NOETHER-LEFSCHETZ LOCUS FOR SURFACES

SUNG-OCK KIM

ABSTRACT. We generalize M. Green's Explicit Noether-Lefschetz Theorem to the family of smooth complete intersection surfaces in the higher dimensional projective spaces. Moreover, we give a new proof of the Density Theorem due to C. Ciliberto, J. Harris, and R. Miranda [5].

1. INTRODUCTION

Let $\mathbb{P}^n$ be the complex projective space of dimension $n$. The Noether-Lefschetz Theorem says that a general surface $S$ of degree $d$ in $\mathbb{P}^3$ contains only curves which are complete intersections of $S$ with another hypersurface in $\mathbb{P}^3$ for $d \geq 4$. The word "general" is used in the following sense: A property is said to hold at a general point of a projective variety $V$, if there exists a countable union $\Sigma$ of proper subvarieties of $V$ such that the property holds at all points of $V - \Sigma$. In [21], Lefschetz proved an even more general version: A general complete intersection surface $S$ of $n - 2$ hypersurfaces in $\mathbb{P}^n$, $n \geq 3$, contains only curves that are themselves complete intersections unless $S$ is an intersection of two quadric 3-folds in $\mathbb{P}^4$ or degree $S \leq 3$ in $\mathbb{P}^3$. We denote by $Y_n$ the space of smooth complete intersection surfaces of type $(d_1, \ldots, d_{n-2})$ in $\mathbb{P}^n$, where $2 \leq d_1 \leq d_2 \leq \cdots \leq d_{n-2}$. Let $E = \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^n}(d_i)$. $Y_n$ is parametrized by an open subset, which is also denoted by $Y_n$, by abuse of notation, of the Grassmannian of 1-dimensional subspaces of $H^0(\mathbb{P}^n, E)$. The Noether-Lefschetz locus $\Sigma_n$ in $Y_n$ is the set of smooth surfaces in $Y_n$ containing curves which are not complete intersections, i.e.,

$$\Sigma_n = \{ S \in Y_n \mid \text{Pic}(S) \text{ is not generated by the hyperplane class} \}.$$

Since the fundamental work of Noether and Lefschetz, their results have been improved in a number of interesting directions (see, e.g., [3, 7, 11, 12, 23, 26, 27]). For $n = 3$, by a mixture of Hodge-theoretic and algebraic techniques, Green [8, 10] showed that every irreducible component of $\Sigma_3$ has codimension at least $d_1 - 3$ in the family of smooth surfaces of degree $d_1$ in $\mathbb{P}^3$ for $d_1 \geq 3$,
which is called the explicit Noether-Lefschetz Theorem. A generalization of this theorem to the case \( n \geq 4 \) is given in §2 (cf. Theorem 1). There is one new phenomenon in this case which is not present in the case of surfaces in \( \mathbb{P}^3 \). For example, the analog of Green's result in \( \mathbb{P}^4 \) holds only when a general element of a component is the intersection of two smooth 3-folds.

On the other hand, an upper bound for the codimension of the irreducible components in the case \( n = 3 \) is \( p_g = (d_i - 1) \). Ciliberto, Harris and Miranda [5] proved that over an algebraically closed field of any characteristic, for \( d_i \geq 4 \), the Noether-Lefschetz locus in the family \( Y_3 \) of smooth surfaces of degree \( d_1 \) in \( \mathbb{P}^3 \) contains infinitely many components having maximal codimension \( p_g \) and the union of these components is Zariski dense in \( Y_3 \). Following M. Green's idea, they showed that over the complex numbers, the existence of one such component implies that the union of the components having maximal codimension \( p_g \) is dense in \( Y_3 \) in the classical topology. We will give a rather simple infinitesimal proof of this without constructing such components directly in §3.

2. A GENERALIZATION OF THE EXPLICIT NOETHER-LEFSCHETZ THEOREM

Let \( \Sigma_L \subset \Sigma_n \) denote the subvariety of surfaces containing lines, i.e., curves of degree 1 in \( \mathbb{P}^n \). Let \( G = \text{Grassmannian of lines in } \mathbb{P}^n \). Then \( \Sigma_L \) is the image under projection on the second factor of the incidence correspondence

\[
\tilde{\Sigma}_L = \{(C, S) \mid C \subset S \} \subset G \times Y_n.
\]

For a line \( l \subset \mathbb{P}^n \), we have an exact sequence

\[
0 \rightarrow \mathcal{I}_l|\mathbb{P}^n 
\rightarrow \mathcal{O}_{\mathbb{P}^n} 
\rightarrow \mathcal{O}_l 
\rightarrow 0
\]

where \( r \) is the restriction map. Tensoring with \( E \) and taking the long exact sequence of cohomology, we have

\[
0 \rightarrow H^0(\mathbb{P}^n, \mathcal{I}_l|\mathbb{P}^n \otimes E) \rightarrow H^0(\mathbb{P}^n, E) \rightarrow H^0(\mathbb{P}^n, E \otimes \mathcal{O}_l) \rightarrow \cdots.
\]

From this sequence, we can see that the fiber of the projection map \( \pi_1: \tilde{\Sigma}_L \rightarrow G \) over \( l \) is contained in \( H^0(\mathbb{P}^n, \mathcal{I}_l|\mathbb{P}^n \otimes E) \). Since \( H^0(\mathbb{P}^n, E) \rightarrow H^0(\mathbb{P}^n, E \otimes \mathcal{O}_l) \) is surjective,

\[
\dim \text{fiber of } \pi_1 = \dim H^0(\mathbb{P}^n, E) - \sum_{i=1}^{n-2} (d_i + 1) - 1.
\]

So

\[
\text{codim}_{Y_n} \Sigma_L = \sum_{i=1}^{n-2} d_i - n.
\]

We have the following explicit Noether-Lefschetz Theorem for \( n \geq 4 \), which generalizes the theorem of Green [10] for \( n = 3 \).
Theorem 1. Let \( n \geq 4 \). An irreducible component \( \Sigma' \) of \( \Sigma_n \) has codimension at least \( \sum_{i=1}^{n-2} d_i - n \) in \( Y_n \) if for a general point of \( \Sigma' \), the corresponding surface \( S = \bigcap_{i=1}^{k} H_i \) has the property that \( \bigcap_{i=1}^{k} H_i \) has no singularity for any \( k \) with \( d_k < d_{k+1} \).

