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HOW POROUS IS THE GRAPH OF BROWNIAN MOTION?

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ABSTRACT. We prove that the graph of Brownian motion is almost surely porous, and determine the Hausdorff dimension of sets with a given porosity index. In particular we show that the porosity index of the graph is $\gamma_0 \doteq 0.6948$.

1. INTRODUCTION AND STATEMENT OF RESULTS

Real function theory has seen an increase in interest recently in the study of porous and σ-porous sets. These are sets which are "small" in a certain sense. They were first introduced by Denjoy [4] and later rediscovered by Dolzhenko [5]. Much of the recent interest centers around the work of Zajicek [12]. For further references see [6 and 11].

Since we are primarily interested in the graph of Brownian motion, we will only define porosity for the graph of a real valued function. Given $f : [0, \infty) \to \mathbb{R}$ let G be its graph, $G = \{(t, f(t)) : t \geq 0\}$. Using Λ to denote a square in $\mathbb{R}^2$ with sides parallel to the coordinate axes, we define

$$\Lambda_t(h) = \{(x, s) : t - h \leq s \leq t + h,\ f(t) - h \leq x \leq f(t) + h\},$$

and

$$\delta_t(h) = \sup\{|\Lambda|^{1/2} : \Lambda \subseteq \Lambda_t(h),\ \Lambda \cap G = \varnothing\},$$

where $|\cdot|$ denotes Lebesgue measure. Thus $\delta_t(h)$ is the supremum of the side lengths of all squares contained in $\Lambda_t(h)$, with sides parallel to the coordinate axes, which do not intersect G. The function f, or more precisely its graph G, is said to be *porous at* t if

$$\limsup_{h \downarrow 0} \frac{\delta_t(h)}{h} > 0. \tag{1.1}$$

It is *porous* if (1.1) holds for all $t \geq 0$. The definition of porosity from the right is obtained by replacing $\Lambda_t(h)$ and $\delta_t(h)$ with

$$\Lambda_t^+(h) = \{(s, x) : t \leq s \leq t + h,\ f(t) - h \leq x \leq f(t) + h\}$$

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and

$$\delta_t^+(h) = \sup\{|\Lambda|^{1/2} : \Lambda \subseteq \Lambda_t^+(h),\ \Lambda \cap G = \varnothing\}$$

respectively. The definition of porosity from the left should be clear.

Our original interest in porosity is due to the following question raised by Goffman [7]:

Is the graph of Brownian motion porous with probability one?

More precisely, let $W(t)$, $t \geq 0$, be a standard one-dimensional Brownian motion, and let δ_t be defined for the function $f(t) = W(t)$. The question is then whether or not

$$P\left(\limsup_{h \downarrow 0} \frac{\delta_t(h)}{h} > 0 \text{ for all } t \geq 0\right) = 1. \tag{1.2}$$

The analogous question for porosity from the right is whether or not

$$P\left(\limsup_{h \downarrow 0} \frac{\delta_t^+(h)}{h} > 0 \text{ for all } t \geq 0\right) = 1. \tag{1.3}$$

As we will see the analysis of porosity for Brownian motion essentially reduces to the analysis of right porosity, and thus we turn out attention to this first.

We will show that the answer to Goffman's question is yes by showing that (1.3) holds. Our original proof in fact gave the stronger result that there exists a (nonrandom) $\varepsilon > 0$ for which

$$P\left(\limsup_{h \downarrow 0} \frac{\delta_t^+(h)}{h} \geq \varepsilon \text{ for all } t \geq 0\right) = 1. \tag{1.4}$$

This then raises the question as to the largest value of ε for which (1.4) holds. This value is called the right porosity index of the graph. Analogously the value of the lim sup in (1.1), with δ_t replaced by δ_t^+, is called the right porosity index at t.

Before addressing this problem we will make a few simple observations. First, it follows from the definitions that one always has for every $h > 0$

$$\delta_t^+(h)/h \leq 1. \tag{1.5}$$

It is not hard to show (see the end of §3) that for every $t \geq 0$

$$P\left(\limsup_{h \downarrow 0} \frac{\delta_t^+(h)}{h} = 1\right) = 1. \tag{1.6}$$

Hence by Fubini

$$P\left(\limsup_{h \downarrow 0} \frac{\delta_t^+(h)}{h} = 1 \text{ for a.e. } t\right) = 1. \tag{1.7}$$

Thus the right porosity index at almost all t is 1 with probability one. However, the value of the right porosity index of the graph is *not* 1, as can be seen from our first result.

Theorem 1.1. $P(\inf_{t\geq 0} \limsup_{h\downarrow 0} \delta_t^+(h)/h = \frac{1}{2}) = 1$

From Theorem 1.1 and (1.7) we see that

$$A^+ = \left\{ t : \limsup_{h\downarrow 0} \frac{\delta_t^+(h)}{h} < 1 \right\}$$

satisfies

$$P(A^+ \neq \varnothing) = 1 \quad \text{and} \quad P(|A|^+ = 0) = 1 . \tag{1.8}$$

Thus A^+ is a set of one-dimensional Lebesgue measure zero but it is still quite "large". More precisely, if for any set B we let

$$d(B) = \text{Hausdorff dimension of } B ,$$

then we are able to show

Theorem 1.2. $P(d(A^+) = 1) = 1$.

Our final result on right porosity concerns the size of the set

$$A_\gamma^+ = \left\{ t : \limsup_{h\downarrow 0} \frac{\delta_t^+(h)}{h} \leq \gamma \right\} \tag{1.9}$$

where $0 \leq \gamma < 1$. If $\gamma < \frac{1}{2}$ then $A_\gamma^+ = \varnothing$ by Theorem 1.1. If $\gamma \geq \frac{1}{2}$ we again ask for its Hausdorff dimension. To answer this question, we define $\alpha(\gamma)$ for $\gamma \in (0, 1)$ to be the unique solution α of

$$\int_1^{(1-\gamma)^{-1}} \frac{t^\alpha - 1}{(t-1)^{\frac{3}{2}}}\, dt = 2\left(\frac{1-\gamma}{\gamma}\right)^{\frac{1}{2}} . \tag{1.10}$$

This expression has its origin in the determination of the moments of a certain stopping time (see Lemma 2.1 below). We then have

Theorem 1.3. *For* $\frac{1}{2} \leq \gamma < 1$

$$P(d(A_\gamma^+) = 1 - \alpha(\gamma)) = 1 .$$

To state the analogous results to Theorems 1.1–1.3 for porosity, define

$$A = \left\{ t : \limsup_{h\downarrow 0} \frac{\delta_t(h)}{h} < 1 \right\}$$

and

$$A_\gamma = \left\{ t : \limsup_{h\downarrow 0} \frac{\delta_t(h)}{h} \leq \gamma \right\} .$$

We then have

Theorem 1.4. *Let* $\gamma_0 \doteq 0.6948$ *be the unique value of* γ *for which* (1.10) *holds with* $\alpha = \frac{1}{2}$. *Then*

$$P\left(\inf_{t\geq 0}\limsup_{h\downarrow 0}\frac{\delta_t(h)}{h} = \gamma_0\right) = 1\,.$$

Theorem 1.5. $P(d(A) = 1) = 1$.

Theorem 1.6. *For* $\gamma_0 \leq \gamma < 1$

$$P(d(A_\gamma) = 1 - 2\alpha(\gamma)) = 1\,.$$

We are also able to obtain the Hausdorff dimension of the intersection of A_γ and A_γ^+ with an arbitrary analytic set. These results are stated in §2, where we will reformulate our results in a more convenient form (see Theorems 2.1 and 2.2). The proofs are given in §§3 and 4.

In view of (1.6), Theorems 1.1–1.6 can be considered as results concerning points on the graph of Brownian motion which are "less porous" than they should be. The study of such points turns out to be closely related to the study of Brownian slow points (see [1–3, 8–10]). In particular, the methods developed by Davis in [2] for finding slow points, and Perkins in [10] for computing Hausdorff dimensions, play a very important role in our analysis. The reader familiar with their work will note the close correspondence between the statements of our results and those in [10] on which they are modelled.

