 2.3. We see that $L = \langle [\partial B_i^-] | B_i^-$ is a component of $B^- \rangle_{\mathbf{Z}_2}$, and ∂B_i^- is embedded in W two-sidedly. Hence, by the definition (see [4]) of our Rokhlin form q (recall §0), we have

$$q([\partial B_i^-]) = (2 \cdot 2(-\chi(B_i^-))_{\mathrm{mod}\, 4}) = 0$$

as in [9, Proposition 1]. Namely, q vanishes on L.

Remark 2.4. (i) For the elements $[{}_{jk}E_a^i]$, we see the following:

$$\begin{aligned}
(5) \qquad & [{}_{01}E_1^1] = [S_{\imath+2}], \\
& [{}_{01}E_2^1] = [S_{\imath+2}] + [S_{\imath+3}], \\
& [{}_{01}E_3^1] = [S_{\imath+3}] + [S_{\imath+4}], \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
& [{}_{01}E_{r_{101}}^1] = [S_{\imath+r_{101}}] + [S_{\imath+r_{101}+1}], \\
& [{}_{02}E_1^1] = [S_{\imath+r_{101}+3}], \\
& [{}_{02}E_2^1] = [S_{\imath+r_{101}+3}] + [S_{\imath+r_{101}+4}], \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
& [{}_{02}E_{r_{102}}^1] = [S_{\imath+r_{101}+r_{102}+1}] + [S_{\imath+r_{101}+r_{102}+2}], \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Set $M_1 = \{m | \imath + 1 \le m \le \imath + l' - (\sum p_j) + 1$ and $m \in M\}$. Then, remark that $[S_m]$ appears in (5) if and only if $m \in M_1$. Hence, by Remark 2.1, $\{[S_m] | m \in M_1\}$ is a basis of $\langle [{}_{jk}E_a^i] \rangle_{\mathbf{Z}_2}$. We see that

$$\dim \langle [{}_{jk}E_a^i] \rangle_{\mathbf{Z}_2} = \sum_j (P^{2j} - p_j) + \sum_{i\,:\,\text{odd}} N^i.$$

(ii) For the elements $[E_b^{2j}]$ and $[\partial R_i]$, we see the following, where we set

$q_j = \imath + l' - (p_j + p_{j+1} + \cdots + p_{l''/2})$ $(j = 1, \ldots, l''/2)$:

$$
\begin{aligned}
(6) \qquad & [E_1^2] = [S_{q_1+2}] + [S_{q_1+3}], \\
& [E_2^2] = [S_{q_1+3}] + [S_{q_1+4}], \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
& [E_{p_1}^2] = [S_{q_2+1}] + [S_{q_2+2}], \\
& [E_{l'+2}] + [E_{l'+3}] = [S_{q_2+2}] + [S_{q_2+4}], \\
& [E_1^4] = [S_{q_2+4}] + [S_{q_2+5}], \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
& [E_{p_2}^4] = [S_{q_3+3}] + [S_{q_3+4}]; \\
& [E_{l'+4}] + [E_{l'+5}] = [S_{q_3+4}] + [S_{q_3+6}], \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& [E_{p_{l''/2}}^{l''}] = [S_{\imath+l_A-1}] + [S_{\imath+l_A}], \\
& [E_{l_A}] + [E_{l'+1}] = [S_{\imath+l_A}] + [S_{q_1+2}]
\end{aligned}
$$

Set $M_2 = M \backslash M_1$. Then, remark that $[S_m]$ appears in (6) if and only if $m \in M_2$. Hence, by Remark 2.1 again, if we remove $[E_{l_A}] + [E_{l'+1}]$ from (6), then the remainder forms a basis of $\langle [E_b^{2j}] \rangle_{\mathbf{Z}_2} + \langle [\partial R_i] | i\colon \text{odd} \rangle_{\mathbf{Z}_2}$. We see that

$$\dim(\langle E_b^{2j}] \rangle_{\mathbf{Z}_2} + \langle [\partial R_i] | i\colon \text{odd} \rangle_{\mathbf{Z}_2}) = (\textstyle\sum p_j) + l''/2 - 1.$$

(iii) Thus we have

$$\dim L = \begin{cases} \#(M) = N \quad (\text{if } l'' = 0), \\ \#(M) - 1 = \displaystyle\sum_{i\,:\,\text{even}} P^i + \sum_{i\,:\,\text{odd}} N^i + l''/2 - 1 \quad (\text{if } l'' > 0). \end{cases}$$

We set $\lambda = \dim L$ and

$$n = \#(M_1) = \sum_j (P^{2j} - p_j) + \sum_{i\,:\,\text{odd}} N^i.$$

Let u_i (resp. v_i) $(i = 1, \ldots, n)$ denote the elements of $\{[S_m]$ (resp. $[\overline{S}_m])| m \in M_1\}$ in order. Then the following lemma is useful in calculating $\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W)$.

Lemma 2.1. *There exist elements* u_i, v_i $(i = n+1, \ldots, \lambda)$, *and* $v_{\lambda+1}$ *of* $\bigoplus_{m \in M_2} \langle [S_m], [\overline{S}_m] \rangle_{\mathbf{Z}_2}$ *such that*

(i) $\{u_i | i = 1, \ldots, \lambda\}$ *is a basis of* L,

(ii) *the intersection form* $\circ$ *is represented by* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *or* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ *with respect to each* $\{u_i, v_i\}$ $(i = n+1, \ldots, \lambda)$,

(iii) $+_{i=n+1}^{\lambda}\langle u_i, v_i\rangle_{\mathbf{Z}_2} + \langle [S_{\iota+l_A}], v_{\lambda+1}\rangle_{\mathbf{Z}_2}$ *is an orthogonal sum, and*