Example. The hypothesis in Theorem 1 is necessary. For example, let \( n = 4 \) and \( F_1 = z_0^d_1 + z_0 z_2^{d_2-1} + z_1^{d_1} + z_2 z_3^{d_3-1} \). Then \( H_1 = \{ F_1 = 0 \} \) has an isolated singularity at \((0, 0, 0, 0, 1)\) and has no other singularities. It is a cone over a smooth surface in \( \mathbb{P}^3 \) containing a line. \( H_1 \) contains the plane \( z_0 = z_1 = 0 \). The intersection of \( H_1 \) with any 3-fold \( H_2 \) of degree \( d_2 \) contains a plane curve of degree \( d_2 \) and therefore is in the Noether-Lefschetz locus. The component \( \Sigma' \) containing these complete intersection surfaces has codimension depending only on \( d_1 \), i.e., codimension of \( \Sigma' \) is at most \( (\frac{d_1+4}{4}) \). If \( d_2 > (\frac{d_1+4}{4}) + 4 - d_1 \), then \( \text{codim}_{\mathbb{P}_4} \Sigma' < d_1 + d_2 - 4 \).

We will give a proof of Theorem 1 using similar techniques to Green’s [10]. First, we will show the following simple algebraic fact and then reduce our theorem to this.

**Proposition 1.** Let \( W \subseteq H^0(\mathbb{P}^n, E) \) be a subspace such that the evaluation map

\[
W \otimes \mathcal{O}_{\mathbb{P}^n,x} \rightarrow E_x
\]

is surjective for all \( x \in \mathbb{P}^n \). Then the map

\[
W \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \rightarrow H^0(\mathbb{P}^n, E(k))
\]

is surjective if \( k \geq \text{codim} \ W \).

**Proof.** Let \( c = \text{codim} \ W \). We can choose an increasing sequence of linear subspaces

\[
W_c = W \subset W_{c-1} \subset \ldots \subset W_1 \subset W_0 = H^0(\mathbb{P}^n, E)
\]

so that \( \text{dim} \ W_{i-1}/W_i = 1 \) for \( i = 1, 2, \ldots, c \). Since the evaluation map \( W \otimes \mathcal{O}_{\mathbb{P}^n,x} \rightarrow E_x \) is surjective at all \( x \in \mathbb{P}^n \), the kernel \( M_i \) of the map \( W_i \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow E \) is a vector bundle on \( \mathbb{P}^n \) sitting in the exact sequence

\[
0 \rightarrow M_i \rightarrow W_i \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow E \rightarrow 0
\]

for \( i = 1, 2, \ldots, c \), and it is enough to show that

\[
H^1(M_c \otimes \mathcal{O}_{\mathbb{P}^n}(k)) = 0 \quad \text{if} \quad k \geq c = \text{codim} \ W,
\]

which follows from the following lemma.

**Lemma.** For all \( i = 0, 1, \ldots, c \), \( H^q(\mathbb{P}^n, \bigwedge^p M_i(k)) = 0 \) if \( q \geq 1 \) and \( k + q \geq p + i \).

**Proof.** We note that the \( M_i \)'s sit in the exact sequence

\[
0 \rightarrow M_i \rightarrow M_{i-1} \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow 0
\]
and thus we have an exact sequence

$$0 \to \bigwedge^{p+1} M_i \to \bigwedge^{p+1} M_{i-1} \to \bigwedge^p M_i \to 0$$

for each $i$. Tensoring by $\mathcal{O}_{\mathbb{P}^n}(k)$ and taking the long exact sequence on cohomology, we have

$$\cdots \to H^q\left(\mathbb{P}^n, \bigwedge^{p+1} M_{i-1}(k)\right) \to H^q\left(\mathbb{P}^n, \bigwedge^p M_i(k)\right) \to H^{q+1}\left(\mathbb{P}^n, \bigwedge^{p+1} M_i(k)\right) \to \cdots.$$

Let $q \geq 1$ and $k + q \geq p + i$, $i = 0, 1, \ldots, c$. We will use induction on $i$ and $p$ to prove the lemma. First, notice that if $p \geq \text{rank } M_i$, then $H^q(\mathbb{P}^n, \bigwedge^{p+1} M_i(k)) = 0$ for all $q \geq 0$ and for any $k \geq 0$.

**Sublemma.** For $i = 0$, $H^q(\mathbb{P}^n, \bigwedge^p M_0(k)) = 0$ if $q \geq 1$ and $k + q \geq p$.

To see this, we first recall (cf. [24, Lecture 14]) that a coherent sheaf $F$ on $\mathbb{P}^n$ is said to be $m$-regular, if $H^q(\mathbb{P}^n, F(m-q)) = 0$ for $q > 0$.

From the exact sequence

$$0 \to M_0 \to H^0(\mathbb{P}^n, E) \otimes \mathcal{O}_{\mathbb{P}^n} \to E \to 0$$

tensored with $\mathcal{O}_{\mathbb{P}^n}(1-q)$, we have the long exact sequence on the cohomology

$$\cdots \to H^{q-1}(\mathbb{P}^n, E(1-q)) \to H^0(\mathbb{P}^n, M_0(1-q)) \to H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1-q)) \to \cdots.$$

If $q = 1$, then $H^0(\mathbb{P}^n, E) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \to H^0(\mathbb{P}^n, E)$ is an isomorphism and $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0$, hence $H^1(\mathbb{P}^n, M_0) = 0$.

For $q > 1$, $H^{q-1}(\mathbb{P}^n, E(1-q)) = 0$ and $H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1-q)) = 0$, and hence

$$H^q(\mathbb{P}^n, M_0(1-q)) = 0.$$

Thus $M_0$ is 1-regular. Then $\bigwedge^p M_0$ is $p$-regular (see, e.g., [20, Lemma 2.7]). Since $p$-regularity implies $(p+1)$-regularity [24, loc.cit], the sublemma follows.