2. Preliminaries

Recall that $W(s)$ is a standard one-dimensional Brownian motion. For $\beta > 1$ and $\varepsilon \in (0, \beta - 1)$ let $Z(\varepsilon, \beta)$ be a random variable with distribution given by

$$P(Z(\varepsilon, \beta) = 0) = \frac{2}{\pi}\operatorname{arc\,sine}\left(\frac{\varepsilon}{\beta - 1}\right)^{1/2}, \tag{2.1}$$

$$P(1 + \varepsilon \leq Z(\varepsilon, \beta) \leq t) = 1 - \frac{2}{\pi}\operatorname{arc\,sine}\left(\frac{\varepsilon}{t - 1}\right)^{1/2}, \qquad t \in [1 + \varepsilon, \beta]\,.$$

The way in which this distribution arises is as follows; fix $h > 0$ and let

$$\tau = \inf\{s \geq (1 + \varepsilon)h : W(s) = W(h)\}\,.$$

Then the *arcsine law* and scaling imply that

$$Z(\varepsilon, \beta) \stackrel{d}{=} \tfrac{\tau}{h}\, 1\left(\tfrac{\tau}{h} \leq \beta\right),$$

where we have used $\stackrel{d}{=}$ to denote equality in distribution, and $1(\cdot)$ for indicator function.

Differentiating in (2.1), one sees that the density of $Z(\varepsilon, \beta)$ on $[1 + \varepsilon, \beta]$ is given by

$$f(t) = \frac{1}{\pi}\left(\frac{\varepsilon}{t - 1 - \varepsilon}\right)^{1/2}\frac{1}{t - 1}\,.$$

Thus for $p \geq 0$

$$E[Z^p(\varepsilon, \beta)] = \frac{\varepsilon^{1/2}}{\pi} \int_{1+\varepsilon}^{\beta} \frac{t^p}{t-1} \frac{dt}{(t-1-\varepsilon)^{1/2}}.$$

Lemma 2.1. *Fix $p \geq 0$ and $\beta > 1$. Then*

$$E[Z^p(\varepsilon, \beta)] = 1 + \varepsilon^{1/2} \frac{1}{\pi} \left[\int_1^{\beta} \frac{t^p - 1}{(t-1)^{3/2}} dt - \frac{2}{(\beta-1)^{1/2}} \right] + o(\varepsilon^{1/2})$$

as $\varepsilon \to 0$.

Proof. For any $p \geq 0$ and $\beta > 1$,

$$\begin{aligned} E[Z^p(\varepsilon, \beta)] &= \frac{\varepsilon^{1/2}}{\pi} \int_{1+\varepsilon}^{\beta} \frac{t^p - 1}{(t-1)} \frac{dt}{(t-1-\varepsilon)^{1/2}} + \frac{\varepsilon^{1/2}}{\pi} \int_{1+\varepsilon}^{\beta} \frac{1}{t-1} \frac{dt}{(t-1-\varepsilon)^{1/2}} \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

An application of dominated convergence shows that

$$\int_{1+\varepsilon}^{\beta} \frac{t^p - 1}{t-1} \frac{dt}{(t-1-\varepsilon)^{1/2}} \to \int_1^{\beta} \frac{t^p - 1}{(t-1)^{3/2}} dt < \infty$$

as $\varepsilon \to 0$. Thus

$$\mathrm{I} = \frac{\varepsilon^{1/2}}{\pi} \int_1^{\beta} \frac{t^p - 1}{(t-1)^{3/2}} dt + o(\varepsilon^{1/2}). \tag{2.2}$$

The integral in II can be explicitly evaluated, giving

$$\begin{aligned} \mathrm{II} &= 1 - \frac{2}{\pi} \operatorname{arc\,sine} \left(\frac{\varepsilon}{\beta - 1} \right)^{1/2} \\ &= 1 - \frac{2}{\pi} \left(\frac{\varepsilon}{\beta - 1} \right)^{1/2} + o(\varepsilon^{1/2}). \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3) finishes the proof. □

Now fix $\beta > 1$ and define

$$g(p) = \int_1^{\beta} \frac{t^p - 1}{(t-1)^{3/2}} dt, \qquad p \geq 0. \tag{2.4}$$

Clearly g is continuous, strictly increasing, $g(0) = 0$, and $g(p) \to \infty$ as $p \to \infty$. In particular, for each $\beta \in (1, \infty)$, there exists a unique $p(\beta)$ such that

$$g(p(\beta)) = 2(\beta - 1)^{-1/2}. \tag{2.5}$$

The following observations will be used frequently throughout the paper:

(2.6)
- (i) $p(\cdot)$ is continuous,
- (ii) $p(\cdot)$ is strictly decreasing,
- (iii) $p(\beta) \uparrow \infty$ as $\beta \downarrow 1$, $p(\beta) \downarrow 0$ as $\beta \uparrow \infty$,
- (iv) $p(2) = 1$,
- (v) if $p_1 < p(\beta) < p_2$, then for all $\varepsilon > 0$ sufficiently small
$$E[Z^{p_1}(\varepsilon, \beta)] < 1 \quad \text{and} \quad E[Z^{p_2}(\varepsilon, \beta)] > 1.$$

Properties (i)–(iv) are all easy consequences of the definition of $p(\beta)$, and (v) follows from Lemma 2.1.

Observe that (1.10) is just a reparameterization of (2.5) obtained by setting $\gamma = 1 - \beta^{-1}$. It is more convenient however to work directly with $p(\beta)$ and so we now reformulate our results in these terms. For $\beta \in (1, \infty)$ define

$$B_\beta^+ = \left\{ t \in [0, 1] : \limsup_{h \downarrow 0} \frac{\delta_t^+(h)}{h} \le 1 - \beta^{-1} \right\},$$

$$B_\beta = \left\{ t \in [0, 1] : \limsup_{h \downarrow 0} \frac{\delta_t(h)}{h} \le 1 - \beta^{-1} \right\}.$$

We will prove

Theorem 2.1. *Let $B \subseteq [0, 1]$ be an analytic set, then*

$$P(d(B \cap B_\beta^+) = d(B) - p(\beta)) = 1 \quad \text{if } p(\beta) \le d(B),$$
$$P(B \cap B_\beta^+ = \varnothing) = 1 \quad \text{if } p(\beta) > d(B).$$

Theorem 2.2. *Let $B \subseteq [0, 1]$ be an analytic set, then*

$$P(d(B \cap B_\beta) = d(B) - 2p(\beta)) = 1 \quad \text{if } 2p(\beta) \le d(B),$$
$$P(B \cap B_\beta = \varnothing) = 1 \quad \text{if } 2p(\beta) > d(B).$$

In Theorems 2.1 and 2.2 we consider $t \in [0, 1]$; it follows from the Markov property that the same conclusions hold for $t \in [n, n+1]$ for any positive integer n, and hence for $t \in [0, \infty)$. Thus, Theorem 1.1 follows from Theorem 2.1 by using (2.6)(ii) and (iv); Theorem 1.2 follows from Theorem 2.1 by letting $\beta \to \infty$ and using (2.6)(iii); and Theorem 1.3 is just a special case of Theorem 2.1. Similarly, Theorems 1.4–1.6 follow from Theorem 2.2.

3. Upper bounds on $d(B \cap B_\beta^+)$ and $d(B \cap B_\beta)$

For $\eta > 0$ define

$$B_{\beta,\eta}^+ = \left\{ t \in [0, 1] : \frac{\delta_t^+(h)}{h} \le (1 - \beta^{-1}) \text{ for all } h \in (0, \eta] \right\},$$
$$B_{\beta,\eta}^- = \left\{ t \in [0, 1] : \frac{\delta_t^-(h)}{h} \le (1 - \beta^{-1}) \text{ for all } h \in (0, \eta] \right\}, \tag{3.1}$$

and

$$B_{\varepsilon,\beta,\eta}^+ = \{ t \in [0, 1] : \forall h \in (0, \eta\beta^{-1}] \exists s \in [(1+\varepsilon)h, \beta h] \text{ with } W(t+s) = W(t) \},$$
$$B_{\varepsilon,\beta,\eta}^- = \{ t \in [0, 1] : \forall h (0, \eta\beta^{-1}] \exists s \in [(1+\varepsilon)h, \beta h] \text{ with } W(t-s) = W(t) \}.$$

In order that the definition of $B^-_{\varepsilon,\beta,\eta}$ make sense we adopt the convention that $W(u) = -\infty$ for $u < 0$. For $\eta_1 < \eta_2$

$$B^+_{\beta,\eta_1} \supseteq B^+_{\beta,\eta_2}, \tag{3.2}$$

while for $\beta_1 < \beta_2$

$$\lim_{\eta\downarrow 0} B^+_{\beta_1,\eta} \subseteq B^+_{\beta_1} \subseteq \lim_{\eta\downarrow 0} B^+_{\beta_2,\eta}. \tag{3.3}$$