(iv) *the intersection form* $\circ$ *is represented by* $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ *(resp.* $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$*) with respect to* $\{[S_{\iota+l_A}], v_{\lambda+1}\}$ *in the case* A/τ *is orientable and* $\hat{l}$ *is even (resp. odd).*

Proof. Let $S'_{n+1} + S'_{n+2}$, $S'_{n+2} + S'_{n+3}$, $\ldots$, $S'_{\lambda+1} + S'_{n+1}$ denote the right-hand sides of (6) in order. If S'_j denotes $[S_m]$, then let $\overline{S}'_j$ denote $[\overline{S}_m]$. We set $u_i = S'_i + S'_{i+1}$ for any i $(= n+1, \ldots, \lambda)$. We define $v_{n+1} = \overline{S}'_{n+1}$ and for any i $(= n+2, \ldots, \lambda+1)$,

$$v_i = \begin{cases} v_{i-1} + \overline{S}'_i + S'_{i-1} & (\text{if } v_{i-1} \circ v_{i-1} = 1), \\ v_{i-1} + \overline{S}'_i & (\text{if } v_{i-1} \circ v_{i-1} = 0). \end{cases}$$

Then (i) is obvious by Remark 2.4. We can verify (ii) and (iii) by Remark 2.2. Finally, since $v_i \circ v_i = v_{i-1} \circ v_{i-1} + \overline{S}'_i \circ \overline{S}'_i$, we have

$$\begin{aligned} v_{\lambda+1} \circ v_{\lambda+1} &= \begin{cases} 0 & (\text{if } \#\{i | n+1 \leq i \leq \lambda+1 \text{ and } \overline{S}'_i \circ \overline{S}'_i = 1\} \text{ is even}) \\ 1 & (\text{if otherwise}) \end{cases} \\ &= (\hat{l})_{\mathrm{mod}\, 2} \end{aligned}$$

by the convention (4). Thus (iv) follows. Q.E.D.

Remark 2.5. If we set $\widetilde{L} = \bigoplus_{i=1}^{\lambda}\langle u_i, v_i\rangle_{\mathbf{Z}_2}$, then

$$\dim \widetilde{L}^{\perp} = \begin{cases} \iota & (\text{if } l'' = 0), \\ \iota + 2 & (\text{if } l'' > 0). \end{cases}$$

Hence we have

$$\widetilde{L}^{\perp} = \begin{cases} \langle [S_m] | m = 1, \ldots, \iota\rangle_{\mathbf{Z}_2} & (\text{if } l'' = 0), \\ \langle [S_m] | m = 1, \ldots, \iota\rangle_{\mathbf{Z}_2} \oplus \langle [S_{\iota+l_A}], v_{\lambda+1}\rangle_{\mathbf{Z}_2} & (\text{if } l'' > 0) \end{cases} \tag{7}$$

and $H_1(W; \mathbf{Z}_2) = \widetilde{L} \oplus \widetilde{L}^{\perp}$.

Recall that $\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W)$ is the Brown invariant of our Rokhlin form $q: H_1(W; \mathbf{Z}_2) \to \mathbf{Z}_4$. The Brown invariant has the following properties (see [2]):

(8) Let V be a finite dimensional $\mathbf{Z}_2$-vector space with an inner product and a *quadratic* function (see [2]) $\varphi: V \to \mathbf{Z}_4$. (i) If $V = V_1 \oplus V_2$, then for the Brown invariant $\sigma(\varphi)$ $(\in \mathbf{Z}_8)$ we have $\sigma(\varphi) = \sigma(\varphi|V_1) + \sigma(\varphi|V_2)$. (ii) If $\dim V = 1$, then $\sigma(\varphi) = \pm 1$. (iii) If the inner product is represented by $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ on V, then $\sigma(\varphi) = 0, 4$.

Remark 2.6. Since q is a quadratic function in the sense of [2], we have $q(u+v) = q(u) + q(v) + 2(u \circ v)$. See [4]. Hence, by Remark 2.3, Lemma 2.1(ii) and the definition of the Brown invariant, we see $\sigma(q|\langle u_i, v_i\rangle_{\mathbf{Z}_2}) = 0$ for any i $(= 1, \ldots, \lambda)$. Thus, we conclude that $\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W) = \sigma(q|\widetilde{L}^{\perp})$.

Proof of Theorem. In the case $l'' = 0$, by Remark 2.5, $\dim \widetilde{L}^{\perp} = \iota$. Hence, by Remark 2.6, we see the following. (1) If $\mathbf{R}A$ is an M-curve, then

$\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W) = 0$. (2) If $\mathbf{R}A$ is an $(M-1)$-curve, then by (8)(ii), $\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W) = \pm 1$. (3) If $\mathbf{R}A$ is a not-dividing $(M-2)$-curve (i.e., A/τ is nonorientable and $\imath = 2$), then by Remark 2.2 and (7), $\circ$ is represented by $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ on $\widetilde{L}^{\perp}$. Hence, by (8)(i) and (ii), $\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W) = 0, \pm 2$. (4) If $\mathbf{R}A$ is a dividing curve (i.e., A/τ is orientable), then by Remark 2.2 and (7), $\circ$ is represented by $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \oplus \cdots \oplus \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ on $\widetilde{L}^{\perp}$. Hence, by (8)(i) and (iii), $\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W) = 0, 4$. Next, in the case $l'' > 0$, by Remark 2.5, $\dim \widetilde{L}^{\perp} = \imath + 2$. Hence, we see the following. (1′) If $\mathbf{R}A$ is an M-curve and $\hat{l}$ is odd, then, by Lemma 2.1(iv), $\circ$ is represented by $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$ ($\sim \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$) on $\widetilde{L}^{\perp}$. Hence, $\alpha(\mathbf{C}P^1 \times \mathbf{C}P^1/\tau, W) = 0, \pm 2$. (2′) If $\mathbf{R}A$ is a dividing curve and $\hat{l}$ is even, then we have the required result by Remark 2.2 and Lemma 2.1(iv) again. This completes the proof of our theorem.

