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ON THE GROWTH OF SOLUTIONS OF $f'' + gf' + hf = 0$

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ABSTRACT. Suppose g and h are entire functions with the order of h less than the order of g. If the order of g does not exceed $\frac{1}{2}$, it is shown that every (necessarily entire) nonconstant solution f of the differential equation $f'' + gf' + hf = 0$ has infinite order. This result extends previous work of Ozawa and Gundersen.

I. INTRODUCTION

Recently Ozawa [14] proved the following result:

Theorem A. *Let g be a transcendental entire function of order $\rho < \frac{1}{2}$ and let h be a polynomial. Then any nonconstant solution f of*

$$f'' + gf' + hf = 0 \tag{1.1}$$

has infinite order.

Here the order $\rho(f)$ and lower order $\mu(f)$ of an entire function f are defined by

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log\log M(r, f)}{\log r} \quad \text{and} \quad \mu(f) = \underline{\lim_{r\to\infty}} \frac{\log\log M(r, f)}{\log r},$$

where $M(r, f) = \max_{|z|=r} |f(z)|$.

We extend Ozawa's result by proving

Theorem. *If g and h are entire functions with $\rho(h) < \rho(g) \le \frac{1}{2}$, then any nonconstant solution of* (1.1) *has infinite order.*

Gundersen [5, Theorem 6] has extended Theorem A to obtain the conclusion of our theorem under the more restrictive hypothesis $\rho(h) < \rho(g) < \frac{1}{2}$. Thus our contribution is to treat the case $\rho(g) = \frac{1}{2}$. The main ingredient in the proof when $\rho(g) < \frac{1}{2}$ is the classical $\cos \pi\rho$ theorem. The case $\rho(g) = \frac{1}{2}$ seems to be more delicate.

In §5 we remark that our conclusion also holds under the hypothesis $\rho(h) < \mu(g) \le \frac{1}{2}$. For ease of exposition we treat in detail only the above theorem and

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in §5 confine ourselves to highlighting the modifications of our proof necessary to obtain the lower order result.

It is a simple consequence of the lemma on the logarithmic derivative that any nonconstant solution of (1.1) has infinite order if $\rho(h) > \rho(g)$. Our result complements this fact for $\rho(g) \leq \frac{1}{2}$.

If $\rho(g) = \rho(h)$, the conclusion of our theorem is in general false. Indeed, if p is any polynomial, then $f = e^p$ solves (1.1) for arbitrary g with $h = -p' - (p')^2 - gp'$.

If $\rho(h) < \rho(g) = 1$, (1.1) may have nonconstant solutions of finite order. Such an example is $f'' + e^z f' - f = 0$, which has the solution $f(z) = e^{-z} - 1$. The possibility of nonconstant solutions of finite order of (1.1) remains open in the case $\rho(h) < \rho(g)$ with $\frac{1}{2} < \rho(g) < 1$.

We assume familiarity with the notation and fundamental results of Nevanlinna theory. In addition to the Nevanlinna characteristic $T(r, f)$, we will have occasion to use the Ahlfors-Shimizu characteristic

$$T_0(r, f) = \int_0^r \frac{A(t, f)}{t}\, dt,$$

where $A(t, f)$ is the average number of solutions of $f(z) = a$ in $|z| \leq t$ as a varies over the Riemann sphere. In view of

$$|T_0(r, f) - T(r, f)| = O(1), \qquad r \to \infty$$

(see [7, p. 13]), the two characteristics are interchangeable for many purposes.

Throughout this paper the order of f will be denoted by λ, the order of g by ρ, and the order of h by ρ'. We will consistently choose parameters ρ_j, $1 \leq j \leq 4$, satisfying

$$\rho' < \rho_1 < \rho_2 < \rho_3 < \rho < \rho_4. \tag{1.2}$$

We will represent the counting functions $n(r, 0, F)$ and $N(r, 0, F)$ of the zeros of an entire function F by $n(r)$ and $N(r)$ when it seems unnecessary to specify the function.

II. Known results

Our proof depends on some results of $\cos \pi\rho$ type. Before stating these results, we recall the concepts of density and logarithmic density of subsets of $[1, \infty)$. For $E \subset [1, \infty)$, define the linear measure of E by $m(E) = \int_1^\infty \chi_E(t)\, dt$ where χ_E is the characteristic function of E, and define the logarithmic measure of E by

$$m_l(E) = \int_1^\infty \frac{\chi_E(t)}{t}\, dt.$$

The upper density and upper logarithmic density of E are defined by

$$\overline{\text{dens}} E = \overline{\lim_{r\to\infty}} \frac{m(E \cap [1, r])}{r - 1} \quad \text{and} \quad \overline{\log}\,\overline{\text{dens}} E = \overline{\lim_{r\to\infty}} \frac{m_l(E \cap [1, r])}{\log r}.$$

The lower density and lower logarithmic density, $\underline{\text{dens}}\, E$ and $\underline{\log\ \text{dens}}\, E$, are defined similarly with lim sup replaced by lim inf. It is easy to verify [16, p. 121]

$$0 \le \underline{\text{dens}} E \le \underline{\log\ \text{dens}} E \le \overline{\log\ \text{dens}} E \le \overline{\text{dens}} E \le 1 \tag{2.1}$$

for any $E \subset [1, \infty)$.