By ascending induction on $i$, we may assume

$$H^q\left(\mathbb{P}^n, \bigwedge^{p+1} M_{i-1}(k)\right) = 0$$

since $k + q \geq (p + 1) + (i - 1) = p + i$. By descending induction on $p$, we may assume

$$H^{q+1}\left(\mathbb{P}^n, \bigwedge^{p+1} M_i(k)\right) = 0$$

since $k + q \geq p + i$ which is equivalent to $k + (q + 1) \geq (p + 1) + i$. Hence

$$H^q\left(\mathbb{P}^n, \bigwedge^p M_i(k)\right) = 0,$$

and the lemma follows.
For a compact complex manifold $M$ of dimension $n$ with the associated $(1,1)$-form $\omega$, we recall that the primitive cohomology is

$$H^{n-k}_{pr}(M) = \ker(\omega^{k+1} : H^{n-k}(M) \to H^{n+k+2}(M)).$$

We denote $H^{p,q}_{pr}(M) = H^{p,q}(M) \cap H^{p+q}_{pr}(M)$. For a smooth hypersurface $X$ of degree $d$ in $\mathbb{P}^n$ with defining equation $F(z_0, \ldots, z_n) = 0$, it is known (cf. [4, 15]) that there are natural Poincaré residue isomorphisms

$$(2.1) \quad H^{n-k-1}_{pr,k}(X) \simeq S^{d(k+1)-n-1}/J_{F,d(k+1)-n-1}$$

where $S = \bigoplus_{k \geq 0} S^k$ is the graded ring $\mathbb{C}[z_0, \ldots, z_n]$ and $J_F = \bigoplus_{k \geq d-1} J_{F,k}$ denotes the Jacobian ideal of $F$ generated by the first partial derivatives of $F$.

In the proof of Theorem 1, we will use this kind of algebraic representations of $H^{2,0}(S)$ and $H^{1,1}_{pr}(S)$ for $S \in Y_n$.

We need the following special cases of the Bott Vanishing Theorem (cf. [2]):

**Bott Vanishing Theorem.** $H^p(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(k)) = 0$ unless

(i) $p = q$ and $k = 0$,
(ii) $p = 0$ and $k > q$, or
(iii) $p = n$ and $k < q - n$.

We will also use the following well-known fact (see, e.g., [18, pp. 445–446]):

$$(2.2) \text{Let}$$

$$0 \to \mathcal{E}^0 \to \cdots \to \mathcal{E}^m \to 0$$

be an exact sequence of sheaves on a topological space $X$. Then there is a spectral sequence abutting to zero with $E_{i}^{p,q} = H^q(X, \mathcal{E}^p)$.

Let $B = \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^n}(-d_i)$. Then for $S \in Y_n$, there is a Koszul complex

$$(2.3) \quad 0 \to \bigwedge_{i=1}^{n-2} B \to \bigwedge_{i=1}^{n-3} B \to \cdots \to \bigwedge_{i=1}^{2} B \to B \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_S \to 0,$$

which is exact since $S$ is a complete intersection (see, e.g., [18, p. 688]). We denote $\mu = \sum_{i=1}^{n-2} d_i - n - 1$. For an algebraic representation of $H^{2,0}(S)$, tensoring (2.3) with $\mathcal{O}_{\mathbb{P}^n}(\mu)$ and applying (2.2), we obtain a spectral sequence abutting to zero with $E_{1}^{p,q} = 0$ unless $q = 0$, $q = n$, or $p = n - 1$. There is no nonzero differential other than the differentials in $E_1$ coming into the position $(p, 0)$ for $p = 0, 1, \ldots, n - 1$. So we obtain an exact sequence

$$\cdots \to H^0\left(\mathbb{P}^n, \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^n}(\mu - d_i)\right) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mu)) \to H^0(\mathbb{P}^n, \mathcal{O}_S(\mu)) \to 0,$$

and hence

$$H^{2,0}(S) \simeq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mu))/\text{im} \ H^0\left(\mathbb{P}^n, \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^n}(\mu - d_i)\right).$$
For an algebraic representation of $H_{pr}^{1,1}(S)$ for $S \in Y_n$, we take the long exact sequence on the cohomology of the short exact sequence

$$0 \to \Theta_S \otimes K_S \to \Theta_{p^n|S} \otimes K_S \to N_{S|p^n} \otimes K_S \to 0,$$

where $\Theta_S$ and $N_{S|p^n}$ denote the holomorphic tangent bundle of $S$ and the normal bundle of $S$ in $\mathbb{P}^n$, respectively. Then we get

$$\to H^0(S, \Theta_{p^n|S} \otimes K_S) \to H^0(S, N_{S|p^n} \otimes K_S) \to H^1(S, \Theta_S \otimes K_S)$$

$$\to H^1(S, \Theta_{p^n|S} \otimes K_S) \to \cdots.$$

So

$$\frac{H^0(S, N_{S|p^n} \otimes K_S)}{\text{im} H^0(S, \Theta_{p^n|S} \otimes K_S)} \simeq \left( \frac{H^1(S, \Omega^1_S)}{\text{im} H^1(S, \Omega_{p^n|S})} \right)^*$$

by Serre duality. We will show that

$$(2.4) \quad H_{pr}^{1,1}(S) \simeq \frac{H^1(S, \Omega^1_S)}{\text{im} H^1(S, \Omega_{p^n|S})}.$$}

