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INFINITESIMAL RIGIDITY FOR THE ACTION OF $\mathbf{SL}(n, \mathbb{Z})$ ON $\mathbb{T}^n$

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ABSTRACT. Let $\Gamma = \mathbf{SL}(n, \mathbb{Z})$ or any subgroup of finite index. Then the action of Γ on $\mathbb{T}^n$ by automorphisms is infinitesimally rigid for $n \geq 7$, i.e., the cohomology $\mathrm{H}^1(\Gamma, \mathrm{Vec}(\mathbb{T}^n)) = 0$, where $\mathrm{Vec}(\mathbb{T}^n)$ denotes the module of C^∞ vector fields on $\mathbb{T}^n$.

1. INTRODUCTION

Let G be a connected semisimple Lie group with trivial center and without compact factors, $\Gamma \subset G$ a lattice. Then if Γ is irreducible and G is not isomorphic to $\mathbf{PSL}(2, \mathbb{R})$, there are a number of well-known results reflecting the "rigidity" of Γ in G. Classical results established rigidity of Γ acting on a finite-dimensional vector space; the present work is part of a more recent program of understanding the realizations of Γ as a group of smooth transformations on a compact manifold.

Early results established local rigidity. For any finitely-generated group Γ and topological group G, let $R(\Gamma, G)$ denote the set of homomorphism of Γ into G with the compact/open topology. (Recall that any lattice in a connected Lie group is finitely-generated.) A homomorphism $\phi: \Gamma \to G$ is said to be locally rigid if its orbit in $R(\Gamma, G)$ under the action of G by conjugation is open. Selberg [S] first proved that the inclusion of a uniform lattice in $\mathbf{SL}(n, \mathbb{R})$, $n \geq 3$, is locally rigid. Weil [W1, W2] obtained the same result for an irreducible uniform lattice in a connected, semisimple Lie group without compact factors and not locally isomorphic to $\mathbf{SL}(2, \mathbb{R})$. Weil also observed [W3] that if $\phi \in R(\Gamma, G)$ such that $\mathrm{H}^1(\Gamma, \mathrm{Ad}_G \circ \phi)$ vanishes, then ϕ is locally rigid. Matsushima and Murakami [M-M] obtained vanishing theorems for $\mathrm{H}^p(\Gamma, \rho)$ in a range of p (depending on ρ) for uniform Γ and ρ an irreducible representation of the ambient Lie group. Borel [B1, B2] has obtained comparable results for the case where Γ is an arithmetic lattice in an algebraic group defined over $\mathbb{Q}$, not necessarily uniform.

Suppose $\Gamma' \subset G'$ satisfy the the same hypotheses as Γ and G, above, and $\pi: \Gamma \to \Gamma'$ is an isomorphism. Then the celebrated rigidity theorem first established by Mostow [Mo2] in the uniform case and extended to the nonuniform case by Margulis [Ma2] ($\mathbb{R}$-rank$(G) \geq 2$) and Prasad [P] ($\mathbb{R}$-rank$(G) = 1$,

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$G \neq \mathbf{PSL}(2, \mathbb{R})$) asserts that π extends to an isomorphism $\pi: G \to G'$. This theorem (for $\mathbb{R}$-rank$(G) \geq 2$) follows directly from a more general result of Margulis [Ma1] ("superrigidity") which provides essentially complete information about (finite-dimensional) representations of Γ in semisimple groups.

Recently, attention has focused on the realizations of G and Γ as smooth transformation groups on compact manifolds M. Work to date suggests that the finite-dimensional rigidity phenomena described above are reflected quite strongly in this context. Zimmer has obtained a number of results asserting rigidity for actions that preserve some special class of geometric structures on M (e.g. [Zi3, Zi4]). All of these results are essentially nonlinear generalizations of Margulis's superrigidity theorem and are established in part by similar arguments together with Kazhdan's "property T."

In this paper, we obtain a generalization of the infinitesimal method which preceded Margulis's technique. Let Vec(M) denote the space of smooth vector fields on M, Diff(M) the group of smooth diffeomorphisms of M with the standard C^∞ topology (uniform convergence of k-jets, $k < \infty$, on compact sets). Any action of Γ on M by diffeomorphisms induces a natural action of Γ on Vec(M). The Γ action on M is called *infinitesimally rigid* if $H^1(\Gamma, \mathrm{Vec}(M)) = 0$. Our main result provides an affirmative answer (for large enough n) to a question posed by Zimmer in his 1986 address to the International Congress of Mathematicians [Zi1]:

Theorem. *Let $\Gamma = \mathbf{SL}(n, \mathbb{Z})$ or any subgroup of finite index. Then the action of Γ on $\mathbb{T}^n$ by automorphisms is infinitesimally rigid for $n \geq 7$.*

(The term "infinitesimal rigidity" is meant to suggest an analogy with the local rigidity theorem of Weil described above. However, in the present context, the connection between infinitesimal and local rigidity has not been established.)

Recently, Zimmer [Zi2] obtained the following result:

Theorem. *Suppose G and H are semisimple Lie groups, with G connected with finite center, no compact factors, and no simple factor locally isomorphic to $\mathbf{SO}(1, n)$ or $\mathbf{SU}(1, n)$, $\Gamma \subset G$ and $\Lambda \subset H$ are uniform lattices, and $\phi: G \to H$ a homomorphism so that the action of Γ on the compact manifold $M = H/\Lambda$ is ergodic. Then if* (i) *$\phi(\Gamma)$ is dense in H or* (ii) *$H = H_1 \times H_2$ and Γ projects densely into H_1 and trivially into H_2, the Γ action on M is infinitesimally rigid.*

The proof has two main ingredients. The first is Hodge theory on a suitable foliated manifold and yields L^2 solutions for coboundary equations with cocycles taking values in Vec(M). (This part of the argument works in general, without assuming (i) or (ii).) The second ingredient is essentially a regularity result, which makes the passage from L^2 solutions to smooth ones.

The argument we give here has the same overall structure: We begin by constructing solutions in L^2 for H^1-coboundary equations with cofficients in $C^\infty(M; E)$ (where M is now the torus $\mathbb{T}^n$) for a fairly large class of finite-dimensional vector bundles E, then apply the L^2 result to obtain smooth solutions for cocycles in $\mathrm{Vec}(M)$. The construction of the L^2 solutions is a direct extension of the ideas in [Zi2], although the situation is considerably more delicate owing to the noncompactness of $\Gamma\backslash G$. The passage from L^2 to smooth solutions is essentially new; it was suggested in part by a paper of Veech [Ve].

Most of this work was carried out while I was a graduate student at the University of Chicago. I would like to thank the faculty there, and especially my thesis advisor, Robert Zimmer, for their generous help and encouragement.

2. Preliminaries

Let G be a connected Lie group, $K \subset G$ a maximal compact subgroup, $\Gamma \subset G$ a torsion-free lattice. Let $X = G/K$, $Y = \Gamma\backslash X = \Gamma\backslash G/K$, which is a manifold with $\pi_1(Y) \simeq \Gamma$.

Suppose M is a manifold with volume density μ; let μ denote the corresponding smooth measure as well. If $E \to M$ is a finite-dimensional vector bundle, let $\mathrm{C}^k(M; E)$ denote the space of C^k sections of E. If E is endowed with a Riemannian fiber metric, $\mathrm{L}^2(M; E)$ will denote the Hilbert space of square-integrable sections with respect to μ. Since any two Riemannian metrics over a compact base are equivalent, we define $\mathrm{L}^2(M; E)$ when M is compact, and $\mathrm{L}^2_{\mathrm{loc}}(M; E)$ for arbitrary M, even in the absence of a preferred fiber metric. Here $\mathrm{L}^2_{\mathrm{loc}}(M; E)$ denotes the space of locally-L^2 sections, i.e., sections each of whose restrictions to a compact neighborhood in M is square-integrable. $\mathrm{C}^k(M)$ will denote the space of C^k $\mathbb{R}$-valued functions on M, and $\mathrm{Vec}^k(M) = \mathrm{C}^k(M; \mathrm{T}M)$ the space of C^k vector fields. We denote the space of $\mathbb{R}$-valued p-forms on M by $\mathrm{A}^p(M)$, and by $\mathrm{A}^p(M; E)$ the space of E-valued p-forms. I.e., $\mathrm{A}^p(M; E) = \mathrm{C}^\infty(M; \mathrm{Alt}^p(\mathrm{T}M, E))$, where $\mathrm{Alt}^p(E, E')$ denotes the bundle of alternating p-linear maps from the fibers of E to the fibers of E'.

If E has the structure of a flat vector bundle, i.e., admits a reduction to a countable subgroup of the relevant general linear group, then there is a natural exterior derivative $d\colon \mathrm{A}^p(M; E) \to \mathrm{A}^{p+1}(M; E)$ with $d^2 = 0$. Denote the corresponding cohomology group by $\mathrm{H}^\bullet_{\mathrm{DR}}(M; E)$.

If T is any left Γ-space, we can form the associated bundle to the principal Γ-bundle $G/K = X \to Y = \Gamma\backslash G/K$. Denote this bundle by B_T and the projection $B_T \to Y$ by π_T. Thus $B_T = \Gamma\backslash(X \times T)$ where Γ acts on $X \times T$ by $\gamma(x, t) = (\gamma x, \gamma t)$. The fiber of this bundle is T, and the images of sets of the form $X \times \{t\}$ in B_T yield a foliation $\mathscr{F}_T$ of B_T which is transverse to the fibers. If $E \to M$ is a vector bundle with fiber V on which Γ acts by vector-bundle automorphisms, then $B_E \to B_M$ is again a vector bundle with fiber V.

We continue under the assumption that T is a left Γ-space. Let $I_T = \Gamma\backslash(G \times T)$ so that G acts on the right of I_T by $\Gamma(g, t)h = \Gamma(gh, t)$. The stabilizers in G of points of I_T are conjugates in G of stabilizers in Γ of points in T, so in particular the action of G on I_T is locally free (has discrete stabilizers). I_T is a bundle over $\Gamma\backslash G$ with fiber T, and the natural map $\tilde{\pi}_T\colon I_T \to \Gamma\backslash G$ is a G-map. If Γ preserves a finite measure μ on T, G will preserve the finite measure $\nu = \int_{z\in\Gamma\backslash G}^{\oplus} \mu_z dm$ on I_T, where μ_z is the measure on the fiber over z defined by μ and m is the G-invariant volume on $\Gamma\backslash G$.

If $E \to M$ is a vector bundle with fiber V, so is $I_E \to I_M$, and if Γ preserves a flat structure or Riemannian metric on E, G will preserve a structure of the same type on I_E. We have a commuting diagram:

$$\begin{array}{ccc} I_E & \longrightarrow & I_E/K \simeq B_E \\ \downarrow & & \downarrow \\ I_M & \longrightarrow & I_M/K \simeq B_M \end{array}$$

The orbits of G in I_M project onto the leaves of $\mathscr{F}_M$.

We will be concerned with the special case $E = M \times V$, where V is a vector space, and the Γ-action is given by $\gamma(m, v) = (\gamma m, \rho(\gamma)v)$, where $\rho\colon \Gamma \to \mathbf{GL}(V)$ is a (finite-dimensional) linear representation. Then $E \to M$ is a flat vector bundle and Γ acts by automorphisms preserving the flat structure. In this case $\mathrm{C}^\infty(M; E) \simeq \mathrm{C}^\infty(M) \otimes V$ as Γ-modules. We denote the corresponding bundle B_E by B_ρ. In this case we also have associated to ρ a flat vector bundle $E_\rho \to Y = \Gamma\backslash G/K$, and $B_\rho = \pi_M^*(E_\rho)$, the pull-back under $\pi_M\colon B_M \to Y$. Similarly, we have an associated flat vector bundle $\tilde{E}_\rho$ on $\Gamma\backslash G$, and denoting $I_{M\times V}$ by I_ρ, we have $I_\rho = \tilde{\pi}_M^*(\tilde{E}_\rho)$, where $\tilde{\pi}_M\colon I_M \to \Gamma\backslash G$.

If the representation ρ is actually the restriction to Γ of a representation $\rho\colon G \to \mathbf{GL}(V)$, then $E_\rho \to Y$ can be identified with the bundle associated to the principal K-bundle $\Gamma\backslash G \to Y$ and the representation $\rho|K\colon K \to \mathbf{GL}(V)$. In the present context, we have the following

Lemma [Zi2]. *If $\rho\colon G \to \mathbf{GL}(V)$ is a (finite-dimensional linear) representation, then $B_\rho \to B_M$ is isomorphic to the vector bundle over B_M associated to the principal K-bundle $I_M \to B_M$ and the representation $\rho \mid K$.*

If $\Xi \to B_M$ is a vector bundle, let $\mathrm{A}^p(\mathscr{F}_M; \Xi)$ denote the space of Ξ-valued p-forms along the leaves of $\mathscr{F}_M$, i.e., $\mathrm{A}^p(\mathscr{F}_M; \Xi) = \mathrm{C}^\infty(B_M; \mathrm{Alt}^p(\mathrm{T}\mathscr{F}_M, \Xi))$, where $\mathrm{T}\mathscr{F}_M$ is the tangent bundle to $\mathscr{F}_M$. If $\Xi = B_E$, where E is a vector bundle over M on which Γ acts by vector bundle automorphisms, then we have a natural exterior derivative along the leaves of $\mathscr{F}_M$, $d_{\mathscr{F}}\colon \mathrm{A}^p(\mathscr{F}_M; B_E) \to \mathrm{A}^{p+1}(\mathscr{F}_M; B_E)$, with $d_{\mathscr{F}}^2 = 0$. Denote the corresponding cohomology group by $\mathrm{H}^\bullet_{\mathrm{DR}}(\mathscr{F}_M; B_E)$.