For entire g, let $L(t, g) = \min_{|z|=t} |g(z)|$. The method of Denjoy-Kjellberg [12, pp. 193–196] shows for entire g with $g(0) = 1$ and for $0 < \alpha < 1$ that there exist positive constants $k_1(\alpha)$ and $k_2(\alpha)$ such that

$$\begin{aligned} \int_r^R (\log L(t, g) - (\cos \pi\alpha) \log M(t, g)) \frac{dt}{t^{1+\alpha}} \\ \ge k_1(\alpha) \frac{\log M(r, g)}{r^\alpha} - k_2(\alpha) \frac{\log M(4R, g)}{R^\alpha} \end{aligned} \tag{2.2}$$

when $0 < r < R$.

If $\mu(g) < \rho(g) \le \frac{1}{2}$, let ρ_3 and α satisfy $\mu(g) < \rho_3 < \alpha < \rho(g) \le \frac{1}{2}$. Choose $r_n \to \infty$ with $R_n > r_n$ satisfying $r_n^\alpha = o(\log M(r_n, g))$ and $\log M(4R_n, g) = o(R_n^\alpha)$ to conclude

$$\int_{r_n}^{R_n} \frac{\log L(t, g)}{t^{1+\alpha}}\, dt \to \infty$$

as $n \to \infty$. Thus there exists $s_n \to \infty$ with

$$\log L(s_n, g) > s_n^{\rho_3}. \tag{2.3}$$

If $\mu(g) = \rho(g) < \frac{1}{2}$ and $\rho_3 < \rho(g)$, the classical $\cos \pi\rho$ theorem (contained in (2.2)) yields $s_n \to \infty$ satisfying (2.3). In both of the above cases results of Barry [1, p. 294 and 2, Theorem 4] also imply the existence of unbounded s_n satisfying (2.3).

If $\rho_3 < \mu(g) = \rho(g) = \frac{1}{2}$, then either there exists $s_n \to \infty$ satisfying

$$\log L(s_n, g) > \varepsilon \log M(s_n, g) \tag{2.4}$$

for some $\varepsilon > 0$ and hence also satisfying (2.3), or

$$\log L(r, g) = o(\log M(r, g)). \tag{2.5}$$

In this final case, namely $\mu(g) = \rho(g) = \frac{1}{2}$ with g satisfying (2.5), g is extremal for the $\cos \pi\rho$ theorem. Such functions were studied extensively in [4]. A portion of Theorem 8.1 of [4], when specialized to the case $\rho = \frac{1}{2}$, becomes

Theorem B. *Suppose g is an entire function of order $\frac{1}{2}$ and satisfies (2.5). There exists a set G of logarithmic density 1, a set H of density 0, a real-valued function $\varphi(r)$, and a positive function $\mathscr{L}(r)$ varying slowly in the sense that*

$$\lim_{\substack{r \to \infty \\ r \in G}} \frac{\mathscr{L}(\sigma r)}{\mathscr{L}(r)} = 1 \tag{2.6}$$

for all $\sigma > 0$, *such that for* $r \in G - H$

$$\log|g(re^{i(\varphi+\varphi(r))})| = (\cos(\varphi/2) + o(1))r^{1/2}\mathscr{L}(r), \qquad r \to \infty, \tag{2.7}$$

uniformly for $\varphi \in [-\pi, \pi]$.

By (2.1) we note that $G^* = G - H$ has logarithmic density 1. For $\rho_3 < \frac{1}{2}$, we conclude from (2.6), (2.7), and the fact that G^* has logarithmic density 1 that

$$\mathscr{L}(r)r^{1/2-\rho_3} \to \infty \tag{2.8}$$

as $r \to \infty$. Defining

$$K_r = \{\theta \in [0, 2\pi] : \log|g(re^{i\theta})| < r^{\rho_3}\}, \tag{2.9}$$

we conclude from (2.7) and (2.8) that

$$m(K_r) \to 0, \qquad r \in G^*, \qquad r \to \infty. \tag{2.10}$$

In summary, if $\rho(g) \le \frac{1}{2}$ then for all $\rho_3 < \rho(g)$ we either have $s_n \to \infty$ satisfying (2.3), or we have (2.10) for some set G^* of logarithmic density 1 where K_r is defined in (2.9).