The following observation is central to our methods for obtaining bounds on the size of $B^+_{\beta,\eta}$ and hence on B^+_β: if $\beta_1 < \beta_2$, and $\varepsilon > 0$ is small enough that $1 - (1+\varepsilon)\beta_2^{-1} > 1 - \beta_1^{-1}$, then

$$B^+_{\beta_1,\eta} \subseteq B^+_{\varepsilon,\beta_2,\eta}. \tag{3.4}$$

To prove this, suppose $t \notin B^+_{\varepsilon,\beta_2,\eta}$, and hence that for some $h \in (0, \eta\beta_2^{-1}]$ we have $W(t+s) \neq W(t)$ for all $s \in [(1+\varepsilon)h, \beta_2 h]$. Then either

$$\Lambda = \{(t+s, x) : (1+\varepsilon)h \le s \le \beta_2 h \text{ and } W(t) - (\beta_2 - (1+\varepsilon))h \le x \le W(t)\}$$

or

$$\Lambda = \{(t+s, x) : (1+\varepsilon)h \le s \le \beta_2 h \text{ and } W(t) \le x \le W(t) + (\beta_2 - (1+\varepsilon))h\}$$

satisfies

$$\Lambda \subseteq \Lambda^+_t(\beta_2 h)\backslash G$$

by continuity of the sample paths (G is the graph of $W(s)$). Consequently, $h' = \beta_2 h \in (0, \eta]$ satisfies

$$\frac{\delta^+_t(h')}{h'} = \frac{\delta^+_t(\beta_2 h)}{\beta_2 h} \ge \frac{\beta_2 - (1+\varepsilon)}{\beta_2} > 1 - \beta_1^{-1}.$$

So $t \notin B^+_{\beta_1,\eta}$, and (3.4) follows.

Similarly, the analogous versions of (3.2)–(3.4) for B^-_β, $B^-_{\beta,\eta}$, and $B^-_{\varepsilon,\beta,\eta}$ are valid. We now give an estimate on the size of $B^+_{\varepsilon,\beta,\eta}$. Let

$$\mathscr{F}(t) = \sigma\{W(s) : 0 \le s \le t\}.$$

Proposition 3.1. *Fix $\beta > 1$, $\varepsilon \in (0, \beta - 1)$ and $p \ge 0$, and assume that $E[Z^p(\varepsilon, \beta)] < 1$. Then for any $\eta > 0$ there exists $C < \infty$ such that for all $v \ge 0$ and all ζ sufficiently small,*

$$P([v, v+\zeta] \cap B^+_{\varepsilon,\beta,\eta} \neq 0 \mid \mathscr{F}(v+\zeta)) \le C\zeta^p \quad a.s. \tag{3.5}$$

Proof. By the Markov property, it suffices to consider the case $v = 0$. Choose $q > p$ and $\beta_1 > \beta$ such that $E[Z^q(\varepsilon, \beta_1)] < 1$. Fix $\lambda \in (0, 1)$ such that $\lambda q \ge p$, and assume that

$$\zeta \le \left(\frac{\beta_1 - \beta}{\beta - 1}\right)^{1/(1-\lambda)} \wedge \frac{1}{2}.$$

Finally let $r_\zeta = (-2\zeta p \log \zeta)^{1/2}$. Then with $M(t) = \sup\{|W(s)| : 0 \le s \le t\}$ we have

$$
\begin{aligned}
&P([0, \zeta] \cap B^+_{\varepsilon,\beta,\eta} \neq \varnothing \mid \mathscr{F}(\zeta)) \\
&\quad \le P(\exists b \in (-\infty, \infty) \forall h \in (\zeta, \eta\beta^{-1}] \exists s \\
&\qquad\qquad \in [(1+\varepsilon)h, \beta h] \text{ with } W(s) = b \mid \mathscr{F}(\zeta)) \\
&\quad \le P(\exists b \in (-\infty, \infty) \forall h \in (2\zeta, \eta\beta^{-1}] \exists s \\
&\qquad\qquad \in [(1+\varepsilon)h, \beta h] \text{ with } W(s) = b \mid \mathscr{F}(\zeta)) \\
&\quad = P(\exists b \in (-\infty, \infty) \forall h \in (\zeta, \eta\beta^{-1} - \zeta] \exists s \\
&\qquad\qquad \in [(1+\varepsilon)(h+\zeta) - \zeta, \beta(h+\zeta) - \zeta] \text{ with } W(s) = b) \text{ a.s.} \\
&\quad \le P(M(\zeta) \ge r_\zeta) \\
&\qquad + P(\exists b, |b| \le r_\zeta \text{ such that } \forall h \in (\zeta, \eta\beta^{-1} - \zeta], \\
&\qquad\qquad \exists s \in [(1+\varepsilon)(h+\zeta) - \zeta, \beta(h+\zeta) - \zeta] \text{ with } W(s) = b) \\
&\quad = \mathrm{I} + \mathrm{II}.
\end{aligned}
$$

We will estimate I and II separately.

The first term is simple:

$$
\begin{aligned}
\mathrm{I} &\le 4P(W(\zeta) \ge r_\zeta) = 4P(W(1) \ge \zeta^{-1/2} r_\zeta) \\
&\le 2(-\pi p \log \zeta)^{-1/2} \exp(p \log \zeta) \le C\zeta^p
\end{aligned}
$$

for some C.

To bound the second term we use scaling to obtain,

$$
\begin{aligned}
\mathrm{II} &\le P(\forall h \in [\zeta^\lambda, \eta\beta^{-1} - \zeta] \exists s \in [(1+\varepsilon)(h+\zeta) - \zeta, \beta(h+\zeta) - \zeta] \\
&\qquad\qquad \text{with } |W(s)| \le r_\zeta) \\
&= P(\forall u \in [1, \zeta^{-\lambda}(\eta\beta^{-1} - \zeta)] \exists s \in [(1+\varepsilon)u + \varepsilon\zeta^{1-\lambda}, \beta u + (\beta - 1)\zeta^{1-\lambda}] \\
&\qquad\qquad \text{with } |W(s)| \le \varepsilon_\zeta) \\
&\le P(\forall u \in [1, \zeta^{-\lambda}(\eta\beta^{-1} - \zeta)] \exists s \in [(1+\varepsilon)u, \beta_1 u] \text{ with } |W(s)| \le \varepsilon_\zeta) \\
&= P(\Gamma_\zeta)
\end{aligned}
$$

say, where $\varepsilon_\zeta \overset{\text{def}}{=} \zeta^{-\lambda/2} r_\zeta \to 0$ as $\zeta \downarrow 0$. To estimate this last quantity we introduce stopping times τ_k, which are finite a.s., defined by

$$
\tau_{-1} = 1, \quad \tau_k = \inf\{s \ge (1+\varepsilon)\tau_{k-1} : |W(s)| \le \varepsilon_\zeta\}, \qquad k \ge 0.
$$

Also define

$$
V_k(\varepsilon, \beta_1) = \frac{\tau_k}{\tau_{k-1}} 1\left(\frac{\tau_k}{\tau_{k-1}} \le \beta_1\right), \qquad k \ge 0,
$$

$$
V^*(\varepsilon, \beta_1) = \sup_{k \ge 0} \prod_{m=0}^{k} V_m(\varepsilon, \beta_1).
$$

Observe that

$$\Gamma_\zeta \subset \{V^*(\varepsilon, \beta_1) \geq \zeta^{-\lambda}(\eta\beta^{-1} - \zeta)\}. \tag{3.6}$$

Now if $u \geq 1$ and $|x| \leq \varepsilon_\zeta$, then for any $t \in [1+\varepsilon, \beta_1]$ and any $k \geq 0$

$$\begin{aligned} &P(V_{k+1}(\varepsilon, \beta_1) \in [1+\varepsilon, t] \mid \tau_k = u,\ W(\tau_k) = x) \\ &\quad = P(\exists s \in [(1+\varepsilon)u, tu] \text{ with } |W(s)| \leq \varepsilon_\zeta | W(u) = x) \\ &\quad = P(\exists s \in [(1+\varepsilon), t] \text{ with } |W(s)| \leq \varepsilon_\zeta u^{-1/2} | W(1) = xu^{-1/2}) \\ &\quad \to P(Z(\varepsilon, \beta_1) \in [1+\varepsilon, t]) \end{aligned}$$

uniformly in u, x, and k as $\zeta \downarrow 0$. Since the random variables are bounded, this means that for all $\alpha \geq 0$, all $k \geq 0$ and all $\delta > 0$

$$E[V_{k+1}^\alpha(\varepsilon, \beta_1) | \mathscr{F}(\tau_k)] \leq E[Z^\alpha(\varepsilon, \beta_1)] + \delta \quad \text{a.s.}$$

provided ζ is sufficiently small. In particular since $E[Z^q(\varepsilon, \beta_1)] < 1$, there exists an $r < 1$ such that for all ζ sufficiently small and all $k \geq 0$

$$E[V_{k+1}^q(\varepsilon, \beta_1) \mid \mathscr{F}(\tau_k)] \leq r \quad \text{a.s.}$$

