ON COMPLETING UNIMODULAR POLYNOMIAL VECTORS
OF LENGTH THREE

RAVI A. rAO

Abstract. It is shown that if \( R \) is a local ring of dimension three, with \( \frac{1}{3} \in R \), then a polynomial three vector \( (v_0(X), v_1(X), v_2(X)) \) over \( R[X] \) can be completed to an invertible matrix if and only if it is unimodular. In particular, if \( \frac{1}{3!} \in R \), then every stably free projective \( R[X_1, \ldots, X_n] \)-module is free.

1. INTRODUCTION

In [6] A. Suslin queries

A. Suslin's question \( (S_r(R)) \). Let \( R \) be a local ring. If \( \frac{1}{r!} \in R \), can every unimodular \( (r + 1) \)-vector over \( R[X] \) be completed to an invertible matrix?

In this note we settle \( S_r(R) \) when \( R \) is a noetherian local ring of Krull dimension three.

Let us briefly recapitulate known results on \( S_r(R) \). Let \( R \) be a two dimensional noetherian local ring. A beautiful theorem of L. N. Vaserstein in [8] identifies the set \( \text{Um}_3(R[X])/\text{E}_3(R[X]) \) with the Elementary Symplectic Witt group \( \text{WE}(R[X]) \). If \( \frac{1}{2} \in R \), a well-known theorem of M. Karoubi asserts that any invertible alternating matrix over a polynomial ring \( R[X] \) is stably congruent to its constant form. In particular, the Symplectic Witt group \( \text{W}(R[X]) \equiv 0 \).

M. P. Murthy had remarked that these two facts could be used to prove that every \( v \in \text{Um}_3(R[X]) \) can be completed to an invertible matrix. We expanded on this theme of M. P. Murthy, in [3], to show that \( S_d(R) \) holds. Here we extend the methods in [3] to prove

Theorem. Let \( R \) be a noetherian, local ring of Krull dimension three with \( \frac{1}{2} \in R \). Then every unimodular 3-vector over \( R[X] \) can be completed to an invertible matrix.

The reader can also find some very interesting results on A. Suslin's question, due to M. Roitman, in positive prime characteristics in [5]. The present approach had its genesis in [2], (of course, with roots in Vaserstein theory developed in [8], and guided by M. P. Murthy's remark), where I could extend some of M. Roitman's results in dimensions \( \leq 4 \).

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2. Preliminary remarks and calculations

All rings $A$ considered in this article will be commutative with an identity element and noetherian. A vector $v = (v_0, v_1, \ldots, v_r) \in A^{r+1}$ is said to be unimodular if there is a vector $w = (w_0, w_1, \ldots, w_r) \in A^{r+1}$ such that $v_0w_0 + \cdots + v_rw_r = 1$. $\text{Um}_{r+1}(A)$ will denote the set of all unimodular vectors $v \in A^{r+1}$. The group $GL_{r+1}(A)$ of invertible matrices acts on $A^{r+1}$ in a natural way: if $v \in A^{r+1}$, $\sigma \in GL_{r+1}(A)$ then $\sigma$ will map $v$ to $v\sigma$. Under this action $\text{Um}_{r+1}(A)$ is mapped onto itself; and so $GL_{r+1}(A)$ acts on $\text{Um}_{r+1}(A)$. We let $\sim$ denote equivalence of two vectors under this action. Let $E_{r+1}(A)$ denote the subgroup of $GL_{r+1}(A)$ consisting of all the elementary matrices, i.e. those matrices which are a finite product of matrices of the form $E_{ij}(\lambda)$, $i \neq j$, $\lambda \in A$, which has all its diagonal entries one, has one off-diagonal entry in the $(i, j)$th position equal $\lambda$, and has all other entries zero. $v \sim w$ will denote that $v$ can be elementarily transformed to $w$. Let $\text{Um}_{r+1}(A)/E_{r+1}(A)$ be the set of equivalence classes of vectors $v$ under the equivalence $\sim$ induce by the action of $E_{r+1}(A)$ on $\text{Um}_{r+1}(A)$; and let $[v]$ denote the equivalence class of $v \in \text{Um}_{r+1}(A)$ in $\text{Um}_{r+1}(A)/E_{r+1}(A)$.