III. Preliminaries

Our proof depends heavily on the behavior of the logarithmic derivative of an entire function. For entire F we consider the differentiated Poisson-Jensen representation where $\{a_\mu\}$ denotes the zeros of F and where $|z| = r < R$:

$$\begin{aligned}\frac{zF'(z)}{F(z)} &= \frac{1}{2\pi}\int_0^{2\pi} \log|F(Re^{i\varphi})|\frac{2zRe^{i\varphi}}{(Re^{i\varphi}-z)^2}\,d\varphi \\ &\quad + \sum_{|a_\mu|\le r/e}\left(\frac{z}{z-a_\mu} + \frac{\bar{a}_\mu z}{R^2-\bar{a}_\mu z}\right) \\ &\quad + \sum_{r/e<|a_\mu|\le r}\left(\frac{z}{z-a_\mu} + \frac{\bar{a}_\mu z}{R^2-\bar{a}_\mu z}\right) \\ &\quad + \sum_{r<|a_\mu|<R}\left(\frac{z}{z-a_\mu} + \frac{\bar{a}_\mu z}{R^2-\bar{a}_\mu z}\right) \\ &= F_1(z) + F_2(z) + F_3(z) + F_4(z).\end{aligned} \tag{3.1}$$

For future use we collect the following observations. It is elementary that

$$\operatorname{Re}\left(\frac{re^{i\theta}}{re^{i\theta}-a} + \frac{\bar{a}re^{i\theta}}{R^2-\bar{a}re^{i\theta}}\right) = \frac{d}{d\theta}\arg\frac{R(re^{i\theta}-a)}{R^2-\bar{a}re^{i\theta}}. \tag{3.2}$$

Since $w = R(z-a)/(R^2-\bar{a}z)$ maps $|z| = r$ onto a circle, we conclude that if $|a| < r$ then

$$\operatorname{Re}\left(\frac{re^{i\theta}}{re^{i\theta}-a} + \frac{\bar{a}re^{i\theta}}{R^2-\bar{a}re^{i\theta}}\right) > 0 \tag{3.3}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{re^{i\theta}}{re^{i\theta}-a} + \frac{\bar{a}re^{i\theta}}{R^2-\bar{a}re^{i\theta}}\right) d\theta = 1. \tag{3.4}$$

Trivially if $|a| \le r/e$, then

$$\left|\frac{re^{i\theta}}{re^{i\theta}-a} + \frac{re^{i\theta}}{R^2/\bar{a}-re^{i\theta}}\right| \le \frac{e+1}{e-1} < 4. \tag{3.5}$$

For $r < |a| < R$ it follows from (3.2) that

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\operatorname{Re}\left(\frac{re^{i\theta}}{re^{i\theta}-a} + \frac{\bar{a}re^{i\theta}}{R^2-\bar{a}re^{i\theta}}\right)\right| d\theta \le 1. \tag{3.6}$$

Since $w = z/(z-a)$ maps the circle $|z| = |a|$ to the line $\operatorname{Re} w = \frac{1}{2}$, for $r < |a| < R$ we have that

$$\operatorname{Re}\left(\frac{re^{i\theta}}{re^{i\theta}-a} + \frac{re^{i\theta}}{R^2/\bar{a}-re^{i\theta}}\right) < \frac{1}{2} + \frac{r}{R-r}. \tag{3.7}$$

Finally we notice from (3.1), (3.3), (3.4), and (3.6) with $R = 3r$ that for $r > r_0$

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} &\left|\operatorname{Re} re^{i\theta}\frac{F'(re^{i\theta})}{F(re^{i\theta})}\right| d\theta \\ &\le 3T(R, F) + n(R, 0, F) + O(1) < 4T(eR, F) = 4T(3er, F).\end{aligned} \tag{3.8}$$

We now prove a sequence of lemmas. The conclusion of Lemma 1 is immediate from Ahlfors's covering surface theory (see [7, Theorem 5.2]). We include an elementary proof for completeness.