Lemma 2.1 [Zi2]. *Let $E \to M$ be a vector bundle on which Γ acts by vector bundle automorphisms. Then there is a natural isomorphism*

$$\mathrm{H}^\bullet(\Gamma, \mathrm{C}^\infty(M; E)) \simeq \mathrm{H}^\bullet_{\mathrm{DR}}(\mathscr{F}_M; B_E).$$

We again specialize to the case $E = M \times V$ with action given $\rho: \Gamma \to \mathbf{GL}(V)$. Let $q: X \times M \to B_M = \Gamma \backslash (X \times M)$ denote the natural map, $p_X: X \times M \to X$ the projection. Γ acts via vector bundle automorphisms on $\mathrm{Alt}^p(\mathrm{T}X, X \times V)$; for $\phi_x \in \mathrm{Alt}^p_x(\mathrm{T}X, X \times V)$, $\gamma\phi_x = \rho(\gamma) \circ \phi_x \circ (\gamma^{-1})_* \in \mathrm{Alt}^p_{\gamma x}(\mathrm{T}X, X \times V)$. The corresponding action on $\mathrm{C}^\infty(X; \mathrm{Alt}^p(\mathrm{T}X, X \times V)) = \mathrm{A}^p(X; X \times V) \simeq \mathrm{A}^p(X) \otimes V$ is given by $\gamma\omega = \rho(\gamma) \circ (\gamma^{-1})^*\omega$. Since p_X is Γ-equivariant, we obtain an action, again by automorphisms, on

$$p_X^*\mathrm{Alt}^p(\mathrm{T}X, X \times V) \simeq \mathrm{Alt}^p(p_X^*\mathrm{T}X, X \times M \times V),$$

the bundle of V-valued p-forms on $X \times M$ "along X." We have an isomorphism of the quotient bundle $\Gamma \backslash p_X^*\mathrm{Alt}^p(\mathrm{T}X, X \times V) \simeq \mathrm{Alt}^p(\mathrm{T}\mathscr{F}_M, B_\rho)$ and an isomorphism of the space of Γ-invariants

$$\mathrm{C}^\infty(X \times M; p_X^*\mathrm{Alt}^p(\mathrm{T}X, X \times V))^\Gamma \simeq \mathrm{A}^p(\mathscr{F}_M; B_\rho).$$

We can consider $\omega \in \mathrm{C}^\infty(X \times M; p_X^*\mathrm{Alt}^p(\mathrm{T}X, X \times V))$ as a family of V-valued p-forms on X parameterized by M, $m \mapsto \omega_m \in \mathrm{A}^p(X; X \times V)$, $m \in M$. Thus we can apply both $\gamma \in \Gamma$ and $d_{\mathscr{F}}$ to sections of $p_X^*\mathrm{Alt}^p(\mathrm{T}X, X \times V)$ for which each ω_m, $m \in M$, is smooth, but for which ω_m need not vary smoothly in M.

Lemma 2.2 [Zi2]. *Let $\rho: \Gamma \to \mathbf{GL}(V)$ be a linear representation. Assume M is compact. Suppose that for each $\omega \in \mathrm{A}^1(\mathscr{F}_M; B_\rho)$ there is a measurable section θ of $B_\rho \to B_M$ with corresponding family of functions $\theta_m: X \to V$, $m \in M$, satisfying*:

(1) $\theta \in \mathrm{L}^2_{\mathrm{loc}}(B_M; B_\rho)$.
(2) *θ is smooth along the leaves of $\mathscr{F}_M$, or, equivalently, each θ_m, $m \in M$, is smooth.*
(3) *For a.e. $m \in M$, $d\theta_m = \omega_m$.*

Then the map $\mathrm{H}^1(\Gamma, \mathrm{C}^\infty(M, V)) \to \mathrm{H}^1(\Gamma, \mathrm{L}^2(M, V))$ is zero.

3. Formal Hodge decomposition

Henceforth, we adopt the following hypotheses. M is a compact manifold upon which Γ acts, preserving a volume form μ. $\rho: G \to \mathbf{GL}(V)$ is a finite-dimensional linear representation. We adopt notation E_ρ, B_M, B_ρ, etc., as defined above.

Recall the classical construction of a Γ-invariant metric on $\mathrm{Alt}^\bullet(\mathrm{T}X, X \times V)$: Fix a positive-definite, K-invariant, symmetric inner product on V. This determines a Γ-invariant fiber metric on $X \times V \to X$. Namely, for $x = \bar{g} \in X$; $g \in G$; $v_1, v_2 \in V$, define $\langle (x, v_1), (x, v_2) \rangle = \langle \rho(g)^{-1}v_1, \rho(g)^{-1}v_2 \rangle$. X is endowed with a natural G-invariant Riemannian metric, which determines, via the standard construction, a G-invariant Riemannian fiber metric on $\mathrm{Alt}^\bullet(\mathrm{T}X, X \times \mathbb{R})$. There is a natural Γ-equivariant isomorphism of vector bundles $\mathrm{Alt}^\bullet(\mathrm{T}X, X \times V) \simeq \mathrm{Alt}^\bullet(\mathrm{T}X, X \times \mathbb{R})\hat{\otimes}(X \times V)$ ("$\hat{\otimes}$" denotes tensor product of vector bundles), so we obtain a metric on $\mathrm{Alt}^\bullet(\mathrm{T}X, X \times V)$.

Since p_X is Γ-equivariant, the Γ-invariant metric on $\mathrm{Alt}^\bullet(\mathrm{T}X, X \times V)$ determines a Γ-invariant metric on the pull-back bundle $p_X^*\mathrm{Alt}^\bullet(\mathrm{T}X, X \times V)$, i.e., a metric on the quotient bundle $\Gamma\backslash p_X^*\mathrm{Alt}^\bullet(\mathrm{T}X, X \times V) \simeq \mathrm{Alt}^\bullet(\mathrm{T}\mathscr{F}_M, B_\rho)$.

As above, a smooth section $\omega \in \mathrm{C}^\infty(X \times M; p_X^*\mathrm{Alt}^\bullet(\mathrm{T}X, X \times V))$ can be viewed as a $\mathrm{C}^\infty(X; p_X^*\mathrm{Alt}^\bullet(\mathrm{T}X, X \times V))$-valued function on M, so that any differential operator on $\mathrm{C}^\infty(X; p_X^*\mathrm{Alt}^\bullet(\mathrm{T}X, X\times V))$ determines an operator on $\mathrm{C}^\infty(X\times M; p_X^*\mathrm{Alt}^\bullet(\mathrm{T}X, X\times V))$. Since p_X is Γ-equivariant, this construction preserves Γ-invariance. Thus Γ-invariant operators on

$$\mathrm{C}^\infty(X; p_X^*\mathrm{Alt}^\bullet(\mathrm{T}X, X \times V))$$

determine operators on

$$\mathrm{C}^\infty(X\times M; p_X^*\mathrm{Alt}^\bullet(\mathrm{T}X, X\times V))^\Gamma \simeq \mathrm{C}^\infty(B_M; \mathrm{Alt}^\bullet(\mathrm{T}\mathscr{F}_M, B_\rho)) = \mathrm{A}^\bullet(\mathscr{F}_M; B_\rho).$$

In particular, this applies to the G-invariant operators on

$$\mathrm{C}^\infty(X; p_X^*\mathrm{Alt}^\bullet(\mathrm{T}X, X \times V)) \simeq \mathrm{A}^\bullet(X) \otimes V$$

obtained as tensor products of G-invariant operators on $\mathrm{A}^\bullet(X)$ with the identity operator on V. For example, the operator $d_\mathscr{F}$ corresponds to the exterior derivative d on $\mathrm{A}^\bullet(X)$. In general, for D an invariant operator on $\mathrm{A}^\bullet(X)$, let $D_\mathscr{F}$ denote the corresponding operator on $\mathrm{A}^\bullet(\mathscr{F}_M; B_\rho)$.

Recall that M is endowed with a Γ-invariant volume form. Together with the G-invariant Riemannian volume form on X, this yields a Γ-invariant volume form on $X \times M$, hence a volume form ν on B_M. We set $\mathrm{A}_2^\bullet(\mathscr{F}_M; B_\rho) = \mathrm{L}^2(B_M; \mathrm{Alt}^\bullet(\mathrm{T}\mathscr{F}_M, B_\rho))$, a Hilbert space with inner product $\langle\alpha, \beta\rangle = \int_{B_M}\langle\alpha, \beta\rangle d\nu$, where the integrand is defined via the metric on $\mathrm{Alt}^\bullet(\mathrm{T}\mathscr{F}_M, B_\rho)$. Let $\mathrm{A}_\mathrm{C}^\bullet(\mathscr{F}_M; B_\rho) \subset \mathrm{A}^\bullet(\mathscr{F}_M; B_\rho)$ denote the subspace of forms with compact support. As usual, $\mathrm{A}_2^\bullet(\mathscr{F}_M; B_\rho)$ is identified with the completion of $\mathrm{A}_\mathrm{C}^\bullet(\mathscr{F}_M; B_\rho)$ with respect to the L^2-norm.

If D is a differential operator on $\mathrm{A}^\bullet(X)$, let D^* denote the formal adjoint to D with respect to the invariant metric on $\mathrm{Alt}^\bullet(\mathrm{T}X, X\times\mathbb{R})$ and the measure corresponding to the invariant Riemannian volume form on X. Likewise, if D is a differential operator on $\mathrm{A}^\bullet(\mathscr{F}_M; B_\rho)$, let D^* denote its formal adjoint with respect to the metric on $\mathrm{Alt}^\bullet(\mathrm{T}\mathscr{F}_M, B_\rho)$ and the volume ν defined above. Then if D is an invariant operator on $\mathrm{A}^\bullet(X)$, $(D_\mathscr{F})^* = (D^*)_\mathscr{F}$.

Let Δ denote the Laplace operator on $\mathrm{A}^\bullet(X)$, $\Delta = dd^* + d^*d$, where d is the exterior derivative. We define a metric on the space $\mathrm{A}_\mathrm{C}^\bullet(\mathscr{F}_M; B_\rho)$ of compactly-supported forms by the formula $\|\omega\|_{2,r} = \langle(I+\Delta_\mathscr{F})^r\omega, \omega\rangle^{1/2}$, where $\langle\ ,\ \rangle$ is the L^2 inner product. We set $\mathrm{A}_{2,r}^\bullet(\mathscr{F}_M; B_\rho)$ equal to the completion of $\mathrm{A}_\mathrm{C}^\bullet(\mathscr{F}_M; B_\rho)$ with respect to $\|\cdot\|_{2,r}$. As noted above, $\mathrm{A}_{2,0}^\bullet$ is identified with $\mathrm{A}_2^\bullet$. Also, the usual argument shows that the natural map $\mathrm{A}_{2,r}^\bullet \to \mathrm{A}_{2,s}^\bullet$ is an injection $(r > s)$, so that we identify $A^\bullet_{2,r}$ with a dense subspace of $A^\bullet_{2,s}$. In particular, elements of all the $A^\bullet_{2,r}$ are represented by measurable sections of the bundle $\mathrm{Alt}^\bullet(T\mathscr{F}_M, B_\rho)$.

In addition to the "Sobolev-type" spaces $A^\bullet_{2,r}$ for nonnegative integers r, we define the "negative norm" spaces by setting $A^\bullet_{2,-r}(\mathscr{F}_M; B_\rho)$ equal to the normed dual to $A^\bullet_{2,r}(\mathscr{F}_M; B_\rho)$. Extending by continuity from $A^\bullet_C$, the differential operator $\Delta_{\mathscr{F}}$ on $A^\bullet$ uniquely determines a continuous linear map

$$\Delta_{\mathscr{F}} : A^\bullet_{2,r}(\mathscr{F}_M; B_\rho) \to A^\bullet_{2,r-2}(\mathscr{F}_M; B_\rho)$$

for $r \geq 2$. When (as usual) we identify $A^\bullet_2$ with its dual via the inner product, this definition extends to the cases $r = 0, 1$. Finally, since $\Delta_{\mathscr{F}}$ is (formally) selfadjoint, we denote the adjoint maps $\Delta_{\mathscr{F}} : A^\bullet_{2,-r} \to A^\bullet_{2,-r-2}$, $r \geq 0$, by the same notation. Note that for $r = 0$, the two definitions coincide. In particular, we have the following:

Proposition 3.1. *For each $p \geq 0$, let*

$$\Delta^{(1)}_{\mathscr{F}} = \Delta_{\mathscr{F}} : A^p_{2,2}(\mathscr{F}_M; B_\rho) \to A^p_2(\mathscr{F}_M; B_\rho),$$
$$\Delta^{(2)}_{\mathscr{F}} = \Delta_{\mathscr{F}} : A^p_2(\mathscr{F}_M; B_\rho) \to A^p_{2,-2}(\mathscr{F}_M; B_\rho).$$

Then both maps are continuous and

$$A^p_2(\mathscr{F}_M; B_\rho) = \overline{\mathrm{im}(\Delta^{(1)}_{\mathscr{F}})} \oplus \ker(\Delta^{(2)}_{\mathscr{F}})$$

is an orthogonal direct sum.