(2.1) W. Van der Kallen's group structure on $\text{Um}_{d+1}(A)/E_{d+1}(A)$. If $A$ is a ring whose maximal spectrum $\text{Max}(A)$ is a finite union of subsets $V_i$ where each $V_i$, when endowed with the (topology induced from the) Zariski topology is a space of Krull dimension $\leq d$ we shall say that $A$ is essentially of dimension $d$. For instance, a ring of Krull dimension $d$ is obviously essentially of dimension $\leq d$; a local ring of dimension $d$ is essentially of dimension $0$; whereas a polynomial extension $R[X]$ of a local ring $R$ of dimension $d \geq 1$ has dimension $d + 1$ but is essentially of dimension $d$ as $\text{Max}(R[X]) = \text{Max}(R/(a)[X]) \cup \text{Max}(R_a[X])$ for any non-zero-divisor $a \in R$.

In [9, Theorem 3.6], W. Van der Kallen has described how one could have an abelian group structure on $\text{Um}_{d+1}(A)/E_{d+1}(A)$. In the sequel we shall always refer to this group structure on $\text{Um}_{d+1}(A)/E_{d+1}(A)$; and let $*$ denote the group multiplication henceforth. One has

(2.1.1) Remark. Let $A$ be essentially of dimension $d \geq 2$, and let $C_{d+1}(A)$ denote the set of all completable $(d + 1)$-vectors in $\text{Um}_{d+1}(A)$. Then,

(i) The map $\sigma \rightarrow [e_i\sigma]$, where $e_i = (1, 0, \ldots, 0) \in \text{Um}_{d+1}(A)$, is a group homomorphism $SL_{d+1}(A) \rightarrow \text{Um}_{d+1}(A)/E_{d+1}(A)$.

(ii) $C_{d+1}(A)/E_{d+1}(A)$ is a subgroup of $\text{Um}_{d+1}(A)/E_{d+1}(A)$.

Proof. (i) follows from [9, Theorem 3.16(iv)]. Since any $v \in C_{d+1}(A)$ can be completed to a matrix of determinant one, $C_{d+1}(A)/E_{d+1}(A)$ is the image of $SL_{d+1}(A)$ under the homomorphism mentioned in (i); whence it is a subgroup of $\text{Um}_{d+1}(A)/E_{d+1}(A)$. 
(2.2) On A. Suslin’s procedure for completing \((a_0, a_1, a_2, \ldots, a_r)\). In [6, Proposition 1.6] A. Suslin shows that if \((a_0, a_1, \ldots, a_r) \in \text{Um}_{r+1}(A)\) then \((a_0, a_1, a_2^2, \ldots, a_r^2)\) can be completed. His proof, as observed by M. P. Murthy in [1, Chapter V, Proposition 1.2], actually demonstrates,

(2.2.1) Proposition. Let \((a_0, a_1, \ldots, a_r) \in \text{Um}_{r-1}(A)\). Suppose that \((\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_{r+1})\) is completable in \(\overline{A} = A/(a_r)\). Then \((a_0, a_1, \ldots, a_r')\) is completable.

As an application of this proposition we have

(2.2.2) Proposition. Let \(R\) be a local ring of dimension 3 with \(1/2 \in R\). Let \(v = (v_0, v_1, v_2, v_3) \in \text{Um}_4(R[X])\). Then \(v\) is completable if and only if \(v^{(2)} = (v_0^2, v_1, v_2, v_3)\) is completable.

Proof. By [3, Example 1.5.3 and Lemma 1.3.1],

\[ [v^{(2)}] = [v] \ast [v] \]

in \(\text{Um}_4(R[X])/E_4(R[X])\). By Remark 2.1.1, \(v\) is completable implies that \(v^{(2)}\) is also completable.