Lemma 1. *For nonconstant entire f, let*

$$\varphi(r) = \frac{1}{2\pi}\int_0^{2\pi} n(r, e^{i\alpha}, f)\, d\alpha$$

be the mean covering number of the unit circle under the map f restricted to $\{z : |z| \le r\}$. There exists a set E_1 with $m_l(E_1) < \infty$ such that

$$\lim_{\substack{r\to\infty \\ r\notin E_1}} \frac{\varphi(r)}{A(r, f)} = 1.$$

Proof. The result is trivial for polynomials, and consequently we restrict our attention to transcendental f. In this case $A(r, f)$ is strictly increasing, unbounded, and continuous. Clearly $\varphi(r)$ is nondecreasing and, by Cartan's identity [7, p. 8],

$$\left|\int_0^r \frac{\varphi(t)}{t}\, dt - \int_0^r \frac{A(t, f)}{t}\, dt\right| = O(1), \qquad r \to \infty. \tag{3.9}$$

For $\varepsilon > 0$, define t_n by requiring $A(t_n, f) = (1+\varepsilon)^n$ for $n = 0, 1, 2, \dots$, and let $I_n = (t_n, t_{n+1})$. Let

$$E_1^* = \bigcup_n ((t_n, t_n \exp(1+\varepsilon)^{-n/2}) \cup (t_{n+1} \exp(-(1+\varepsilon)^{-n/2}), t_{n+1})).$$

Evidently $m_l(E_1^*) < \infty$.

Suppose for some $t' \in I_n - E_1^*$ that $\varphi(t') < (1-\varepsilon)^2 A(t', f)$. For $t_n < t < t'$ we have

$$\varphi(t) < \varphi(t') < (1-\varepsilon)^2 A(t', f) < (1-\varepsilon) A(t_n, f) < (1-\varepsilon) A(t, f).$$

Thus

$$\begin{aligned}(3.10)\quad & \int_{t_n}^{t'} \frac{A(t, f)}{t}\, dt - \int_{t_n}^{t'} \frac{\varphi(t)}{t}\, dt \\ & > \varepsilon \int_{t_n}^{t'} \frac{A(t, f)}{t}\, dt > \varepsilon A(t_n, f) \log(t'/t_n) > \varepsilon (1+\varepsilon)^n (1+\varepsilon)^{-n/2},\end{aligned}$$

where in the last step we have used $t' \notin E_1^*$. Clearly (3.9) and (3.10) are incompatible, implying

$$\varliminf_{\substack{r \to \infty \\ r \notin E_1^*}} \frac{\varphi(r)}{A(r, f)} \geq 1.$$

A reversal of the roles of the nondecreasing functions $\varphi(r)$ and $A(r, f)$ now yields the result.

From the argument principle it follows for entire f that

$$\frac{1}{2\pi} \int_0^{2\pi} n(r, e^{i\alpha}, f)\, d\alpha = \frac{1}{2\pi} \int_{B_r} \operatorname{Re} re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})}\, d\theta,$$

where

$$B_r = \{\theta \in [0, 2\pi] : |f(re^{i\theta})| > 1\}.$$

In conjunction with Lemma 1, we deduce

$$(3.11)\qquad A(r, f) \leq \frac{(1+o(1))}{2\pi} \int_0^{2\pi} \left(\operatorname{Re} re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right)^+ d\theta, \qquad r \notin E_1,$$

for some set E_1 with $m_l(E_1) < \infty$.

Lemma 2. *Suppose f is an entire function of order $\lambda < \infty$. Let $n(r) = n(r, 0, f)$ and suppose $n(1) \geq 1$. For $K > 1$, let*

$$(3.12)\qquad E_2(K) = \{r > e : n(r) > K n(r/e)\}.$$

Then

$$(3.13)\qquad \overline{\log}\,\overline{\operatorname{dens}} E_2(K) \leq 4\lambda / \log K.$$

Proof. For $n = 0, 1, 2, \ldots$, let $r_n = e^n$ and $I_n = [r_n, r_{n+1})$. For $m \geq 2$, let

$$J_m = \{j \in [1, m] : n(r_j) > \sqrt{K} n(r_{j-1})\}$$

and

$$L_m = \{j \in [1, m-1] : I_j \cap E_2(K) \neq \varnothing\}.$$

Denote the number of elements of J_m and L_m by α_m and β_m respectively. Evidently $n(r_m) > K^{\alpha_m/2}$. Hence if $\lambda' > \lambda$ then

$$\frac{\alpha_m}{m} < \frac{2 \log n(r_m)}{(\log K) \log r_m} < \frac{2\lambda'}{\log K}$$

for $m > m_0(\lambda')$.

We note that if $j \in L_m$, then either $j \in J_m$ or $j + 1 \in J_m$. It follows that $\beta_m \leq 2\alpha_m$. Thus

$$\frac{\beta_m}{m} < \frac{4\lambda'}{\log K} \tag{3.14}$$

for large m. We deduce (3.13) from (3.14) and the fact that each I_n has logarithmic measure 1.

Lemma 3. *Let f be entire of order λ, $0 < \lambda < \infty$. For $K > 1$, let*

$$E_3(K) = \{r > 1 : A(r, f)/T_0(r, f) > K\lambda\}.$$

Then $\overline{\log}\,\overline{\text{dens}} E_3(K) \leq 1/K$.