Lemma 3.2. *Suppose $\omega \in L^2_{\mathrm{loc}}(B_M; \mathrm{Alt}^\bullet(T\mathscr{F}_M, B_\rho))$ and $\Delta_{\mathscr{F}}\omega$ is smooth along the leaves. Then ω is smooth along the leaves.*

(Here $\Delta_{\mathscr{F}}\omega$ is defined distributionally, as a linear functional on $A^\bullet_C$. The hypothesis is that $\Delta_{\mathscr{F}}\omega$ is represented by a measurable section which is smooth along the leaves.)

Proof. Let $\pi_M : B_M = \Gamma\backslash(X \times M) \to Y = \Gamma\backslash X$ denote the map induced by $p_X : X \times M \to X$. Cover Y with a countable collection U_i of open sets such that $\pi_M^{-1}(U_i) \simeq U_i \times M$. Then on $\pi_M^{-1}(U_i)$, ω corresponds to a family of E_ρ-valued forms on U_i parameterized by $m \in M$, say ω_i^m. Since $\omega \in L^2_{\mathrm{loc}}$ and the measure ν on B_M restricts to the product measure on $\pi_M^{-1}(U_i) \simeq U_i \times M$, $\omega_i^m \in L^2_{\mathrm{loc}}(U_i; \mathrm{Alt}^\bullet(T\mathscr{F}_M, B_\rho)|U_i)$ for a.e. $m \in M$ by Fubini's theorem. Similarly, $\Delta(\omega_i^m) = (\Delta_{\mathscr{F}}\omega_i)^m$ for a.e. $m \in M$, and $(\Delta_{\mathscr{F}}\omega_i)^m$ is smooth by hypothesis. Then since Δ is elliptic, ω_i^m has a smooth representative for a.e. $m \in M$ by elliptic regularity. Let $M_0 \subset M$ be the null set for which ω_i^m is not smooth; then ΓM_0 is also null. Taking the union over the countable collection of U_i, conclude that ω is smooth on a conull set of leaves. Set $\omega = 0$ on the complementary null set.

The following two propositions correspond to well-known results about complete Riemannian manifolds (cf. [Vs]). The proofs carry over to the present context essentially verbatim, using completeness of the metric on Y.

Proposition 3.3. *$\mathrm{A}^\bullet_{2,1}(\mathscr{F}_M; B_\rho)$ can be identified with $W^\bullet = \{\omega \in \mathrm{A}^\bullet_2(\mathscr{F}_M; B_\rho) \mid d\omega, d^*\omega \in \mathrm{A}^\bullet_2(\mathscr{F}_M; B_\rho)\}$.*

(Here $d\omega$ and $d^*\omega$ are again defined distributionally; their representatives in $\mathrm{A}^\bullet_2$ are unique. Note that the containment $\mathrm{A}^\bullet_{2,1} \subset W^\bullet$ is obvious; completeness is needed to show that the compactly-supported forms are dense in $W^\bullet$.)

Proposition 3.4 (Stampacchia inequality). *For every measurable section*

$$\omega\colon B_M \to \mathrm{Alt}^\bullet(\mathrm{T}\mathscr{F}_M, B_\rho)$$

which is smooth along the leaves and every $\sigma > 0$,

$$\|d_{\mathscr{F}}\omega\|^2 + \|d^*_{\mathscr{F}}\omega\|^2 \le \sigma\|\Delta_{\mathscr{F}}\omega\|^2 + \frac{1}{\sigma}\|\omega\|^2.$$

Proposition 3.5. *Suppose that $\exists$ a constant $C > 0$ such that for every compactly-supported form $\omega \in \mathrm{A}^p_{\mathrm{C}}(\mathscr{F}_M; B_\rho)$ we have $\langle\Delta_{\mathscr{F}}\omega, \omega\rangle \ge C\|\omega\|^2$. Then, with the notation of Proposition 3.1, above, we have $\ker(\Delta^{(2)}_{\mathscr{F}}) = 0$ and $\mathrm{im}(\Delta^{(1)}_{\mathscr{F}})$ is closed.*

Proof. (1) $\ker(\Delta^{(2)}_{\mathscr{F}}) = 0$: Suppose $\omega \in \mathrm{A}^p_2(\mathscr{F}_M; B_\rho)$ and $\Delta_{\mathscr{F}}\omega = 0 \in \mathrm{A}^p_{2,-2}$. Then ω is smooth along the leaves by Lemma 3.2. Hence $d_{\mathscr{F}}\omega = d^*_{\mathscr{F}}\omega = 0$ by Proposition 3.4. By Proposition 3.3, $\exists\omega_i \in \mathrm{A}^p_{\mathrm{C}}$ such that $\omega_i \to \omega$ in $\mathrm{A}^p_{2,1}$; in particular, $\|d_{\mathscr{F}}\omega_i\|^2 \to 0$, $\|d^*_{\mathscr{F}}\omega_i\|^2 \to 0$, and $\|\omega_i\|^2 \to \|\omega\|^2$. But by hypothesis, $\|d_{\mathscr{F}}\omega_i\|^2 + \|d^*_{\mathscr{F}}\omega_i\|^2 = \langle\Delta_{\mathscr{F}}\omega_i, \omega_i\rangle \ge C\|\omega_i\|^2$. Thus $\omega = 0$.

(2) $\mathrm{im}(\Delta^{(1)}_{\mathscr{F}})$ is closed: (The proof of this statement does not use completeness of the metric on Y.) We will establish the inequality

$$\langle(\mathrm{I} + \Delta_{\mathscr{F}})^2\omega, \omega\rangle \le \left(1 + \frac{2}{C} + \frac{1}{C^2}\right)\langle\Delta_{\mathscr{F}}\omega, \Delta_{\mathscr{F}}\omega\rangle$$

for $\omega \in \mathrm{A}^p_{\mathrm{C}}$. The claim follows since $\mathrm{A}^p_{\mathrm{C}}$ is dense in $\mathrm{A}^p_{2,2}$. By the Schwarz inequality,

$$\langle\Delta_{\mathscr{F}}\omega, \omega\rangle \le \langle\omega, \omega\rangle^{1/2}\langle\Delta_{\mathscr{F}}\omega, \Delta_{\mathscr{F}}\omega\rangle^{1/2}.$$

Combined with the hypothesis, this yields

$$\langle\omega, \omega\rangle \le \frac{1}{C^2}\langle\Delta_{\mathscr{F}}\omega, \Delta_{\mathscr{F}}\omega\rangle.$$

Substituting in the Schwarz inequality, obtain

$$\langle\Delta_{\mathscr{F}}\omega, \omega\rangle \le \frac{1}{C}\langle\Delta_{\mathscr{F}}\omega, \Delta_{\mathscr{F}}\omega\rangle.$$

Thus

$$\begin{aligned}\langle (\mathrm{I}+\Delta_{\mathscr{F}})^2\omega, \omega\rangle &= \langle\omega, \omega\rangle + 2\langle\Delta_{\mathscr{F}}\omega, \omega\rangle + \langle\Delta_{\mathscr{F}}^2\omega, \omega\rangle \\ &= \langle\omega, \omega\rangle + 2\langle\Delta_{\mathscr{F}}\omega, \omega\rangle + \langle\Delta_{\mathscr{F}}\omega, \Delta_{\mathscr{F}}\omega\rangle \\ &\le \left(\frac{1}{C^2} + \frac{2}{C} + 1\right)\langle\Delta_{\mathscr{F}}\omega, \Delta_{\mathscr{F}}\omega\rangle.\end{aligned}$$

Proposition 3.6. *Suppose that for $p = 1$ the following two conditions are satisfied:*

(1) *Every cohomology class in $\mathrm{A}^p(\mathscr{F}_M; B_\rho)$ has a representative which is square-integrable.*

(2) *$\exists$ a constant $C > 0$ such that $\langle\Delta_{\mathscr{F}}\omega, \omega\rangle \ge C\langle\omega, \omega\rangle$ for every compactly-supported form $\omega \in \mathrm{A}^p_{\mathrm{C}}(\mathscr{F}_M; B_\rho)$.*

Then the map $\mathrm{H}^1(\Gamma, \mathrm{C}^\infty(M, V)) \to \mathrm{H}^1(\Gamma, \mathrm{L}^2(M, V))$ is zero.

Proof. Given an element of $\mathrm{H}^1(\Gamma, \mathrm{C}^\infty(M, V))$, fix a closed, square-integrable form $\omega \in \mathrm{A}^1(\mathscr{F}_M; B_\rho)$ representing it. By Propositions 3.1 and 3.5, condition (2) implies that $\exists\eta \in \mathrm{A}^1_{2,2}(\mathscr{F}_M; B_\rho)$ such that $\Delta_{\mathscr{F}}\eta = \omega$. By Lemma 3.2, η is smooth along the leaves. Now $\omega = \Delta_{\mathscr{F}}\eta = d_{\mathscr{F}}d^*_{\mathscr{F}}\eta + d^*_{\mathscr{F}}d_{\mathscr{F}}\eta$ and $d_{\mathscr{F}}\omega = d_{\mathscr{F}}d^*_{\mathscr{F}}d_{\mathscr{F}}\eta = 0$. Thus

$$\langle d_{\mathscr{F}}d^*_{\mathscr{F}}d_{\mathscr{F}}\eta, d_{\mathscr{F}}\eta\rangle = \langle d^*_{\mathscr{F}}d_{\mathscr{F}}\eta, d^*_{\mathscr{F}}d_{\mathscr{F}}\eta\rangle = 0 \Rightarrow \Delta_{\mathscr{F}}\eta = d_{\mathscr{F}}d^*_{\mathscr{F}}\eta.$$

Thus $\omega = d_{\mathscr{F}}\theta$, where $\theta = d^*_{\mathscr{F}}\eta \in \mathrm{A}^0_{2,1}(\mathscr{F}_M; B_\rho) \subset \mathrm{L}^2(\mathscr{F}_M; B_\rho)$ and θ is smooth along the leaves of $\mathscr{F}_M$. The proposition follows by Lemma 2.2.

4. Application of the Matsushima-Murakami computations

Our objective in this section is to verify condition (2) of Proposition 3.6. The compactness (or lack thereof) of $\Gamma\backslash G$ is irrelevant to this portion of the argument, and the discussion in [Zi2] applies essentially without modification. Zimmer's development is in turn patterned directly after the classical computations of Matsushima-Murakami, detailed accounts of which are given in [Mu] and [R1]. We shall accordingly limit ourselves to a brief summary of the results.

We retain the hypotheses and notation built up in previous sections; in particular, we assume that the representation ρ of Γ is obtained by restriction from a representation ρ of the ambient group G, and employ the notation I_ρ, I_M defined at the outset.

The vector bundle $B_\rho \to B_M$ is naturally isomorphic to the vector bundle over B_M associated to the principal K-bundle $I_M \to B_M$ and the representation $\rho|K$. The action of G on I_M is locally free; let $\mathscr{O}$ denote the orbit foliation of this action, so that $\mathrm{T}\mathscr{O} = \tilde{\pi}^*_M(\mathrm{T}(\Gamma\backslash G))$. In order to simplify the notation, we will write $\mathrm{Alt}^\bullet(\mathrm{T}\mathscr{O}, V)$ in place of $\tilde{\pi}^*_M\mathrm{Alt}^\bullet(\mathrm{T}(\Gamma\backslash G), (\Gamma\backslash G)\times V) = \mathrm{Alt}^\bullet(\mathrm{T}\mathscr{O}, I_M \times V)$. The diffeomorphism

$$\Phi\colon G\times M\times V \to G\times M\times V; \quad (g, m, v) \mapsto (g, m, \rho(g)v)$$

induces an injection

$$\Psi\colon \mathrm{C}^\infty(X\times M;\, p_X^*\mathrm{Alt}^\bullet(\mathrm{T}X,\, X\times V))^\Gamma \hookrightarrow \mathrm{C}^\infty(I_M;\, \mathrm{Alt}^\bullet(\mathrm{T}\mathscr{O},\, V))^K.$$

If $A\in\mathfrak{g}$, A naturally determines a vector field on I_M, which we also denote by A, taking values in $\mathrm{T}\mathscr{O}\subset \mathrm{T}I_M$ at each point of I_M. The image of Ψ is exactly the set $\{\eta^0 | i(A)\eta^0 = 0 \text{ for every } A\in K\}$; we will denote this set by $\mathrm{C}^\infty(I_M;\, \mathrm{Alt}^\bullet(\mathrm{T}\mathscr{O},\, V))^K_0$. (Here $i(A)$ denotes the interior multiplication by A, which maps p-forms to $(p-1)$-forms.)