Thus by the strong Markov property, for any $j \geq 1$ and ζ small

$$E\left[\prod_{m=0}^{j} V_m^q(\varepsilon, \beta_1)\right] = EE\left[\prod_{m=0}^{j} V_m^q(\varepsilon, \beta_1) \mid \mathscr{F}(\tau_{j-1})\right] \leq rE\left[\prod_{m=0}^{j-1} V_m^q(\varepsilon, \beta_1)\right].$$

Hence, since $V_0(\varepsilon, \beta_1) \leq \beta_1$, we have for all small ζ

$$E[V^*(\varepsilon, \beta_1)^q] \leq \sum_{k=0}^{\infty} E\left[\prod_{m=0}^{k} V_m^q(\varepsilon, \beta_1)\right] \leq \beta_1^q \sum_{k=0}^{\infty} r^k < \infty.$$

Thus by (3.6)

$$P(\Gamma_\zeta) \leq \frac{E[V^*(\varepsilon, \beta_1)^q]}{\zeta^{-\lambda q}(\eta\beta^{-1} - \zeta)^q} \leq C\zeta^p$$

for some constant C. □

We note that by integration in (3.5), for all $v \geq 0$ and all ζ sufficiently small

$$P([v, v+\zeta] \cap B^+_{\varepsilon, \beta, \eta} \neq \varnothing) \leq C\zeta^p,$$

and by considering the Brownian motion $\widetilde{W}(s) = W(1) - W(1-s)$, that

$$P([v, v+\zeta] \cap B^-_{\varepsilon, \beta, \eta} \neq \varnothing) \leq C\zeta^p.$$

With Proposition 3.1 at hand, the proof of Theorem 3.1 is now similar to that of the upper bound in Theorem 4 of Perkins [10].

Theorem 3.1. *Assume* $B \subseteq [0, 1]$. *Then*

$$\begin{aligned} &P(d(B \cap B^+_{\beta_1}) \leq d(B) - p(\beta_1)) = 1, \qquad d(B) \geq p(\beta_1); \\ &P(B \cap B^+_{\beta_1} = \varnothing) = 1, \qquad d(B) < p(\beta_1); \end{aligned} \tag{3.7}$$

and

(3.8)

$$P(d(B \cap B^+_{\beta_1} \cap B^-_{\beta_2}) \le d(B) - p(\beta_1) - p(\beta_2)) = 1\,, \qquad d(B) \ge p(\beta_1) + p(\beta_2)\,;$$

$$P(B \cap B^+_{\beta_1} \cap B^-_{\beta_2} = \varnothing) = 1\,, \qquad d(B) < p(\beta_1) + p(\beta_2)\,.$$

Proof. We start with the proof of (3.8). For $i = 1, 2$ fix $p_i < p(\beta_i)$, and if $d(B) < p(\beta_1) + p(\beta_2)$ let the p_i also satisfy $p_1 + p_2 > d(B)$. By (2.6) we can find $\xi_i > \beta_i$ so that $p_i < p(\xi_i) < p(\beta_i)$ for $i = 1, 2$. We can also choose $\varepsilon > 0$ sufficiently small such that

$$1 - (1+\varepsilon)\xi_i^{-1} > 1 - \beta_i^{-1} \tag{3.9}$$

and

$$E[Z^{p_i}(\varepsilon, \xi_i)] < 1\,. \tag{3.10}$$

It follows from (3.3) and (3.4) that

$$B^+_{\beta_i} \subseteq \lim_{\eta \downarrow 0} B^+_{\varepsilon, \xi_i, \eta} \tag{3.11}$$

with the analogous result holding if $+$ is replaced by $-$.

Next, fix $\lambda > d(B)$ (if $d(B) < p(\beta_1) + p(\beta_2)$ let $\lambda = p_1 + p_2$). For n sufficiently large we can choose a cover $\{S_i\}$ of B such that

$$\rho(S_i) \le n^{-1}\ \forall i, \quad \text{and} \quad \sum_1^\infty \rho(S_i)^\lambda \le n^{-1}\,, \tag{3.12}$$

where $\rho(S) = \sup\{|x - y| : x, y \in S\}$ is the diameter of S.

Now consider the cover of $B \cap B^+_{\varepsilon,\xi_1,\eta} \cap B^-_{\varepsilon,\xi_2,\eta}$ given by $\{\widetilde{S}_i\}$, where

$$\widetilde{S}_i = \begin{cases} S_i, & \text{if } S_i \cap B^+_{\varepsilon,\xi_1,\eta} \cap B^-_{\varepsilon,\xi_2,\eta} \ne \varnothing, \\ \varnothing, & \text{otherwise}. \end{cases}$$

Note that

$$E\left[\sum_{i=1}^\infty \rho(\widetilde{S}_i)^{\lambda - p_1 - p_2}\right] = \sum_{i=1}^\infty \rho(S_i)^{\lambda - p_1 - p_2} P(S_i \cap B^+_{\varepsilon,\xi_1,\eta} \cap B^-_{\varepsilon,\xi_2,\eta} \ne \varnothing)\,.$$

Since $S_i \subseteq [v_i, v_i + s_i]$ for some v_i, where $s_i = \rho(S_i)$, we have for sufficiently large n

$$\begin{aligned} &P(S_i \cap B^+_{\varepsilon,\xi_1,\eta} \cap B^-_{\varepsilon,\xi_2,\eta} \ne \varnothing) \\ &\quad \le E[E[1(S_i \cap B^-_{\varepsilon,\xi_2,\eta} \ne \varnothing) 1(S_i \cap B^+_{\varepsilon,\xi_1,\eta} \ne \varnothing) \mid \mathscr{F}(v_i + s_i)]] \\ &\quad = E[1(S_i \cap B^-_{\varepsilon,\xi_2,\eta} \ne \varnothing) P(S_i \cap B^+_{\varepsilon,\xi_1,\eta} \ne \varnothing \mid \mathscr{F}(v_i + s_i))] \\ &\quad \le C\rho(S_i)^{p_1} P(S_i \cap B^-_{\varepsilon,\xi_2,\eta} \ne \varnothing) \\ &\quad \le C^2 \rho(S_i)^{p_1 + p_2} \end{aligned}$$

by Proposition 3.1 and the remarks following it. Hence by (3.12) and Fatou

$$\liminf_{n\to\infty}\sum_{i=1}^{\infty}\rho(\widetilde{S}_i)^{\lambda-p_1-p_2}=0 \quad \text{a.s.} \tag{3.13}$$

We now consider the cases $d(B)\geq p(\beta_1)+p(\beta_2)$ and $d(B)<p(\beta_1)+p(\beta_2)$ separately. In the former case, since $\lambda>d(B)$ was arbitrary, (3.13) implies that for any $\eta>0$

$$P(d(B\cap B^+_{\varepsilon,\xi_1,\eta}\cap B^-_{\varepsilon,\xi_2,\eta})\leq d(B)-p_1-p_2)=1\,.$$

By letting $\eta\downarrow 0$ and noting (3.11), this gives

$$P(d(B\cap B^+_{\beta_1}\cap B^-_{\beta_2})\leq d(B)-p_1-p_2)=1\,.$$

Since $p_i<p(\beta_i)$ are arbitrary, this proves the first part of (3.8). If $d(B)<p(\beta_1)+p(\beta_2)$, then since we chose $\lambda=p_1+p_2$, (3.13) implies for all $\eta>0$

$$P(B\cap B^+_{\varepsilon,\xi_1,\eta}\cap B^-_{\varepsilon,\xi_2,\eta}\neq\varnothing)=0\,.$$

Letting $\eta\downarrow 0$ completes the proof of (3.8).