Applying (2.2) to the exact sequence (2.3) tensored with $\Omega^1_{p^n}$, we get a spectral sequence abutting to zero. By the Bott Vanishing Theorem, $E^p,q = 0$ unless $q = 0, n$, or $p = n - 1$, or $(p, q) = (n - 2, 1)$. Moreover, no nonzero differential except the differential in $E_1$ comes into or goes out of the position $(n - 2, 1)$ or $(n - 1, 1)$. So $H^1(\mathbb{P}^n, \Omega^1_{p^n}) = H^1(S, \Omega^1_{p^n|S})$. From the exact sequence (2.3) tensored with the dual $E^*$ of $E$, we get a spectral sequence abutting to zero with $E^p,q = 0$ unless $q = 0$, or $q = n$, or $p = n - 1$. No nonzero differential comes into the position $(n - 1, 1)$. So $H^1(S, \mathcal{O}_S \otimes E^*) = 0$. We note that $N_{S|p^n} = \mathcal{O}_S$. Thus

$$\text{im} H^1(S, \Omega^1_{p^n|S}) \simeq H^1(S, \Omega^1_{p^n|S}) \simeq H^1(\mathbb{P}^n, \Omega^1_{p^n}) \simeq (\omega),$$

where $\omega$ is the associated (1,1) form of $\mathbb{P}^n$ (i.e., $\omega$ is the first Chern class $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ of $\mathcal{O}_{\mathbb{P}^n}(1)$). By Lefschetz decomposition, $H^1(\Omega^1_S) \simeq H_{pr}^{1,1}(S) \otimes \omega|_S \cdot H^0,0(S)$. Hence we get (2.4). From the spectral sequence attached to the exact sequence (2.3) tensored with $E(\mu)$, we can see that

$$0 \to H^0(\mathbb{P}^n, E(-n - 1)) \to \cdots \to H^0(\mathbb{P}^n, E(\mu)) \to H^0(S, N_{S|p^n} \otimes K_S) \to 0$$

is exact. Hence

$$\frac{H^0(S, N_{S|p^n} \otimes K_S)}{\text{im} H^0(S, \Theta_{p^n|S} \otimes K_S)} \simeq \frac{H^0(\mathbb{P}^n, E(\mu))}{r^{-1}(\text{im} H^0(\Theta_{p^n|S} \otimes K_S))}.$$

Summarizing the above computations, we obtain the following identifications:
Proposition 2.

(i) $H^{2,0}(S) \simeq \frac{H^0(P^n, \mathcal{O}_{P^n}(\mu))}{\text{im } H^0(P^n, \bigoplus_{i=1}^{n-2} \mathcal{O}_{P^n}(\mu - d_i))}$

(ii) $H^{1,1}_{pr}(S) \simeq \frac{H^0(S, N_{S|P^n} \otimes K_S)}{\text{im } H^0(S, \Theta_{P^n}|_S \otimes K_S)} \simeq \frac{H^0(P^n, E \otimes \mathcal{O}_{P^n}(\mu))}{r^{-1}(\text{im } H^0(S, \Theta_{P^n}|_S \otimes K_S))}$.

We also need an algebraic representation of the subspace of $H^1(S, \Theta_S)$ parametrizing the deformations of $S$ in $P^n$, that is, the image of the Zariski tangent space $T_S(Y_n)$ of $Y_n$ at $S$ under the Kodaira-Spencer map $\rho: T_S(Y_n) \to H^1(S, \Theta_S)$. Let $S = \bigcap_{i=1}^{n-2}\{F_i = 0\}$. $T_S(Y_n)$ is naturally isomorphic to

$\text{Hom}((S), H^0(P^n, E)/(S)) \simeq H^0(P^n, E)/(S)$,

where $(S)$ denotes the 1-dimensional subspace of $H^0(P^n, E)$ generated by $(F_1, \ldots, F_{n-2})$. So the map $T_S(Y) \to H^0(S, N_{S|P^n})$ is surjective and

$\rho(T_S(Y_n)) = \text{im}\{H^0(S, N_{S|P^n}) \to H^1(S, \Theta_S)\}$.

Tensoring the exact sequence (2.3) with $\Theta_{P^n}$ and applying (2.2), we have a spectral sequence abutting to zero. By Serre duality and the Bott Vanishing Theorem, $H^0(P^n, \Theta_{P^n}(\kappa))$ vanishes unless $-k - n - 1 < 1 - n$. Hence $E_1^{p, q} = 0$ unless $(p, q) = (n - 2, 0), (n - 1, 0), (n - 2, 1), (n - 1, 1)$, or $q = n$. No nonzero differential except the differentials in $E_1$ comes into the position $(n - 2, 0)$ or $(n - 1, 0)$. Hence the map $\gamma_1: H^0(P^n, \Theta_{P^n}) \to H^0(S, \Theta_{P^n}|_S)$ is an isomorphism. From the spectral sequence attached to the exact sequence (2.3) tensored with $E$, we can see that the map $\gamma_2: H^0(P^n, E) \to H^0(S, E \otimes \Theta_S)$ is surjective. From the short exact sequence

$0 \to \Theta_S \to \Theta_{P^n}|_S \to N_{S|P^n} \to 0$,

we get the following long exact sequence which fits into a diagram:

$\begin{array}{ccccccc}
0 & \to & \Theta_S & \to & \Theta_{P^n}|_S & \to & N_{S|P^n} \to 0 \\
\downarrow & & & & & & \\
\longrightarrow & H^0(P^n, \Theta_{P^n}) & \longrightarrow & H^0(S, \Theta_{P^n}|_S) & \alpha \\
\downarrow & & & & & \\
\longrightarrow & H^0(P^n, E) & \longrightarrow & H^0(S, N_{S|P^n}) & \longrightarrow & 0 & \beta \\
\downarrow & & & & & & \\
& H^1(S, \Theta_S) & & & & & \\
\end{array}$
Hence

\[ \rho(T_S(Y_n)) \simeq \frac{H^0(S, N_{S|P^n})}{\alpha \circ \gamma_1(H^0(P^n, \Theta_{P^n}))} \simeq \frac{H^0(P^n, E)}{\gamma^*_1(\alpha(H^0(S, \Theta_{P^n}|S)))}. \]

Another preliminary fact we will use is the description of the Zariski tangent space to

\[ \tilde{Y}_n = \{(S, L) \mid S \in Y_n, \ L \in \text{Pic}(S)\}. \]