For each $y\in I_M$, there is a natural identification of $\mathrm{T}\mathscr{O}_y$ with $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, where $\mathfrak{p}$ is the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. Thus $\mathrm{C}^\infty(I_M;\, \mathrm{Alt}^\bullet(\mathrm{T}\mathscr{O},\, V))$ is identified with $\mathrm{C}^\infty(I_M;\, \mathrm{Alt}^\bullet(\mathfrak{g},\, V))$, and the G-action on the latter space is given by the G-action on I_M, the representation ρ on V, and Ad_G on $\mathfrak{g}$. By the condition $i(A)(\eta^0)=0$ for every $A\in\mathfrak{k}$, $\mathrm{C}^\infty(I_M;\, \mathrm{Alt}^\bullet(\mathrm{T}\mathscr{O},\, V))^K_0$ is identified with $\mathrm{C}^\infty(I_M;\, \mathrm{Alt}^\bullet(\mathfrak{p},\, V))^K$. Fix an orthonormal basis $Y_1,\ldots,Y_N$ for $\mathfrak{p}$ with respect to the Killing form, and set $\eta^0_{i_1,\ldots,i_p}=\eta^0(Y_{i_1},\ldots,Y_{i_p})\in \mathrm{C}^\infty(I_M;\, V)$ for $\eta^0\in \mathrm{C}^\infty(I_M;\, \mathrm{Alt}^\bullet(\mathfrak{p},\, V))^K$.

Recall that the construction of the metric on $\mathrm{Alt}^\bullet(\mathrm{T}\mathscr{F}_M,\, B_\rho)$ began by fixing a positive-definite, K-invariant, symmetric inner product on V. Henceforth we impose the additional hypothesis that $\rho(Y)$ is symmetric for every $Y\in\mathfrak{p}$; such a metric exists by [Mo1].

Lemma. *For $\eta\in \mathrm{C}^\infty(X\times M;\, p_X^*\mathrm{Alt}^\bullet(\mathrm{T}X,\, X\times V))^\Gamma$, let*

$$\eta^0=\Psi(\eta)\in \mathrm{C}^\infty(I_M;\, \mathrm{Alt}^\bullet(\mathfrak{p},\, V))^K.$$

(1) *For $\xi,\eta\in \mathrm{C}^\infty(X\times M;\, p_X^*\mathrm{Alt}^\bullet(\mathrm{T}X,\, X\times V))^\Gamma$,*

$$\langle\xi,\eta\rangle=\sum_{i_1<\cdots<i_p}\int_{I_M}\langle\xi^0_{i_1,\ldots,i_p}\eta^0_{i_1,\ldots,i_p}\rangle d\nu.$$

(2) *Under the isomorphism Ψ, $\Delta_{\mathscr{F}}\eta$ corresponds to $(\Delta_D+\Delta_\rho)(\eta^0)$, where Δ_D and Δ_ρ are differential operators on $\mathrm{C}^\infty(I_M;\, \mathrm{Alt}^\bullet(\mathfrak{p},\, V))$ such that $\langle\Delta_D\eta^0,\eta^0\rangle\ge 0$, $\langle\Delta_\rho\eta^0,\eta^0\rangle\ge 0$ whenever η (and hence also η^0) has compact support. (Here $<,>$ denotes the ordinary L^2 inner product on I_M.)*

(3) *There is a linear operator $H_\rho=\sum_p^\oplus H_\rho^p$ on $\mathrm{Alt}^\bullet(\mathfrak{p},\, V)$ such that at each $y\in I_M$, $\Delta_\rho(\eta^0)(y)=H_\rho(\eta^0(y))$.*

Lemma [R2]. *For $p=1$, ρ nontrivial and irreducible, and for $\mathfrak{g}$ semisimple with no simple factors isomorphic to any $\mathfrak{so}(1,n)$ or $\mathfrak{su}(1,n)$, H_ρ^1 is a positive-definite, symmetric operator on $\mathrm{Hom}(\mathfrak{p},\, V)$.*

Corollary. *Under the hypotheses of the preceding lemma, $\exists$ a constant $C>0$ such that for every compactly-supported form $\omega\in \mathrm{A}^1_{\mathrm{C}}(\mathscr{F}_M;\, B_\rho)$, $\langle\Delta_{\mathscr{F}}\omega,\omega\rangle\ge C\langle\omega,\omega\rangle$.*

5. Square-integrability criterion

In this section we obtain conditions under which condition (1) of Proposition 3.6 is verified, i.e., under which each cohomology class in $\mathrm{A}^1(\mathscr{F}_M; B_\rho)$ must have a square-integrable representative. The argument is an extension of one which appears in [B2] and makes use of the manifold with corners construction of [B-S]. We shall accordingly adopt the notation and employ the results of these papers to the maximum extent possible, indicating the modifications which are necessary to obtain analogous results in our present context.

In addition to the hypotheses and notation built up in previous sections, we will require the additional structure of algebraic $\mathbb{Q}$-group on the ambient group G and arithmeticity of the lattice Γ. More specifically, we establish the following hypotheses for the remainder of this section. $G = \mathbf{G}(\mathbb{R})^0$, the connected component of the real points of a connected, semisimple $\mathbb{Q}$-group $\mathbf{G}$. The finite-dimensional representation $\rho: G \to \mathbf{GL}(V)$ is $\mathbb{R}$-rational. The lattice Γ is arithmetic and net, in the sense of [R1, Chapter 6]. We shall also assume that $\mathbf{G}$ has $\mathbb{Q}$-rank ≥ 1, otherwise, we would have Γ co-compact in $\mathbf{G}(\mathbb{R})$, and the entire discussion would be vacuous.

In outline, the argument goes as follows. We consider a subcomplex of $\mathrm{A}^\bullet(\mathscr{F}_M; B_\rho)$ consisting of forms which, together with their leafwise exterior differentials, satisfy growth conditions along the leaves of $\mathscr{F}_M$. We show that the inclusion into $\mathrm{A}^\bullet(\mathscr{F}_M; B_\rho)$ induces an isomorphism in cohomology. The proof is essentially the same as that given in [B2] and makes use of sheaf theory in a compactification $\overline{B}_M$ of B_M corresponding to the compact manifold with corners $\Gamma\backslash\overline{X}$ constructed in [B-S]. These forms are square-integrable up to a constant $C(G, \rho^*)$ determined by the Lie algebra of G and the representation ρ.

We begin by summarizing essential notation from [B1, B2, and B-S]. As above, K is a maximal compact subgroup of $G = \mathbf{G}(\mathbb{R})$. θ will denote the corresponding Cartan involution of $\mathbf{G}$ or of $\mathfrak{g}$. Let $\mathbf{P}$ be a parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$, $\mathbf{U}$ the unipotent radical of $\mathbf{P}$. To avoid conflict with our existing notation we will write "$\mathbf{L}$" in place of Borel's "$\mathbf{M}$"; $\mathbf{L}$ is the (unique) Levi $\mathbb{Q}$-subgroup of $\mathbf{P}$ stable under θ. $\mathbf{S_P} = \mathbf{L} \cap R_d\mathbf{P}$ is the maximal torus of the split radical $R_d\mathbf{P}$ of $\mathbf{P}$ stable under θ and $A = \mathbf{S_P}(\mathbb{R})^0$.

For any connected $\mathbb{Q}$-group $\mathbf{H}$, ${}^0\mathbf{H} = \bigcap_{a \in X(\mathbf{H})_\mathbb{Q}} \ker(a^2)$, where $X(\mathbf{H})_\mathbb{Q}$ denotes the group of $\mathbb{Q}$-morphisms of $\mathbf{H}$ into $\mathbf{GL}_1$ (cf. [B-S]). Then $\mathbf{L}(\mathbb{R}) = {}^0\mathbf{L}(\mathbb{R}) \times A$, and $Z = {}^0\mathbf{L}(\mathbb{R})/(K \cap \mathbf{L})$.

Henceforth, we abandon our previous notation "$Y = \Gamma\backslash X$" and let $Y = \mathbf{U}(\mathbb{R}) \times Z \times A$ in conformity with [B2]. Let $\sigma: G \to X$ denote the canonical projection and $o = \sigma(K)$. Then σ induces a diffeomorphism $\mu_0: Y \xrightarrow{\sim} X$.

Let Δ be a basis of the set of $\mathbb{Q}$-roots of $\mathbf{G}$ with respect to some maximal $\mathbb{Q}$-split torus $\mathbf{S}$. The conjugacy classes over $\mathbf{G}_\mathbb{Q}$ of parabolic $\mathbb{Q}$-subgroups are parameterized by the subsets of Δ. Let $I = I(\mathbf{P}) \subset \Delta$ be the type of $\mathbf{P}$. There is a canonical isomorphism $A \xrightarrow{\sim} (\mathbb{R}_+^*)^{\Delta - I}$, which we extend to an isomorphism of manifolds with corners $\overline{A} \xrightarrow{\sim} (\overline{\mathbb{R}}_+^*)^{\Delta - I}$, where $\overline{\mathbb{R}}_+^* = \mathbb{R}_+^* \cup \{\infty\}$.

Let $\mathbf{\Phi_P}$ be the set of roots of $\mathbf{P}$ with respect to $\mathbf{S_P}$. For $\beta \in \mathbf{\Phi_P}$, let $\mathfrak{u}_\beta = \{X \in \mathfrak{g}(\mathbb{R}) | \operatorname{Ad} a X = a^\beta X \ (a \in A)\}$. Then $\mathfrak{u}(\mathbb{R}) = \bigoplus_\beta \mathfrak{u}_\beta$. Let $\Delta - I = \{\alpha_1, \dots, \alpha_l\}$, $l = \dim A$. Let $n = \dim X$, and fix a frame $(\omega^i)_{1 \le i \le n}$ on Y, where ω^i is lifted, under one of the natural projections, from $d \log \alpha_i$ $(1 \le i \le l)$ on A, from an orthonormal frame on Z, and from a set of left-invariant one-forms on $\mathbf{U}(\mathbb{R})$ which, at the origin, span the various subspaces $\mathfrak{u}_\beta^*$. Borel refers to (ω^i) as a *special frame.*

Let $I_m = \{1, \dots, m\}$. For a subset J of I_m, we let $\omega^J = \bigwedge_{i \in J} \omega^i$. Let $\eta \in \mathrm{A}^p(X; X \times V)$. With respect to a special frame, we write $\mu_0^* \eta = \sum_{|J|=p} \eta_J \omega^J$, where the coefficients η_J are smooth V-valued functions.

$X(A)$ will denote the group of continuous homomorphisms of A into $\mathbb{R}_+^*$. Any $\lambda \in X(A)$ is a linear combination of the α_is and can be written $\lambda = \sum c_i \alpha_i$. λ is dominant (respectively, dominant regular) if every $c_i \ge 0$ ($c_i > 0$), in which case we write $\lambda \ge 0$ ($\lambda > 0$). For $\lambda, \mu \in X(A)$, we write $\lambda \ge \mu$ (respectively, $\lambda > \mu$) if $\lambda - \mu \ge 0$ ($\lambda - \mu > 0$).

Again to avoid conflict with existing notation, we will write "τ" in place of Borel's "ρ." $\tau_\mathbf{P}$ is defined by $a^{2\tau_\mathbf{P}} = \det(\operatorname{Ad} a|_\mathfrak{u})$, i.e., $\tau_\mathbf{P}$ is one-half the sum of the weights of A on $\mathfrak{u}$, with multiplicities.

Fix a minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_0$, and let $A_0 = \mathbf{S}_{\mathbf{P}_0}(\mathbb{R})^0$ denote the corresponding split component. Given $\lambda \in X(A_0)$, we denote by $C(G, \lambda)$ the greatest integer q such that $\tau = \tau_{\mathbf{P}_0} > \lambda + \mu$ for every weight of μ of A_0 in $\bigwedge^q \mathfrak{u}$. For a finite-dimensional representation ρ of G, we let $C(G, \lambda, \rho) = \inf_\mu C(G, \lambda + \mu)$, where μ ranges over the weights of ρ restricted to A_0. For $\lambda = 0$, we denote $C(G, \lambda, \rho)$ more simply by $C(G, \rho)$.

For $t > 0$, let $A_t = \{a \in A \mid a^\alpha \ge t \ (\alpha \in \Delta - I)\}$, $\overline{A}_t = \{a \in \overline{A} \mid a^\alpha \ge t \ (\alpha \in \Delta - I)\}$. A Siegel set $\mathscr{S}_{t,\omega}$ in X with respect to $\mathbf{P}$, o is a set of the form $\mathscr{S}_{t,\omega} = \mu_0(\omega \times A_t)$, where ω is a relatively compact set in $\mathbf{U}(\mathbb{R}) \times Z$.