The proof of (3.7) is similar, but easier. Fix $p_1<p(\beta_1)$, and if $d(B)<p(\beta_1)$ let p_1 also satisfy $p_1>d(B)$. Choose $\xi_1>\beta_1$ so that $p_1<p(\xi_1)<p(\beta_1)$, and choose $\varepsilon>0$ sufficiently small such that

$$1-(1+\varepsilon)\xi_1^{-1}>1-\beta_1^{-1} \tag{3.9$'$}$$

and

$$E[Z^{p_1}(\varepsilon,\xi_1)]<1\,. \tag{3.10$'$}$$

Next, fix $\lambda>d(B)$ (if $d(B)<p(\beta_1)$ let $\lambda=p_1$). For n sufficiently large we can choose a cover $\{S_i\}$ of B satisfying (3.12). Define $\{\widetilde{S}_i\}$ by

$$\widetilde{S}_i=\begin{cases} S_i, & \text{if } S_i\cap B^+_{\varepsilon,\xi_1,\eta}\neq\varnothing,\\ \varnothing, & \text{otherwise}.\end{cases}$$

By the remarks following Proposition 3.1 we have

$$P(S_i\cap B^+_{\varepsilon,\xi_1,\eta}\neq\varnothing)\leq C\rho(S_i)^{p_1}\,.$$

Thus

$$E\left[\sum_{i=1}^{\infty}\rho(\widetilde{S}_i)^{\lambda-p_1}\right]\to 0\,,$$

and so by Fatou

$$\liminf_{n\to\infty}\sum_{i=1}^{\infty}\rho(\widetilde{S}_i)^{\lambda-p_1}=0 \quad \text{a.s.} \tag{3.13$'$}$$

The remainder of the argument is as before. □

Remark. The estimate (3.5) can be used to give a proof (though not necessarily the most elementary one) of (1.6). Fix $t\geq 0$, and note that for any $\beta>1$ and

$\varepsilon \in (0, \beta - 1)$ we can find $p > 0$ such that $E[Z^p(\varepsilon, \beta)] < 1$. By Proposition 3.1, for any $\eta > 0$ there exists a finite constant C such that for all sufficiently small ζ

$$P([t, t+\zeta] \cap B^+_{\varepsilon,\beta,\eta} \neq \varnothing) \leq C\zeta^p.$$

This implies

$$P(t \in B^+_{\varepsilon,\beta,\eta}) = 0,$$

and consequently we have

$$P\left(t \in \bigcup_{\eta>0} B^+_{\varepsilon,\beta,\eta}\right) = 0.$$

This means that a.s. for every $\eta > 0$ there exists $h \in (0, \eta]$ with $\delta_t^+(h) \geq h(1-(1+\varepsilon)\beta^{-1})$. Thus

$$P\left(\limsup_{h\downarrow 0} \frac{\delta_t^+(h)}{h} \geq 1-(1+\varepsilon)\beta^{-1}\right) = 1.$$

Letting $\beta \uparrow \infty$ completes the proof of (1.6).

4. Lower Bounds on $d(B \cap B^+_\beta)$ and $d(B \cap B_\beta)$

For $\beta > 1$, $\varepsilon \in (0, \beta - 1)$ and $\lambda > 0$, define

$$\tau_0(\varepsilon) = \inf\{s \geq 1 : W(s) = W(0)\}, \quad X_0(\varepsilon, \beta, \lambda) = \tau_0(\varepsilon) 1(\tau_0(\varepsilon) \leq \beta), \tag{4.1}$$

(τ_0 and X_0 do not actually depend on ε and λ, but it simplifies the notation below to include these parameters in their definitions) and for $k \geq 1$

(4.2)

$$\begin{aligned}
&\tau_k(\varepsilon) = \inf\{s \geq (1+\varepsilon)\tau_{k-1}(\varepsilon) : W(s) = W(0)\},\\
&C_k(\varepsilon, \lambda) = \{\exists u, v \in [\tau_{k-1}(\varepsilon), (1+\varepsilon)\tau_{k-1}(\varepsilon)]\\
&\qquad\qquad \text{with } W(u) = W(0) + \lambda\tau_{k-1}^{1/2}(\varepsilon) \text{ and } W(v) = W(0) - \lambda\tau_{k-1}^{1/2}(\varepsilon)\},\\
&Y_k(\varepsilon, \beta, \lambda) = \frac{\tau_k(\varepsilon)}{\tau_{k-1}(\varepsilon)} 1\left(\left\{\frac{\tau_k(\varepsilon)}{\tau_{k-1}(\varepsilon)} \leq \beta\right\} \cap C_k(\varepsilon, \lambda)\right),\\
&X_k(\varepsilon, \beta, \lambda) = X_0(\varepsilon, \beta, \lambda) \prod_{m=1}^{k} Y_m(\varepsilon, \beta, \lambda),\\
&X^*(\varepsilon, \beta, \lambda) = \sup_{k\geq 0} X_k(\varepsilon, \beta, \lambda).
\end{aligned}$$

First observe that $\tau_k(\varepsilon) < \infty$ a.s. for all k. Next, by the scaling and strong Markov properties, $Y_k(\varepsilon, \beta, \lambda)$, $k \geq 1$ are i.i.d. with common distribution

$$\begin{aligned}
&P(Y_1(\varepsilon, \beta, \lambda) \in [1+\varepsilon, t])\\
&\quad = P(\exists u, v \in [1, 1+\varepsilon] \text{ with } W(u) = W(1) + \lambda \text{ and } W(v) = W(1) - \lambda,\\
&\qquad\qquad \text{and } \exists s \in [(1+\varepsilon), t] \text{ with } W(s) = W(1))
\end{aligned}$$

for $1+\varepsilon \le t \le \beta$, with all remaining mass at the origin. In particular, if $\Rightarrow$ denotes weak convergence, then

$$Y_1(\varepsilon, \beta, \lambda) \Rightarrow Z(\varepsilon, \beta) \quad \text{as } \lambda \downarrow 0.$$

Thus for all $\alpha \ge 0$

$$E[Y_1^\alpha(\varepsilon, \beta, \lambda)] \to E[Z^\alpha(\varepsilon, \beta)]. \tag{4.3}$$

Also by the scaling and strong Markov properties $X_0(\varepsilon, \beta, \lambda)$ is independent of $\{Y_k(\varepsilon, \beta, \lambda); k \ge 1\}$, and hence $X_k(\varepsilon, \beta, \lambda)$ is a product of independent random variables.

In what follows we will often consider two independent Brownian motions W_1 and W_2 defined on the same probability space Ω. We denote the corresponding random variables in (4.2) by $Y_k(i; \varepsilon, \beta, \lambda)$, $X_k(i; \varepsilon, \beta, \lambda)$, etc. for $i = 1, 2$. We will use the notation $a_n \approx b_n$ to mean a_n/b_n is bounded above and below by finite, nonzero constants which are independent of n.

Lemma 4.1. *Fix $\delta > 0$, $\beta_1 > 1$, $\beta_2 > 1$, and assume that $p(\beta_1) + p(\beta_2) < \delta$. Then for all $\varepsilon > 0$ sufficiently small there exists a $\lambda > 0$ and a $p \in (0, \delta)$ such that for all $b \ge \delta$,*

$$E[(X^*(1; \varepsilon, \beta_1, \lambda) \wedge X^*(2; \varepsilon, \beta_2, \lambda) \wedge n)^b] \approx n^{b-p},$$

where $a \wedge b = \min(a, b)$.

Proof. Let $q_1 > p(\beta_1)$ and $q_2 > p(\beta_2)$ satisfy $q_1 + q_2 < \delta$. Then by (2.6) and (4.3) for all $\varepsilon > 0$ sufficiently small we can find $\lambda > 0$ so that $E[Y_1^{q_i}(i; \varepsilon, \beta_i, \lambda)] > 1$ for $i = 1, 2$. So by reducing q_1 and q_2 we can find $p_1 > 0$ and $p_2 > 0$ satisfying $p_1 + p_2 < \delta$ and

$$E[Y_1^{p_i}(i; \varepsilon, \beta_i, \lambda)] = 1.$$

Thus, recalling that $X_k(i; \varepsilon, \beta_i, \lambda)$ is a product of independent random variables, it follows that $X_k^{p_i}(i; \varepsilon, \beta_i, \lambda)$ is a nonnegative martingale. Hence $X_k^{p_i}(i; \varepsilon, \beta_i, \lambda)$ converges a.s., necessarily to zero, as $k \to \infty$. Next, for $r > 1$ set

$$T_r(i) = \inf\{k : X_k^{p_i}(i; \varepsilon, \beta_i, \lambda) > r\}.$$

Since

$$X_{T_r(i)\wedge k}^{p_i}(i; \varepsilon, \beta_i, \lambda) \le \beta_i r,$$

dominated convergence implies

$$\begin{aligned} E[X_0^{p_i}(i; \varepsilon, \beta_i, \lambda)] &= E[X_{T_r(i)\wedge k}^{p_i}(i; \varepsilon, \beta_i, \lambda)] \\ &\to E[X_{T_r(i)}^{p_i}(i; \varepsilon; \beta_i, \lambda); T_r(i) < \infty] \end{aligned}$$

as $k \to \infty$. Observe that on $\{T_r(i) < \infty\} = \{X^*(i; \varepsilon, \beta_i, \lambda)^{p_i} > r\}$ we have

$$r \le X_{T_r(i)}^{p_i}(i; \varepsilon, \beta_i, \lambda) \le \beta_i r.$$

Thus

$$\mu_i(\beta_i r)^{-1} \le P(X^*(i;\varepsilon,\beta_i,\lambda)^{p_i} > r) \le \mu_i r^{-1}$$

where

$$\mu_i = E[X_0^{p_i}(i;\varepsilon,\beta_i,\lambda)] > 0.$$

Now let $p = p_1 + p_2$, and assume that $b \ge \delta$. Then since $p < \delta$

$$\begin{aligned} &E[(X^*(1;\varepsilon,\beta_1,\lambda) \wedge X^*(2;\varepsilon,\beta_2,\lambda) \wedge n)^b] \\ &\quad = \int_0^n b r^{b-1} P(X^*(1;\varepsilon,\beta_1,\lambda) > r) P(X^*(2;\varepsilon,\beta_2,\lambda) > r)\, dr \\ &\quad \approx n^{b-p}. \quad \square \end{aligned}$$