Conversely, let \(v^{(2)}\) be completable. By [3, Proposition 1.4.4],

\[ v \sim (w_0, w_1, w_2, c) \]

with \(c \in R\) a non-zero-divisor. As mentioned in the introduction (or cf. [3, Theorem 2.5]), since \(\dim R/(c) = 2\) and \(1/2 \in R\),

\[ (w_0, w_1, w_2) \in e_1S_3(R/(c)[X]). \]

By Proposition 2.2.1, \((w_0, w_1, w_2, c^3)\) is completable. Thus,

(i) \(v_0, v_1, v_2, v_3 \sim (w_0, w_1, w_2, c^3)\) by [10, Theorem],

(ii) \([v]^n = [(v_0, v_1, v_2, v_3^2)]\) for all \(n\) by [3, Example 1.5.3 and Lemma 1.3.1].

Hence \([v]^2 = [v^{(2)}] \in C_4(R[X])/E_4(R[X])\), and \([v]^3 = [(w_0, w_1, w_2, c^3)] \in C_4(R[X])/E_4(R[X])\). By Remark 2.1.1, \([v] \in C_4(R[X])/E_4(R[X])\), i.e. \(v\) is completable.

(2.3) The elementary symplectic Witt group \(W_{E}(A)\). If \(\alpha \in M_r(A)\), \(\beta \in M_s(A)\) are matrices then \(\alpha \perp \beta\) denotes the matrix \((\begin{smallmatrix} 0 & 0 \\ 0 & \alpha \end{smallmatrix}) \in M_{r+s}(A)\). \(\psi_1\) will denote \((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) \in E_2(Z)\), and \(\psi_r\) is inductively defined by \(\psi_r = \psi_{r-1} \perp \psi_1 \in E_{2r}(Z)\), for \(r \geq 2\).

A skew-symmetric matrix whose diagonal elements are zero is called an alternating matrix. If \(\varphi \in M_{2r}(A)\) is alternating then \(\det(\varphi) = (\text{pf}(\varphi))^2\) where \(\text{pf}\) is a polynomial (called the Pfaffian) in the matrix elements with coefficients \(\pm 1\). Note that we need to fix a sign in the choice of \(\text{pf}\); so insist \(\text{pf}(\psi_r) = 1\) for all \(r\). For any \(\alpha \in M_{2r}(A)\) and any alternating matrix \(\varphi \in M_{2r}(A)\) we have \(\text{pf}(\alpha' \varphi \alpha) = \text{pf}(\varphi) \det(\alpha)\). For alternating matrices \(\varphi, \psi\) it is easy to check that \(\text{pf}(\varphi \perp \psi) = (\text{pf}(\varphi))(\text{pf}(\psi))\).
Two matrices \( \alpha \in M_r(A) \), \( \beta \in M_s(A) \) are said to be equivalent (w.r.t. \( EA \)) if there is a \( \epsilon \in E_{2(r+s+l)}(A) \), for some \( l \), such that \( \alpha \perp \psi_{s+l} = \epsilon^t(\beta \perp \psi_{s+l})\epsilon \), (the \( t \) stands for ‘transpose’). Denote this by \( \alpha \sim \beta \). \( \sim \) is an equivalence relation; denote by \( [\alpha] \) the orbit of \( \alpha \) under this relation. Moreover, a matrix equivalent to an alternating matrix is itself alternating and has the same Pfaffian.

It is easy to see (cf. [8, p. 945]) that \( \perp \) induces the structure of an abelian group on the set of all equivalence classes of alternating matrices with Pfaffian 1; this group is called the Elementary Symplectic Witt group and is denoted by \( W_E(A) \).

