ON COMPLETING UNIMODULAR POLYNOMIAL VECTORS OF LENGTH THREE

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Abstract. It is shown that if \( R \) is a local ring of dimension three, with \( \frac{1}{2} \in R \), then a polynomial three vector \((v_0(X), v_1(X), v_2(X))\) over \( R[X] \) can be completed to an invertible matrix if and only if it is unimodular. In particular, if \( \frac{1}{3!} \in R \), then every stably free projective \( R[X_1, \ldots, X_n] \)-module is free.

1. Introduction

In [6] A. Suslin queries A. Suslin's question \((S_r(R))\). Let \( R \) be a local ring. If \( \frac{1}{1!} \in R \), can every unimodular \((r + 1)\)-vector over \( R[X] \) be completed to an invertible matrix?

In this note we settle \( S_r(R) \) when \( R \) is a noetherian local ring of Krull dimension three.

Let us briefly recapitulate known results on \( S_r(R) \). Let \( R \) be a two dimensional noetherian local ring. A beautiful theorem of L. N. Vaserstein in [8] identifies the set \( \text{Um}_3(R[X])/E_3(R[X]) \) with the Elementary Symplectic Witt group \( W_E(R[X]) \). If \( \frac{1}{2} \in R \), a well-known theorem of M. Karoubi asserts that any invertible alternating matrix over a polynomial ring \( R[X] \) is stably congruent to its constant form. In particular, the Symplectic Witt group \( W(R[X]) \equiv 0 \). M. P. Murthy had remarked that these two facts could be used to prove that every \( v \in \text{Um}_3(R[X]) \) can be completed to an invertible matrix. We expanded on this theme of M. P. Murthy, in [3], to show that \( S_d(R) \) holds. Here we extend the methods in [3] to prove

Theorem. Let \( R \) be a noetherian, local ring of Krull dimension three with \( \frac{1}{2} \in R \). Then every unimodular 3-vector over \( R[X] \) can be completed to an invertible matrix.

The reader can also find some very interesting results on A. Suslin's question, due to M. Roitman, in positive prime characteristics in [5]. The present approach had its genesis in [2], (of course, with roots in Vaserstein theory developed in [8], and guided by M. P. Murthy's remark), where I could extend some of M. Roitman's results in dimensions \( \leq 4 \).

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2. Preliminary Remarks and Calculations

All rings $A$ considered in this article will be commutative with an identity element and noetherian. A vector $v = (v_0, v_1, \ldots, v_r) \in A^{r+1}$ is said to be unimodular if there is a vector $w = (w_0, w_1, \ldots, w_r) \in A^{r+1}$ such that $v_0w_0 + \cdots + v_rw_r = 1$. $\text{Um}_{r+1}(A)$ will denote the set of all unimodular vectors $v \in A^{r+1}$. The group $\text{Gl}_{r+1}(A)$ of invertible matrices acts on $A^{r+1}$ in a natural way: if $v \in A^{r+1}$, $\sigma \in \text{Gl}_{r+1}(A)$ then $\sigma$ will map $v$ to $v\sigma$. Under this action $\text{Um}_{r+1}(A)$ is mapped onto itself; and so $\text{Gl}_{r+1}(A)$ acts on $\text{Um}_{r+1}(A)$. We let $\sim$ denote equivalence of two vectors under this action. Let $E_{r+1}(A)$ denote the subgroup of $\text{Gl}_{r+1}(A)$ consisting of all the elementary matrices, i.e. those matrices which are a finite product of matrices of the form $E_{ij}(\lambda)$, $i \neq j$, $\lambda \in A$, which has all its diagonal entries one, has one off-diagonal entry in the $(i, j)$th position equal $\lambda$, and has all other entries zero. $v \sim w$ will denote that $v$ can be elementarily transformed to $w$. Let $\text{Um}_{r+1}(A)/E_{r+1}(A)$ be the set of equivalence classes of vectors $v$ under the equivalence $\sim$ induce by the action of $E_{r+1}(A)$ on $\text{Um}_{r+1}(A)$; and let $[v]$ denote the equivalence class of $v \in \text{Um}_{r+1}(A)$ in $\text{Um}_{r+1}(A)/E_{r+1}(A)$.