We will need to consider slightly smaller sets than B_β^+ and B_β^- defined as follows: for $a \ge 1$ let

$$\begin{aligned} {}^a\Lambda_t^+(h) &= \{(s,x) : t \le s \le t+h,\ f(t) - ah \le x \le f(t) + ah\}, \\ {}^a\delta_t^+(h) &= \sup\{|\Lambda|^{1/2} : \Lambda \subseteq {}^a\Lambda_t^+(h),\ \Lambda \cap G = \varnothing\}, \\ {}^aB_\beta^+ &= \left\{ t \in [0,1] : \limsup_{h \downarrow 0} \frac{{}^a\delta_t^+(h)}{h} \le 1 - \beta^{-1} \right\}, \end{aligned}$$

with analogous definitions if $+$ is replaced by $-$. Observe that since $a \ge 1$, ${}^a\delta_t^+(h) \ge \delta_t^+(h)$ and ${}^a\delta_t^-(h) \ge \delta_t^-(h)$, hence

$$(4.4) \qquad {}^aB_\beta^+ \subseteq B_\beta^+ \quad \text{and} \quad {}^aB_\beta^- \subseteq B_\beta^-.$$

For $i = 1, 2$, let ${}^a\Lambda_t^+(i;h)$, ${}^a\delta_t^+(i;h)$, ${}^aB_\beta^+(i)$ and $B_\beta^+(i)$ be the quantities analogous to ${}^a\Lambda_t^+(h)$, ${}^a\delta_t^+(h)$, ${}^aB_\beta^+$ and B_β^+ defined relative to W_i, and similarly with $+$ replaced by $-$. Define

$$\mathscr{F}_i(t) = \sigma\{W_i(s) : 0 \le s \le t\}, \quad \mathscr{G}(t) = \sigma\{\mathscr{F}_1(t) \cup \mathscr{F}_2(t)\}.$$

Given a subset Γ of $[0,1] \times \Omega$, let $\Gamma(\omega) = \{t \in [0,1] : (t,\omega) \in \Gamma\}$. In what follows we will often write Γ for $\Gamma(\omega)$ when the meaning is clear. The following proposition is based on Proposition 14 in [10].

Proposition 4.1. *Let Γ be a $\mathscr{G}(t)$ progressively measurable subset of $[0,1] \times \Omega$ such that Γ is closed a.s. Then for any $a \ge 1$, and $\xi_1, \xi_2 > 1$*

$$P(\{d(\Gamma) > p(\xi_1) + p(\xi_2)\} \backslash \{\Gamma \cap {}^aB_{\xi_1}^+(1) \cap {}^aB_{\xi_2}^+(2) \ne \varnothing\}) = 0.$$

Proof. Since the proof of this result is rather lengthy, we divide it into six steps. In the first two steps certain parameters are fixed and various sequences of random variables are defined. Steps 3 and 4 are concerned with some purely geometric arguments used to describe certain events in terms of these random variables. In Step 5 the problem is reduced to equation (4.15), which is then verified in Step 6.

Step 1. We may assume $p(\xi_1) + p(\xi_2) < 1$, else the result is trivial. Fix $\delta \in (p(\xi_1) + p(\xi_2), 1)$ and let $\beta_i \in (1, \xi_i)$ for $i = 1, 2$ satisfy

$$p(\beta_1) + p(\beta_2) < \delta. \tag{4.5}$$

Choose $\varepsilon > 0$, $\lambda > 0$ and $p \in (0, \delta)$, so that the conclusion of Lemma 4.1 holds, and for $i = 1, 2$

$$(1 + \varepsilon)\beta_i \le \xi_i. \tag{4.6}$$

Let $\beta = \beta_1 \vee \beta_2$ and $\beta' = \beta_1 \wedge \beta_2$, and choose M so that

$$M \ge \frac{a\beta^2(1+\varepsilon)^2}{\beta'(1+\varepsilon) - 1}. \tag{4.7}$$

Finally, let

$$\Delta \le \beta\lambda^2 M^{-2}. \tag{4.8}$$

The parameters δ, β_1, β_2, p, ε, λ, M and Δ have now been fixed.

Step 2. For $n \ge 1$ (an integer) and $i = 1, 2$ define

$$\tau_0(i) = \inf\{s \ge n^{-1} : W_i(s) = W_i(0)\}, \quad X_0(i) = \tau_0(i) 1(\tau_0(i) \le \beta_i),$$

and for $k \ge 1$

$$\begin{aligned}
\tau_k(i) &= \inf\{s \ge (1+\varepsilon)\tau_{k-1}(i) : W_i(s) = W_i(0)\}, \\
C_k(i) &= \{\exists u, v \in [\tau_{k-1}(i), (1+\varepsilon)\tau_{k-1}(i)] \\
&\qquad \text{with } W_i(u) = W_i(0) + \lambda\tau_{k-1}^{1/2}(i) \text{ and } W_i(v) = W_i(0) - \lambda\tau_{k-1}^{1/2}(i)\}, \\
Y_k(i) &= \frac{\tau_k(i)}{\tau_{k-1}(i)} 1\left(\left\{\frac{\tau_k(i)}{\tau_{k-1}(i)} \le \beta_i\right\} \cap C_k(i)\right), \\
X_k(i) &= X_0(i) \prod_{m=1}^{k} Y_m(i), \quad X^*(i) = \sup_{k \ge 0} X_k(i), \quad X^* = X^*(1) \wedge X^*(2).
\end{aligned}$$

Then by scaling

$$nX^*(i) \stackrel{d}{=} X^*(i; \varepsilon, \beta_i, \lambda), \quad nX^* \stackrel{d}{=} X^*(1; \varepsilon, \beta_1, \lambda) \wedge X^*(2; \varepsilon, \beta_2, \lambda). \tag{4.9}$$

Step 3. For any $\eta \le \Delta$

$$\begin{aligned}
\{\beta X^* > \eta\} \subseteq \{\forall h \in [\beta n^{-1}, \beta^{-2}\eta] \exists u(i), v(i) \in [h, h(1+\varepsilon)\beta_i] \\
\text{with } W_i(u(i)) = W_i(0) + Mh \text{ and} \\
W_i(v(i)) = W_i(0) - Mh, \ i = 1, 2\}.
\end{aligned} \tag{4.10}$$

To see this, suppose $\beta X^* > \eta$, and $h \in [\beta n^{-1}, \beta^{-2}\eta]$. Then for $i = 1, 2$ we can find a $k = k(i)$ such that $X_k(i) > \beta h$. But this means that for some $m < k$,

$$h \le \tau_m(i) \le \beta h \tag{4.11}$$

and $C_{m+1}(i)$ occurs. In particular there exist

$$u(i), v(i) \in [\tau_m(i), \tau_m(i)(1+\varepsilon)] \subseteq [h, h(1+\varepsilon)\beta]$$

such that

$$W_i(u(i)) = W(0) + \lambda\tau_m(i)^{1/2},$$
$$W_i(v(i)) = W(0) - \lambda\tau_m(i)^{1/2}.$$

But by (4.8) and (4.11), since $h \le \beta^{-2}\eta \le \beta^{-2}\Delta$

$$\lambda\tau_m(i)^{1/2} \ge \lambda h^{1/2} \ge M(\Delta\beta^{-1})^{1/2}h^{1/2} \ge Mh\beta^{1/2} \ge Mh$$

since $\beta > 1$. Thus (4.10) holds. The reason for using βX^* rather than X^* on the left-side of (4.10) is that clearly $\{X^* \le u\} \in \mathcal{G}(\beta u)$, and so βX^* is a stopping time.