3. Some other results

In this section we consider the following model of a double covering of $\mathbf{C}P^1 \times \mathbf{C}P^1$ branched along A. We set $\widetilde{U}_i = \{[X_0 : X_1 : X_2] \in \mathbf{C}P^2 | X_i \neq 0\}$ for i $(= 0, 1, 2)$ and $E = (\widetilde{U}_0 \cup \widetilde{U}_1) \times (\widetilde{U}_0 \cup \widetilde{U}_1)$. Let $E/\sim$ be the quotient space of E with respect to the equivalence relation:

$$([X_0 : X_1 : X_2], [Y_0 : Y_1 : Y_2]) \sim ([X'_0 : X'_1 : X'_2], [Y'_0 : Y'_1 : Y'_2])$$

if $([X_0 : X_1], [Y_0 : Y_1]) = ([X'_0 : X'_1], [Y'_0 : Y'_1])$ in $\mathbf{C}P^1 \times \mathbf{C}P^1$ and $(X_2/X_i)^{d/2} (Y_2/Y_j)^{r/2} = (X'_2/X'_i)^{d/2}(Y'_2/Y'_j)^{r/2}$ for any i $(= 0, 1)$ and j $(= 0, 1)$. Let $\pi : E/\sim \to \mathbf{C}P^1 \times \mathbf{C}P^1$ be the natural projection. This is nothing but the line bundle $p_1^*\mathcal{O}_{\mathbf{C}P^1}(\frac{d}{2}) \otimes p_2^*\mathcal{O}_{\mathbf{C}P^1}(\frac{r}{2})$, where p_i $(i = 1, 2)$: $\mathbf{C}P^1 \times \mathbf{C}P^1 \to \mathbf{C}P^1$ is the projection to the ith component. In fact, trivializations $\widetilde{U}_i \times \widetilde{U}_j/\sim \to U_i \times U_j \times \mathbf{C}$ are defined to be the maps

$$\langle [X_0 : X_1 : X_2], [Y_0 : Y_1 : Y_2] \rangle \mapsto ([X_0 : X_1], [Y_0 : Y_1], (X_2/X_i)^{d/2}(Y_2/Y_j)^{r/2}),$$

where we set $U_i = \{[X_0 : X_1] \in \mathbf{C}P^1 | X_i \neq 0\}$ and $\langle [X_0 : X_1 : X_2], [Y_0 : Y_1 : Y_2] \rangle$ is the class containing $([X_0 : X_1 : X_2], [Y_0 : Y_1 : Y_2])$. Hence, transition functions $U_i \times U_j \cap U_{i'} \times U_{j'} \to \mathbf{C}$ are given by the maps:

$$([X_0 : X_1], [Y_0 : Y_1]) \mapsto (X_{i'}/X_i)^{d/2}(Y_{j'}/Y_j)^{r/2}. \tag{1}$$

We set $Y = \{F(X_0, X_1; Y_0, Y_1) + X_2^d Y_2^r = 0\}$ $(\subset E/\sim)$. Then Y is a compact connected complex surface and the restriction $\pi : Y \to \mathbf{C}P^1 \times \mathbf{C}P^1$ is a required double covering branched along A. Similarly to [10, p. 65], Y has the following properties:

$$\pi_1(Y) = 0, \quad \chi(Y) = 6 + 2(d-1)(r-1), \quad \text{and} \quad \operatorname{Sign}(Y) = -dr. \tag{2}$$

Let θ be the fiber-preserving involution on $E/\sim$ that acts as -1 on each fiber, and T^- be the involution on $E/\sim$ induced by the complex conjugation on E. We set $T^+ = \theta \circ T^-$ $(= T^- \circ \theta)$. The restriction $\theta: Y \to Y$ gives the nontrivial covering transformation. Since F is a real polynomial, we also have $T^{\pm}(Y) = Y$. T^+ and T^- are antiholomorphic involutions on Y. Let $\mathbf{R}Y^{\pm}$ be the fixed point set of $T^{\pm}$. Then we have $\mathbf{R}Y^{\pm} = \{\langle [X_0 : X_1 : X_2], [Y_0 : Y_1 : Y_2]\rangle \in E/\!\sim\, | ([X_0 : X_1], [Y_0 : Y_1]) \in \mathbf{R}P^1 \times \mathbf{R}P^1,\ F(X_0, X_1; Y_0, Y_1) + X_2^d Y_2^r = 0$, and $X_2^{d/2} Y_2^{r/2}$ is purely imaginary (respectively, real)$\} = \pi^{-1}(B^{\pm})$. Hence, $\mathbf{R}Y^+$ and $\mathbf{R}Y^-$ are possibly disconnected closed surfaces. We see that $\theta(\mathbf{R}Y_i^{\pm}) = \mathbf{R}Y_i^{\pm}$ for every connected component $\mathbf{R}Y_i^{\pm}$ of $\mathbf{R}Y^{\pm}$ and $\pi: \mathbf{R}Y^{\pm} \to B^{\pm}$ gives a bijection between the connected components of $\mathbf{R}Y^{\pm}$ and those of $B^{\pm}$.

Remark 3.1. From this, we see that in the case $l'' = 0$, $\mathbf{R}A$ is an $(M-\iota)$-curve if and only if (Y, T^-) is an $(M-\iota)$-manifold in the sense of [10]. By applying [10, Theorems 3.3 and 3.11], we can prove (1) and (2) of our theorem again.