(2.1) W. Van der Kallen's group structure on $\text{Um}_{d+1}(A)/E_{d+1}(A)$. If $A$ is a ring whose maximal spectrum $\text{Max}(A)$ is a finite union of subsets $V_i$ where each $V_i$, when endowed with the (topology induced from the) Zariski topology is a space of Krull dimension $\leq d$ we shall say that $A$ is essentially of dimension $d$. For instance, a ring of Krull dimension $d$ is obviously essentially of dimension $\leq d$; a local ring of dimension $d$ is essentially of dimension 0; whereas a polynomial extension $R[X]$ of a local ring $R$ of dimension $d \geq 1$ has dimension $d+1$ but is essentially of dimension $d$ as $\text{Max}(R[X]) = \text{Max}(R/(a)[X]) \cup \text{Max}(R_a[X])$ for any non-zero-divisor $a \in R$.

In [9, Theorem 3.6], W. Van der Kallen has described how one could have an abelian group structure on $\text{Um}_{d+1}(A)/E_{d+1}(A)$. In the sequel we shall always refer to this group structure on $\text{Um}_{d+1}(A)/E_{d+1}(A)$; and let $\ast$ denote the group multiplication henceforth. One has

(2.1.1) Remark. Let $A$ be essentially of dimension $d \geq 2$, and let $C_{d+1}(A)$ denote the set of all completable $(d+1)$-vectors in $\text{Um}_{d+1}(A)$. Then,

(i) The map $\sigma \rightarrow [e_i \sigma]$, where $e_i = (1, 0, \ldots, 0) \in \text{Um}_{d+1}(A)$, is a group homomorphism $\text{Sl}_{d+1}(A) \rightarrow \text{Um}_{d+1}(A)/E_{d+1}(A)$.

(ii) $C_{d+1}(A)/E_{d+1}(A)$ is a subgroup of $\text{Um}_{d+1}(A)/E_{d+1}(A)$.

Proof. (i) follows from [9, Theorem 3.16(iv)]. Since any $v \in C_{d+1}(A)$ can be completed to a matrix of determinant one, $C_{d+1}(A)/E_{d+1}(A)$ is the image of $\text{Sl}_{d+1}(A)$ under the homomorphism mentioned in (i); whence it is a subgroup of $\text{Um}_{d+1}(A)/E_{d+1}(A)$.
(2.2) On A. Suslin’s procedure for completing \((a_0, a_1, \ldots, a_r)\). In [6, Proposition 1.6] A. Suslin shows that if \((a_0, a_1, \ldots, a_r) \in \text{Um}_{r+1}(A)\) then \((a_0, a_1, a_2, \ldots, a_r)\) can be completed. His proof, as observed by M. P. Murthy in [1, Chapter V, Proposition 1.2], actually demonstrates,

(2.2.1) **Proposition.** Let \((a_0, a_1, \ldots, a_r) \in \text{Um}_{r-1}(A)\). Suppose that \((\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{r+1})\) is completable in \(A = A/(a_r)\). Then \((a_0, a_1, \ldots, a_r')\) is completable.

As an application of this proposition we have

(2.2.2) **Proposition.** Let \(R\) be a local ring of dimension 3 with \(1/2 \in R\). Let \(v = (v_0, v_1, v_2, v_3) \in \text{Um}_4(R[X])\). Then \(v\) is completable if and only if \(v^{(2)} = (v_0^2, v_1, v_2, v_3)\) is completable.

**Proof.** By [3, Example 1.5.3 and Lemma 1.3.1],

\[[v^{(2)}] = [v] * [v]\]

in \(\text{Um}_4(R[X])/E_4(R[X])\). By Remark 2.1.1, \(v\) is completable implies that \(v^{(2)}\) is also completable.