Step 4. Let

$$D_{n,\eta} = \{t \in \Gamma : \forall h \in [\beta n^{-1}, \beta^{-2}\eta] \exists u(i),\ v(i) \in [h, h(1+\varepsilon)\beta_i],$$
$$\text{with } W_i(t+u(i)) = W_i(t) + Mh$$
$$\text{and } W_i(t+v(i)) = W_i(t) - Mh,\ i = 1, 2\},$$

and let Θ_t be the usual shift operator. Then for every ω

$$\{t \in \Gamma : \beta X^* \circ \Theta_t > \eta\} \subseteq D_{n,\eta}, \tag{4.12}$$

$$\bigcup_{\eta>0} \bigcap_{n=1}^{\infty} D_{n,\eta} \subseteq \Gamma \cap {}^aB^+_{\xi_1}(1) \cap {}^aB^+_{\xi_2}(2). \tag{4.13}$$

Assertion (4.12) follows immediately from (4.10). To see (4.13), fix $\eta > 0$ and suppose that $t \in \bigcap_{n=1}^{\infty} D_{n,\eta}$. Let $h \in (0, \beta^{-2}\eta]$, and assume that for either $i = 1$ or 2

$$\Lambda \subseteq {}^a\Lambda^+_t(i; h)$$

where

$$\Lambda = (t, W_i(t)) + ([s_1, s_2] \times [x_1, x_2])$$

and

$$s_2 - s_1 = x_2 - x_1 = h(1 - ((1+\varepsilon)\beta_i)^{-1}).$$

We will show that the graph of W_i intersects Λ, and hence

$$\frac{{}^a\delta^+_t(i; h)}{h} \le 1 - \frac{1}{(1+\varepsilon)\beta_i} \le 1 - \frac{1}{\xi_i},$$

which implies $t \in \Gamma \cap {}^aB^+_{\xi_1}(1) \cap {}^aB^+_{\xi_2}(2)$. To do this, observe that

$$s_2 \ge \left(1 - \frac{1}{(1+\varepsilon)\beta_i}\right)h, \quad s_1 \le \frac{1}{(1+\varepsilon)\beta_i}h,$$

and

$$\frac{1}{(1+\varepsilon)\beta_i}s_2 \ge s_1.$$

This last fact follows from

$$\frac{s_2}{(1+\varepsilon)\beta_i} - s_1 = \left(1 - \frac{1}{(1+\varepsilon)\beta_i}\right)\left(\frac{h}{(1+\varepsilon)\beta_i} - s_1\right) \geq 0.$$

Now let $h' = ((1+\varepsilon)\beta_i)^{-1}s_2$ and note that $s_1 \leq h' < s_2 \leq h$. Since $t \in \bigcap_{n=1}^{\infty} D_{n,\eta}$ there must exist $u(i)$, $v(i) \in [h', h'(1+\varepsilon)\beta_i] \subseteq [s_1, s_2]$ with

$$W_i(t+u(i)) = W_i(t) + Mh', \qquad W_i(t+v(i)) = W_i(t) - Mh'.$$

By the definition of M and the previous inequalities we see that $Mh' \geq ah$, and thus path continuity implies the graph of W_i must intersect Λ. This establishes (4.13).

Step 5. Let

$$H_\eta = \inf\left\{\sum_{i=1}^{k} \rho(S_i)^\delta : \Gamma \subseteq \bigcup_{i=1}^{k} S_i,\ S_i \text{ are closed intervals with } \rho(S_i) \leq \eta,\ k \geq 1\right\},$$

and

$$E = \{H_\eta \to \infty \text{ as } \eta \downarrow 0\}.$$

Thus

$$E \supseteq \{d(\Gamma) > \delta\}. \tag{4.14}$$

Finally let

$$G_{n,\eta} = \{D_{n,\eta} \neq \varnothing\}.$$

In the next step we will show that

$$P\left(E \setminus \bigcup_{\eta>0} \bigcap_n G_{n,\eta}\right) = 0. \tag{4.15}$$

Now clearly $\bigcup_{\eta>0} \bigcap_n G_{n,\eta} \supseteq \{\bigcup_{\eta>0} \bigcap_n D_{n,\eta} \neq \varnothing\}$, and since a.s. $D_{n,\eta}$ is compact and $D_{n+1,\eta} \subseteq D_{n,\eta}$

$$P\left(\bigcup_{\eta>0} \bigcap_n G_{n,\eta} \setminus \left\{\bigcup_{\eta>0} \bigcap_n D_{n,\eta} \neq \varnothing\right\}\right) = 0.$$

Thus by (4.13), (4.14) and (4.15)

$$P(\{d(\Gamma) > \delta\} \setminus \{\Gamma \cap {}^a B^+_{\xi_1}(1) \cap {}^a B^+_{\xi_2}(2) \neq \varnothing\}) = 0.$$

Letting $\delta \downarrow p(\xi_1) + p(\xi_2)$ then completes the proof of the proposition. Thus it remains to prove (4.15).

Step 6. Define

$$\zeta_1 = \inf\{t \geq 0 : t \in \Gamma\} \wedge 2, \quad \sigma_1 = \zeta_1 + ((\beta X^* \circ \Theta_{\zeta_1}) \wedge 1),$$

and for $k \geq 2$

$$\zeta_k = \inf\{t \geq \sigma_{k-1} : t \in \Gamma\} \wedge 2, \quad \sigma_k = \zeta_k + ((\beta X^* \circ \Theta_{\zeta_k}) \wedge 1)$$

where $\inf(\varnothing) = \infty$. Then ζ_k and σ_k are stopping times since Γ is progressively measurable and βX^* is a stopping time. Observe also that if $\zeta_k \leq 1$, then $\zeta_k \in \Gamma$ a.s. since Γ is closed a.s. Let

$$N_1 = \sup\{k : \zeta_k \leq 1\},$$

where $\sup(\varnothing) = 0$. Then

$$\Gamma \subseteq \bigcup_{k=1}^{N_1} [\zeta_k, \sigma_k] \quad \text{a.s.}$$

Thus by (4.12)

$$\begin{aligned} P(E \cap G_{n,\eta}^c) &\leq P\left(E, \max_{1 \leq k \leq N_1} (\sigma_k - \zeta_k) \leq \eta\right) \\ &\leq P\left(E, \sum_{k=1}^{N_1} (\sigma_k - \zeta_k)^\delta \geq H_\eta\right) \\ &\leq P(E, H_\eta \leq x) + P\left(\sum_{k=1}^{N_1} (\sigma_k - \zeta_k)^\delta \geq x\right) \end{aligned}$$

for any $x > 0$.

We now estimate the last term above. Since $(\sigma_k - \zeta_k) =^d \beta X^* \wedge 1$, and ζ_k is independent of $\sigma_k - \zeta_k$, we have for any $b > 0$

$$\begin{aligned} E\left[\sum_{k=1}^{N_1} (\sigma_k - \zeta_k)^b\right] &= E\left[\sum_{k=1}^{\infty} 1(\zeta_k \leq 1)(\sigma_k - \zeta_k)^b\right] \\ &= \sum_{k=1}^{\infty} P(\zeta_k \leq 1) E[(\sigma_k - \zeta_k)^b] = E[N_1] E[(\beta X^* \wedge 1)^b]. \end{aligned}$$

With $b = \delta$ this implies

$$E\left[\sum_{k=1}^{N_1} (\sigma_k - \zeta_k)^\delta\right] = E[N_1] E[(\beta X^* \wedge 1)^\delta].$$

With $b = 1$ and the fact that $2 \geq \sum_{k=1}^{N_1} (\sigma_k - \zeta_k)$ we get

$$E[N_1] \leq \frac{2}{E[\beta X^* \wedge 1]}.$$

Thus

$$E\left[\sum_{k=1}^{N_1} (\sigma_k - \zeta_k)^\delta\right] \leq \frac{2E[(\beta X^* \wedge 1)^\delta]}{E[\beta X^* \wedge 1]}.$$

Now by (4.5), (4.9), and Lemma 4.1, there exists $p \in (0, \delta)$ such that for $b \geq \delta$,

$$E[(\beta X^* \wedge 1)^b] = n^{-b} E[(X^*(1; \varepsilon, \beta_1, \lambda) \wedge X^*(2; \varepsilon, \beta_2, \lambda) \wedge n)^b] \approx n^{-p}.$$

This implies that there exists a finite constant c independent of n such that

$$E\left[\sum_{k=1}^{N_1} (\sigma_k - \zeta_k)^\delta\right] \leq c,$$

and thus by Chebyshev

$$P(E \cap G^c_{n,\eta}) \leq P(E, H_\eta \leq x) + c/x.$$

For any $r > 0$ we can choose x sufficiently large and η_0 sufficiently small so that

$$P(E \cap G^c_{n,\eta_0}) < r$$

for all n. Now let $G_\eta = \bigcap_n G_{n,\eta}$. Since $G_{n+1,\eta} \subseteq G_{n,\eta}$ we have

$$P(E \cap G^c_{\eta_0}) < r.$$

Equation (4.15) now follows from the fact that $G_\eta \uparrow$ as $\eta \downarrow$. $\square$

Remark. Let $C \subseteq [0, 1]$ be closed. By taking $\Gamma = C \times \Omega$ in Proposition 4.1 we have for every $a \geq 1$

$$P(C \cap {}^a B^+_{\xi_1}(1) \cap {}^a B^+_{\xi_2}(2) \neq \varnothing) = 1 \tag{4.16}$$

provided $p(\xi_1) + p(\xi_2) < d(C)$. This will be used in the next result.