Let $\mathscr{B}$ denote the set of parabolic $\mathbb{Q}$-subgroups of $\mathbf{G}$ and $\pi: X \to \Gamma \backslash X$ the natural projection. Given $\mathbf{P} \in \mathscr{B}$ and ω a relatively compact open set in $\mathbf{U}(\mathbb{R}) \times Z$, there exists $t(0, \omega) < \infty$ such that if $t \ge t(0, \omega)$ the equivalence relations defined by Γ and $\Gamma \cap \mathbf{P}$ on $\mathscr{S}_{t,\omega}$ are the same and consequently $\mu_0' = \pi \circ \mu_0$ is an isomorphism

$$\mu_0' \colon (\Gamma \cap \mathbf{P}) \backslash \omega \times A_t \xrightarrow{\sim} \pi(\mathscr{S}_{t,\omega}).^1$$

In [B-S], X is enlarged to a manifold with corners $\overline{X}$, which is the disjoint union of faces $e(\mathbf{P})$ over $\mathbf{P} \in \mathscr{B}$, where $e(\mathbf{G}) = X$, and the face $e(\mathbf{P})$ is

[1] This is asserted with $\mathbf{U}(\mathbb{R}) \times Z$ in place of ω in [B-S]. As observed in [Zu], the proof there is valid only for relatively compact open subsets. I am grateful to the referee who pointed out the error in my original manuscript.

identified with $\mathbf{U}(\mathbb{R}) \times Z$. The group Γ acts properly and freely on $\overline{X}$, and $\Gamma\backslash\overline{X}$ is a compact manifold with corners, the disjoint union of faces

$$e'(\mathbf{P}) = (\Gamma \cap \mathbf{P})\backslash(\mathbf{U}(\mathbb{R}) \times Z) \simeq (\Gamma \cap \mathbf{P})\backslash e(\mathbf{P})$$

where $\mathbf{P}$ ranges over a set of representatives for the (finite) collection of congruence classes of parabolic $\mathbb{Q}$-subgroups over Γ. Thus we obtain a corresponding compactification $\overline{B}_M$ for $B_M = \Gamma\backslash(X \times M)$, namely $\overline{B}_M = \Gamma\backslash(\overline{X} \times M)$. $\overline{B}_M$ is likewise a compact manifold with corners, the disjoint union of faces

$$e'_M(\mathbf{P}) = (\Gamma \cap \mathbf{P})\backslash(e(\mathbf{P}) \times M) \simeq (\Gamma \cap \mathbf{P})\backslash(\mathbf{U}(\mathbb{R}) \times Z \times M).$$

We need to recall the definition of special neighborhoods in $\Gamma\backslash\overline{X}$ and extend it to $\overline{B}_M$. For $t > 0$, $\mathbf{P} \in \mathscr{B}$, and $I = I(\mathbf{P})$, let $A_{(t)} = \{a \in A \mid a^\alpha > t \ (\alpha \in \Delta - I)\}$, $\overline{A}_{(t)} = \{a \in \overline{A} \mid a^\alpha > t \ (\alpha \in \Delta - I)\}$. Thus $A_{(t)}$ and $\overline{A}_{(t)}$ are open in the corner $\overline{A}$.

Let $y \in e(\mathbf{P})$ and y' its image in $e'(\mathbf{P})$. Let ω be an open, relatively compact neighborhood of y in $e(\mathbf{P})$ on which $e(\mathbf{P}) \to e'(\mathbf{P})$ is injective. Then $\mu_0(\omega \times A_{(t)})$ is an open Siegel set on which π is injective, and μ'_0 extends to an isomorphism of manifolds with corners of $\omega \times \overline{A}_{(t)}$ onto an open neighborhood U of y' in $\Gamma\backslash\overline{X}$. Borel calls U a *special neighborhood* of y'. If $\mathbf{P} = \mathbf{G}$, so that $y' \in \Gamma\backslash X$, a special neighborhood of y' is just the isomorphic image under π of an open, relatively compact neighborhood of y in X.

Now suppose $m \in M$, and let $b \in e'_M(\mathbf{P}) = (\Gamma \cap \mathbf{P})\backslash(e(\mathbf{P}) \times M)$ denote the image of $(y, m) \in e(\mathbf{P}) \times M$. If U is an open neighborhood of m in M, then with ω, t as above, $\mu_0(\omega \times \overline{A}_{(t)}) \times U$ projects isomorphically onto an open neighborhood W of b in $\overline{B}_M = \Gamma\backslash(\overline{X} \times M)$. We will refer to W as a special neighborhood of b. Clearly, b has a fundamental system of special neighborhoods.

Fix $\lambda \in X(A_0)$. (In practice, the case of interest will be $\lambda = 0$.) Suppose $\eta \in \mathrm{A}^p(\mathscr{S}_{t,\omega}; \mathscr{S}_{t,\omega} \times V)$ is a V-valued form defined on a Siegel set $\mathscr{S}_{t,\omega} = \mu_0(\omega \times A_t)$. As above, we write $\mu_0^*\eta = \sum_{|J|=p} \eta_J \omega^J$ with respect to a special frame. Borel calls η *weakly λ-bounded* if there exists a polynomial Q in $l = \dim A$ variables (with real coefficients) and a constant $c > 0$ such that

$$\|\eta_J(q, a)\| < ca^\lambda |Q(\ln a^{\alpha_1}, \ldots, \ln a^{\alpha_l})| \quad \text{for } a \in A_t,\ q \in \omega, \text{ and } |J| = p,$$

where $\|\cdot\|$ refers to the norm on V corresponding to the previously determined admissible metric. A form η' on $\pi(\mathscr{S}_{t,\omega})$ is weakly λ-bounded if $\eta' \circ \pi$ is weakly λ-bounded on $\mathscr{S}_{t,\omega}$. For η defined on an open set $U \subset X$, η is said to be weakly λ-bounded if, given a parabolic subgroup and associated Siegel set $\mathscr{S}$, there exists a polynomial Q and constant c so that η satisfies the above condition on $\mathscr{S} \cap U$. If U is an open set in $\Gamma\backslash\overline{X}$, a form η defined on $U \cap (\Gamma\backslash X)$ is weakly λ-bounded near the boundary if every $y \in U \cap (\Gamma\backslash\partial\overline{X})$ has a special neighborhood W in U such that η is weakly λ-bounded on $W \cap (\Gamma\backslash X)$.

Now suppose U is an open set of M (necessarily relatively compact, since we are assuming that M is compact), $\mathscr{S}_{t,\omega}$ as above, so that $W = \mathscr{S}_{t,\omega} \times U$ is an open set of $X \times M$. Suppose $\eta \in \mathrm{C}^\infty(W; p_X^* \mathrm{Alt}^p(\mathrm{T}X, X \times V))$ is a V-valued p-form on W along X. As usual, for $m \in U \subset M$, $x \in \mathscr{S}_{t,\omega} \subset X$, set $\eta_m(x) = \eta(x, m)$ so that $\eta_m \in \mathrm{A}^p(\mathscr{S}_{t,\omega}; \mathscr{S}_{t,\omega} \times V)$. We will say that η is weakly λ-bounded if there exists a polynomial Q and $c > 0$ such that

$$\|(\eta_m)_J(q, a)\| < ca^\lambda |Q(\ln a^{\alpha_1}, \dots, \ln a^{\alpha_l})| \qquad \text{for } m \in U,\ a \in A_t,\ q \in \omega, \text{ and } |J| = p,$$

i.e., η_m satisfies the condition for weak λ-boundedness uniformly in $m \in U$. As above, we extend the definition to forms which are defined on arbitrary open sets in $X \times M$ and to forms defined on $B_M = \Gamma\backslash(X \times M)$. Finally, if U is an open set in $\overline{B}_M = \Gamma\backslash(\overline{X} \times M)$, a form η defined on $U \cap B_M$ is weakly λ-bounded near the boundary if every $b \in U \cap (\partial \overline{B}_M) = U \cap (\Gamma\backslash(\partial\overline{X} \times M))$ has a special neighborhood W in U such that η is weakly λ-bounded on $W \cap B_M$.

We define a subcomplex

$$\mathrm{A}^\bullet_\lambda(\mathscr{F}_M; B_\rho) \subset \mathrm{A}^\bullet(\mathscr{F}_M; B_\rho) = \mathrm{C}^\infty(X \times M; p_X^* \mathrm{Alt}^\bullet(\mathrm{T}X, X \times V))^\Gamma$$

consisting of those forms $\eta \in \mathrm{A}^\bullet(\mathscr{F}_M; B_\rho)$ such that both η and $d_{\mathscr{F}}\eta$ are weakly λ-bounded.

Theorem. *Assume $\lambda \in X(A_0)$ is dominant. Then the inclusion*

$$\mathrm{A}^\bullet_\lambda(\mathscr{F}_M; B_\rho) \hookrightarrow \mathrm{A}^\bullet(\mathscr{F}_M; B_\rho)$$

induces an isomorphism in cohomology.

Proof. Let $\mathbb{E}$ (respectively $\overline{\mathbb{E}}$) denote the locally-constant sheaf over $\Gamma\backslash X$ (respectively $\Gamma\backslash\overline{X}$) corresponding to the local system defined by the action Γ on $\mathrm{C}^\infty(M; E)$; $\mathbb{E}$ (respectively $\overline{\mathbb{E}}$) is the sheaf of germs of locally-constant sections of the (infinite-dimensional) vector bundle

$$\Gamma\backslash(X \times \mathrm{C}^\infty(M; E)) \to \Gamma\backslash X \quad (\text{respectively } \Gamma\backslash(\overline{X} \times \mathrm{C}^\infty(M; E)) \to \Gamma\backslash\overline{X}).$$

The finite-dimensional vector bundle

$$B_\rho = \Gamma\backslash(X \times M \times V) \to B_M = \Gamma\backslash(X \times M)$$

extends to a vector bundle

$$\overline{B}_\rho = \Gamma\backslash(\overline{X} \times M \times V) \to \overline{B}_M = \Gamma\backslash(\overline{X} \times M),$$

and the foliation $\mathscr{F}_M$ on B_M by images of sets of the form $X \times \{m\}$, $m \in M$ extends to a foliation $\overline{\mathscr{F}}_M$ on $\overline{B}_M$ by images of sets $\overline{X} \times \{m\}$. Let $\mathscr{E}$ (respectively $\overline{\mathscr{E}}$) denote the sheaf of germs of smooth sections of $B_\rho \to B_M$ ($\overline{B}_\rho \to \overline{B}_M$) which are locally-constant along the leaves of $\mathscr{F}_M$ ($\overline{\mathscr{F}}_M$).

There is a topological identification of $\mathscr{E}$ with $\mathbb{E}$, which yields a natural isomorphism

$$\mathrm{H}^\bullet(\Gamma\backslash X; \mathbb{E}) \simeq \mathrm{H}^\bullet(B_M; \mathscr{E}).$$

Similarly, there is a natural isomorphism

$$H^\bullet(\Gamma\backslash\overline{X}; \overline{\mathbb{E}}) \simeq H^\bullet(\overline{B}_M; \overline{\mathscr{E}}).$$

Since the inclusion $\Gamma\backslash X \hookrightarrow \Gamma\backslash\overline{X}$ is a homotopy equivalence, the map

$$H^\bullet(\Gamma\backslash\overline{X}; \overline{\mathbb{E}}) \to H^\bullet(\Gamma\backslash X; \mathbb{E})$$

induced by restriction is an isomorphism, hence

$$H^\bullet(B_M; \mathscr{E}) \to H^\bullet(\overline{B}_M; \overline{\mathscr{E}})$$

(also induced by restriction) is an isomorphism as well.

Since the complex of presheaves of sections of $\mathrm{Alt}^\bullet(\mathrm{T}\mathscr{F}_M, B_\rho) \to B_M$ is a complex of fine sheaves and a resolution of $\mathscr{E}$, the corresponding complex of global sections $\mathrm{A}^\bullet(\mathscr{F}_M; B_\rho)$ computes the cohomology $H^\bullet(B_M; \mathscr{E})$. (Here we are simply recalling the proof of Lemma 2.1, above, which is given in [Zi2].)

We define a complex of presheaves $\mathscr{A}^\bullet$ on $\overline{B}_M$ by assigning to an open set $U \subset \overline{B}_M$ the space of $\mathscr{E}$-valued forms η on $U \cap B_M$ such that η and $d_{\mathscr{F}}\eta$ are weakly λ-bounded near the boundary. Clearly the $\mathscr{A}^\bullet$ are sheaves and $\mathrm{A}^\bullet_\lambda(\mathscr{F}_M; B_\rho)$ is the space of global sections of $\mathscr{A}^\bullet$. We will complete the proof of the theorem by showing that $\mathscr{A}^\bullet$ gives a fine resolution of $\overline{\mathscr{E}}$, whence $\mathrm{A}^\bullet_\lambda(\mathscr{F}_M; B_\rho)$ computes $H^\bullet(\overline{B}_M; \overline{\mathscr{E}})$ and

$$\mathrm{A}^\bullet_\lambda(\mathscr{F}_M; B_\rho) \hookrightarrow \mathrm{A}^\bullet(\mathscr{F}_M; B_\rho)$$

induces the isomorphism

$$H^\bullet(\overline{B}_M; \overline{\mathscr{E}}) \xrightarrow{\sim} H^\bullet(B_M; \mathscr{E}).$$

The proof that $\mathscr{A}^\bullet$ is a fine resolution of $\overline{\mathscr{E}}$ is the same as the proof of Theorem 3.4 in [B2], performed "leafwise"; we refer to the latter for most of the details.