The first prolongation bundle \( P_1(L) \) of \( L \) is defined by an exact sequence

\[ 0 \to \Omega^1_S \otimes L \to P_1(L) \to L \to 0 \]

with the extension class \( c_1(L) \in \text{Ext}^1(L, \Omega^1_S \otimes L) = H^1(S, \Omega^1_S) \). The computation of Zariski tangent space to the set of pairs of curves with line bundles is given in [1]. An analogous argument gives the description for the surface case: For a fixed \( (S, L) \), the Zariski tangent space \( T_{(S, L)}(\tilde{Y}_n) \) of \( Y_n \) at \( (S, L) \) maps into \( H^1(S, P_1(L)^* \otimes L) \) as follows. As a complex manifold, the line bundle \( L \to S \) is given by the data

\[ \{U_\alpha, z_\alpha, f_\alpha, g_\alpha\}, \]

where \( \{U_\alpha\} \) is a finite open covering of \( S \), \( z_\alpha = (z_{\alpha_1}, z_{\alpha_2}) \) are local coordinates in \( U_\alpha \), \( f_\alpha \) is the coordinate transformation on \( U_\alpha \cap U_\beta \), and \( g_\alpha \) is the transition function for \( L \). Thus two cocycle rules \( f_{\alpha\gamma} = f_\alpha \circ f_\gamma \) and \( g_{\alpha\gamma} = g_\alpha g_\gamma \) hold in \( U_\alpha \cap U_\beta \cap U_\gamma \). The first order deformation of \( L \to S \) is given by

\[ \{U_\alpha, z_\alpha, f_\alpha, g_\alpha\}, \]

satisfying

\[ f_{\alpha\gamma}(z_\gamma, t) \equiv f_\alpha(f_\gamma(z_\gamma, t), t) \mod t^2, \]

\[ g_{\alpha\gamma}(z_\gamma, t) \equiv g_\alpha(g_\gamma(z_\gamma, t), t) \cdot g_\gamma(z_\gamma, t) \mod t^2 \]

on \( U_\alpha \cap U_\beta \cap U_\gamma \). Taking derivatives at \( t = 0 \), we can see that \( \tilde{f}_\alpha = \{\frac{\partial f_\gamma}{\partial z_\gamma}\} \) is a cocycle defining a class \( \tilde{f} \in H^1(S, \Theta_S) \) and that \( \tilde{g}_\alpha = \{\frac{\partial g_\gamma}{\partial z_\gamma}\} \) is a 1-cochain with coefficients in \( \Theta_S \). For the coboundary map \( \delta \), \( \delta\{\tilde{g}_\alpha\} \) is the cup product of \( \tilde{f} \) with \( c_1(L) \). Note that \( c_1(L) = \{g_\alpha^{-1} dg_\alpha\} \in H^1(S, \Omega^1_S) \).

\[ \sigma = \{(\tilde{f}_\alpha, \tilde{g}_\alpha)\} \]

defines a 1-cocycle with coefficients in the extension \( M \) of \( \Theta_S \) by \( \Theta_S \), i.e., \( M \) is defined by the exact sequence

\[ 0 \to \Theta_S \to M \to \Theta_S \to 0, \]

with the extension class \( c_1(L) \). But \( M = P_1(L)^* \otimes L \). So

\[ (\sigma) \in H^1(S, P_1(L)^* \otimes L). \]
Proof of Theorem 1. Let $\tilde{\Sigma}_n = \{(S, L) \mid S \in \Sigma_n, \text{and } L \in \text{Pic}(S)\}$, and let $\pi : (S, L) \mapsto S$ be a projection. For $(S, L) \in \tilde{Y}_n$, we have a commutative diagram

$$
\begin{array}{ccc}
T_{(S, L)}(\tilde{Y}_n) & \xrightarrow{\pi_*} & T_S(Y_n) \\
\downarrow & & \downarrow \\
H^1(S, P_1(L)^* \otimes L) & \xrightarrow{h_1} & H^1(S, \Theta_S)
\end{array}
$$

where $h_1$ sits in the long exact sequence on cohomology

$$
\rightarrow H^1(S, P_1(L)^* \otimes L) \xrightarrow{h_1} H^1(S, \Theta_S) \xrightarrow{h_2} \cdots .
$$

Fix $(S, L) \in \tilde{\Sigma}_n$ with $c_1(L) \in H_{pr}^{1,1}(S)$. Let $Z$ be the union of all irreducible components of $\tilde{\Sigma}_n$ containing $(S, L)$. The image $T(Z)$ of the Zariski tangent space $T_S(\pi(Z))$ of $\pi(Z)$ at $S$ under $\rho$ is in the kernel of $h_2$, i.e.,

$$
\rho(T_S(Y_n)) \otimes H_{pr}^{1,1}(S) \xrightarrow{\cup} H^2(S, \Theta_S),
$$

$$
T(Z) \otimes H^0(S, K_S) \mapsto 0.
$$

Equivalently,

$$
\rho(T_S(Y_n)) \otimes H^0(S, K_S) \xrightarrow{\cup} H_{pr}^{1,1}(S)^*,
$$

(2.7)

$$
T(Z) \otimes H^0(S, K_S) \mapsto c_1(L). \downarrow
$$

Using the notations in the diagram (2.5), we set $T' = \gamma_2^{-1} \circ \beta^{-1}(T(Z)) \subset H^0(\mathbb{P}^n, E)$. Then $T' \supset \gamma_2^{-1}(\text{im } \alpha) \supset \ker \gamma_2$ and the following holds:

Claim. If $S = \bigcap_{i=1}^{m-2} \{F_i = 0\}$, and $F_i$ is a homogeneous polynomial of degree $d_i$ such that $\bigcap_{i=1}^{k} \{F_i = 0\}$ is nonsingular for each $k$ with $d_k < d_{k+1}$, then the evaluation map $T' \otimes \mathcal{O}_{\mathbb{P}^n,x} \rightarrow E_x$ is surjective at every $x \in \mathbb{P}^n$.

To see this, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}^{n-2}$ denote the $i$th coordinate vector, for $i = 1, \ldots, n-2$. Then

$$
\ker \gamma_2 \supset \{F_k G_k e_k | G_k \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_k - d_i)) , \quad d_k > d_i, \text{ and } i = 1, \ldots, n-2\}.
$$

We note that

$$
\gamma_2^{-1}(\text{im } \alpha) \supset \left\{ z_i \left( \frac{\partial F_1}{\partial z_j}, \ldots, \frac{\partial F_{n-2}}{\partial z_j} \right) \mid l, j = 0, 1, \ldots, n \right\}.
$$

For a fixed $x \in \mathbb{P}^n$, let $i_0$ denote the smallest number such that $x \in \{F_{i_0} \neq 0\}$. Then (i) $i_0 = 1$, or (ii) $i_0 > 1$ and $d_{i-1} < d_{i}$, or (iii) $i_0 > 1$ and $d_{i-1} = d_{i}$.