Remark 3.2. Since $\theta: \mathbf{R}Y_i^{\pm} \to \mathbf{R}Y_i^{\pm}$ is locally written as the map $(x, y) \mapsto (x, -y)$ on $\mathbf{R}^2$ and the fixed point set is $\mathbf{R}Y_i^{\pm} \cap \mathbf{R}A$, if $\mathbf{R}Y_i^{\pm} \backslash \mathbf{R}A$ is disconnected, then this consists of two connected components that are interchanged by θ (see the dividingness of $\mathbf{R}A$ in §1 and also in [10, §6]). Hence, in this case, $\mathbf{R}Y_i^{\pm}$ is the double of $\pi(\mathbf{R}Y_i^{\pm})$ and therefore orientable because $\pi(\mathbf{R}Y_i^{\pm})$ is orientable. Conversely, if $\mathbf{R}Y_i^{\pm}$ is orientable, then $\mathbf{R}Y_i^{\pm} \backslash \mathbf{R}A$ is necessarily disconnected as the orientability of A/τ in §2. See also [9, p. 52]. On the other hand, we can decide the connectedness of each $\mathbf{R}Y_i^{\pm} \backslash \mathbf{R}A$ by the transition functions (1). We have the following results on the orientability of $\mathbf{R}Y^{\pm}$.

(i) If both $\frac{d}{2}$ and $\frac{r}{2}$ are even, then every component of $\mathbf{R}Y^{\pm}$ is orientable.

(ii) If $\frac{d}{2}$ or $\frac{r}{2}$ is odd and $l'' = 0$, the the component that lies over the intersection of the exteriors of all the ovals is nonorientable and any other components of $\mathbf{R}Y^{\pm}$ are orientable.

(iii) If $\frac{d}{2}$ or $\frac{r}{2}$ is odd, $l'' > 0$ and $\frac{d}{2}t + \frac{r}{2}s$ is even (resp. odd), then the components that lie over the intersection of some R_i and the exteriors of all the ovals in R_i is orientable (resp. nonorientable) and any other components of $\mathbf{R}Y^{\pm}$ are orientable.

Now we consider the case when $\mathbf{R}A$ is dividing. In this case, A/τ is identified with A^+ (recall §1). Hence, W is a submanifold of $\mathbf{C}P^1 \times \mathbf{C}P^1$. We have the following lemma.

Lemma 3.1. *In the case* $\mathbf{R}A$ *is dividing, we have*

$$[W] = \begin{cases} \frac{d}{2}[\infty \times \mathbf{C}P^1] + \frac{r}{2}[\mathbf{C}P^1 \times \infty] & (\text{if } l'' = 0), \\ (\frac{d}{2} + \hat{l}s)[\infty \times \mathbf{C}P^1] + (\frac{r}{2} + \hat{l}t)[\mathbf{C}P^1 \times \infty] & (\text{if } l'' > 0) \end{cases}$$

in $H_2(\mathbf{C}P^1 \times \mathbf{C}P^1; \mathbf{Z}_2)$. (*For the definitions of* s, t, *and* $\hat{l}$, *see* (1) *and* (3) *in* §1.)

Proof. We fix a triangulation of $\mathbf{C}P^1 \times \mathbf{C}P^1$ such that (i) the subspaces arising are all subcomplexes and (ii) the complex conjugation τ is a simplicial map. From now on we allow the following abuse of notation: A^+, for example, may denote either the subspace A^+ or the corresponding $\mathbf{Z}$- or $\mathbf{Z}_2$-chain (i.e., the sum of all the 2-simplexes contained in A^+ oriented sensibly).

We have to show Lemma 3.1 only in the case $l'' > 0$. Actually, if we replace $\hat{l}$ by 0 in the following proof, then we will obtain a proof of the assertion in the case $l'' = 0$.

As $\mathbf{Z}$-chains on $\mathbf{C}P^1 \times \mathbf{C}P^1$, we have that

$$\partial A^+ = \sum \text{(all the components of } \mathbf{R}A \text{ with certain signs } \pm\text{), and}$$
$$\partial B^+ = \sum \text{(the same but perhaps with different signs).}$$

Hence we have

$$\begin{aligned}\partial(A^+ + B^+) = 2\sum &\text{(some of the ovals of } \mathbf{R}A \text{ with certain signs)}\\ &+ 2\sum \text{(some of the nontrivial components of } \mathbf{R}A\\ &\qquad\qquad\text{with certain signs).}\end{aligned}$$

For each oval E_i, there exists a $\mathbf{Z}$-chain D_i in $\mathbf{R}P^1 \times \mathbf{R}P^1$ such that $\partial D_i = E_i$. The nontrivial components that remain in $\partial(A^+ + B^+)$ consist of some pairs, which can be written as $\pm\partial R_j$ (j : even) and $\hat{l}$ components. For each nontrivial component E_k, there exists a $\mathbf{Z}$-chain D'_k in $\mathbf{R}P^1 \times \mathbf{R}P^1$ such that $E_k = \pm(s(\infty \times \mathbf{R}P^1) + t(\mathbf{R}P^1 \times \infty)) + \partial D'_k$. Let $D^\pm$ be the closures of the connected components of $\mathbf{C}P^1 \backslash \mathbf{R}P^1$. We assume that $\partial D^+ = \mathbf{R}P^1$. Then we have

$$\tau(\infty \times D^+) = -(\infty \times D^-), \quad \tau(D^+ \times \infty) = -(D^- \times \infty),$$

and

$$\begin{aligned}\partial(A^+ + B^+) = 2\partial \Bigg(&\sum_i D_i + \sum_j (\pm R_j)\\ &+ \sum_k \{\pm(s(\infty \times D^+) + t(D^+ \times \infty)) + D'_k\}\Bigg),\end{aligned}$$

where we remark that $\#(k) = \hat{l}$. We set

$$\begin{aligned}\widetilde{C} = A^+ + B^+ - 2\Bigg(&\sum_i D_i + \sum_j (\pm R_j)\\ &+ \sum_k \{\pm(s(\infty \times D^+) + t(D^+ \times \infty)) + D'_k\}\Bigg).\end{aligned}$$