Conversely, let \(v^{(2)}\) be completable. By [3, Proposition 1.4.4],

\[v \sim (w_0, w_1, w_2, c)\]

with \(c \in R\) a non-zero-divisor. As mentioned in the introduction (or cf. [3, Theorem 2.5]), since \(\dim R/(c) = 2\) and \(1/2 \in R\),

\[(\bar{w}_0, \bar{w}_1, \bar{w}_2) \in e_1S\ell_3(R/(c)[X]).\]

By Proposition 2.2.1, \((w_0, w_1, w_2, c)\) is completable. Thus,

(i) \((v_0, v_1, v_2, v_3) \sim (w_0, w_1, w_2, c)\) by [10, Theorem],

(ii) \([v]^n = [(v_0, v_1, v_2, v_3^n)]\) for all \(n\) by [3, Example 1.5.3 and Lemma 1.3.1].

Hence \([v]^2 = [v^{(2)}] \in C_4(R[X])/E_4(R[X]),\) and \([v]^3 = [(w_0, w_1, w_2, c^3)] \in C_4(R[X])/E_4(R[X])\) by Remark 2.1.1, \([v] \in C_4(R[X])/E_4(R[X]),\) i.e. \(v\) is completable.

(2.3) **The elementary symplectic Witt group** \(W_E(A)\). If \(\alpha \in M_r(A)\), \(\beta \in M_s(A)\) are matrices then \(\alpha \perp \beta\) denotes the matrix \(\left(\begin{smallmatrix} 0 & \beta \\ -\beta & 0 \end{smallmatrix}\right) \in M_{r+s}(A)\). \(\psi_1\) will denote \(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in E_2(\mathbb{Z}),\) and \(\psi_r\) is inductively defined by \(\psi_r = \psi_{r-1} \perp \psi_1 \in E_{2r}(\mathbb{Z}),\) for \(r \geq 2\).

A skew-symmetric matrix whose diagonal elements are zero is called an alternating matrix. If \(\varphi \in M_{2r}(A)\) is alternating then \(\det(\varphi) = (\text{pf}(\varphi))^2\) where \(\text{pf}\) is a polynomial (called the Pfaffian) in the matrix elements with coefficients \(\pm 1\). Note that we need to fix a sign in the choice of \(\text{pf}\); so insist \(\text{pf}(\psi_r) = 1\) for all \(r\). For any \(\alpha \in M_{2r}(A)\) and any alternating matrix \(\varphi \in M_{2r}(A)\) we have \(\text{pf}(\alpha' \varphi \alpha) = \text{pf}(\varphi) \det(\alpha)\). For alternating matrices \(\varphi, \psi\) it is easy to check that \(\text{pf}(\varphi \perp \psi) = (\text{pf}(\varphi))(\text{pf}(\psi)).\)
Two matrices $\alpha \in M_r(A)$, $\beta \in M_r(A)$ are said to be equivalent (w.r.t. $EA$) if there is a $\varepsilon \in E_{2(r+s+l)}(A)$, for some $l$, such that $\alpha \perp \psi_{s+l} = \varepsilon^t(\beta \perp \psi_{s+l})\varepsilon$, (the $t$ stands for ‘transpose’). Denote this by $\alpha \sim_{E} \beta$. $\sim_{E}$ is an equivalence relation; denote by $[\alpha]$ the orbit of $\alpha$ under this relation. Moreover, a matrix equivalent to an alternating matrix is itself alternating and has the same Pfaffian.

It is easy to see (cf. [8, p. 945]) that $\perp$ induces the structure of an abelian group on the set of all equivalence classes of alternating matrices with Pfaffian 1; this group is called the Elementary Symplectic Witt group and is denoted by $\text{WE}(A)$.

(2.4) M. Karoubi’s theorem and square roots in $\text{WE}(R[X])$. A famous theorem of M. Karoubi asserts that any invertible alternating matrix $V(X)$ over a polynomial ring $R[X]$ is stably congruent to its constant form if $1/2 \in R$, i.e. there is an $l$, and a $\sigma \in SL_{2}(R[X])$, for suitable $s$, such that $\sigma^t(V(X) \perp \psi_l)\sigma = V(0) \perp \psi_l$. The machination of M. Karoubi’s proof (cf. [8, §3]) gives

(2.4.1) Proposition. Let $R$ be a local ring with $1/2k \in R$, and let $[V] \in \text{WE}(R[X])$. Then $[V]$ has a $k$th root, i.e. there is a $[W] \in \text{WE}(R[X])$ such that $[V] = [W]^k$ in $\text{WE}(R[X])$.