We will show that the evaluation map at $x$ is surjective in any case.

Case (i): If $i_0 = 1$, then we can choose $G_k \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_k - d_i))$, $k = 1, \ldots, n-2$ so that $G_k(x) \neq 0$ for each $k$. So $\{F_k(x) G_k(x) e_k | k = 1, \ldots, n-2\}$ are $n-2$ linearly independent elements in $E_x$.
Case (ii): If \( i_0 > 1 \) and \( d_{i_0 - 1} < d_{i_0} \), then by the hypothesis of the claim
\[
\bigcap_{i=1}^{i_0-1} \{ F_i = 0 \}
\]
has no singularity and so there is a nonvanishing \((i_0 - 1) \times (i_0 - 1)\) minor of a matrix
\[
\begin{pmatrix}
\frac{\partial F_i}{\partial z_j} \\
\vdots \\
\frac{\partial F_{i}}{\partial z_j}
\end{pmatrix}
\]
\(i = 1, \ldots, i_0 - 1\)
\(j = 0, 1, \ldots, n\)
say
\[
\begin{pmatrix}
\frac{\partial F_i}{\partial z_j} \\
\vdots \\
\frac{\partial F_{i}}{\partial z_j}
\end{pmatrix}
\]
\(i = 1, \ldots, i_0 - 1\)
\(j = 1, \ldots, j_{i_0 - 1}\)
which has rank \( i_0 - 1 \). Moreover, there is some \( m \) such that \( z_m(x) \neq 0 \). We can choose \( G_k \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_k - d_{i_0})) \) for \( k = i_0, i_0 + 1, \ldots, n - 2 \) so that \( G_k(x) \neq 0 \) for each \( k \). Then
\[
\left\{ z_m \left( \frac{\partial F_{i}}{\partial z_j}, \ldots, \frac{\partial F_{n-2}}{\partial z_j} \right) \mid j = j_1, \ldots, j_{i_0 - 1} \right\}
\]
\[
\cup \left\{ F_i G_k e_k \mid k = i_0, i_0 + 1, \ldots, n - 2 \right\}
\]
provides \( n - 2 \) linearly independent elements in \( E_x \) when evaluated at \( x \).

Case (iii): If \( i_0 > 1 \) and \( d_{i_0 - 1} = d_{i_0} \), let \( i_1 \) be the smallest number such that \( d_{i_1} = \cdots = d_{i_0 - 1} = d_{i_0} \). Then \( d_{i_1 - 1} < d_{i_1} \) and by the hypothesis of the claim, \( \bigcap_{i=1}^{i_1-1} \{ F_i = 0 \} \) has no singularity. So, as in (ii) we can find \( j_1, \ldots, j_{i_1 - 1} \) such that
\[
\begin{pmatrix}
\frac{\partial F_i}{\partial z_j} \\
\vdots \\
\frac{\partial F_{i}}{\partial z_j}
\end{pmatrix}
\]
\(i = 1, \ldots, i_1 - 1\)
\(j = 1, \ldots, j_{i_1 - 1}\)
has rank \( i_1 - 1 \). Let \( G_k \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_k - d_{i_0})) \) be chosen so that \( G_k(x) \neq 0 \) for \( k = i_1, i_1 + 1, \ldots, i_0, \ldots, n - 2 \). Furthermore, there is some \( m \) such that \( z_m(x) \neq 0 \). Hence
\[
\left\{ z_m \left( \frac{\partial F_{i}}{\partial z_j}, \ldots, \frac{\partial F_{n-2}}{\partial z_j} \right) \mid j = j_1, \ldots, j_{i_1 - 1} \right\}
\]
\[
\cup \left\{ F_i G_k e_k \mid k = i_1, i_1 + 1, \ldots, n - 2 \right\}
\]
defines \( n - 2 \) linearly independent vectors in \( E_x \) when evaluated at \( x \).

Thus, in any case, the evaluation map at \( x \) is surjective and the claim follows.

In terms of the identifications in Proposition 2 and (2.6), (2.7) implies that the evaluation map
\[
T' \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mu)) \to H^0(\mathbb{P}^n, E(\mu))
\]
is not surjective. Therefore, by Proposition 1, \( \text{codim} \; T' \geq \sum_{i=1}^{n-2} d_i - n \). But
\[
\text{codim} \; T' \leq \text{codim}_{T_x(Y)} T_S(\pi(Z)) \leq \text{codim}_{X} Z.
\]
Hence the theorem follows.
3. A NEW PROOF OF THE DENSITY THEOREM

In this section, we denote \( d = d_1 \), \( Y = Y_3 \), and \( \text{NL}_d = \Sigma_3 \). Recall (cf. [3]) that the upper bound of the codimension of irreducible components of the Noether-Lefschetz locus \( \text{NL}_d \) in the family \( Y \) of smooth surfaces of degree \( d \) in \( \mathbb{P}^3 \) is the geometric genus \( p_g = \binom{d-1}{3} \) of any surface in \( Y \). We will give a new proof of the following density theorem due to Ciliberto, Harris, and Miranda [5].

**Theorem 2.** For \( d \geq 4 \), the union of all irreducible components of \( \text{NL}_d \) having codimension \( p_g \) in \( Y \) is dense in the classical topology.

Using an infinitesimal method, we will reduce the theorem to the following proposition.

**Proposition 3.** For each \( d \geq 4 \), there are some polynomials \( G \in S^{d-4} \) and a surface \( X \in Y \) with defining equation \( F \) such that the map

\[
g : S^{d-4} \to S^{3d-8} / J_F, 3d-8
\]

defined by multiplication by \( G \) is injective.