(2.4) **M. Karoubi’s theorem and square roots in** \( W_E(R[X]) \). A famous theorem of M. Karoubi asserts that any invertible alternating matrix \( V(X) \) over a polynomial ring \( R[X] \) is stably congruent to its constant form if \( 1/2 \in R \), i.e. there is an \( l \), and a \( \sigma \in Sl_2(R[X]) \), for suitable \( s \), such that \( \sigma^t(V(X) \perp \psi_l)\sigma = V(0) \perp \psi_l \). The machination of M. Karoubi’s proof (cf. [8, §3]) gives

\[ (*) \quad \epsilon^t(\psi_{s+l})\epsilon = \psi_{s+l}^2 = (\beta^2)^{t}\psi_{s+l}^2. \]

Let \( W = \beta^t\psi_{s+l}^2 \). Then applying Whitehead’s lemma one can check that \( W \perp W \perp \cdots \perp W \) \( (k \times) \sim V \), i.e. \( [V] = [W]^k \in W_E(R[X]) \).

(2.5) **The antipodal vectors equality in** \( Um_3(R[X]) \) **in small dimensions.** In [3, Lemma 1.3.1] we showed that if a \( v = (v_0, v_1, \ldots, v_d) \in Um_{d+1}(A) \), where \( A \) is essentially of dimension \( d \), can be elementarily transformed to (its antipodal vector) \(-v = (-v_0, v_1, \ldots, v_d) \) then for all \( n \), \([v^n] = [v] \in Um_{d+1}(A)/E_{d+1}(A) \). There are many examples of vectors which cannot be elementarily transformed to their antipodal vector; but in [3, §1.5] we showed that if \( A = R[X] \), \( R \) a local ring of dimension 2 with \( 1/2 \in R \), then for any \( v \in Um_3(R[X]) \), \( v \sim -v \). Here, by a a different argument, we show that
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(2.5.1) **Proposition.** Let $R$ be a local ring of dimension $\leq 4$ with $1/2 \in R$ and let $v = (v_0, v_1, v_2) \in \text{Um}_3(R[X]).$ Then $v = (v_0, v_1, v_2) \sim (-v_0, -v_1, -v_2) = -v.$

**Proof.** Choose a $w = (w_0, w_1, w_2)$ such that $v_0w_0 + v_1w_2 + v_2w_1 = 1,$ and consider the alternating matrix $V$ with Pfaffian 1 given by

$$V(v, w) = \begin{pmatrix} 0 & v_0 & v_1 & v_2 \\ -v_0 & 0 & w_2 & -w_1 \\ -v_1 & -w_2 & 0 & w_0 \\ -v_2 & w_1 & -w_0 & 0 \end{pmatrix} \in SL_4(R[X]).$$

Since $1/2 \in R,$ by M. Karoubi's theorem (cf. §2.4) there is a $\beta \in SL_4(R[X]),$ for some $l,$ such that $\beta^t(V \downarrow \psi_l)\beta = \psi_{l+2}.$ Since $\text{dim } R \leq 4,$ by [7, Theorem 2.6], $\text{Um}_r(R[X]) = e_1 E_r(R[X])$ for all $r \geq 6.$ Hence on applying [8, Lemma 5.5 and Lemma 5.6] we can find a $\beta^* \in SL_4(R[X])$ such that $\beta^* V \beta^* = \psi_2.$

Let $\delta = \text{diagonal } (-1, 1, -1, 1) \in E_4(R).$ Then $\delta^t \psi_2 \delta = -\psi_2.$ Thus

$$\delta^t (\beta^*)^t V \delta = \delta^t \psi_2 \delta = -\psi_2 = \psi_2^t = [(\beta^*)^t V \beta^*]^t = (\beta^*)^t V^t \beta^*,$$

and so if $\sigma = (\beta^*)^t$ then $(\sigma^{-1} \delta^t \sigma) V (\sigma^{-1} \delta^t \sigma)^t = -V.$

By [7, Corollary 1.4] $\sigma^{-1} \delta^t \sigma \in E_4(R[X]).$ Now the equation (*) will prove the proposition on applying [11, Theorem 10].