Then $\widetilde{C}$ is a $\mathbf{Z}$-cycle in $\mathbf{C}P^1 \times \mathbf{C}P^1$. Here, since $\tau(\infty \times \mathbf{C}P^1) = -(\infty \times \mathbf{C}P^1)$ and $\tau(\mathbf{C}P^1 \times \infty) = -(\mathbf{C}P^1 \times \infty)$, we see that

(3) τ_* acts as -1 on $H_2(\mathbf{C}P^1 \times \mathbf{C}P^1 ; \mathbf{Z})$. Hence, if we set

$$\begin{aligned}\widehat{C} = W &- \left(\frac{d}{2}(\infty \times \mathbf{C}P^1) + \frac{r}{2}(\mathbf{C}P^1 \times \infty)\right) - 2\left(\sum_i D_i + \sum_j (\pm R_j)\right) \\ &- \sum_k \{\pm(s(\infty \times D^+) + t(D^+ \times \infty)) + D'_k\} \\ &+ \sum_k \{\pm(s(\infty \times D^-) + t(D^- \times \infty)) - D'_k\},\end{aligned}$$

then we obtain a $\mathbf{Z}$-boundary $(1+\tau)(\widetilde{C}) + A - (d(\infty \times \mathbf{C}P^1) + r(\mathbf{C}P^1 \times \infty)) = 2\widehat{C}$. Since $H_2(\mathbf{C}P^1 \times \mathbf{C}P^1 ; \mathbf{Z})$ is free, we see that $\widehat{C}$ is also a $\mathbf{Z}$-boundary. Thus, we obtain a $\mathbf{Z}_2$-boundary $W + (\frac{d}{2} + \hat{l}s)(\infty \times \mathbf{C}P^1) + (\frac{r}{2} + \hat{l}t)(\mathbf{C}P^1 \times \infty)$. This completes the proof of Lemma 3.1.

Now we lift our triangulation of $\mathbf{C}P^1 \times \mathbf{C}P^1$ to a triangulation of Y. We define a map

$$\mathrm{tr} : \text{(the chains of } \mathbf{C}P^1 \times \mathbf{C}P^1) \to \text{(the chains of } Y)$$

as follows. If σ is a simplex of A, then $\mathrm{tr}(\sigma)$ is twice the corresponding simplex in Y, and if σ is not in A, then $\mathrm{tr}(\sigma)$ is the sum of the two simplexes lying over it in Y. Then tr is a chain map. Since $H_2(Y ; \mathbf{Z})$ is free, we have

$$[A] = \tfrac{d}{2}\,\mathrm{tr}_*[\infty \times \mathbf{C}P^1] + \tfrac{r}{2}\,\mathrm{tr}_*[\mathbf{C}P^1 \times \infty]$$

in $H_2(Y ; \mathbf{Z})$. As $\mathbf{Z}_2$-chains, $\mathbf{R}Y^{\pm} = \mathrm{tr}\, B^{\pm}$. From Lemma 3.1, we have $A^+ + B^+ \sim (\frac{d}{2} + \hat{l}s)(\infty \times \mathbf{C}P^1) + (\frac{r}{2} + \hat{l}t)(\mathbf{C}P^1 \times \infty)$ as $\mathbf{Z}_2$-cycles. Taking tr gives $2A^+ + \mathbf{R}Y^+ \sim (\frac{d}{2} + \hat{l}s)\,\mathrm{tr}(\infty \times \mathbf{C}P^1) + (\frac{r}{2} + \hat{l}t)\,\mathrm{tr}(\mathbf{C}P^1 \times \infty)$. On the other hand, since $B^+ + B^- = \mathbf{R}P^1 \times \mathbf{R}P^1 = \partial(D^+ \times \mathbf{R}P^1)$ as $\mathbf{Z}$-chains, we have $\mathrm{tr}\, B^+ + \mathrm{tr}\, B^- = \mathrm{tr} \circ \partial(D^+ \times \mathbf{R}P^1) = \partial \circ \mathrm{tr}(D^+ \times \mathbf{R}P^1)$. Hence we have $\mathbf{R}Y^+ \sim \mathbf{R}Y^-$ as $\mathbf{Z}_2$-cycles. Thus, if $\mathbf{R}A$ is dividing, then we have

$$(4)\quad [\mathbf{R}Y^{\pm}] = \begin{cases} \frac{d}{2}\,\mathrm{tr}_*[\infty \times \mathbf{C}P^1] + \frac{r}{2}\,\mathrm{tr}_*[\mathbf{C}P^1 \times \infty] & (\text{if } l'' = 0), \\ (\frac{d}{2} + \hat{l}s)\,\mathrm{tr}_*[\infty \times \mathbf{C}P^1] + (\frac{r}{2} + \hat{l}t)\,\mathrm{tr}_*[\mathbf{C}P^1 \times \infty] & (\text{if } l'' > 0) \end{cases}$$

in $H_2(Y ; \mathbf{Z}_2)$.

Remark 3.3. If $\mathbf{R}Y^+$ is orientable (recall Remark 3.2), from (4) as just shown, we can give another proof of (4) and (2′) of our theorem in a similar way to [10, Proof of Arnol′d's Theorem (6.4)].

Now we consider nondividing curves as well as dividing ones. From Remark 3.2, we see that both $\mathbf{R}Y^+$ and $\mathbf{R}Y^-$ are orientable in the case of degree $(4, 4)$. The following proposition was applied to curves of degree $(4, 4)$ in §1, where P^i_+ denotes the number of positive (recall §1) even ovals in R_i. We define P^i_0, P^i_-, N^i_+, N^i_0, and N^i_- in the same way and set l_{even} (resp. l_{odd}) $= \#\{i :$ even (resp. odd) $|R_i$ contains some ovals$\}$.