Fix a parabolic $\mathbb{Q}$-subgroup and special neighborhood W, the isomorphic image of $\mu_0(\omega\times\overline{A}_{(t)})\times U$, as above. We may take as coordinates along the leaves of $\mathscr{F}_M$ $\beta^i = a^{-\alpha_i}$ $(1 \le i \le l)$ in $\overline{A}_{(t)}$ together with any coordinates x^j $(l < j \le n = \dim X)$ in ω. Hence $d\beta^i$ and the ω^j $(j > l)$ form a local frame for $\mathrm{T}^*\mathscr{F}_M$ on W. If ϕ is a smooth function on $\overline{B}_M$, then $d_{\mathscr{F}}\phi$ is a linear combination of the $d\beta^i$, ω^j with bounded coefficients. Since $d\beta^i/\beta^i = -da^{\alpha_i}/a^{\alpha_i}$, if follows that the coefficients $(d_{\mathscr{F}}\phi)_j$ of $d_{\mathscr{F}}\phi$, expressed as a linear combination of the ω^j, are bounded. Thus if η and $d_{\mathscr{F}}\eta$ are weakly λ-bounded, then so are $\phi\eta$ and $d_{\mathscr{F}}(\phi\eta)$. Consequently, $\mathscr{A}^\bullet$ is fine.

It remains to show that $\mathscr{A}^\bullet$ is a resolution of $\overline{\mathscr{E}}$. Since λ is dominant, $\mathscr{A}^0$ contains the (smooth) leafwise locally-constant $\mathscr{E}$-valued functions, hence $H^0(\mathscr{A}^\bullet) = \overline{\mathscr{E}}$. To check that $H^p(\mathscr{A}^\bullet) = 0$ for $p \ge 1$, we need to verify the Poincaré lemma for weakly λ-bounded forms on special neighborhoods. In [B2], Borel shows that the usual homotopy operator preserves weak λ-boundedness;

the identical argument is applicable in the present context. This completes the proof of the theorem.

Lemma. *Suppose ρ is irreducible, $\lambda \in X(A_0)$. Then if $p \leq C(G, \lambda, \rho^*)$, any weakly λ-bounded p-form $\eta \in \mathrm{A}^p(\mathscr{F}_M; B_\rho)$ is square-integrable.*

Proof. If will suffice to check that η is square-integrable on any set of the form $\pi(\mathscr{S}_{t,\omega} \times M)$ where $\pi\colon X \times M \to B_M$ and $\mathscr{S}_{t,\omega}$ is a Siegel set with respect to some proper parabolic $\mathbb{Q}$-subgroup and such that $\mathscr{S}_{t,\omega}$ projects injectively to an open set in $\Gamma\backslash X$. Lift η to a V-valued p-form

$$\tilde{\eta} \in \mathrm{C}^\infty(\mathscr{S}_{t,\omega} \times M; p_X^*(\mathrm{T}X; X \times V)).$$

We need to see that

$$\int_{\mathscr{S}_{t,\omega}\times M} ||\tilde{\eta}||^2 d\nu < \infty,$$

where ν is the Γ-invariant volume form on $X \times M$, obtained as the product of the Γ-invariant volume form μ on M and the Γ-invariant Riemannian volume form χ on X. But

$$\int_{\mathscr{S}_{t,\omega}\times M} ||\tilde{\eta}||^2 d\nu = \int_{m\in M} \left(\int_{\mathscr{S}_{t,\omega}} ||\tilde{\eta}_m||^2 d\chi\right) d\mu.$$

Since η is weakly λ-bounded on B_M, each $\tilde{\eta}_m$ is weakly λ-bounded on $\mathscr{S}_{t,\omega}$. Thus

$$\int_{\mathscr{S}_{t,\omega}} ||\tilde{\eta}_m||^2 d\chi < \infty$$

by Lemma 3.6 of [B2]. Consequently

$$m \mapsto \int_{\mathscr{S}_{t,\omega}} ||\tilde{\eta}_m||^2 d\chi$$

defines a continuous (even smooth) finite-valued function on the compact manifold M and

$$\int_{\mathscr{S}_{t,\omega}\times M} ||\tilde{\eta}||^2 d\nu < \infty,$$

as desired.

Taking $\lambda = 0$ and combining the preceding results we obtain the desired square-integrability criterion:

Proposition. *For $p \leq C(G, \rho^*)$, every cohomology class in $\mathrm{A}^p(\mathscr{F}_M; B_\rho)$ has a representative which is square-integrable.*

In particular, every class in $\mathrm{A}^1(\mathscr{F}_M; B_\rho)$ will have a square-integrable representative provided $C(G, \rho^*) \geq 1$. Since the weights of ρ^* are the inverses of the weights of ρ, this is simply the condition $\tau + \lambda - \mu > 0$ for every weight

λ of ρ restricted to $\mathbf{A}_0$ and μ of $\mathbf{A}_0$ on $\mathfrak{u}$, where τ and $\mathbf{A}_0$ are as defined above.

6. Smooth solutions for $\mathbf{SL}(n, \mathbb{Z})$ acting on $\mathbb{T}^n$

Theorem. *Let $\Gamma = \mathbf{SL}(n, \mathbb{Z})$ or any subgroup of finite index. Then the action of Γ on $\mathbb{T}^n$ by automorphisms is infinitesimally rigid for $n \geq 7$.*

The natural identification between $\mathbb{R}^n$ and the tangent space to $\mathbb{T}^n$ at each point yields a trivialization $\mathrm{Tan}(\mathbb{T}^n) \simeq \mathbb{T}^n \times \mathbb{R}^n$, so that $\mathrm{Vec}(\mathbb{T}^n)$ is identified with $C^\infty(\mathbb{T}^n, \mathbb{R}^n)$. For $f \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$, the action of $\gamma \in \Gamma$ may be written $(\gamma f)(t) = \gamma \cdot f(\gamma^{-1} t)$ for $t \in \mathbb{T}^n$, where Γ acts via the standard representation on $\mathbb{R}^n$. The assertion of the theorem is that corresponding to each $g \colon \Gamma \to C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ satisfying the cocycle condition

$$g_{\gamma_1\gamma_2} = \gamma_1 g_{\gamma_2} + g_{\gamma_1}, \quad \text{i.e.,} \quad g_{\gamma_1\gamma_2}(t) = \gamma_1 \cdot g_{\gamma_2}(\gamma_1^{-1} t) + g_{\gamma_1}(t) \quad \forall \gamma_1, \gamma_2 \in \Gamma, t \in \mathbb{T}^n,$$

there exists $f \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ such that

$$g_\gamma = \gamma f - f, \quad \text{i.e.,} \quad g_\gamma(t) = \gamma \cdot f(\gamma^{-1} t) - f(t) \quad \forall \gamma \in \Gamma, t \in \mathbb{T}^n.$$

Heretofore we have been operating under the assumption that the lattice Γ is torsion-free. This condition is actually no more than a technical convenience. In particular, every lattice Γ in a linear group has a normal subgroup Γ' of finite index which is net, hence, in particular, is torsion-free (see, for example, [R1, Theorem 6.11]). Recall that if Γ' is any normal subgroup of Γ and T is any Γ-module whatsoever, there is an exact sequence

$$H^1(\Gamma/\Gamma', T^{\Gamma'}) \to H^1(\Gamma, T) \to H^1(\Gamma', T)^\Gamma.$$

(This is part of the well-known inflation-restriction sequence, cf. [Mc, Section 11.10].) In particular, if T is a vector space over a field of characteristic 0,

$$H^1(\Gamma/\Gamma', T^{\Gamma'}) = 0$$

since Γ/Γ' is finite, hence

$$H^1(\Gamma, T) \to H^1(\Gamma', T)^\Gamma \subset H^1(\Gamma', T)$$

is injective. Thus we may assume without loss of generality that Γ is torsion-free, and henceforth we apply the results of preceeding sections without further comment.

By a straightforward computation, the standard representation of $\mathbf{SL}(n, \mathbb{R})$ on $\mathbb{R}^n$ satisfies the square-integrability criterion of the preceding section provided $n \geq 5$, so that the coboundary equation has a solution $f \in L^2(\mathbb{T}^n, \mathbb{R}^n) \simeq \mathrm{Vec}(\mathbb{T}^n)$ by Proposition 3.6. As we shall see below, this solution is unique. We must show that f is smooth (i.e., has a smooth representative).

Fix the standard basis on $\mathbb{R}^n$ and for $f \in L^2(\mathbb{T}^n, \mathbb{C}^n)$, $\alpha \in \mathbb{Z}^n$, let $\hat{f}(\alpha) \in \mathbb{C}^n$ denote the αth Fourier coefficient of f, so that for $t = (z_1, \dots, z_n) \in \mathbb{T}^n$

($z_i \in \mathbb{C}$, $|z_i| = 1$) we have

$$f(t) = \sum_{\alpha \in \mathbb{Z}^n} \hat{f}(\alpha) z_1^{\alpha_1} \cdots z_n^{\alpha_n} = \sum_{\alpha} \hat{f}(\alpha) z^{\alpha}(t).$$

With these coordinates, the natural action of Γ on $\mathrm{L}^2(\mathbb{T}^n)$ is given by

$$\begin{aligned}(\gamma z^{\alpha})(t) &= z^{\alpha}(\gamma^{-1} t) = z_1(\gamma^{-1} t)^{\alpha_1} \cdots z_n(\gamma^{-1} t)^{\alpha_n} = \prod_i z_i(\gamma^{-1} t)^{\alpha_i} \\ &= \prod_{ij} z_j(t)^{(\gamma^{-1})^i_j \alpha_i} = \prod_j z_j(t)^{\sum_i (\gamma^{-1})^i_j \alpha_i} = z^{\gamma^* \alpha}(t),\end{aligned}$$

where γ^* denotes the contragredient representation to the standard one, and $\mathbb{Z}^n$ is identified with a subset of $(\mathbb{R}^n)^*$ via the standard inner product. For f as above, we have

$$f(\gamma^{-1} t) = \sum_{\alpha} \hat{f}(\alpha) z^{\gamma^* \alpha}(t) = \sum_{\alpha} \hat{f}((\gamma^*)^{-1} \alpha) z^{\alpha}(t),$$

and

$$(\gamma f)^{\widehat{}}\,(\alpha) = \gamma \cdot \hat{f}(\gamma^{\mathrm{t}} \alpha),$$

where γ acts on $\mathbb{C}^n$ via the standard representation, and γ^{t} denotes the adjoint (transpose) of the standard action of γ with respect to the standard inner product.

Before proceeding with the proof, it will be convenient to set up some notation. For $\rho \in \mathbb{R}$, define $\mathbf{A}_\rho$ as the space of formal Fourier series

$$\mathbf{A}_\rho = \left\{ f \mid \|f\|_\rho = \sup_{\alpha \in \mathbb{Z}^n} (\|\alpha\| + 1)^{\rho} \|\hat{f}(\alpha)\| < \infty \right\}.$$

Note that, in general, $f \in \mathbf{A}_\rho$ has no significance other than the formal assignment of coefficients $\hat{f}(\alpha)$ for $\alpha \in \mathbb{Z}^n$. We will however adopt the obvious abuse of notation and identify such f with functions on $\mathbb{T}^n$ whenever this makes sense. For example, we will write $\mathrm{L}^2 \subset \mathbf{A}_0$ and $\mathrm{C}^\infty \subset \mathbf{A}_\rho$ for every $\rho \in \mathbb{R}$.

Since $\sum_{\alpha \in \mathbb{Z}^n} \|\alpha\|^{-\rho}$ converges for $\rho > n$, $\mathbf{A}_\rho \subset \mathrm{L}^2$ for $\rho > n/2$. More generally, for $k \in \mathbb{Z}$, $k \geq 0$, if $\rho > k + n/2$, $f \in \mathbf{A}_\rho$, then

$$\sum_{\alpha \in \mathbb{Z}^n} ((1 + \|\alpha\|)^k \|\hat{f}(\alpha)\|)^2 < \infty.$$

Thus by the Sobolev lemma we have $\mathbf{A}_\rho \subset \mathrm{C}^k$ whenever $\rho > k + n/2 + [n/2] + 1$.

Returning to the proof of the theorem, fix a smooth cocycle

$$g \colon \Gamma \to \mathrm{C}^\infty(\mathbb{T}^n, \mathbb{R}^n)$$

and let $f \in \mathrm{L}^2(\mathbb{T}^n, \mathbb{R}^n)$ denote the unique L^2 solution to the corresponding coboundary equation. By the preceding remarks, it will suffice to show $f \in \mathbf{A}_\rho$ for every $\rho \in \mathbb{R}$. As a first step, we show that it will suffice to find $\rho_0 > 1$ with $f \in \mathbf{A}_{\rho_0}$.

Lemma. *Suppose* $\rho_0 > 1$, $f \in \mathbf{A}_{\rho_0}$ *such that* $g_\gamma = \gamma f - f \in \mathrm{C}^\infty$ *for all* $\gamma \in \Gamma$. *Then* $f \in \mathbf{A}_\rho$ *for every* $\rho \in \mathbb{R}$, *hence* $f \in \mathrm{C}^\infty$ (*where the action of* Γ *on* $\mathbf{A}_\rho$ *is defined via* $(\gamma f)^\wedge(\alpha) = \gamma \cdot \hat{f}(\gamma^{\mathrm{t}}\alpha)$, *as above*).