(2.5.2) **Remark.** The above argument can be suitably modified to show that if $[V] \in W_E(R[X]),$ where $R$ is a local ring with $1/2 \in R,$ then $[V] = [-V]$ in $W_E(R[X]).$

(2.6) "Coordinate squares" in $W_E(R[X]).$ Let us say that an invertible alternating matrix $V$ is a "coordinate $k$th power" if the first row of $V$ has the form $(0, v_1^k, v_2, \ldots, v_{2r-1}).$ It would be of interest to know if, under congenial conditions, the above fact, proven in Proposition 2.4.1, that every $[V] \in W_E(R[X])$ is a $k$th power in $W_E(R[X])$ (under suitable hypothesis on $R$) can be translated to read that $[V]$ has a representative $V^*_*$ which is a coordinate $k$th power and which, moreover, has the same size as that of $V.$ We give some evidence for this here.

Firstly recall some multiplicative relations in $W_E(A)$ observed by L. N. Vaserstein in [8, Theorem 5.2(a$_2$)].

(2.6.1) **The Vaserstein Rule.** Let $v_1 = (a_0, a_1, a_2), \ v_2 = (a_0, b_1, b_2)$ be unimodular vectors. Suppose that $a_0 a'_1 + a_1 a'_2 + a_2 a'_2 = 1,$ and that

$$v_3 = (a_0, (b_1, b_2)(a'_1, a'_2)) \in \text{Um}_3(A).$$

Then for any $w_1, w_2, w_3$ such that $v_i \cdot w_i = 1,$ $i = 1, 2, 3,$ we have

$$[V(v_1, w_1)] \perp [V(v_2, w_2)] = [V(v_3, w_3)] \text{ in } W_E(A).$$

(Note. $V(v, w)$ is defined in Proposition 2.5.1, and $[V(v, w)]$ is well defined in $W_E(A)$ via [8, Lemma 5.1].)
(2.6.2) Corollary. (i) Let \( v_1 = (a_0, a_1, a_2) \), \( v_2 = (b_0, a_1, a_2) \) be unimodular vectors. Suppose that \( a_0a'_0 + a_1a'_1 + a_2a'_2 = 1 \) and that \( v_3 = (a_0(b_0 + a'_0) - 1, (b_0 + a'_0)a_1, a_2) \in \text{Um}_3(A) \). Then for any \( w_1, w_2, w_3 \) such that \( v_i \cdot w_i = 1, i = 1, 2, 3 \), we have

\[
[V(v_1, w_1)] \perp [V(v_2, w_2)] = [V(v_3, w_3)] \quad \text{in} \quad \mathcal{W}_E(A).
\]

(ii) Let \( v_1 = (a_0, a_1, a_2) \), \( v_2 = (b_0^2, a_1, a_2) \) be unimodular vectors. Suppose that \( v_3 = (a_0b_0, a_1, a_2) \) and that \( w_1, w_2, w_3 \) are such that \( v_iw_i = 1, i = 1, 2, 3 \), then

\[
[V(v_1, w_1)] \perp [V(v_2, w_2)] = [V(v_3, w_3)] \quad \text{in} \quad \mathcal{W}_E(A).
\]

Proof. (i) is immediate from the Vaserstein Rule. We refer the reader to [9, Theorem 3.16(iii)] for deriving (ii) from (i). Note: You may need the Roitman lemma in [5, Lemma 1].

(2.6.3) The “antipodal vectors equality” lemma in \( \mathcal{W}_E(A) \). Let \( v = (v_0, v_1, v_2) \) be a unimodular vector and assume that \( v \sim E \sim (-v_0, -v_1, -v_2) \). Let \( v^{(2)} = (v_0^2, v_1, v_2) \) and let \( w, w_1 \) be such that \( v \cdot w = v^{(2)} \cdot w_1 = 1 \). Then

\[
[V(v, w)]^2 = [V(v^{(2)}, w_1)] \quad \text{in} \quad \mathcal{W}_E(A).
\]

Proof. Imitate the argument in [3, Lemma 1.3.1] in \( \mathcal{W}_E(A) \). (Note. You will need Corollary 2.6.2(ii) above.)