Proof. Since $R$ is local $\text{WE}(R) = 0$, so we may assume that $V(0) = \psi_r$, for some $r$. Let me describe M. Karoubi’s process showing $V$ is stably congruent to $V(0)$; for details consult [8, §3]. The first step is to “stably make $V(X)$ linear” (known as the “Higman trick”)—i.e. find an $\varepsilon \in E_{2(r+l)}(R[X])$ such that

$$\varepsilon^t(V \perp \psi_l) = \psi_{r+l} + nX,$$

for some $t \geq 0$, some $n \in M_{2(r+l)}(R)$.

Since $\gamma = I_{r+l} - \psi_{r+l}nX \in SL_{2(r+l)}(R[X])$, $\psi_{r+l}n$ is nilpotent, i.e. $(\psi_{r+l}n)^l \equiv 0$ for some $l$. Hence, if $1/2k \in R$, we can extract a $k$th root of $\gamma$ ($= \beta^{2k}$, say) for some $\beta \in SL_{2(r+l)}(R[X])$. Now M. Karoubi pointed out that

$$\varepsilon^t(V \perp \psi_l)\varepsilon = \psi_{r+l}\gamma = \psi_{r+l}\beta^{2k} = (\beta^{k})^t\psi_{r+l}\beta^k.$$

Let $W = \beta^{t}\psi_{r+l}\beta$. Then applying Whitehead’s lemma one can check that $W \perp W \perp \cdots \perp W$ ($k$ times) $\sim_{E} V$, i.e. $[V] = [W]^k$ in $\text{WE}(R[X])$.

(2.5) The antipodal vectors equality in $\text{Um}_3(R[X])$ in small dimensions. In [3, Lemma 1.3.1] we showed that if a $v = (v_0, v_1, \ldots, v_d) \in \text{Um}_{d+1}(A)$, where $A$ is essentially of dimension $d$, can be elementarily transformed to (its antipodal vector) $-v = (-v_0, v_1, \ldots, -v_d)$ then for all $n$, $[v^n]$ in $\text{Um}_{d+1}(A)/E_{d+1}(A)$. There are many examples of vectors which cannot be elementarily transformed to their antipodal vector; but in [3, §1.5] we showed that if $A = R[X]$, $R$ a local ring of dimension 2 with $1/2 \in R$, then for any $v \in \text{Um}_3(R[X])$, $v \sim_{E} -v$. Here, by a a different argument, we show that
(2.5.1) **Proposition.** Let \( R \) be a local ring of dimension \( \leq 4 \) with \( 1/2 \in R \) and let \( v = (v_0, v_1, v_2) \in \text{Um}_3(R[X]) \). Then \( v = (v_0, v_1, v_2) \sim (-v_0, -v_1, -v_2) \) \( = -v \).

**Proof.** Choose a \( w = (w_0, w_1, w_2) \) such that \( v_0w_0 + v_1w_2 + v_2w_2 = 1 \), and consider the alternating matrix \( V \) with Pfaffian 1 given by

\[
V(v, w) = \begin{pmatrix}
0 & v_0 & v_1 & v_2 \\
-v_0 & 0 & w_2 & -w_1 \\
v_1 & -w_2 & 0 & w_0 \\
v_2 & w_1 & -w_0 & 0
\end{pmatrix} \in \text{Sl}_4(R[X]).
\]

Since \( 1/2 \in R \), by M. Karoubi's theorem (cf. §2.4) there is a \( \beta \in \text{Sl}_4(R[X]) \), for some \( \ell \), such that \( \beta^i(V \downarrow_{\psi_i})\beta = \psi_{i+2} \). Since \( \dim R \leq 4 \), by [7, Theorem 2.6], \( \text{Um}_r(R[X]) = e_1E_r(R[X]) \) for all \( r \geq 6 \). Hence on applying [8, Lemma 5.5 and Lemma 5.6] we can find a \( \beta^* \in \text{Sl}_4(R[X]) \) such that \( \beta^*V\beta^* = \psi_2 \).