Proposition 3.1. *We first obtain*

$$\text{(I)} \qquad |\chi(B^+)| \leq (\tfrac{d}{2}-1)(\tfrac{r}{2}-1)+\tfrac{dr}{2}.$$

Next we assume that both $\mathbf{R}Y^+$ *and* $\mathbf{R}Y^-$ *are orientable* (*recall Remark* 3.2) *and consider the following eight inequalities*:

$$\text{(II-1)} \qquad P_- + P_0 \leq \left(\frac{d}{2}-1\right)\left(\frac{r}{2}-1\right),$$

$$\text{(II-2)} \qquad P_+ + P_0 \leq \left(\frac{d}{2}-1\right)\left(\frac{r}{2}-1\right)+\frac{dr}{2}+(P-N),$$

$$\text{(II-3)} \qquad N_- + N_0 \leq \left(\frac{d}{2}-1\right)\left(\frac{r}{2}-1\right)-1,$$

$$\text{(II-4)} \qquad N_+ + N_0 \leq \left(\frac{d}{2}-1\right)\left(\frac{r}{2}-1\right)+\frac{dr}{2}-(P-N),$$

$$\text{(II-1}'\text{)} \qquad \sum_{i:\text{ odd}} P_-^i + \sum_{i:\text{ even}} N_-^i + \sum_{i:\text{ odd}} P_0^i + \sum_{i:\text{ even}} N_0^i + \frac{l''}{2} \leq \left(\frac{d}{2}-1\right)\left(\frac{r}{2}-1\right),$$

$$\text{(II-2}'\text{)} \qquad \sum_{i:\text{ odd}} P_+^i + \sum_{i:\text{ even}} N_+^i + \sum_{i:\text{ odd}} P_0^i + \sum_{i:\text{ even}} N_0^i + \frac{l''}{2} - l_{\text{even}} \leq \left(\frac{d}{2}-1\right)\left(\frac{r}{2}-1\right)+\frac{dr}{2}+\sum_i(-1)^{i+1}(P^i-N^i),$$

$$\text{(II-3}'\text{)} \qquad \sum_{i:\text{ odd}} N_-^i + \sum_{i:\text{ even}} P_-^i + \sum_{i:\text{ odd}} N_0^i + \sum_{i:\text{ even}} P_0^i + \frac{l''}{2} \leq \left(\frac{d}{2}-1\right)\left(\frac{r}{2}-1\right),$$

$$\text{(II-4}'\text{)} \qquad \sum_{i:\text{ odd}} N_0^i + \sum_{i:\text{ even}} P_+^i + \sum_{i:\text{ odd}} N_0^i + \sum_{i:\text{ even}} P_0^i + \frac{l''}{2} - l_{\text{odd}} \leq \left(\frac{d}{2}-1\right)\left(\frac{r}{2}-1\right)+\frac{dr}{2}-\sum_i(-1)^{i+1}(P^i-N^i).$$

Then we have the following.

(i) *If* $l''=0$ (*resp.* $l''>0$) *and* $\mathbf{R}A$ *is not dividing, then* (II-1), (II-2), (II-3), *and* (II-4) (*resp.* (II-1$'$), (II-2$'$), (II-3$'$), *and* (II-4$'$)) *are correct.*

(ii) *If* $\frac{d}{2}$ *and* $\frac{r}{2}$ *are even,* $l''=0$, $\mathbf{R}A$ *is dividing, and* B^+ (*resp.* B^-) *has a component whose Euler characteristic is not zero*; *then* (II-1) *and* (II-2) (*resp.* (II-3) *and* (II-4)) *are correct.*

(ii$'$) (a) *If* $\frac{d}{2}$ *and* $\frac{r}{2}$ *are even,* $l''>0$, $\mathbf{R}A$ *is dividing,* $\hat{l}$ *is even, and* B^+ (*resp.* B^-) *has a component whose Euler characteristic is not zero, then* (II-1$'$) *and* (II-2$'$) (*resp.* (II-3$'$) *and* (II-4$'$)) *are correct.* (b) *If* $s \equiv \frac{d}{2} \pmod 2$,

$t \equiv \frac{r}{2} \pmod 2$, $l'' > 0$, $\mathbf{R}A$ is dividing, $\hat{l}$ is odd, and B^+ (resp. B^-) has a component whose Euler characteristic is not zero, then (II-1$'$) *and* (II-2$'$) *(resp.* (II-3$'$) *and* (II-4$'$)) *are correct.*

(iii) *In the case $l'' = 0$ (resp. $l'' > 0$);* (II-1), (II-2), (II-3), *and* (II-4) *(resp.* (II-1$'$), (II-2$'$), (II-3$'$), *and* (II-4$'$)) *are correct if we add* 1 *to the right-hand side of each of them.*

Proof. We consider the $\mathbf{R}$-vector space $H_2(Y;\mathbf{R})$ and the involutions θ_* and $T_*^{\pm}$. Then we have $H_2(Y;\mathbf{R}) = {}_1V_1 \oplus {}_{-1}V_1 \oplus {}_1V_{-1} \oplus {}_{-1}V_{-1}$, where ${}_{-1}V_1$, for example, denotes the subspace on which T_*^+ acts as 1 and T_*^- acts as -1. By the definition of tr, we have $\operatorname{tr}_*(H_2(\mathbf{C}P^1 \times \mathbf{C}P^1;\mathbf{R})) =$ (the 1-eigenspace of θ_*). On the other hand, ${}_1V_1 \oplus {}_{-1}V_{-1} =$ (the 1-eigenspace of θ_*). Since τ_* acts as -1 on $H_2(\mathbf{C}P^1 \times \mathbf{C}P^1;\mathbf{R})$ (recall (3) in this section) and $T^{\pm}$ are the lifts of τ, we see that ${}_1V_1 = \{0\}$.