**Proof.** Let \( F = z_0^d + z_1^d + z_2^d + z_3^d \) and

\[
G = \sum_{j=0}^{d-2} a_j z_0^j z_1^j z_2^{d-2-j} z_3^{d-2-j},
\]

where the constant coefficients \( a_j \)'s are chosen so that every possible matrix of the form

\[
\begin{pmatrix}
  a_k & a_{k+1} & \cdots & a_{k+m} \\
  a_{k+1} & a_{k+2} & \cdots & a_{k+m+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{k+m} & a_{k+m+1} & \cdots & a_{k+2m}
\end{pmatrix}
\]

(3.1)

has nonzero determinant. Then we claim that \( g \) is injective:

Without loss of generality, we may assume that a nonzero element of the kernel of \( g \) is of the form

\[
P = \sum_{j=m_1}^{m_2} c_j z_0^{p+j} z_1^{q+j} z_2^{r-j} z_3^{s-j},
\]

where \( p + q + r + s = d - 4 \) and \( m_1 < m_2 \). This is because \( G \) belongs to the span of the set of monomials \( z_0^{i_0} z_1^{i_1} z_2^{i_2} z_3^{i_3} \) satisfying the equalities

\[
i_0 = i_1 = d - 2 - i_2 = d - 2 - i_3.
\]

If we therefore break up \( S^{d-4} \) into the span of monomials satisfying

\[
i_0 - i_1 = p - q, \quad i_0 + i_2 = p + r, \quad i_0 + i_3 = p + s,
\]

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where \( p, q, r, s \) vary but add up to \( d - 4 \), and if we expand an element of \( \ker g \) in terms of these subspaces, then each piece also lies in \( \ker g \).

By symmetry of the role of \( z_0 \) and \( z_1 \), and of \( z_2 \) and \( z_3 \), we may assume \( p \geq q \) and \( r \leq s \). Then the limits of the sum above satisfy \( m_1 \geq -q \) and \( m_2 \leq r \). The condition that

\[
P \cdot G = \sum_{j,k} a_j c_k \varepsilon_j^{j+k+q} \varepsilon_2^{j+k+r} \varepsilon_3^{d-j-k+s} \in J_{d,3d-8}
\]

is equivalent to the system of equations

\[
\sum_{j+k=l} a_j c_k = 0 \quad \text{for} \quad s \leq l \leq d - 2 - p.
\]

Since \( m_1 \geq -p \) and \( m_2 \leq s \), the two inequalities \( m_1 \leq k \leq m_2 \) and \( s \leq k+j \leq d - 2 - p \) imply the inequality \( 0 \leq j \leq d - 2 \).

The coefficient matrix for the \( c_k \)’s is

\[
A = \begin{pmatrix}
a_\alpha & a_{\alpha+1} & \cdots & a_\beta \\
a_{\alpha+1} & a_{\alpha+2} & \cdots & a_{\beta+1} \\
\vdots & \vdots & \ddots & \vdots \\
a_\gamma & a_{\gamma+1} & \cdots & a_\delta
\end{pmatrix}
\]

where

\[
\alpha = s - m_2, \quad \beta = s - m_1, \quad \gamma = d - 2 - p - m_2, \quad \delta = d - 2 - p - m_1.
\]

The number of rows is

\[
\gamma - \alpha + 1 = d - 2 - p - s + 1
\]

and the number of columns is

\[
\beta - \alpha + 1 = m_2 - m_1 + 1.
\]

But \( (3.2) - (3.3) = d - 2 - p - s - (m_2 - m_1) \geq d - 2 - p - s - (r + q) = 2 \). Therefore, \( g \) is injective provided that the appropriate minors of the matrix of the \( a_j \)’s of the form \( (3.1) \) are nonvanishing, and this may be arranged by taking the ratios \( |a_{j+1}/a_j| \) to increase very rapidly with \( j \).

**Proof of Theorem 2.** For a smooth surface \( X \in Y \), \( X \in NL_d \) if and only if \( H_{pr}^1(X) \cap H^2(X, \mathbb{Z}) \neq 0 \). If there is a nonzero element \( \gamma \in H_{pr}^1(X) \cap H^2(X, \mathbb{Z}) \), then \( m \cdot \gamma \in H_{pr}^1(X) \cap H^2(X, \mathbb{Z}) \) for some integer \( m \) and hence \( X \in NL_d \). For a given \( \gamma \in H_{pr}^1(X) \cap H^2(X, \mathbb{R}) \), there are some elements of \( H^2(X, \mathbb{Q}) \) that are arbitrarily near to \( \gamma \). We will show that one of these rational classes can be made to have type \((1, 1)\) by making a small deformation of \( X \).

We consider the universal family \( \mathcal{F} \) of smooth surfaces of degree \( d \) in \( \mathbb{P}^3 \):

\[
\mathcal{F} \longrightarrow Y \times \mathbb{P}^3
\]

\[
\pi \downarrow
\]

\[
Y
\]
Since $\pi$ is a proper smooth map with maximal rank everywhere, Ehresmann’s fibration theorem says that on a sufficiently small open neighborhood $U$ of $X$, there is a fiber preserving diffeomorphism

\[(3.4) \quad \phi: \pi^{-1}(X) \times U \cong \pi^{-1}(U)\]

so that $\phi$ defines a diffeomorphism $\phi_S: X \to S$ and the induced map on the cohomology $\phi^*_S: H^2(S, C) \to H^2(X, C)$ is an isomorphism for $S \in U$.

Let $R^2\pi_*C$ be the second direct image sheaf of $\pi: \mathcal{F} \to Y$, which we recall is the sheaf associated to the presheaf

\[U \to H^2(\pi^{-1}(U), C),\]

where $U$ runs through the open subsets of $Y$. Let $R^2_{pr}$ be the kernel of a map

\[L: R^2\pi_*C \to R^4\pi_*C\]

defined as follows: For an open set $U \subset Y$ with $\pi^{-1}(U) \cong \pi^{-1}(X) \times U$ as before,

\[H^2(\pi^{-1}(U), C) \cong H^2(X, C),\]

$L_U: R^2\pi_*C(U) \to R^4\pi_*C(U)$ is the cup product map with the associated $(1, 1)$ form of $X$.

Then $R^2_{pr}$ is a locally constant sheaf and there is a holomorphic vector bundle $\mathcal{K}$ on $Y$ associated to it, whose fiber over $S \in Y$ is $H^2_{pr}(X, C)$. We have a Hodge filtration $F^2 \subset F^1 \subset F^0 = \mathcal{K}$, and Hodge bundles $\mathcal{K}^{1,1} = F^1/F^2$ and $\mathcal{K}^{0,2} = F^0/F^1$, where the $F^p$'s are holomorphic vector bundles.