Let \( \delta = \text{diagonal} \ (-1, 1, -1, 1) \in E_4(R) \). Then \( \delta^i\psi_2\delta = -\psi_2 \). Thus

\[
\delta^i(\beta^*)^iV\beta^*\delta = \delta^i\psi_2\delta = -\psi_2 = \psi_2^i = [(\beta^*)^iV\beta^*]^i = (\beta^*)^iV^i\beta^*,
\]

and so if \( \sigma = (\beta^*)^i \) then \((\sigma^{-1}\delta^i\sigma)V(\sigma^{-1}\delta^i\sigma)^i = -V\).

By [7, Corollary 1.4] \( \sigma^{-1}\delta^i\sigma \in E_4(R[X]) \). Now the equation (*) will prove the proposition on applying [11, Theorem 10].

(2.5.2) **Remark.** The above argument can be suitably modified to show that if \( [V] \in \text{WE}(R[X]) \), where \( R \) is a local ring with \( 1/2 \in R \), then \( [V] = [-V] \) in \( \text{WE}(R[X]) \).

(2.6) **"Coordinate squares"** in \( \text{WE}(R[X]) \). Let us say that an invertible alternating matrix \( V \) is a "coordinate kth power" if the first row of \( V \) has the form \((0, v_1, v_2, \ldots, v_{2r-1})\). It would be of interest to know if, under congenial conditions, the above fact, proven in Proposition 2.4.1, that every \( [V] \in \text{WE}(R[X]) \) is a kth power in \( \text{WE}(R[X]) \) (under suitable hypothesis on \( R \)) can be translated to read that \( [V] \) has a representative \( V^* \) which is a coordinate kth power and which, moreover, has the same size as that of \( V \). We give some evidence for this here.

Firstly recall some multiplicative relations in \( \text{WE}(A) \) observed by L. N. Vaserstein in [8, Theorem 5.2(a_2)].

(2.6.1) **The Vaserstein Rule.** Let \( v_1 = (a_0, a_1, a_2) \), \( v_2 = (a_0, b_1, b_2) \) be unimodular vectors. Suppose that \( a_0a_0' + a_1a_1' + a_2a_2' = 1 \), and that

\[
v_3 = (a_0, (b_1, b_2)(\frac{a_1}{a_1'}, \frac{a_2}{a_2'})) \in \text{Um}_3(A).
\]

Then for any \( w_1, w_2, w_3 \) such that \( v_i \cdot w_i = 1 \), \( i = 1, 2, 3 \), we have

\[
[V(v_i, w_i)] \perp [V(v_2, w_2)] = [V(v_3, w_3)] \text{ in } \text{WE}(A).
\]

(Note. \( V(v, w) \) is defined in Proposition 2.5.1, and \( [V(v, w)] \) is well-defined in \( \text{WE}(A) \) via [8, Lemma 5.1].)
(2.6.2) **Corollary.** (i) Let \( v_1 = (a_0, a_1, a_2), \ v_2 = (b_0, a_1, a_2) \) be unimodular vectors. Suppose that \( a_0a'_0 + a_1a'_1 + a_2a'_2 = 1 \) and that \( v_3 = (a_0(b_0 + a'_0) - 1, (b_0 + a'_0)a_1, a_2) \in \text{Um}_3(A) \). Then for any \( w_1, w_2, w_3 \) such that \( v_i \cdot w_i = 1, \ i = 1, 2, 3 \), we have

\[
[V(v_1, w_1)] \perp [V(v_2, w_2)] = [V(v_3, w_3)] \quad \text{in } W_E(A).
\]

(ii) Let \( v_1 = (a_0, a_1, a_2), \ v_2 = (b_0^2, a_1, a_2) \) be unimodular vectors. Suppose that \( v_3 = (a_0b_0^2, a_1, a_2) \) and that \( w_1, w_2, w_3 \) are such that \( v_iw_i = 1, \ i = 1, 2, 3 \), then

\[
[V(v_1, w_1)] \perp [V(v_2, w_2)] = [V(v_3, w_3)] \quad \text{in } W_E(A).
\]

**Proof.** (i) is immediate from the Vaserstein Rule. We refer the reader to [9, Theorem 3.16(iii)] for deriving (ii) from (i). Note: You may need the Roitman lemma in [5, Lemma 1].