We next consider the intersection form on $H_2(Y;\mathbf{R})$. Remark that $H_2(Y;\mathbf{R}) = {}_{-1}V_1 \oplus {}_1V_{-1} \oplus {}_{-1}V_{-1}$ is an orthogonal decomposition. Moreover, we decompose ${}_{-1}V_1$ into positive and negative definite subspaces ${}_{-1}V_1^{\pm}$, and also decompose ${}_1V_{-1}$ and ${}_{-1}V_{-1}$. Then we see that $\dim {}_{-1}V_{-1}^+ = \dim {}_{-1}V_{-1}^- = 1$. Let α, β, γ, and δ be the dimensions of ${}_{-1}V_1^+$, ${}_{-1}V_1^-$, ${}_1V_{-1}^+$, and ${}_1V_{-1}^-$, respectively. Then, by (2) in this section and similar arguments to [10, p. 69], we have the following.

(5) $\alpha + \beta + \gamma + \delta + 2 = \dim H_2(Y;\mathbf{R}) = 4 + 2(d-1)(r-1)$,

(6) $(\alpha + \gamma + 1) - (\beta + \delta + 1) = \operatorname{Sign}(Y) = -dr$,

(7) $\chi(\mathbf{R}Y^+) = \sum_{i=0}^4 (-1)^i$ trace $T_*^+ | H_i(Y;\mathbf{R})$, namely, $2\chi(B^+) = 1 + (\alpha + \beta - \gamma - \delta - 2) + 1$

(8) (the signature of the involution T^+) $= (\mathbf{R}Y^+ \circ \mathbf{R}Y^+)_Y$, namely, $(\alpha - \beta) - ((\gamma + 1) - (\delta + 1)) = -2\chi(B^+)$.

By (5)–(8), we have

(9) $\alpha = \gamma = (\frac{d}{2} - 1)(\frac{r}{2} - 1)$, $\beta = (\frac{d}{2} - 1)(\frac{r}{2} - 1) + \frac{dr}{2} + \chi(B^+)$, and $\delta = (\frac{d}{2} - 1)(\frac{r}{2} - 1) + \frac{dr}{2} - \chi(B^+)$.

Since $\beta \geq 0$ and $\delta \geq 0$, (I) of Proposition 3.1 follows.

TABLE 3.1

	$l'' = 0$	$l'' > 0$
s_+^+	P_-	$\sum_{i:\text{ odd}} P_-^i + \sum_{i:\text{ even}} N_-^i + l_{\text{even}}$
s_-^+	P_+	$\sum_{i:\text{ odd}} P_+^i + \sum_{i:\text{ even}} N_+^i$
s_0^+	P_0	$\sum_{i:\text{ odd}} P_0^i + \sum_{i:\text{ even}} N_0^i + l''/2 - l_{\text{even}}$
s_+^-	$N_- + 1$	$\sum_{i:\text{ odd}} N_-^i + \sum_{i:\text{ even}} P_-^i + l_{\text{odd}}$
s_-^-	N_+	$\sum_{i:\text{ odd}} N_+^i + \sum_{i:\text{ even}} P_+^i$
s_0^-	N_0	$\sum_{i:\text{ odd}} N_0^i + \sum_{i:\text{ even}} P_0^i + l''/2 - l_{\text{odd}}$

Now, let $\mathbf{R}Y_i^{\pm}$ $(1 \le i \le s_+^{\pm})$, $\mathbf{R}Y_i^{\pm}$ $(s_+^{\pm}+1 \le i \le s_+^{\pm}+s_-^{\pm})$ and $\mathbf{R}Y_i^{\pm}$ $(s_+^{\pm}+s_-^{\pm}+1 \le i \le s_+^{\pm}s_-^{\pm}+s_0^{\pm})$ denote the components of $\mathbf{R}Y^{\pm}$ with positive, negative, and zero self-intersections, respectively. Then $s_+^{\pm}$, $s_-^{\pm}$, and $s_0^{\pm}$ are written as in Table 3.1. Since $\mathbf{R}Y^{\pm}$ are assumed to be orientable, we may consider the homology classes $[\mathbf{R}Y_i^{\pm}]$ in $H_2(Y;\mathbf{R})$ for all the components $\mathbf{R}Y_i^{\pm}$. Remark that every $[\mathbf{R}Y_i^{+}]$ (resp. $[\mathbf{R}Y_i^{-}]$) is an element of ${}_{-1}V_1$ (respectively, ${}_1V_{-1}$), all the $[\mathbf{R}Y_i^{\pm}]$ are pairwise orthogonal, and hence the elements $[\mathbf{R}Y_i^{\pm}]$ $(i = 1, \dots, s_+^{\pm}+s_-^{\pm})$ are linearly independent. If moreover,

(10) the elements $[\mathbf{R}Y_i^{+}]$ $(i = 1, \dots, s_+^{+}+s_-^{+}+s_0^{+})$ are linearly independent, then we have

(11) $s_+^{+}+s_0^{+} \le \alpha$ and $s_-^{+}+s_0^{+} \le \beta$.

In the same say, if

(12) the elements $[\mathbf{R}Y_i^{-}]$ $(i = 1, \dots, s_+^{-}+s_-^{-}+s_0^{-})$ are linearly independent, then we have

(13) $s_+^{-}+s_0^{-} \le \gamma$ and $s_-^{-}+s_0^{-} \le \delta$.