For a sufficiently small open neighborhood $U$ of $X$ as in (3.4), we can define a smooth map $f_C$ on the total space of $\mathcal{K}^{1,1}_U$ as

\[f_C: \mathcal{K}^{1,1}_U \mid U = \{(S, \gamma)|S \in U, \gamma \in H^{1,1}_{pr}(S)\} \to H^2_{pr}(X, C),\]

\[(S, \gamma) \mapsto \phi^*_S(\gamma).\]

Then $f_C$ restricts to a map

\[f: \mathcal{K}^{1,1}_U \mid \mathbb{R} = \{(S, \gamma)|S \in U, \gamma \in H^{1,1}_{pr}(S) \cap H^2(S, \mathbb{R})\} \to H^2_{pr}(X, \mathbb{R}).\]

We note that for the map $\pi_1: \mathcal{K}^{1,1} \to Y$, giving the bundle structure on $\mathcal{K}^{1,1}$,

\[\pi_1(f^{-1}(H^2_{pr}(X, \mathbb{Q}))) = NL_d \cap U.\]

First, we will show that $f$ has maximal rank at some $(S_0, \gamma_0) \in \mathcal{K}^{1,1}_U \mid \mathbb{R}$. Then, by the Implicit Function Theorem, this implies that

\[(3.5) \quad \text{there is an element } \gamma_V \in f(V) \cap H^2_{pr}(X, \mathbb{Q}) \text{ for each small open neighborhood } V \text{ of } (S_0, \gamma_0), \text{ and codim } \pi_1(f^{-1}(\gamma_V)) = p_g.\]

In order to make the necessary computation, it is a good idea to distinguish the real tangent space $T_S(U)_R$, the complexified tangent space $T_S(U)_C$, and
the holomorphic tangent space $T_S(U)$. There is of course a natural $\mathbb{R}$-linear isomorphism $T_S(U) \cong T_S(U)_\mathbb{R}$. Since $df$ takes the tangent space of the fibers of $\pi_1$ to $H^{1,1}_{pr}(X) \cap H^2(X, \mathbb{R})$, we obtain an induced $\mathbb{R}$-linear map

$$\lambda: T_S(U)_\mathbb{R} \rightarrow \frac{H^{2,0}_{pr}(X, \mathbb{R})}{H^{1,1}_{pr}(X) \cap H^2(X, \mathbb{R})} \cong (H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbb{R})$$

having maximal rank if and only if $f$ does. Under the $\mathbb{R}$-linear identifications $T_S(U)_\mathbb{R} \cong T_S(U)$ and

$$(H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbb{R}) \cong H^{0,2}(X),$$

the map $\lambda$ is identified with the derivative of the period map

$$T_S(U) \rightarrow H^{0,2}(X).$$

By the work of Griffiths [14], the derivative of the period map is the composition of the Kodaira-Spencer map $\rho$ with the cup product with $\gamma$, i.e.

$$T_S(Y) \xrightarrow{\rho} H^1(S, \Theta_S) \xrightarrow{\cup \gamma} H^{0,2}(S).$$

Thus $\lambda$, and hence $f$, has maximal rank if and only if

$$\cup \gamma: \rho(T_S(Y)) \rightarrow H^{0,2}(S)$$

is surjective, or equivalently,

$$(3.6) \quad H^{2,0}(S) \xrightarrow{\cup \gamma} \rho(T_S(Y))^* \text{ is injective.}$$

Referring to (2.6),

$$\rho(T_S(Y)) \simeq S^d / J_{F, d},$$

where $F$ is the defining equation of $S$. By Macaulay’s theorem (see, e.g., [9, Theorem 2.15]),

$$(\rho(T_S(Y)))^* \simeq S^{3d-8} / J_{F, 3d-8}.$$

In terms of the identifications in (2.1) and above, the above map (3.6) is injective if the multiplication map

$$g: S^{d-4} \rightarrow S^{3d-8} / J_{F, 3d-8}$$

is injective, where $g$ is the multiplication by $G(\gamma) \in S^{2d-4}$ corresponding to $\gamma$. By Proposition 3, $g$ is injective at $(S_0, \gamma_0)$, where

$$S_0 = \{z_0^d + z_1^d + z_2^d + z_3^d = 0\}$$

and $\gamma_0$ corresponds to $G \in S^{2d-4} / J_{F, 2d-4}$ with some fixed real coefficients $a_i$’s, and hence $\gamma_0 \in H^{1,1}_{pr}(S_0) \cap H^2(S_0, \mathbb{R})$. So $f$ has maximal rank $p_g$ at $(S_0, \gamma_0)$.
In fact, \( \rho \) composed with the cup product map \( \cup \gamma \) gives rise to a holomorphic map of vector bundles on \( \mathcal{H}^{1,1} \) so that we can define a map

\[
\sigma : \mathcal{H}^{1,1} \to \Theta_Y^* \otimes \mathcal{H}^{0,2},
\]

\[
(S, \gamma) \mapsto \sigma(S, \gamma) : \Theta_Y \to \mathcal{H}^{0,2}.
\]

The locus \( A \) where \( \sigma(S, \gamma) \) drops rank is an analytic subvariety of \( \mathcal{H}^{1,1} \). Since \( \sigma(S_0, \gamma_0) \) has maximal rank \( p_g \), \( A \) is proper. Since \( f \) has maximal rank at \( (S_0, \gamma_0) \), \( A \cap \mathcal{H}^{1,1}_R \) is also proper, where \( \mathcal{H}^{1,1}_R = \{(S, \gamma) | \gamma \in H^{1,1}_p(S) \cap H^2(S, \mathbb{R}) \} \). Hence, (3.5) holds for every \( (S, \gamma) \in \mathcal{H}^{1,1} \) and the theorem follows.

References

12. , Griffiths' infinitesimal invariant and the Abel-Jacobi map, Preprint.
13. , Koszul cohomology and geometry, Preprint.


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024

Current address: 56 Stephen Hopkins Court, Providence, R.I. 02904