(2.6.3) **The “antipodal vectors equality” lemma in \( W_E(A) \).** Let \( v = (v_0, v_1, v_2) \) be a unimodular vector and assume that \( v \sim -v \sim (v_0, -v_1, -v_2) \). Let \( v_1^{(2)} = (v_0^{(2)}, v_1, v_2) \) and let \( w, w_1 \) be such that \( v \cdot w = v \cdot w_1 = 1 \). Then

\[
[V(v, w)]^2 = [V(v^{(2)}, w_1)] \quad \text{in } W_E(A).
\]

**Proof.** Imitate the argument in [3, Lemma 1.3.1] in \( W_E(A) \). (Note. You will need Corollary 2.6.2(ii) above.)

Finally, we give some conditions under which we can extract “coordinate squares” in \( W_E(R[X]) \):

(2.6.4) **Corollary.** Let \( R \) be a local ring of dimension \( \leq 4 \) with \( 1/2 \in R \) and let \( v = (v_0, v_1, v_2), \ v^{(2)} = (v_0^{(2)}, v_1, v_2) \) be unimodular \( R[X] \)-vectors. Let \( w, w_1 \) such that \( v \cdot w = v^{(2)} \cdot w_1 = 1 \). Then,

\[
[V(v, w)]^2 = [V(v^{(2)}, w_1)] \quad \text{in } W_E(R[X]).
\]

**Proof.** This will follow from Proposition 2.5.1 and Lemma 2.6.3.

(2.6.5) **Proposition.** Let \( R \) be a local ring of dimension \( \leq 3 \) with \( 1/2 \in R \) and let \( V \in Sl_4(R[X]) \) be an alternating matrix with Pfaffian 1. Then \( [V] = [V^*] \) in \( W_E(R[X]) \) with \( V^* \in Sl_4(R[X]) \) a coordinate square. Consequently, there is a stably elementary \( \gamma \in Sl_4(R[X]) \) such that \( V = \gamma V^* \gamma \).

**Proof.** By Proposition 2.4.1, \( [V] = [W]^2 \) for some \( [W] \in W_E(R[X]) \). By [7, Theorem 2.6] \( \text{Um}_r(R[X]) = e_1E_1(R[X]) \) for all \( r \geq 5 \), and so on applying [8, Lemma 5.3 and Lemma 5.5] a few times, if necessary, we can find an alternating matrix \( W^* \in Sl_4(R[X]) \) (with Pfaffian 1) such that \( [W] = [W^*] \). Now apply Corollary 2.6.4 to find \( V^* \) as required. The last statement follows as above (only applying [8, Lemma 5.5 and Lemma 5.6] instead).
3. The main theorem

(3.1) **Theorem.** Let $R$ be a local ring of Krull dimension three with $1/2 \in R$ and let $v = (v_0, v_1, v_2)$ be a unimodular 3-vector over $R[X]$. Then $v$ can be completed to an invertible matrix.

**Proof.** Choose a $w = (w_0, w_1, w_2)$ such that $v_0w_0 + v_1w_1 + v_2w_2 = 1$, and consider the alternating matrix $V$ with Pfaffian 1 given by

$$V = \begin{pmatrix}
0 & v_0 & v_1 & v_2 \\
-v_0 & 0 & w_2 & -w_1 \\
-v_1 & -w_2 & 0 & w_0 \\
-v_2 & w_1 & -w_0 & 0
\end{pmatrix} \in SL_4(R[X]).$$

Since $1/2 \in R$, by M. Karoubi's theorem (see (*) in Proposition 2.4.1) there is a $\alpha \in SL_4(R[X])$, for some $l$, such that $\alpha'V\alpha = \psi_2$.

Since $\dim R = 3$, by [7, Theorem 2.6] $Um_3(R[X]) = e_4E_4(R[X])$ for all $r \geq 6$. Hence on applying [8, Lemma 5.5 and Lemma 5.6] we can find an $\alpha \in SL_4(R[X])$ such that $\alpha'V\alpha = \psi_2$. Consider $e_4\alpha'$, where $e_4 = (0, 0, 0, 1)$.