Finally, we give some conditions under which we can extract “coordinate squares” in \( \mathcal{W}_E(R[X]) \);

(2.6.4) Corollary. Let \( R \) be a local ring of dimension \( \leq 4 \) with \( 1/2 \in R \) and let \( v = (v_0, v_1, v_2) \), \( v^{(2)} = (v_0^2, v_1, v_2) \) be unimodular \( R[X] \)-vectors. Let \( w, w_1 \) such that \( v \cdot w = v^{(2)} \cdot w_1 = 1 \). Then,

\[
[V(v, w)]^2 = [V(v^{(2)}, w_1)] \quad \text{in} \quad \mathcal{W}_E(R[X]).
\]

Proof. This will follow from Proposition 2.5.1 and Lemma 2.6.3.

(2.6.5) Proposition. Let \( R \) be a local ring of dimension \( \leq 3 \) with \( 1/2 \in R \) and let \( V \in \text{SL}_4(R[X]) \) be an alternating matrix with Pfaffian 1. Then \( [V] = [V^*] \) in \( \mathcal{W}_E(R[X]) \) with \( V^* \in \text{SL}_4(R[X]) \) a coordinate square. Consequently, there is a stably elementary \( \gamma \in \text{SL}_4(R[X]) \) such that \( V = \gamma V^* \gamma \).

Proof. By Proposition 2.4.1, \( [V] = [W]^2 \) for some \( [W] \in \mathcal{W}_E(R[X]) \). By [7, Theorem 2.6] \( \text{Um}_r(R[X]) = E_r \cap \mathcal{E}_r(R[X]) \) for all \( r \geq 5 \), and so on applying [8, Lemma 5.3 and Lemma 5.5] a few times, if necessary, we can find an alternating matrix \( W^* \in \text{SL}_r(R[X]) \) (with Pfaffian 1) such that \( [W] = [W^*] \). Now apply Corollary 2.6.4 to find \( V^* \) as required. The last statement follows as above (only applying [8, Lemma 5.5 and Lemma 5.6] instead).
3. The main theorem

(3.1) **Theorem.** Let $R$ be a local ring of Krull dimension three with $1/2 \in R$ and let $v = (v_0, v_1, v_2)$ be a unimodular 3-vector over $R[X]$. Then $v$ can be completed to an invertible matrix.

**Proof.** Choose a $w = (w_0, w_1, w_2)$ such that $v_0w_0 + v_1w_1 + v_2w_2 = 1$, and consider the alternating matrix $V$ with Pfaffian 1 given by

$$
V = \begin{pmatrix}
0 & v_1 & v_2 \\
v_0 & 0 & -w_1 \\
v_1 & -w_2 & 0
\end{pmatrix} \in SL_4(R[X]).
$$

Since $1/2 \in R$, by M. Karoubi’s theorem (see (*)) in Proposition 2.4.1) there is a $\alpha \in SL_4+(R[X])$, for some $l$, such that $\alpha^t(V \perp \psi_1)\alpha = \psi_{l+2}$.

Since $\dim R = 3$, by [7, Theorem 2.6] $Um_r(R[X]) = e_4 E_r(R[X])$ for all $r \geq 6$. Hence on applying [8, Lemma 5.5 and Lemma 5.6] we can find an $\alpha \in SL_4(R[X])$ such that $\alpha^tV \alpha = \psi_2$. Consider $e_4\alpha^t$, where $e_4 = (0, 0, 0, 1)$.