Proof. In case $\Gamma = \mathbf{SL}(n, \mathbb{Z})$, set $\gamma_0 = \left(\begin{smallmatrix} \mathrm{A} & 0 \\ 0 & \mathbf{I}_{n-2} \end{smallmatrix}\right) \in \Gamma$, where $\mathrm{A} = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in \mathbf{SL}(2, \mathbb{Z})$ and $\mathbf{I}_{n-2}$ is the $(n-2) \times (n-2)$ identity matrix. Note that A is diagonalizable; it is conjugate in $\mathbf{SL}(2, \mathbb{R})$ to the matrix $\left(\begin{smallmatrix} \lambda & \\ & \lambda^{-1} \end{smallmatrix}\right)$, where $\lambda = (3+\sqrt{5})/2 > 1$. More generally, if Γ is a subgroup of finite index in $\mathbf{SL}(n, \mathbb{Z})$, there exists a positive integer N such that $\gamma_0^N \in \Gamma$, A^N is conjugate to $\left(\begin{smallmatrix} \lambda^N & \\ & \lambda^{-N} \end{smallmatrix}\right)$, and we substitute γ_0^N for γ_0 and λ^N for λ. Then we can fix $v_1, v_2 \in \mathbb{R}^n$ so that $\gamma_0 v_1 = \lambda v_1$, $\gamma_0 v_2 = \lambda^{-1} v_2$, and $\{v_1, v_2, e_3, \dots, e_n\}$ is a basis for $\mathbb{R}^n$, where e_i denotes the ith standard basis element. Set $g = g_{\gamma_0} = \gamma_0 f - f \in \mathrm{C}^\infty(\mathbb{T}^n, \mathbb{R}^n)$.

Write $f = f^1 v_1 + f^2 v_2 + \sum_{i=3}^n f^i e_i$; similarly define g^i, $\hat{f}^i$, and $\hat{g}^i$, so that f^i and g^i are $\mathbb{R}$-valued functions on $\mathbb{T}^n$, and $\hat{f}^i$ and $\hat{g}^i$ are the corresponding $\mathbb{C}$-valued Fourier coefficients. For $\alpha \in \mathbb{Z}^n$, define $\pi_1(\alpha)$, $\pi_2(\alpha) \in \mathbb{R}^n$ and $\pi_3(\alpha) \in \mathbb{Z}^n$ via $\alpha = \sum_{i=1}^3 \pi_i(\alpha)$, with $\pi_i(\alpha)$ a multiple of v_i, $i = 1$ or 2, and $\pi_3(\alpha) \in \langle e_3, \dots, e_n \rangle$.

In terms of the Fourier coefficients, and making use of the fact that $\gamma_0 = \gamma_0^{\mathrm{t}}$, the coboundary equation yields $\gamma_0 \hat{f}(\gamma_0\alpha) - \hat{f}(\alpha) = \hat{g}(\alpha)$ for each $\alpha \in \mathbb{Z}^n$. In particular,

$$\lambda \hat{f}^1(\gamma_0\alpha) - \hat{f}^1(\alpha) = \hat{g}^1(\alpha) \tag{6.1}$$

and

$$\lambda^{-1} \hat{f}^2(\gamma_0\alpha) - \hat{f}^2(\alpha) = \hat{g}^2(\alpha). \tag{6.2}$$

Since $\rho > 1$ and the coefficients $\hat{f}^i$ are bounded, it follows that

$$\hat{f}^1(\alpha) = \sum_{k=1}^{\infty} \lambda^{-k} \hat{g}^1(\gamma_0^{-k}\alpha) \tag{6.3}$$

and

$$\hat{f}^2(\alpha) = -\sum_{k=0}^{\infty} \lambda^{-k} \hat{g}^2(\gamma_0^{k}\alpha) \tag{6.4}$$

It may be worth remarking that the fact that the $\hat{g}^i$ are bounded is enough to ensure that (6.3) and (6.4) converge absolutely to give bounded formal solutions (i.e., solutions in $\mathbf{A}_0$) to (6.1) and (6.2), respectively. But $f \in \mathbf{A}_\rho \subset \mathbf{A}_0$ by hypothesis, and any bounded solution to (6.1) or (6.2) is obviously unique.

It will be convenient, for the present, to use the norm on $\mathbb{R}^n$ which is the sum of the norms on the eigenspaces for γ_0, so that $\|\alpha\| = \|\pi_1(\alpha)\| + \|\pi_2(\alpha)\| + \|\pi_3(\alpha)\|$. We will also extend the notation defined above to scalar functions and coefficients, so that $\|g^i\|_\rho = \sup_{\alpha \in \mathbb{Z}^n}(\|\alpha\| + 1)^\rho \|\hat{g}(\alpha)\|$, and so on. Thus

$f \in \mathbf{A}_\rho \Leftrightarrow f^i \in \mathbf{A}_\rho$ for $i = 1, \ldots, n$. The definition of the $\mathbf{A}_\rho$ is obviously invariant with respect to the choice of equivalent norms on $\mathbb{R}^n$.

We now obtain a pair of inequalities bounding the $\hat{f}^1(\alpha)$ under the alternative hypotheses (i) $\|\pi_2(\alpha) + \pi_3(\alpha)\| > \|\alpha\|/3$ and (ii) $\|\pi_1(\alpha)\| > \|\alpha\|/3$. Consider first the case (i) $\|\pi_2(\alpha) + \pi_3(\alpha)\| > \|\alpha\|/3$:

$$
\begin{aligned}
|\hat{f}^1(\alpha)| &= \left|\sum_{k=1}^{\infty} \lambda^{-k} \hat{g}^1(\gamma_0^{-k}\alpha)\right| \leq \sum \lambda^{-k} \|g^1\|_\rho (\|\gamma_0^{-k}\alpha\| + 1)^{-\rho} \\
&\leq \sum \lambda^{-k} \|g^1\|_\rho (\|\pi_2(\gamma_0^{-k}\alpha)\| + \|\pi_3(\gamma_0^{-k}\alpha)\| + 1)^{-\rho} \\
&= \sum \lambda^{-k} \|g^1\|_\rho (\lambda^k \|\pi_2(\alpha)\| + \|\pi_3(\alpha)\| + 1)^{-\rho} \\
&\leq \sum \lambda^{-k} \|g^1\|_\rho (\|\pi_2(\alpha) + \pi_3(\alpha)\| + 1)^{-\rho} \\
&\leq \sum \lambda^{-k} \|g^1\|_\rho (\|\alpha\|/3 + 1)^{-\rho} \\
&\leq (\sum \lambda^{-k} \|g^1\|_\rho \cdot 3^\rho)(\|\alpha\| + 1)^{-\rho} \qquad \text{for } \rho > 0.
\end{aligned} \tag{6.5}
$$

Now consider the case (ii) $\|\pi_1(\alpha)\| > \|\alpha\|/3$:

$$
\begin{aligned}
|\lambda^k \hat{f}^1(\gamma_0^k\alpha)| &\leq \lambda^k \|f^1\|_\rho \|\gamma_0^k\alpha\|^{-\rho} \leq \lambda^k \|f^1\|_\rho \|\pi_1(\gamma_0^k\alpha)\|^{-\rho} \\
&= \lambda^k \|f^1\|_\rho \|\lambda^k \pi_1(\alpha)\|^{-\rho} = \lambda^{(1-\rho)k} \|f^1\|_\rho \|\pi_1(\alpha)\|^{-\rho} \\
&< \lambda^{(1-\rho)k} \|f^1\|_\rho \cdot 3^\rho \|\alpha\|^{-\rho} \qquad \text{for } \rho > 0.
\end{aligned}
$$

In particular, this inequality holds with $\rho = \rho_0 > 1$, and $\|f^1\|_{\rho_0} < \infty$ by hypothesis. Thus $\lim_{k\to\infty} |\lambda^k \hat{f}^1(\gamma_0^k\alpha)| = 0$. Coupled with equation (6.1), this implies

$$
\hat{f}^1(\alpha) = -\sum_{k=0}^{\infty} \lambda^k \hat{g}^1(\gamma_0^k\alpha).
$$

Thus

$$
\begin{aligned}
|\hat{f}^1(\alpha)| &= |\sum \lambda^k \hat{g}^1(\gamma_0^k\alpha)| \\
&\leq \sum \lambda^k \|g^1\|_\rho (\|\gamma_0^k\alpha\| + 1)^{-\rho} \\
&\leq \sum \lambda^k \|g^1\|_\rho (\lambda^k \|\pi_1(\alpha)\| + 1)^{-\rho} \\
&\leq (\sum \lambda^{(1-\rho)k} \|g^1\|_\rho \cdot 3^\rho) \|\alpha\|^{-\rho} \qquad \text{for } \rho > 0.
\end{aligned} \tag{6.6}
$$

The series in (6.5) always converges, that in (6.6) converges provided $\rho > 1$. Since g is smooth, $\|g^1\|_\rho < \infty$ for every $\rho \in \mathbb{R}$. Thus (6.5) and (6.6) together yield $\|f^1\|_\rho < \infty$ for $\rho > 1$, hence for every $\rho \in \mathbb{R}$.

An argument entirely analogous to the preceding yields $\|f^2\|_\rho < \infty$. Since v_1 and v_2 span the same space as e_1 and e_2, we can conclude that the e_1 and e_2 components of f are smooth. If $n = 2$, we are done. Otherwise, we can adjust the choice of γ_0 above and apply the same argument to obtain smoothness for the other components. To be precise, set $\gamma_i \in \mathbf{SL}(n, \mathbb{Z})$ equal to the $(2, i)$-permutation matrix for $3 \le i \le n$. The preceding argument with $\gamma_i \gamma_0 \gamma_i^{-1}$ in place of γ_0 yields smoothness of the e_i component. This completes the proof of the lemma.

To complete the proof of the theorem, it remains to find $\rho_0 > 1$ with $f \in \mathbf{A}_{\rho_0}$. We will in fact show that f has formal second derivatives in L^2, so that $f \in \mathbf{A}_2$. In outline, the argument goes as follows: The natural identifications $\mathrm{Tan}(\mathbb{T}^n) \simeq \mathbb{T}^n \times \mathbb{R}^n$ and $\mathrm{Tan}(\mathbb{R}^n) \simeq \mathbb{R}^n \times \mathbb{R}^n$ yield an identification

$$\mathrm{C}^\infty(\mathrm{Tan}(\mathbb{T}^n), \mathrm{Tan}(\mathbb{R}^n)) \simeq \mathrm{C}^\infty(\mathbb{T}^n, \mathbb{R}^n) \times \mathrm{C}^\infty(\mathbb{T}^n, (\mathbb{R}^n)^* \otimes \mathbb{R}^n),$$

where on the left-hand side "C^∞" means "smooth vector-bundle morphisms." Composing projection on the second factor with the natural map

$$\mathrm{C}^\infty(\mathbb{T}^n, \mathbb{R}^n) \to \mathrm{C}^\infty(\mathrm{Tan}(\mathbb{T}^n), \mathrm{Tan}(\mathbb{R}^n)),$$

we obtain a differential operator

$$\mathrm{D}\colon \mathrm{C}^\infty(\mathbb{T}^n, \mathbb{R}^n) \to \mathrm{C}^\infty(\mathbb{T}^n, (\mathbb{R}^n)^* \otimes \mathbb{R}^n);$$

D is just the classical multivariate derivative. By iterating this procedure we obtain

$$\mathrm{D}\colon \mathrm{C}^\infty(\mathbb{T}^n, (\mathbb{R}^n)^* \otimes \mathbb{R}^n) \to \mathrm{C}^\infty(\mathbb{T}^n, (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes \mathbb{R}^n)$$

and

$$\mathrm{D}^2\colon \mathrm{C}^\infty(\mathbb{T}^n, \mathbb{R}^n) \to \mathrm{C}^\infty(\mathbb{T}^n, \mathrm{S}^2((\mathbb{R}^n)^*) \otimes \mathbb{R}^n).$$

(Here "S^2" means the second symmetric tensor power.)

Each of the maps D, above, commutes with the naturally defined Γ actions, so that the cocycle condition on g leads to appropriate cocycle conditions on $\mathrm{D}g$ and $\mathrm{D}^2 g$. By Proposition 3.6, the corresponding coboundary equations have solutions in L^2. On the other hand, the operators D extend to "formal differentiation" operators on the spaces $\mathbf{A}_\rho$, $\mathrm{D}\colon \mathbf{A}_\rho \to \mathbf{A}_{\rho-1}$. Formally differentiating f, the L^2 solution to the original coboundary equation, we obtain formal solutions $\mathrm{D}f \in \mathbf{A}_{-1}$ and $\mathrm{D}^2 f \in \mathbf{A}_{-2}$ to the equations corresponding to $\mathrm{D}g$ and $\mathrm{D}^2 g$, respectively. *A priori*, these formal solutions need not exhibit any additional smoothness; in particular, they need not correspond to functions on $\mathbb{T}^n$. But they will satisfy certain symmetry conditions expressing the fact that they are "integrable," i.e., in the image of D. Roughly speaking, we will show that the two sets of solutions coincide, hence $\mathrm{D}^2 f \in \mathrm{L}^2 \subset \mathbf{A}_0 \Rightarrow f \in \mathbf{A}_2$.