By [3, Proposition 1.4.4] $e_4\alpha' \sim (a_0(X), a_1(X), a_2(X), c)$, where $c \in R$ is a non-zero-divisor in $R$. Let the ‘overbar’ denote ‘modulo (c)’. By [3, Proposition 2.2], $(a_0(X), a_1(X), a_2(X)) \sim (b_0(X)^2, b_1(X), b_2(X))$, for some $b_0(X), b_1(X), b_2(X) \in R[X]$. On “lifting” this elementary map, and after an appropriate elementary transformation further, we can arrange that $e_4\alpha' \sim (b_0(X)^2, b_1(X), b_2(X), c)$.

By Proposition 2.2.2, $(b_0(X), b_1(X), b_2(X), c)$ can be completed to an invertible matrix, say $\beta \in SL_4(R[X])$ with $e_4\beta = (b_0(X), b_1(X), b_2(X), c)$.

Via Remark 1.1.1 follows that $e_4\beta^{-2}\alpha' = [e_4\beta^{-2}] * [e_4\alpha'] = [e_4\beta^{-2}] * [e_4\alpha']$

$$= (\{(b_0(X), b_1(X), b_2(X), c)\})^2 * [e_4\alpha'] = [e_4\alpha']^{-1} * [e_4\alpha'] \equiv 1,$$
the last equality being deduced via [3, Example 1.5.3 and Lemma 1.3.1]. Thus, $\beta^{-2}\alpha' = \epsilon' \delta'$ for some $\epsilon' \in E_4(R[X])$ and $\delta' = \left(\begin{smallmatrix} 1 & \delta \end{smallmatrix}\right)$ with $\delta \in SL_3(R[X]).$

Now $\psi_2 = \alpha'V\alpha = (\beta^2 \epsilon' \delta')V(\beta^2 \epsilon' \delta') = \beta^2 V^*(\beta^2)^{-1}$, where $e_1V^* = (0, v \delta \epsilon)$ for some $\epsilon \in E_3(R[X])$—this will follow as $\delta' = \left(\begin{smallmatrix} 1 & \delta \end{smallmatrix}\right)$ and via [11, Theorem 10].

By Proposition 2.6.5 there is a stably elementary $\gamma \in SL_4(R[X])$ such that $\beta V^* \beta^t = \gamma' V^* \gamma$, with $V^* \in SL_4(R[X])$ a coordinate square. Let $e_1V^* = (0, a^2, b, c)$, and let $\alpha_0$ (cf. §2.2) be a completion of $(a^2, b, c)$.

Since

$$c_1V^* = e_1 \left(\begin{array}{c}
1 \\
0 \\
a_0 \\
0
\end{array}\right)^t \psi_2 \left(\begin{array}{c}
1 \\
0 \\
a_0 \\
0
\end{array}\right)$$

it follows via [8, Lemma 5.1] that

$$V^* = e_1 \left(\begin{array}{c}
1 \\
0 \\
a_0 \\
0
\end{array}\right)^t \psi_2 \left(\begin{array}{c}
1 \\
0 \\
a_0 \\
0
\end{array}\right) e_1$$
for some $e_1 \in E_4(R[X])$. Thus,

$$
\beta V^* \beta^t = \gamma^t V^{**} \gamma = \gamma^t e_1 \left( \begin{array}{cc} 1 & 0 \\ 0 & \alpha_0 \end{array} \right)^t \psi_2 \left( \begin{array}{cc} 1 & 0 \\ 0 & \alpha_0 \end{array} \right) e_1 \gamma.
$$

Hence,

$$
\beta^{-1} \left[ \left( \begin{array}{cc} 1 & 0 \\ 0 & \alpha_0 \end{array} \right)^t \left( e_1^{-1} \right)^t \left( \gamma^{-1} \right)^t \right] \beta V^* \beta^t \left[ \gamma^{-1} e_1^{-1} \left( \begin{array}{cc} 1 & 0 \\ 0 & \alpha_0 \end{array} \right) \right] (\beta^{-1})^t
$$

$$
= \beta^{-1} \psi_2 (\beta^{-1})^t = \beta^{-1} (\beta^2 V^* (\beta^2)^t) (\beta^{-1})^t = \beta V^* \beta^t = \gamma^t V^{**} \gamma;
$$

and so if

$$
\theta = \beta^t \gamma^{-1} e_1^{-1} \left( \begin{array}{cc} 1 & 0 \\ 0 & \alpha_1 \end{array} \right) (\beta^t)^{-1} \gamma^{-1},
$$

then $\theta^t V^* \theta = V^{**}$.