By [3, Proposition 1.4.4] $e_4\alpha^t \sim (a_0(X), a_1(X), a_2(X), c)$, where $c \in R$ is a non-zero-divisor in $R$. Let the ‘overbar’ denote ‘modulo (c)’. By [3, Proposition 2.2], $(\overline{a_0(X)}, \overline{a_1(X)}, \overline{a_2(X)}) \sim (b_0(X)^2, b_1(X), b_2(X))$, for some $b_0(X), b_1(X), b_2(X) \in R[X]$. On “lifting” this elementary map, and after an appropriate elementary transformation further, we can arrange that $e_4\alpha^t \sim (b_0(X)^2, b_1(X), b_2(X), c)$.

By Proposition 2.2.2, $(b_0(X), b_1(X), b_2(X), c)$ can be completed to an invertible matrix, say $\beta \in SL_4(R[X])$ with $e_4\beta = (b_0(X), b_1(X), b_2(X), c)$.

Via Remark 1.1.1 follows that

$$
e_4\beta^{-2}\alpha^t = [e_4\beta^{-2}] \ast [e_4\alpha^t] = [e_4\beta^{-2}] \ast [e_4\alpha^t] = ([((b_0(X), b_1(X), b_2(X), c))^2]^{-1} \ast [e_4\alpha^t] = [e_4\alpha^t]^{-1} \ast [e_4\alpha^t] = 1,
$$

the last equality being deduced via [3, Example 1.5.3 and Lemma 1.3.1]. Thus, $\beta^{-2}\alpha^t = \delta' \delta$ for some $\delta \in SL_3(R[X])$.

Now $\psi_2 = \alpha^t V \alpha = (\beta^2 \varepsilon \delta') V (\beta^2 \varepsilon \delta') = \beta^2 V^* (\beta^2)^t$, where $e_1 V^* = (0, 0, \delta^t \varepsilon)$ for some $\delta \in E_3(R[X])$—this will follow as $\delta' = (1, 0, \delta)$ and via [11, Theorem 10].

By Proposition 2.6.5 there is a stably elementary $\gamma \in SL_4(R[X])$ such that $\beta V^* \beta^t = \gamma V^* \gamma$, with $V^* \in SL_4(R[X])$ a coordinate square. Let $e_1 V^* = (0, a^2, b, c)$, and let $\alpha_0$ (cf. §2.2) be a completion of $(a^2, b, c)$.

Since

$$c_1 V^* = e_1 \begin{pmatrix} 1 & 0 \\ 0 & \alpha_0 \end{pmatrix}^t \varepsilon_2 \begin{pmatrix} 1 & 0 \\ 0 & \alpha_0 \end{pmatrix},$$

it follows via [8, Lemma 5.1] that

$$V^* = e_1 \begin{pmatrix} 1 & 0 \\ 0 & \alpha_0 \end{pmatrix}^t \varepsilon_2 \begin{pmatrix} 1 & 0 \\ 0 & \alpha_0 \end{pmatrix} e_1.$$
for some \( e_1 \in E_4(R[X]) \). Thus,
\[
\beta V^* \beta^t = \gamma^t V^{**} \gamma = \gamma^t e_1 \left( \begin{array}{cc} 1 & 0 \\ 0 & \alpha_0 \end{array} \right)^t \psi_2 \left( \begin{array}{cc} 1 & 0 \\ 0 & \alpha_0 \end{array} \right) e_1 \gamma.
\]

Hence,
\[
\beta^{-1} \left[ \left( \begin{array}{cc} 1 & 0 \\ 0 & \alpha_0 \end{array} \right)^t (e_1^{-1} (\gamma^{-1}))^t \right] \beta V^* \beta^t \left[ \left( \begin{array}{cc} 1 & 0 \\ 0 & \alpha_0 \end{array} \right)^t \right] (\beta^{-1})^t
\]
\[
= \beta^{-1} \psi_2 (\beta^{-1})^t = \beta^{-1} (\beta^2 V^* (\beta^2)^t)^t (\beta^{-1})^t = \beta V^* \beta^t = \gamma^t V^{**} \gamma;
\]

and so if
\[
\theta = \beta^t \gamma^{-1} e_1 \left( \begin{array}{cc} 1 & 0 \\ 0 & \alpha_1^{-1} \end{array} \right)^t (\beta^t)^{-1} \gamma^{-1}, \quad \text{then} \quad \theta^t V^* \theta = V^{**},
\]