We begin by observing that the standard representation of $\mathbf{SL}(n, \mathbb{R})$ on $\mathrm{S}^2((\mathbb{R}^n)^*) \otimes \mathbb{R}^n$, which is $n^2(n+1)/2$-dimensional, decomposes as the direct sum of two irreducible subrepresentations, of dimensions $n(n-1)(n-2)/2$ and n, respectively. As above, we let e_i denote the ith standard basis element for $\mathbb{R}^n$, and let η^i denote the corresponding element of the dual basis for $(\mathbb{R}^n)^*$ so that $\eta^i(\mathrm{e}_j) = \delta^i_j$. Set $\xi^{ij}_k = \eta^i \otimes \eta^j \otimes \mathrm{e}_k$. Then $\mathrm{S}^2((\mathbb{R}^n)^*) \otimes \mathbb{R}^n$ has a basis

$$\{\xi^{ii}_k, \xi^{ij}_k + \xi^{ji}_k \mid 1 \le i, j, k \le n;\ i < j\}.$$

The n vectors

$$\zeta_i = 2\xi_i^{ii} + \sum_{j \neq i} (\xi_j^{ij} + \xi_j^{ji})$$

span the n-dimensional invariant, and with respect to this basis the action is precisely the standard contragredient action. Let V denote the remaining $n(n-1)(n+2)/2$-dimensional component. The two components are orthogonal with respect to the standard inner product on $(\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes \mathbb{R}^n$, and V is spanned by the vectors

$$\xi_k^{ij} + \xi_k^{ji}, \qquad i < j,\ k \neq i \text{ or } j,$$

and

$$\xi_j^{ii},\ \xi_i^{ii} - (\xi_j^{ij} + \xi_j^{ji}), \qquad i \neq j.$$

As usual, the action of Γ on $\mathrm{S}^2((\mathbb{R}^n)^*) \otimes \mathbb{R}^n$ gives rise to actions on the various spaces of sections of the trivial bundle $\mathbb{T}^n \times (\mathrm{S}^2((\mathbb{R}^n)^*) \otimes \mathbb{R}^n)$. For $h \in \mathrm{C}^\infty(\mathbb{T}^n, \mathrm{S}^2((\mathbb{R}^n)^*) \otimes \mathbb{R}^n)$ or $\mathrm{L}^2(\mathbb{T}^n, \mathrm{S}^2((\mathbb{R}^n)^*) \otimes \mathbb{R}^n)$, $\gamma \in \Gamma$, $t \in \mathbb{T}^n$, we have $(\gamma h)(t) = \gamma \cdot h(\gamma^{-1} t)$. Proceeding as above, we define Fourier coefficients $\hat{h}(\alpha)$ for $\alpha \in \mathbb{Z}^n$, so that $(\gamma h)\hat{}\ (\alpha) = \gamma \cdot \hat{h}(\gamma^{\mathrm{t}} \alpha)$, and spaces

$$\mathbf{A}_\rho = \mathbf{A}_\rho(\mathbb{T}^n, \mathrm{S}^2((\mathbb{R}^n)^*) \otimes \mathbb{R}^n),$$

each with an action of Γ defined by the same formula.

Lemma. *The action of Γ on $\mathbf{A}_{-2}(\mathbb{T}^n, V)$ has no nonzero fixed vector.*

Proof. Fix γ_0, v_1, and v_2 as above. Set $e_i' = v_i$, $i = 1$ or 2; $e_i' = e_i$, $i > 2$; $\eta^{i\prime}(e_j') = \delta_j^i$; $\xi_k^{ij\prime} = \eta^{i\prime} \otimes \eta^{j\prime} \otimes e_k'$. Note that the $\xi_k^{ij\prime}$ are eigenvectors for the action of γ_0 on $(\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes \mathbb{R}^n$; the maximum eigenvalue, λ^3, corresponds to $\xi_1^{22\prime}$. Since $\xi_1^{22\prime} \in \mathrm{S}^2((\mathbb{R}^n)^*) \otimes \mathbb{R}^n$ and the maximum eigenvalues for γ_0 in the contragredient representation is λ, $\xi_1^{22\prime} \in V$. Let $a_{ij}^{k\prime}$ denote the coordinate functions on $(\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes \mathbb{R}^n$ corresponding to $\xi_k^{ij\prime}$, so that $v = \sum a_{ij}^{k\prime}(v) \xi_k^{ij\prime}$. Now for any fixed v, $a_{22}^{1\prime}(\gamma v)$ is a polynomial in $\gamma \in \mathbf{SL}(n, \mathbb{R})$. If $v \neq 0 \in V$, then $\exists \gamma \in \mathbf{SL}(n, \mathbb{R})$ such that $a_{22}^{1\prime}(\gamma v) \neq 0$, by irreducibility. Thus by the Borel density theorem, $\exists \gamma \in \Gamma$ such that $a_{22}^{1\prime}(\gamma v) \neq 0$.

Now suppose that $h \in \mathbf{A}_{-2}(\mathbb{T}^n, V)$ is fixed under the action of Γ, i.e., $\hat{h}(\gamma^* \alpha) = \gamma \cdot \hat{h}(\alpha)$ for every $\gamma \in \Gamma$, $\alpha \in \mathbb{Z}^n$. Suppose $\alpha_0 \in \mathbb{Z}^n$ and $\hat{h}(\alpha_0) \neq 0$. Fix $\gamma_1 \in \Gamma$ as above, so that $a_{22}^{1\prime}(\gamma_1 \hat{h}(\alpha_0)) \neq 0$, and set $\gamma_2 = \gamma_1^{-1} \gamma_0 \gamma_1$. Consider the ratio

$$\frac{\|\hat{h}((\gamma_2^k)^* \alpha_0)\|}{(1 + \|(\gamma_2^k)^* \alpha_0\|)^2} = \frac{\|\gamma_2^k \cdot \hat{h}(\alpha_0)\|}{(1 + \|(\gamma_2^*)^k \alpha_0\|)^2}$$

for large k. By our choice of γ_2, the growth of the numerator in k is on the order of λ^{3k}. On the other hand, the maximum eigenvalue for γ_2^* is λ, so that the growth of the denominator is of order at most λ^{2k}. In particular,

$$\lim_{k\to\infty} \frac{\|\hat{h}((\gamma_2^k)^*\alpha_0)\|}{(1+\|(\gamma_2^k)^*\alpha_0\|)^2} = \infty,$$

contradicting $h \in \mathbf{A}_{-2}$. Thus $\hat{h}(\alpha) = 0$ for every $\alpha \in \mathbb{Z}^n$; i.e., $h = 0$. This completes the proof of the lemma.

(An analogous, though somewhat simpler argument shows that the action of Γ on $\mathbf{A}_0(\mathbb{T}^n, \mathbb{R}^n)$ (with $\mathbf{A}_0(\mathbb{T}^n, \mathbb{R}^n)$ and the Γ-action defined in the obvious way) has no nonzero fixed vector. Thus our solution $f \in \mathrm{L}^2(\mathbb{T}^n, \mathbb{R}^n) \subset \mathbf{A}_0(\mathbb{T}^n, \mathbb{R}^n)$ to the original coboundary equation is unique in $\mathbf{A}_0$, hence is *a fortiori* unique in L^2, as claimed above.)

In coordinates, the map

$$\mathrm{D}^2 \colon \mathrm{C}^\infty(\mathbb{T}^n, \mathbb{R}^n) \to \mathrm{C}^\infty(\mathbb{T}^n, \mathrm{S}^2((\mathbb{R}^n)^*) \otimes \mathbb{R}^n)$$

is given by

$$\mathrm{D}^2\left(\sum_k f^k e_k\right) = \sum_{ijk}\left(\frac{\partial^2 f^k}{\partial e_i \partial e_j}\right)\xi_k^{ij}.$$

Thus

$$(\mathrm{D}^2 f)\widehat{{}_{ij}}^{\,k}(\alpha) = -(2\pi)^2\alpha_i\alpha_j\hat{f}_{ij}^k(\alpha).$$

One can verify directly that the coboundary equation

$$\hat{g}_\gamma(\alpha) = \gamma\hat{f}(\gamma^{\mathrm{t}}\alpha) - \hat{f}(\alpha)$$

implies

$$(\mathrm{D}^2 g_\gamma)\widehat{\ }(\alpha) = \gamma(\mathrm{D}^2 f)\widehat{\ }(\gamma^{\mathrm{t}}\alpha) - (\mathrm{D}^2 f)\widehat{\ }(\alpha)$$

(with the appropriate action of Γ in each case), or reason from the fact, noted above, that D^2 commutes with the Γ actions on the modules of smooth functions. In any case, it follows that $\mathrm{D}^2 f$ gives a solution in $\mathbf{A}_{-2}$ to the coboundary equation corresponding to the smooth cocycle $\mathrm{D}^2 g$, where g and f are the smooth cocycle and L^2 solution that we started with.

Consider, in particular, the V component of $\mathrm{D}^2 f$, which is a solution to the V component of the $\mathrm{D}^2 g$ equation in $\mathbf{A}_{-2}$. It is again a simple matter to check that the representation of $\mathbf{SL}(n, \mathbb{R})$ on V satisfies the square-integrability criterion for $n \geq 7$, so by Proposition 3.6, this equation has a solution L^2. But by the lemma, the $\mathbf{A}_{-2}$ solution is unique. Thus the two solutions coincide, and we may conclude that the V component of D^2 is in L^2.

It remains to show that the remaining n-dimensional component of $\mathrm{D}^2 f$ is also in L^2. Clearly D^2 satisfies the "integrability" conditions

$$\alpha_l\alpha_m(\mathrm{D}^2 f)\widehat{{}_{ij}}^{\,k}(\alpha) = \alpha_i\alpha_j(\mathrm{D}^2 f)\widehat{{}_{lm}}^{\,k}(\alpha) \qquad \forall i, j, k, l, m.$$

Reading off the components of D^2 with respect to the bases for V and its complement chosen above (which is easy since the ξ_j^{ki} are orthonormal), we see that it will suffice to show that $(D^2f)^i_{ii} \in L^2$ for each i. Note also that $(D^2f)^i_{jj}$ and $(D^2f)^i_{ii} - ((D^2f)^j_{ij} + (D^2f)^j_{ji})$ for $i \neq j$ are coordinates of the V component and must therefore lie in L^2. The integrability conditions yield in particular

$$\alpha_j^2 (D^2f)\widehat{{}_{ii}}^{\,i}(\alpha) = \alpha_i^2 (D^2f)\widehat{{}_{jj}}^{\,i}(\alpha) \tag{6.7}$$

and

$$\alpha_i ((D^2f)\widehat{{}_{ij}}^{\,j} + (D^2f)\widehat{{}_{ji}}^{\,j})(\alpha) = 2\alpha_j (D^2f)\widehat{{}_{ii}}^{\,j}. \tag{6.8}$$

Fix i and $j_0 \neq i$. The $n-1$ sets $\{\alpha \mid |\alpha_i|/|\alpha_j| < 1\}$, $j \neq i$, together with $\{\alpha \mid |\alpha_{j_0}|/|\alpha_i| < 1\}$ cover $\mathbb{Z}^n - \{0\}$. In each set of the first kind, use (6.7) to compare the coefficients of $(D^2f)^i_{ii}$ with those of the L^2 function $(D^2f)^i_{jj}$, and so conclude that the corresponding subsequences of $(D^2f)\widehat{{}_{ii}}^{\,i}$ are square-summable. Similarly, use (6.8) together with

$$(D^2f)^i_{ii} - ((D^2f)^j_{ij} + (D^2f)^j_{ji}) \in L^2$$

and the Schwarz inequality to conclude that $|(D^2f)\widehat{{}_{ii}}^{\,i}(\alpha)|^2$ sums over $\{\alpha \mid |\alpha_{j_0}|/|\alpha_i| < 1\}$. Thus $(D^2f)^i_{ii} \in L^2$, as required.

Finally, it is obvious that $D^2f \in L^2 \subset \mathbf{A}_0 \Rightarrow f \in \mathbf{A}_2$, so the theorem is proved.

It is probably worth remarking that the square-integrability criterion obtained by Borel in [B2] has been strengthened by Zucker in [Zu] (Zucker's criterion is indentical to one which was originally announced in [G-H]). The analogous result, in the present context, would establish the preceding theorem for $n \geq 4$. Unfortunately, certain difficulties arise in attempting to adapt the argument in [Zu] which have not been encountered in adapting [B2], as above. However, it is likely that these obstacles are of a purely technical nature and will be overcome in the future. In case $n = 2$, the group $\mathbf{PSL}(2, \mathbb{Z})$ has a subgroup Γ of finite index isomorphic to the free group on two generators, and a standard argument (cf. [Mc]) shows that $H^1(\Gamma, \mathrm{Vec}(\mathbb{T}^n))$ is not only nonzero, but infinite-dimensional. In case $n = 3$, it is natural to conjecture (extending the analogy with the finite-dimensional case) that H^1 vanishing should still hold, but the argument corresponding to that given above fails even in the finite-dimensional situation, so that the question is still very much an open one.

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