Compute $e_4 \theta^t$ in the abelian group $U_{p_4}(R[X])/E_4(R[X])$ via Remark 2.1.1 to get $[e_4 \theta^t] = [e_4 (\gamma')^{-1}]^2$. But $\gamma$ is stably elementary and so via [3, Proposition 2.6] $[e_4 (\gamma')^{-1}]^2 = 1$; hence $[e_4 \theta^t] = 1$, i.e. $e_4 \theta^t \sim e_4$. Hence

$$
\theta^t e' = \begin{pmatrix} 1 & 0 \\ 0 & (\theta')^t \end{pmatrix}
$$

for some $\theta' \in Sl_3(R[X])$, $e' \in E_4(R[X])$.

Now

$$
\theta^t V^* \theta = \begin{pmatrix} 1 & 0 \\ 0 & (\theta')^t \end{pmatrix} (\varepsilon')^{-1} V^* ((\varepsilon')^{-1})^t \begin{pmatrix} 1 & 0 \\ 0 & \theta' \end{pmatrix} = V^{**},
$$

and so via [11, Theorem 10] we can deduce that there is an $e'' \in E_3(R[X])$ such that $ve'' \theta' = (a^2, b, c)$. Since $(a^2, b, c)$ is completable, so is $v$.

**Remark.** Let us, following M. Krusemeyer, say that a vector $v \in U_{p_4}(A)$ is skew-completable if there is an invertible alternating matrix $V \in Sl_{r+1}(A)$ with its first row $e_1 V = (0, v)$. By making some appropriate modifications in the argument used to prove Theorem 3.1 one can show that,

(3.2) **Theorem.** Let $R$ be a local ring of Krull dimension $d$ with $1/2 \in R$, and let $v = (v_0, v_1, \ldots, v_{d-1})$ be a skew-completable vector over $R[X]$. Then $v$ can be completed to an invertible matrix.

Finally, using the well-known "Quillen-Suslin" Monic inversion and Local-Global principles, one can derive from $S_q(R)$ and Theorem 3.1 that,

(3.3) **Corollary.** Let $R$ be a noetherian ring of dimension 3 with $1/6 \in R$. Then any stably extended projective module over $R[X_1, \ldots, X_n]$ is extended.

**Note added in proof.** The contents (especially the mode of proof of the main result) of this note seems of interest in connection with the following problem:

(i) Let $V : U_{p_4}(A)/E_3(A) \rightarrow W_4(A)$ be the Vaserstein symbol. Is this map injective if $\dim A = 3$?
I also hope that, after incorporation of some additional theories, the techniques used here will provide some insight towards settling,

(a) Let $R$ be a local ring with $\frac{1}{r} \in R$. Is every $v \in \text{Um}_3(R[X])$ completable?

(b) Let $A$ be a smooth affine algebra over the field $\mathbb{C}$ of complex numbers of dimension $d$. Is a stably free $A$-module of rank $(d - 1)$ a free module?

In an article entitled On some actions of stably elementary matrices on alternating matrices we prove that

"Let $A$ have Krull dimension $\leq 5$, and let $V \in \text{Sl}_4(A) \cap E_5(A)$ be a stably elementary alternating matrix of Pfaffian one. Then $V^8 \in E_4(A)$.

Note. One needs to show that $V \in E_4(A)$ to settle (i) above.

We also give some examples of 3 dimensional affine algebras for which the Vaserstein symbol $V$ is bijective.

REFERENCES


7. _____, On the structure of the special linear group over polynomial rings, Math. USSR-Izv. 11 (1977), 221–238.