Proof. For $r > 1$, let $E_3(K, r) = E_3(K) \cap [1, r]$. Then

$$\int_{E_3(K,r)} \frac{dt}{t} \leq \frac{1}{K\lambda} \int_{E_3(K,r)} \frac{A(t, f)}{tT_0(t, f)}\, dt$$
$$\leq \frac{1}{K\lambda} \int_1^r \frac{A(t, f)}{tT_0(t, f)}\, dt = \frac{\log T_0(r, f) - \log T_0(1, f)}{K\lambda}.$$

Thus

$$\varlimsup_{r\to\infty} \frac{m_l(E_3(K, r))}{\log r} \leq \varlimsup_{r\to\infty} \frac{\log T_0(r, f)}{K\lambda \log r} = \frac{1}{K},$$

proving the lemma.

Lemma 4. *Suppose f is entire of order $\lambda < \lambda' < \infty$. If $\rho_1 > 0$, there exists a set $E(\rho_1) \subset [1, \infty)$ satisfying*

$$m(E(\rho_1) \cap [r/e, er]) < 2r^{\lambda'} e^{-r^{\rho_1}}, \qquad r > r_0(\lambda'), \tag{3.15}$$

and such that if $|z| = r \notin E(\rho_1)$, then

$$\left|\frac{f''(z)}{f(z)}\right| < r^{2\lambda'} e^{2(9r)^{\rho_1}}, \qquad r > r_0(\lambda'). \tag{3.16}$$

In particular, $E(\rho_1)$ has logarithmic density 0.

Proof. Let $\Delta(a, \delta) = \{z : |z - a| < \delta\}$ and let $\{a_\mu\}$ denote the zeros of f. Define

$$B = \bigcup_\mu \Delta(a_\mu, \exp(-(3|a_\mu|)^{\rho_1})).$$

Let

$$E^*(\rho_1) = \{t \geq 1 : B \cap \{z : |z| = t\} \neq \varnothing\}.$$

Evidently

$$m(E^*(\rho_1) \cap [r/e, er]) < n(3r)e^{-r^{\rho_1}} < r^{\lambda'} e^{-r^{\rho_1}}, \qquad r > r_0(\lambda'). \tag{3.17}$$

For $r \notin E^*(\rho_1)$, we apply (3.1) with $F = f$ and $R = 3r$ to obtain for $|z| = r$

$$\left|\frac{zf'(z)}{f(z)}\right| \leq 3T(3r, f) + 2rn(3r)e^{(9r)^{\rho_1}} + O(1),$$

implying

$$\left|\frac{f'(z)}{f(z)}\right| < r^{\lambda'} e^{(9r)^{\rho_1}}, \qquad r > r_0(\lambda'), \ r \notin E^*(\rho_1). \tag{3.18}$$

Applying the same argument to f', we obtain a set $E^{**}(\rho_1)$ satisfying (3.17) and satisfying (3.18) with f replaced by f'. We obtain the required $E(\rho_1)$ by setting $E(\rho_1) = E^*(\rho_1) \cup E^{**}(\rho_1)$. Evidently

$$m_l(E(\rho_1) \cap [r/e, er]) \leq \frac{em(E(\rho_1) \cap [r/e, er])}{r} = o(1), \qquad r \to \infty.$$

Since $m_l([r/e, er]) = 2$, we deduce the logarithmic density of $E(\rho_1)$ is 0.

Lemma 5. *Let f be an entire function of finite order $\lambda > 0$. If $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ and a set $E_5(\varepsilon) \subset [1, \infty)$ with lower logarithmic density at least $2c(\varepsilon)$ such that for all $r \in E_5(\varepsilon)$, there exists $h = h_r > 0$ such that if $R' = re^h$, then*

$$T_0(R', f) < h(e + \varepsilon)A(r, f) \tag{3.19}$$

and for all $K > K_0(\varepsilon)$,

$$T_0(R', f) < h^2 K\lambda(e + \varepsilon)A(r, f). \tag{3.20}$$

Proof. Our proof depends on growth lemmas developed in [6 and 10] and later extended in [9 and 8]. For a previous application of these growth lemmas to the present context, see [13, pp. 386–387]. Set $\varphi(x) = T_0(e^x, f)$ and note that $\varphi'(x) = A(e^x, f)$. Let

$$h = \varphi(x)/\varphi'(x) = T_0(e^x, f)/A(e^x, f).$$

Applying Lemmas 1 and 6 of [6], we conclude there exists a set $E_5^*(\varepsilon)$ of lower density at least $3c(\varepsilon)$ for some $c(\varepsilon) > 0$ such that

$$T_0(e^{x+h}, f) < h(e + \varepsilon)A(e^x, f), \qquad x \in E_5^*(\varepsilon). \tag{3.21}$$