From the exact sequence of the pair $(Y, \mathbf{R}Y^{\pm})$, we see that the condition $H_3(Y, \mathbf{R}Y^{\pm}; \mathbf{R}) = 0$ is sufficient for (10) (resp. (12)). We can estimate $\dim H_3(Y, \mathbf{R}Y^{\pm}; \mathbf{R})$ as follows. The covering transformation θ preserves $\mathbf{R}Y^{\pm}$, hence induces an involution on $Y/\mathbf{R}Y^{\pm}$. From the Smith theory sequence for this space (see [10]), we have

$$(14)\qquad 0 \to \mathbf{Z}_2 \to H_3(\mathbf{C}P^1 \times \mathbf{C}P^1, B^{\pm} \cup A) \to H_3(Y, \mathbf{R}Y^{\pm}) \to 0.$$

By the exact sequence of the pair $(\mathbf{C}P^1 \times \mathbf{C}P^1/B^{\pm}, A/\mathbf{R}A)$ with $\mathbf{Z}_2$ coefficients, we have

$$(15)\quad 0 \to H_3(\mathbf{C}P^1 \times \mathbf{C}P^1, B^{\pm} \cup A) \to H_2(A, \mathbf{R}A) \to H_2(\mathbf{C}P^1 \times \mathbf{C}P^1, B^{\pm}).$$

On the other hand, we have

$$(16)\qquad H_2(A, \mathbf{R}A, \mathbf{Z}_2) \simeq \begin{cases} \mathbf{Z}_2 \oplus \mathbf{Z}_2 & (\text{if } \mathbf{R}A \text{ is dividing}), \\ \mathbf{Z}_2 & (\text{if } \mathbf{R}A \text{ is not dividing}). \end{cases}$$

From (14)–(16). If $\mathbf{R}A$ is not dividing, then $H_3(\mathbf{C}P^1 \times \mathbf{C}P^1, B^{\pm} \cup A) \simeq \mathbf{Z}_2$, hence $H_3(Y, \mathbf{R}Y^{\pm}; \mathbf{Z}_2) = 0$. By the universal coefficient theorem, we have $H_3(Y, \mathbf{R}Y^{\pm}; \mathbf{R}) = 0$. Hence both (11) and (13) become correct. Thus, (i) of Proposition 3.1 follows from (9) and Table 3.1. Now if $\mathbf{R}A$ is dividing, then from (14)–(16), $\dim H_3(\mathbf{C}P^1 \times \mathbf{C}P^1, B^{\pm} \cup A, \mathbf{Z}_2) = 1$ or 2. Hence we see that $\dim H_3(Y, \mathbf{R}Y^{\pm}; \mathbf{R}) \le 1$. Thus, (iii) follows. On the other hand, in the case $\mathbf{R}A$ is dividing, by (4) in this section, we have $[\mathbf{R}Y^{\pm}] = 0$ (in $H_2(Y; \mathbf{Z}_2)$) in the following three cases.

(ii) $\frac{d}{2}$ and $\frac{r}{2}$ are even and $l'' = 0$

(ii′)(a) $\frac{d}{2}$ and $\frac{r}{2}$ are even, $l'' > 0$ and $\hat{l}$ is even

(ii′)(b) $s \equiv \frac{d}{2} \pmod 2$, $t \equiv \frac{r}{2} \pmod 2$, $l'' > 0$, and $\hat{l}$ is odd.

Hence, in these cases, if $B^{\pm}$ has a component whose Euler characteristic is not zero, then $H_3(Y, \mathbf{R}Y^{\pm}; \mathbf{R}) = 0$ as in [10, p. 71]. This completes the proof of Proposition 3.1.

Remark 3.4. (i) Since $H_3(\mathbf{C}P^1 \times \mathbf{C}P^1, \mathbf{R}P^1 \times \mathbf{R}P^1, \mathbf{Z}_2) \simeq \mathbf{Z}_2$, we cannot pursue further consideration as in [10, p. 71, lines 8–12] of the previous proposition. (ii) When $\mathbf{R}Y^+$ or $\mathbf{R}Y^-$ has some nonorientable components, we can prove a weakened version of this proposition. But we omit the statement of those results.

References

1. V. I. Arnol′d, *On the arrangement of the ovals of real plane curves, involutions of 4-dimensional smooth manifolds, and the arithmetic of integral quadratic forms*, Functional Anal. Appl. **5** (1971), 169–176.
2. E. H. Brown, Jr., *Generalizations of the Kervaire invariant*, Ann. of Math. **95** (1972), 368–384.
3. D. A. Gudkov, *The topology of real projective algebraic manifolds*, Russian Math. Surveys **29** (1974), 1–79.
4. L. Guillou and A. Marin, *Une extension d'un théorème de Rohlin sur la signature*, C. R. Acad. Sci. Paris **285** (1977), 95–98.
5. A. Harnack, *Über die Vieltheiligkeit der ebenen algebraischen Curven*, Math. Ann. **10** (1876), 189–199.
6. D. Hilbert, *Über die reellen Züge algebraischer Curven*, Math. Ann. **38** (1891), 115–138.
7. G. Ishikawa, *Construction of extremal real correspondences of* $\mathbf{P}^1$ (preprint, 1986).
8. M. Letizia, *Quotients by complex conjugation of nonsingular quadrics and cubics in* $\mathbf{P}^3_{\mathbf{c}}$ *defined over* $\mathbf{R}$, Pacific J. Math. **110** (1984), 307–314.
9. A. Marin, *Quelques remarques sur les courbes algébriques planes réelles*, Publ. Math. Univ. Paris VII **9** (1980), 51–68.
10. G. Wilson, *Hilbert's sixteenth problem*, Topology **17** (1978), 53–73.
11. V. I. Zvonilov, *Complex topological characteristics of real algebraic curves on a surface*, Functional Anal. Appl. **16** (1982), 202–204.

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