Compute \( e_4 \theta^t \) in the abelian group \( \text{Um}_4(R[X])/E_4(R[X]) \) via Remark 2.1.1 to get \( [e_4 \theta^t] = [e_4 (\gamma')^{-1}]^2 \). But \( \gamma \) is stably elementary and so via [3, Proposition 2.6] \( [e_4 (\gamma')^{-1}]^2 = 1 \); hence \( [e_4 \theta^t] = 1 \), i.e. \( e_4 \theta^t \sim e_4 \). Hence
\[
\theta^t e' = \left( \begin{array}{cc} 1 & 0 \\ 0 & (\theta')^t \end{array} \right)
\]

for some \( \theta' \in \text{SL}_3(R[X]), e' \in E_4(R[X]) \).

Now
\[
\theta^t V^* \theta = \left( \begin{array}{cc} 1 & 0 \\ 0 & (\theta')^t \end{array} \right)^t (\theta')^{-1} V^* ((\theta')^{-1})^t \left( \begin{array}{cc} 1 & 0 \\ 0 & \theta' \end{array} \right) = V^{**},
\]

and so via [11, Theorem 10] we can deduce that there is an \( e'' \in E_3(R[X]) \) such that \( ve'' \theta' = (a^2, b, c) \). Since \( (a^2, b, c) \) is completable, so is \( v \).

**Remark.** Let us, following M. Krusemeyer, say that a vector \( v \in \text{Um}_r(A) \) is skew-completable if there is an invertible alternating matrix \( V \in \text{Sl}_{r+1}(A) \) with its first row \( e_1 V = (0, v) \).

By making some appropriate modifications in the argument used to prove Theorem 3.1 one can show that,

(3.2) **Theorem.** Let \( R \) be a local ring of Krull dimension \( d \) with \( 1/2 \in R \), and let \( v = (v_0, v_1, \ldots, v_{d-1}) \) be a skew-completable vector over \( R[X] \). Then \( v \) can be completed to an invertible matrix.

Finally, using the well-known “Quillen-Suslin” Monic inversion and Local-Global principles, one can derive from \( S_d(R) \) and Theorem 3.1 that,

(3.3) **Corollary.** Let \( R \) be a noetherian ring of dimension 3 with \( 1/6 \in R \). Then any stably extended projective module over \( R[X_1, \ldots, X_n] \) is extended.

**Note added in proof.** The contents (especially the mode of proof of the main result) of this note seems of interest in connection with the following problem:

(i) Let \( V : \text{Um}_3(A)/E_3(A) \to W_E(A) \) be the Vaserstein symbol. Is this map injective if \( \dim A = 3 \)?
I also hope that, after incorporation of some additional theories, the techniques used here will provide some insight towards settling,

(a) Let $R$ be a local ring with $\frac{1}{2} \in R$. Is every $v \in \text{Um}_3(R[X])$ completable?

(b) Let $A$ be a smooth affine algebra over the field $\mathbb{C}$ of complex numbers of dimension $d$. Is a stably free $A$-module of rank $(d - 1)$ a free module?

In an article entitled On some actions of stably elementary matrices on alternating matrices we prove that

"Let $A$ have Krull dimension $\leq 5$, and let $V \in \text{Sl}_4(A) \cap E_5(A)$ be a stably elementary alternating matrix of Pfaffian one. Then $V^8 \in E_4(A)$.”

Note. One needs to show that $V \in E_4(A)$ to settle (i) above.

We also give some examples of 3 dimensional affine algebras for which the Vaserstein symbol $V$ is bijective.

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SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, BOMBAY 400 005, INDIA