(See also [13, p. 387].) For $K > 1/c(\varepsilon)$, we apply Lemma 3 to conclude on a set $E_5^{**}(\varepsilon)$ of lower density at least $2c(\varepsilon)$ that in addition to (3.21) we also have $h = T_0(e^x, f)/A(e^x, f) > 1/K\lambda$ and thus

$$T_0(e^{x+h}, f) < h^2 K\lambda(e+\varepsilon)A(e^x, f). \tag{3.22}$$

Writing $r = e^x$ and $R' = re^h$, we obtain (3.19) and (3.20) from (3.21) and (3.22) for a set $E_5(\varepsilon)$ of r-values of lower logarithmic density at least $2c(\varepsilon)$.

Lemma 6. *Suppose g is an entire function of order $\rho \in (0, \infty)$ and suppose $0 < \rho_2 < \rho_3 < \rho < \rho_4 < \infty$. Suppose $\log L(s, g) > s^{\rho_3}$ where $L(s, g) = \min_{|z|=s} |g(z)|$. For $s < r < 2s$, let*

$$C_r = \{\theta \in [0, 2\pi] : \log|g(re^{i\theta})| < r^{\rho_2}\}. \tag{3.23}$$

For s sufficiently large we have

$$m(C_r) < s^{\rho_4 - \rho_3 - 1}(r - s). \tag{3.24}$$

Proof. For $\theta \in C_r$ and $s > s_0$ we have

$$\frac{s^{\rho_3}}{2} < s^{\rho_3} - r^{\rho_2} < \int_s^r \left|\frac{\partial}{\partial t}\log|g(te^{i\theta})|\right| dt = \int_s^r \frac{1}{t}\left|\operatorname{Re} te^{i\theta}\frac{g'(te^{i\theta})}{g(te^{i\theta})}\right| dt.$$

Thus

$$\begin{aligned}
\frac{m(C_r)s^{\rho_3}}{2} &= \int_{C_r} \frac{s^{\rho_3}}{2}\, d\theta \le \int_{C_r}\int_s^r \frac{1}{t}\left|\operatorname{Re} te^{i\theta}\frac{g'(te^{i\theta})}{g(te^{i\theta})}\right| dt\, d\theta \\
&= \int_s^r \frac{1}{t}\int_{C_r}\left|\operatorname{Re} te^{i\theta}\frac{g'(te^{i\theta})}{g(te^{i\theta})}\right| d\theta\, dt \\
&\le \int_s^r \frac{1}{t}\int_0^{2\pi}\left|\operatorname{Re} te^{i\theta}\frac{g'(te^{i\theta})}{g(te^{i\theta})}\right| d\theta\, dt \\
&\le 8\pi T(3er, g)\int_s^r \frac{dt}{t} < \frac{s^{\rho_4 - 1}}{2}(r - s), \qquad s > s_0,
\end{aligned} \tag{3.25}$$

where we have applied (3.8) with $g = F$. We deduce (3.24) immediately from (3.25).

IV. Proof of the theorem

We presume a nonconstant entire f of finite order $\lambda < \lambda'$ satisfies (1.1) and seek a contradiction. We choose parameters ρ_j, $1 \le j \le 4$, satisfying (1.2). By the final remark of §2, we may consider two cases:

(I) there are arbitrarily large s satisfying

$$\log|g(se^{i\theta})| > s^{\rho_3}, \qquad 0 \le \theta \le 2\pi, \tag{4.1}$$

or

(II) there exists $G^* \subset [1, \infty)$ with logarithmic density 1 such that

$$m(K_r) = m\{\theta \in [0, 2\pi] : \log|g(re^{i\theta})| < r^{\rho_3}\} = o(1)$$

as r tends to infinity through values in G^*.

We first consider Case I. For large s satisfying (4.1), we apply (3.15) to obtain

$$r \in (s, s + 3s^{\lambda'} e^{-s^{\rho_1}}) - E(\rho_1). \tag{4.2}$$

Defining C_r as in (3.23), we conclude from (3.24) that

$$m(C_r) < 3s^{\lambda' + \rho_4 - \rho_3 - 1} e^{-s^{\rho_1}}. \tag{4.3}$$

From (1.1) we have

$$\frac{f'(z)}{f(z)} = -\frac{h(z) + f''(z)/f(z)}{g(z)}. \tag{4.4}$$

From (1.2), (3.16), (3.23), (4.2), and (4.4) it follows that there exists an unbounded set H for which $\theta \notin C_r$ implies

$$\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| \le \frac{e^{r^{\rho_1}} + r^{2\lambda'} e^{2(9r)^{\rho_1}}}{e^{r^{\rho_2}}} = o(1), \qquad r \to \infty, \ r \in H. \tag{4.5}$$