Theorem 4.1. *For every analytic set* $B \subseteq [0, 1]$ *and every* $a \geq 1$

$$\begin{aligned} P(d(B \cap {}^a B^+_\beta) \geq d(B) - p(\beta)) &= 1, \\ P(d(B \cap {}^a B^-_\beta) \geq d(B) - p(\beta)) &= 1. \end{aligned} \tag{4.17}$$

Proof. It suffices to prove the ${}^a B^+_\beta$ case since we obtain the ${}^a B^-_\beta$ case by considering $\widetilde{W}(t) = W(1) - W(1-t)$. We may assume that $p(\beta) < d(B)$, else there is nothing to prove. Since B is analytic, for any $\varepsilon > 0$ there exists a compact set $C \subseteq B$ such that $d(C) > d(B) - \varepsilon$. Assume that ε is sufficiently small that $d(C) > p(\beta)$. Then for any β_1 satisfying $p(\beta_1) < d(C) - p(\beta)$ we have by the remark above

$$P(C \cap {}^a B^+_{\beta_1}(1) \cap {}^a B^+_\beta(2) \neq \varnothing) = 1.$$

Thus

$$P(C \cap {}^a B^+_{\beta_1}(1) \cap {}^a B^+_\beta(2) \neq \varnothing \mid W_2) = 1 \quad \text{a.s.} \tag{4.18}$$

Since for a given W_2, the random set $C \cap {}^a B^+_\beta(2)$ may be considered fixed under $P(\cdot \mid W_2)$, and ${}^a B^+_\beta(1) \subseteq B^+_\beta(1)$, we can apply (3.7) with $B = C \cap {}^a B^+_\beta(2)$ to obtain

$$P(d(C \cap {}^a B^+_\beta(2)) \geq p(\beta_1) \mid W_2) = 1 \quad \text{a.s.}$$

Thus

$$P(d(C \cap {}^a B_\beta^+(2)) \geq p(\beta_1)) = 1$$

and hence

$$P(d(B \cap {}^a B_\beta^+(2)) \geq p(\beta_1)) = 1 .$$

Since p is continuous, β_1 is arbitrary subject to the requirement that $p(\beta_1) < d(C) - p(\beta)$, and $\varepsilon > 0$ is arbitrary, we are done. □

Setting $a = 1$ in (4.17) together with (3.7) then gives Theorem 2.1. Next we turn to Theorem 2.2.

Theorem 4.2. *For every analytic set* $B \subseteq [0, 1]$ *and every* $a \geq 1$

$$P(d(B \cap {}^a B_{\beta_1}^+ \cap {}^a B_{\beta_2}^-) \geq d(B) - p(\beta_1) - p(\beta_2)) = 1 .$$

Proof. We may assume $p(\beta_1) + p(\beta_2) < d(B)$. Fix $\varepsilon > 0$ sufficiently small that $p(\beta_1) + p(\beta_2) < d(B) - \varepsilon$. Choose $C \subseteq B$ compact such that $d(C) > d(B) - \varepsilon$, and let β_3 satisfy

$$p(\beta_1) + p(\beta_2) + p(\beta_3) < d(C) . \tag{4.19}$$

Arguing as in Theorem 4.1, it suffices to show

$$P(C \cap {}^a B_{\beta_1}^+(1) \cap {}^a B_{\beta_2}^-(1) \cap {}^a B_{\beta_3}^+(2) \neq \varnothing \mid W_1) = 1 \quad \text{a.s.} \tag{4.20}$$

(let $B = C \cap {}^a B_{\beta_1}^+(1) \cap {}^a B_{\beta_2}^-(1)$ in (3.7), etc.). To prove this we will show that if η is sufficiently small, then

$$P(C \cap {}^a B_{\beta_1}^+(1) \cap {}^a B_{\beta_2,\eta}^-(1) \cap {}^a B_{\beta_3}^+(2) \neq \varnothing) = 1 \tag{4.21}$$

where

$${}^a B_{\beta_2,\eta}^-(1) = \left\{ t \in [0, 1] : \frac{{}^a\delta_t^-(1; h)}{h} \leq (1 - \beta_2^{-1}) \text{ for all } h \in (0, \eta] \right\} .$$

Since ${}^a B_{\beta_2,\eta}^-(1) \subseteq {}^a B_{\beta_2}^-(1)$, (4.20) then follows.

In order to prove (4.21) all we need to do is show that

$$P(d(C \cap {}^a B_{\beta_2,\eta}^-(1)) > p(\beta_1) + p(\beta_3)) = 1 \tag{4.22}$$

and apply Proposition 4.1 (Γ defined by $\Gamma(\omega) = C \cap {}^a B_{\beta_2,\eta}^-(1)$ is a.s. closed and progressively measurable). To do this choose $\beta < \beta_2$ such that $p(\beta_1) + p(\beta) + p(\beta_3) < d(C)$. By Theorem 4.1, since $d(C) - p(\beta) > p(\beta_1) + p(\beta_3)$,

$$P(d(C \cap {}^a B_\beta^-(1)) > p(\beta_1) + p(\beta_3)) = 1 .$$

But $\lim_{\eta \downarrow 0} {}^a B_{\beta_2,\eta}^-(1) \supseteq {}^a B_\beta^-(1)$, and hence

$$\lim_{\eta \downarrow 0} d(C \cap {}^a B_{\beta_2,\eta}^-(1)) \geq d(C \cap {}^a B_\beta^-(1)) > p(\beta_1) + p(\beta_3) \quad \text{a.s.}$$

Thus (4.22) holds. □

Combining the results in Theorem 3.1 and Theorem 4.2 with (4.4) we see that for every analytic set $B \subseteq [0, 1]$ and every $a \geq 1$, $B \cap {}^a B_\beta^+ \cap {}^a B_\beta^- = \varnothing$ a.s. if $2p(\beta) > d(B)$, while if $2p(\beta) \leq d(B)$ then

$$P(d(B \cap {}^a B_\beta^+ \cap {}^a B_\beta^-) = d(B) - 2p(\beta)) = 1 . \tag{4.23}$$

Thus to complete the proof of Theorem 2.2 it suffices to prove

Proposition 4.2. *If* $a \geq 2\beta(\beta-1)^{-1}$ *then for every* ω

$$ {}^{a}B_{\beta}^{+} \cap {}^{a}B_{\beta}^{-} \subseteq B_{\beta} \subseteq B_{\beta}^{+} \cap B_{\beta}^{-} . $$

Proof. Clearly $B_{\beta} \subseteq B_{\beta}^{+} \cap B_{\beta}^{-}$. To prove the other inclusion we argue by contradiction. Thus assume that $t \in ({}^{a}B_{\beta}^{+} \cap {}^{a}B_{\beta}^{-}) \backslash B_{\beta}$. Since $a \geq 1$, $t \in (B_{\beta}^{+} \cap B_{\beta}^{-}) \backslash B_{\beta}$. Hence there exists a sequence $h_n \downarrow 0$ and a sequence of squares Λ_n such that

$$ \Lambda_n \subseteq \Lambda_t(h_n) \backslash G, $$
$$ \Lambda_n \cap \Lambda_t^{+}(h_n) \neq \varnothing, \qquad \Lambda_n \cap \Lambda_t^{-}(h_n) \neq \varnothing, $$
$$ |\Lambda_n|^{1/2} h_n^{-1} \geq \nu > 1 - \beta^{-1} . $$

By taking a further subsequence if necessary we may assume that at least half of Λ_n intersects $\Lambda_t^{+}(h_n)$, i.e.

$$ |\Lambda_n \cap \Lambda_t^{+}(h_n)|^{1/2} h_n^{-1} \geq \nu/\sqrt{2}, \tag{4.24} $$

(if not, argue using $\Lambda_t^{-}(h_n)$ in place of $\Lambda_t^{+}(h_n)$ in what follows). But then by the definition of a,

$$ {}^{a}\delta_t^{+}\left(\frac{\nu h_n}{2}\right) \geq \frac{\nu h_n}{2}, $$

which contradicts $t \in {}^{a}B_{\beta}^{+}$. □

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