We now estimate

$$\frac{1}{2\pi} \int_{C_r} \left(\operatorname{Re} re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})}\right)^+ d\theta$$

from above for $r \in H$ by appealing to (3.1) with $F = f$ and $R = 3r$. Letting f_j, $1 \le j \le 4$, bear the same relationship to f as do F_j, $1 \le j < 4$, to F in (3.1), we have

$$\frac{1}{2\pi} \int_{C_r} |f_1(re^{i\theta})|\, d\theta \le (3T(3r, f) + O(1))m(C_r) = o(1), \qquad r \to \infty, r \in H, \tag{4.6}$$

by (4.2) and (4.3).

If f has no zeros, we conclude from (4.5) and (4.6) that the total variation of $\arg f(re^{i\theta})$ on $[0, 2\pi]$ is $o(1)$. Since f is nonconstant, this is incompatible with the Casorati-Weierstrass theorem and the argument principle, providing the desired contradiction.

If f does have zeros, we conclude from (3.5) and (4.3) for $r \in H$ that

$$\frac{1}{2\pi} \int_{C_r} |\operatorname{Re} f_2(re^{i\theta})|\, d\theta \le \frac{4}{2\pi} n(r/e) m(C_r) = o(1), \qquad r \to \infty. \tag{4.7}$$

From (3.3) and (3.4) we have

$$\frac{1}{2\pi} \int_{C_r} (\operatorname{Re} f_3(re^{i\theta}))^+ \, d\theta \le n(r) - n(r/e). \tag{4.8}$$

From (3.7) and (4.3) we conclude that for $r \in H$

$$\frac{1}{2\pi}\int_{C_r}(\operatorname{Re} f_4(re^{i\theta}))^+\,d\theta \le \frac{1}{2\pi}(n(3r)-n(r))m(C_r) = o(1), \qquad r\to\infty. \tag{4.9}$$

Combining (4.5), (4.6), (4.7), (4.8), and (4.9), we obtain

(4.10)

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left(\operatorname{Re} re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right)^+ d\theta &\le n(r)-n(r/e)+o(1)\\ &\le n(r)-1+o(1), \qquad r>r_0,\ r\in H.\end{aligned}$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re} re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\,d\theta = n(r) \tag{4.11}$$

by the argument principle, (4.10) provides the required contradiction.

We now consider Case II. Suppose $\varepsilon > 0$. Applying Lemma 1, Lemma 4, and Lemma 5, we conclude there exists a set $E \subset [1,\infty)$ such that

$$E\cap E_1 = \varnothing, \tag{4.12}$$

$$E\cap E(\rho_1) = \varnothing, \tag{4.13}$$

$$E\subset (G^*\cap E_5(\varepsilon)), \tag{4.14}$$

and

$$\underline{\log\,\operatorname{dens}} E > c(\varepsilon). \tag{4.15}$$

By (4.13) for large $r\in E$ we have as in (4.5) for $\theta\notin K_r$

$$\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| \le \frac{e^{r^{\rho_1}}+r^{2\lambda'}e^{2(9r)^{\rho_1}}}{e^{r^{\rho_3}}} = o(1), \qquad r\to\infty. \tag{4.16}$$

We choose $K>1$ so large that (3.19) and (3.20) hold for $r\in E$ and estimate

$$\frac{1}{2\pi}\int_{K_r}\left(\operatorname{Re} re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right)^+ d\theta$$

using (3.1) with $F=f$ and $R=re^{h/2}$ where $R'=re^h$ in Lemma 5. We have

$$\begin{aligned}|f_1(re^{i\theta})| &\le (4T(R,f)+O(1))\frac{e^{h/2}}{(e^{h/2}-1)^2}\\ &< \frac{17T_0(R',f)}{h^2} < 17K\lambda(e+\varepsilon)A(r,f)\end{aligned}$$

since $r\in E_5(\varepsilon)$. Consequently

$$\frac{1}{2\pi}\int_{K_r}|f_1(re^{i\theta})|\,d\theta = o(A(r,f)), \qquad r\in E,\ r\to\infty, \tag{4.17}$$

by (2.10) and (4.14). If f has no zeros, the combination of (3.11), (4.16), and (4.17) provides the desired contradiction.

If f does have zeros, we may presume $n(1, 0, f) = n(1) \geq 1$. Applying Lemma 2, we may choose $K > 1$ so large that in addition to (4.12), (4.13), (4.14), and (4.15), E also satisfies

$$E \cap E_2(K) = \varnothing. \tag{4.18}$$

We conclude from (2.10) and (3.5) that

$$\begin{aligned} \frac{1}{2\pi} \int_{K_r} |\operatorname{Re} f_2(re^{i\theta})|\, d\theta \\ \leq \frac{4}{2\pi} n(r/e) m(K_r) = o(n(r/e)), \qquad r \in E,\ r \to \infty. \end{aligned} \tag{4.19}$$

From (3.3) and (3.4) we have

$$\frac{1}{2\pi} \int_{K_r} |\operatorname{Re} f_3(re^{i\theta})|\, d\theta \leq n(r) - n(r/e). \tag{4.20}$$

From (3.7), (3.19), and (3.20) we conclude

$$\begin{aligned} \frac{1}{2\pi} \int_{K_r} (\operatorname{Re} f_4(re^{i\theta}))^+\, d\theta \\ &\leq \frac{1}{2\pi}(n(R) - n(r)) \left(\frac{1}{2} + \frac{1}{e^{h/2} - 1}\right) m(K_r) \\ &< \frac{n(R)}{2\pi} \left(\frac{1}{2} + \frac{2}{h}\right) m(K_r) \leq \frac{N(R')}{\pi h} \left(\frac{1}{2} + \frac{2}{h}\right) m(K_r) \\ &\leq \frac{T(R', f)}{\pi h} \left(\frac{1}{2} + \frac{2}{h}\right) m(K_r) = o(A(r, f)), \qquad r \in E,\ r \to \infty. \end{aligned} \tag{4.21}$$

We conclude from (3.12), (4.17), (4.18), (4.19), (4.20), and (4.21) that

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \left(\operatorname{Re} re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})}\right)^+ d\theta \\ &\leq n(r) - n(r/e) + o(A(r, f) + n(r/e)) \\ &\leq (1 + o(1) - 1/K) n(r) + o(A(r, f)), \qquad r \in E,\ r \to \infty. \end{aligned} \tag{4.22}$$

The combination of (3.11) with (4.22) is incompatible with (4.11), providing the desired contradiction.

V. Concluding remarks

The conclusion of our theorem also holds under the hypothesis $\rho(h) < \mu(g) \leq \frac{1}{2}$. For $\rho_3 < \mu(g)$, we first establish as before that either (2.3) holds for some sequence $s_n \to \infty$ or (2.10) holds for a set G^* of upper logarithmic density 1. If $\mu(g) < \frac{1}{2}$, we conclude from Kjellberg's lower order extension [12] of the $\cos \pi\rho$ theorem (essentially (2.2)) that for each $\rho_3 < \mu(g)$ there exists $s_n \to \infty$ satisfying (2.3). Thus we suppose $\mu(g) = \frac{1}{2}$. Either (2.4) holds for

some $s_n \to \infty$ and hence (2.3) also holds, or alternatively (2.5) holds. By the remarks following Theorem 8.1 in [4, p. 283], (2.5) implies the set K_r defined in (2.9) satisfies (2.10) for all r in some set G^* of upper logarithmic density 1. (We note in this case (2.8) holds as $r \to \infty$ through values in G^* by (2.7) and the fact that $\mu(g) = \frac{1}{2}$.)

The proof proceeds from this point as before with only the trivial modification that logarithmic density is replaced by upper logarithmic density. It is also necessary to observe, before applying Lemma 6, that we may presume g to be of finite order. It follows from the elements of Nevanlinna theory that (1.1) cannot have nonconstant solutions of finite order if $\rho(g) = \infty$ and $\rho(h) < \infty$.

Added in proof. Recently Rossi [15] considered the differential equation

$$w'' + Aw = 0, \tag{5.1}$$

where A is entire of order $\rho(A) < 1$. If w_1 and w_2 are linearly independent solutions of (5.1) and $E = w_1 w_2$, then in [15] it is shown using harmonic measure estimates that the exponent of convergence $\lambda(E)$ of the zero set of E is infinite if $\rho(A) \leq \frac{1}{2}$. We indicate a second proof of this fact using the techniques of this paper.

By differentiating (5.1), it can be verified directly that E satisfies

$$E''' + 4AE' + 2A'E = 0. \tag{5.2}$$

(See also [11, Example 9, p. 395].) We suppose by way of contradiction that $\lambda(E) < \infty$. We first conclude the order of E is finite, since it follows from elementary considerations [15, equation (2.3)] that

$$T(r, E) = N(r, 1/E) + \tfrac{1}{2}T(r, A) + O(\log rT(r, E)), \qquad r \to \infty, \ r \notin G,$$

for some set G of finite measure.

Setting $F = E^4 A^2$, we observe that F has finite order and from (5.2) that

$$F'/F = -\frac{E'''/E}{A}. \tag{5.3}$$

A contradiction can now be obtained by repeating with no essential modification our argument in §4 with (4.4) replaced by (5.3).

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